Logical Localism in the Context of Combining Logics

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Tesi doctoral

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Abstract

Logical localism is a claim in the philosophy of logic stating that different logics are correct in different domains. There are different ways in which this thesis can be motivated and I will explore the most important ones. However, localism has an obvious and major challenge which is known as 'the problem of mixed inferences'. The main goal of this dissertation is to solve this challenge and to extend the solution to the related problem of mixed compounds for alethic pluralism. My approach in order to offer a solution is one that has not been considered in the literature as far as I am aware. I will study different methods for combining logics, concentrating on the method of juxtaposition, by Joshua Schechter, and I will try to solve the problem of mixed inferences by making a finer translation of the arguments and using combination mechanisms as the criterion of validity. One of the most intriguing aspects of the dissertation is the synergy that is created between the philosophical debate and the technical methods with the problem of mixed inferences at the center of that synergy. I hope to show that not only the philosophical debate benefits from the methods for combining logics, but also that these methods can be developed in new and interesting ways motivated by the philosophical problem of mixed inferences. The problem suggests that there are relevant interactions between connectives, justified by the philosophical considerations for conceptualising different logic systems, that the methods for combining logics should allow to emerge. The recognition of this fact is what drives the improvements on the method of juxtaposition that I develop. That is, in order to allow for the emergence of desirable interaction principles, I will propose alternative ways of combining logic systems -specifically classical and intuitionistic logics- that go beyond the standard for combinations, which is based on minimality conditions so as to avoid the so-called collapse theorems.

Acknowledgements

I am quite attracted to the idea that all intelligence is collective intelligence, in the sense that there is no such thing as an indivisible unit of intelligence that we can pinpoint. So, although the neurons of my brain have been the ones struggling to fire in the right paths in order to ultimately produce this dissertation, other neurons from other brains have certainly been of an invaluable help for the correct firings to occur. I am, therefore, thankful to everyone who, in some way or another, interacted with me.

Among those interactions, the first person that showed me a true passion for logic and philosophy was Jesus Mari Larrazabal. He was my logic teacher and my bachelor's thesis supervisor, but, most of all, he was a mentor and a friend. I cannot express how indebted I am to him. I just would like to leave, to whom it may concern, a glimpse of his mind: 'Philosophy is an intellectual activity that will often lead you to loneliness, but this loneliness is necessary if you want to think carefully, freely, and not be affected by all the mess. However, you must know that you are not alone and that other people also deeply care about the same things as you do. Ultimately, the owl of Minerva always spreads its wings with the falling of the dusk to bring order to chaos'. Hopefully, this and other future works will be a worthy testament of his intellectual honesty and passion.

A clear proof of the influence that Jesus Mari had on me and my career is that he introduced me to my supervisors, José Martínez and Elia Zardini. I met them in a workshop that Jesus Mari was organizing at Donostia. It was my last year as an undergraduate and I was in the process of deciding what to do next. Since Jesus Mari knew them both and highly appreciated and respected them, he encouraged me to go and talk to them and seek their advice. I ended up studying the master in Analytic Philosophy, where Pepe taught me the course on Philosophical Logic and supervised my Master's Thesis, with an eye already on a PhD. He introduced me to every topic that

this dissertation is concerned with and has guided me through every thread to get at where I am, while giving me enough freedom. I am deeply thankful for all this. Also, for the time, effort and rigour that he has dedicated to improving this dissertation and for giving me the confidence on myself and my work when I most needed it. I feel lucky to have had the opportunity of working together with someone whom I have admired since he was my teacher.

Elia became my co-supervisor right on time, but I certainly wish he would have joined us before. Since the first talk that I witnessed at Donostia, I have thought that he is one of the sharpest and most brilliant philosophers that I know. Very few other times I have met someone who is able to offer you some helpful and deep insights in almost any topic, from cuisine to philosophical logic. When Pepe informed me about the possibility of having Elia as a supervisor I was really excited, but also a little bit intimidated. In the end, he has been an indispensable part of the last period of my work, when most of the coolest ideas have originated. Not only has he given me outstanding insights to improve my research, but also has had the most kind and supportive words to inspire me at the lowest points of the process. Coming from him, they meant the world to me.

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During my PhD I completed a research stay at the Buenos Aires Logic Group. Given the situation with COVID-19, it was quite difficult to make the stay possible and I had to postpone it almost two years. But, finally, at the end of February 2022 I could travel to Argentina. There were many reasons for choosing this particular research group. First of all, they are an outstanding team, formed by very talented and brilliant researchers who are able to systematically publish in the top level international journals like a well-oiled machine. Second, I was lucky enough to know some of the members of the group before my stay. I met Damian Szmuc at the ‘PhDs in Logic’ celebrated in Prague and had him afterwards in the tiny guest room of my flat at Barcelona (sorry for that). I also knew Lucas Rosenblatt, who was at Logos when I joined and left shortly after to return home, and Eduardo Barrio, whom I had met in a workshop that Pepe organized. Damian and Eduardo were very supportive and helpful with all the paperwork previous to the stay and, most importantly, keeping the hope that I could finally travel. I am deeply grateful for that. Lastly, let me be completely honest, I wanted to travel to Argentina to climb and trek at Patagonia, which I definitely did.

Those months with the Buenos Aires Logic Group were a gift and completely surpassed my expectations. I want to thank all the members of the group for making me feel at home and for all the great discussions which were a true inspiration for my work. Special thanks go to Eduardo, Damian, Lucas, Natalia Buacar and Paula Teijeiro. I cannot thank Mariela Rubin (Maru) enough for everything she did for me. You are my favourite person in the American continent.

I have had the chance of presenting parts of my dissertation in various venues. I would like to thank the people that have attended and helped me with questions or comments. Among all, let me mention Pablo Cobreros, who might not know that he has always given me wise advise whenever we have encountered and Antonio Yuste-Ginel who has become a friend with whom I can share philosophy, logic, climbing and beers.

Of course, there is life outside the academia too and most of the people that are essential to me and to what I am are outsiders. I am deeply grateful to many of my friends: Adur, Raúl, Fynn, Álvaro, Andrea, Gonzalo, Laura and Guillermo. Also to the friends of my partner who have welcomed me and become friends of mine too: Alejandro, María, Mario, Sandra, Mauro, Laura and Sara. And to the ones who know me since I was a 5 years old kid: Iñaki Arbide, Julen Garmendia and Julen Arruti. The other day I realized that you have never asked me what my thesis was about, and I am truly grateful for that. You are in many of my dearest and happiest memories.

I cannot forget to thank my family too: Julio, Carmen, Jorge and Maria, thanks for all the warmth and support. I know that my grandfather, Eugenio Monsalvo, will also be proud and shed some tears when reading his name, despite not knowing what to say when asked what his grandson does for a living. I am lucky to have a big family and there are many other members of it to whom I feel deeply grateful. If you are reading this, I hope you know who you are. And, finally, my parents, Alegría Monsalvo and Jerónimo Benito, and my brother/friend, Alberto Benito. You have given me each one of the virtues that I might have and the only thing that you have asked me in return is to live the life that is meaningful to me. That is the most generous and beautiful action that one can expect. I can only hope to give you back the love that I feel from you.

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A quienes me han hecho

Contents

Chapter 1

Introduction

There seems to be a strong intuition in favour of the idea that reasoning is universal. I, certainly, would not want to be the case that the validity of the arguments in this dissertation depended on whether it was read in Spain or in Nepal, by a Buddhist or an atheist, or even by a Hegelian. This is a sense of universality in which the standard of correct reasoning is independent of the peculiarities of the human being who is reasoning[1].

There is another sense of universality, though, that does not have to do with the characteristics of the subject who is reasoning, but with the subject-matter, or the object, of that reasoning. This is a sense of universality in which the rules of reasoning are topic-neutral, i.e., independent of the subject-matter about which one is reasoning. Contrary to the previous situation, I would not be bothered if it happened to be the case that people used different rules for different domains or topics occurring in the dissertation. If, for instance, someone came across something like ‘everything written in this page is false’ and later on had to reason about a moral dilemma, a property of the real numbers, the property of ‘being bald’ or about whether Hitler would have been racist had he been admitted to art school, I would not mind if that person -in fact, any person- used different rules of reasoning for those various scenarios.

One might draw an analogy between that situation and science. While we expect and work under the assumption that the laws of science are universal, in the sense that anyone and anywhere can verify and apply them, we do not

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[1] I leave aside the question of whether it is independent also of any being who is reasoning, i.e., of whether any AI or intelligent non-human life will abide by the same rules of reasoning.

assume (except, maybe, hardcore reductionists) that the set of scientific laws is universal in the sense of being the same for physics, chemistry or biology. There are properties of those domains that ask for different tools, laws and analyses.

Now, logic is often conceived as a theory of what follows from what; of correct reasoning (or, at least, deductive reasoning). So, in the same way that the heterogeneity of physical, chemical and biological phenomena asks for different sciences, one might argue that the diversity of rules of reasoning asks for different logic systems. This is, very roughly, the thesis that inspires the development of my dissertation.

There is a straightforward challenge for anyone who wants to justify and argue for the validity of such a thesis, though. Imagine that it is the case that different parts of my dissertation require different sets of rules of reasoning and that, therefore, in order to systematically account for what conclusions follow from what premisses, we formalize those parts employing different logics. I, certainly, would not like if there was no interaction between those domains and if the dissertation resulted in separate, disconnected niches of reasoning, each one with its own conclusions, but totally unrelated with each other.

The fact that would justify the awkwardness of such disconnection is that we do reason across domains. That is, even if one has the intuition that some rules of reasoning might hold in some situations but not others, it is undeniable that we reason about a variety of things at the same time. Assume, for instance, that the logic that better captures reasoning within the ethical or moral domain is $\mathcal{L}_1$ and that the logic capturing correct reasoning about everyday middle-sized objects[2] is $\mathcal{L}_2$. Consider the following argument:

1 Waterboarding is morally wrong.

2 The U.S. government has used waterboarding.

3 Therefore, the U.S. government has done something morally wrong.

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[2] As we will see later on in chapter 2, whether the domain of middle-sized objects (or any domain, in general) is characterized by the objects themselves or by the relevant properties or relations that are under discussion, is a matter in dispute. For instance, one could argue that the proposition 'the number π is beautiful' does not belong to the domain of mathematics but to the aesthetic domain. Despite the object being a mathematical object, the way of predicating about the object makes the proposition belong to another domain in that case.

The first premiss and the conclusion are moral claims and, by assumption, are governed by logic $\mathcal{L}_1$. The second premiss, on the other hand, is a claim about the observable fact of someone (acting as an executor of the power of the U.S. government) pouring water over a cloth covering someone else's face. So, it belongs to the domain of middle-sized objects or events formalized by $\mathcal{L}_2$. Nevertheless, the argument seems to be intuitively valid. But, the questions are, which is the logic that accounts for the validity of the argument? And which are the principles of reasoning that allow us to reason across domains? These are not minor issues for a philosophical thesis claiming that the correct application of logic systems is local, since reasoning across domains is something quite pervasive.

Thus, the philosophical thesis that I am going to analyse is *logical localism* and the challenge to localism that I will try to address, the main challenge of localism, indeed, is *the problem of mixed inferences.* Logical localism is often considered as a form of logical pluralism. That is, as a thesis claiming that there is more than one legitimate relation of logical consequence. These types of philosophical positions regarding logic were not a serious option before the appearance and consolidation of non-classical logics at the beginning of the 20th century. These logics reject some of the classical principles, thereby allowing to question the general validity or uniqueness of classical logic.

The emergence of non-classical logics -notably, relevant, intuitionistic and many-valued logics- gave rise to some of the most important questions in the philosophy of logic: are alternative logics really 'logic'? Do they deserve the same status as classical logic? Is there just one correct logic or are there many? Some of the proponents of alternative logics continued defending monist theses, now against classical logic and in favour of one of the alternatives. Michael Dummett, for example, argued for the correctness of intuitionistic logic, claiming even that classical connectives had no meaning at all.

However, the fact of having a plurality of systems and of being able to check out that some of them appeared to have applications where they excelled the most, made pluralistic proposals more and more plausible. These proposals have had various forms and I will present the most relevant ones for my purposes later on.

The problem of mixed inferences is the challenge around which the philosophical and the technical aspects of the dissertation revolve. After presenting the philosophical framework I will introduce the challenge to localism and I will mostly focus on the version that Chase Wrenn offers, which is, to my mind, the best and most detailed presentation of the problem of mixed inferences for logical localism.

My approach in order to offer a solution to the challenge is one that has not been explored, or considered, in the literature as far as I am aware. The strategy will be to explore different methods for combining logics, concentrating specially on the method of *juxtaposition*, by Joshua Schechter, and trying to solve the problem of mixed inferences by making a finer translation of the arguments and using combination mechanisms as the criterion of validity. But combinations of logics bring about other potentially problematic issues regarding interactions between connectives and, in the limit case, they bring about what are known as *collapse theorems*. These problems will be analysed and I will develop the combination mechanisms having in mind that the collapse has to be avoided.

One of the most intriguing aspects of the dissertation is the synergy that is created between the philosophical debate and the technical methods with the problem of mixed inferences at the center of that synergy. I hope to show that not only the philosophical debate benefits from the methods for combining logics, but also that these methods can be developed in new and interesting ways motivated by the philosophical problem of mixed inferences. The problem suggests that there are relevant interactions between connectives, justified by the philosophical considerations for conceptualising different logic systems, that the methods for combining logics should allow to emerge. The recognition of this fact is what drives the improvements on the method of juxtaposition that I develop. That is, in order to allow for the emergence of desirable interaction principles, I will propose alternative ways for combining logic systems, specifically classical and intuitionistic logics, that go beyond the standard for combinations, which is based on minimality conditions so as to avoid collapse theorems.

Besides the technical developments, the new combination mechanisms introduce further subtleties within the philosophical debate around localism and allow for a more fine-grained analysis of alternative kinds of localisms and even of domains. That is, the analysis suggests that there are some properties of the domains that are not captured only by the logic that corresponds to a given domain. Instead, those properties are revealed when the domain interacts with another, i.e. when reasoning across domains. Thus, it might be the case that reasoning across the domain of middle-sized objects and the mathematical domain requires different interactions from those required by reasoning across the domains of middle-sized objects and ethics, even if one believes that the logic of the mathematical and the ethical domain is the same, say, intuitionistic logic. Thus, I think that combining logics can bring us closer to a solution to the problem of mixed inferences and, moreover, help us discern with more accuracy the intricacies of the philosophical debate.

The structure in which I will unravel these ideas is the following: in chapter 2, I present the conceptual framework in which logical localism is going to be characterized. In this conceptual framework, the topics of logical and alethic pluralisms play a crucial role, so it will be relevant to clarify what they are and how they relate to localism. Then, I will conclude the chapter by presenting the main challenge for logical localism, namely, the problem of mixed inferences and I will go through some of the most notable attempts to solve it. The last part will be devoted to Chase Wrenn's version of the problem, which is the most elaborate version of the problem in the literature.

Chapter 3 is meant to establish the connection between the problem of mixed inferences and the field of combining logics. In order to justify that bridge, I start by focusing on some logical conceptions about mixed reasoning and I introduce the notions of 'interaction principles', 'bridge principles' and 'collapse theorems', existent in the literature, and elaborate them. Then, I present some popular methods for combining logics, paying special attention to the one upon which I am going to build my own mechanisms, namely, juxtaposition.

In chapter 4 I develop a solution to the problem of mixed inferences. First, I approach the problem by applying the original method of juxtaposition and discuss its virtues and potential shortcomings. Based on the limitations of the method, I argue that the combination mechanisms should allow for more interaction between the logics being combined, in order to get a more encompassing solution to the problem of mixed inferences. Therefore, I propose some new alternative mechanisms in which the desired bridge principles can naturally emerge in the combination process.

Finally, chapter 5 concludes the dissertation by looking into some promising future work. As I will try to show, the analysis suggests that there are many interesting philosophical and logical/algebraic issues to be sorted out in the vicinity.

Chapter 2

Logical Localism

2.1 What is Logical Localism? A conceptual framework

In this section I will try to set the framework of the discussion that concerns me for the dissertation. As I already advanced in the introduction, the topic of localism is very closely related to, if not included in, the debate around logical pluralism, which is a central debate within the philosophy of logic. Given its centrality, there are many other issues in the field that affect the discussion, such as the debate around which the chief aim of logic is, the normative status of logic, the sense in which logic is formal, whether logical consequence should be spelled out model-theoretically, proof-theoretically or kept primitive, and so on and so forth.

Those topics are huge and each one of them deserves more than a dissertation, as shown, for instance, by J. G. MacFarlane with his brilliant thesis on formality (MacFarlane (2000)). However, it is beyond the scope of my thesis to deepen on those topics and I reckon that it would not even be helpful for fulfilling my more modest and specific aim, namely, to work out methods for combining logics in order to find possible solutions to the problem of mixed inferences that challenges the philosophical position of localism.

Nevertheless, in characterizing logical localism those crucial issues will inevitably come up, since, as I said, they affect how one thinks about the nature of logic[1]. Thus, I will try to make explicit, where applicable, what I

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[1] Notice that there are, at least, three senses of 'logic': a consequence relation, a particular logic system or the discipline. I believe that there will be no confusion throughout

am assuming or how those issues might affect the characterization of localism that I will be developing.

Let me, then, before going into the details of localism, start by laying down a general assumption I will make concerning the chief aim of logic. Following the mainstream tradition and, more concretely, Graham Priest (Priest (2006)) and J. C. Beall & G. Restall (Beall and Restall (2000, 2001, 2006)), which are arguably the most important and influential defenders of monism and pluralism, respectively, I will assume that the chief subject matter of logic is *logical consequence*. Beall and Restall very nicely put it at the beginning of their book *Logical Pluralism*:

> Logical consequence is the heart of logic; it is also at the centre of philosophy and many theoretical and practical pursuits besides. Logical consequence is a relation among claims (sentences, statements, propositions) expressed in a language. An account of logical consequence is an account of what follows from what -of what claims follow from what claims (in a given language, whether it is formal or natural). An account of logical consequence yields a way of *evaluating* the connections between a series of claims- or, more specifically, of evaluating *arguments*. (Beall and Restall, 2006, p. 3)

And also in Beall and Restall (2000):

> The chief aim of logic is to account for consequence, to say, accurately and systematically, what *consequence* amounts to, which is normally done by specifying which arguments (in a given language) are valid. All of this, at least today, is common ground.
> (Beall and Restall, 2000, p. 475)

Similarly, Priest writes:

> What is logic? Uncontroversially, logic is the study of reasoning.[...]The study of reasoning, in the sense in which logic is interested, concerns the issue of what follows from what. Less cryptically, some things -call them *premises*- provide reasons for others -call them *conclusions*.[...]The relationship between premise and conclusion in each case is, colloquially, an argument, implication, or *inference*. Logic is the investigation of that relationship. A good inference may be called a *valid* one. Hence, logic is, in a nutshell, the study of validity.
> (Priest, 2006, p. 176)

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the text since I usually use those more specific words instead of the more general ‘logic’. In any case, the context should suffice in order to disambiguate the generic uses.

Thus, I will adhere to this widely accepted tradition. Logic is the systematic study of what follows from what; of which premises stand in the logical consequence relation to which conclusions. That is, the aim of logic is to account for the validity of arguments.

This is not, however, the only existing position regarding what logic is about. J. van Benthem, for instance, has a more 'liberal' conception of logic and argues that the view of logic as being about consequence relations may have had some sense when it was thought to provide the foundations of mathematics. But, since the 1930s the field has changed and broadened its scope. Logic is now, van Benthem claims, about *definability, computation* and more (van Benthem, 2008, p. 183). Indeed, van Benthem defends that the main issue of logic is "the variety of *informational* tasks performed by intelligent interacting agents, of which inference is only one among many, involving observation, memory, questions and answers, dialogue, or general communication" (van Benthem, 2008, p. 182).

I do not have any particular concern with this conception and I believe that the discussion on whether logic is X or Y is not very fruitful. However, it does affect the plausibility of pluralism and localism how broad the domain of application of logic is. To put it simply, if logic is about so many things beyond logical consequence, as van Benthem claims, it will be more probable that there is more than one 'correct logic' and it will be less likely that one logic does all the job.

When I use 'correct logic' I mean, roughly, the logic that is most fruitful, most adequate to the data, overall simplest, etc. Thus, in this case, I do not aim to imply any metaphysical view on whether there is, or not, an objective reality that logic seeks to capture. Thus, it is a sense of 'correctness' available both to a realist and an instrumentalist (in the sense of Haack (1978)).The difference between the realist and the instrumentalist arises, though, with respect to which the True logic is. Since for the instrumentalist there is no extra-systemic validity, but just valid-in-L, there is no True logic. For the realist on the other hand the intra-systemic notion of validity is trying to capture 'real' external validity. But, then, it is logically possible to conceive a world in which the correct logic, after weighing the theoretical virtues, is not the True logic. Imagine, for instance, that the True logic is one in which every inference rule has some counterexamples. Still, it could be the case that the most fruitful and adequate logic to be applied, i.e., the correct logic, was one with universal inference rules.

On this note, and in order to continue laying down some assumptions, it is relevant to the discussion that we clarify a bit more the notion of 'application'. It is largely uncontentious, nowadays, that the *pure/applied* distinction holds when speaking about logics. Notably, Priest (2003, 2006) justifies the distinction by drawing an analogy with geometry and arithmetic (an idea due to Łukasiewicz). The analogy tries to establish that, in the same way that there are many pure geometries (Euclidean, Riemannian, Lobachevskian, etc.), there are many pure logics (classical, intuitionistic, paraconsistent, connexivist, etc.). These are "well-defined mathematical structure[s] with a proof-theory, model theory, etc." (Priest, 2006, p. 195). We define them, we study their properties, prove results about them, relations between them, and so on. With respect to pure logics, much like pure geometries, there is no doubt about pluralism. It is an uncontentious fact that there is a plurality of pure logics.

There are, however, other aspects of doing logic that have to do with the application of pure logics to different domains and problems. This is a common practice within philosophical logic, for instance, where pure logics are often applied in order to deal with paradoxes, to systematically account for reasoning about knowledge, necessity, obligation, morality, etc. But, it is also the case, as R. Cook points out, that "logics have been central to the study of a number of phenomena, including many that have, at best, an indirect connection to human reasoning such as electronic circuit design, database management, and internet security"(Cook, 2010, p. 494).

Despite the variety of applications that logics have been used for, though, some people argue that there is a privileged application of logic, a *canonical application*, which is the analysis of reasoning. Quoting Priest,

> the most important and traditional application of a pure logic [is] the *canonical application*: the application of a logic in the analysis of reasoning [...]. The central purpose of an analysis of reasoning is to determine what follows from what -what premises support what conclusions- and why. An argument where this is, in fact, the case is *valid*. (Priest, 2006, p. 196)

Thus, the more interesting pluralist thesis would be with respect to its canonical application. It is not enough the mere existence of a variety of pure logics, not even the fact that there might be various *rival* logics competing for being the best codification of reasoning (what Priest (2006) calls *theoretical pluralism*). As Cook stresses,

> if logical pluralism is to be a substantial and controversial thesis, something more must be intended. That something more is the notion of logical consequence -that is, a logic is 'correct', or 'acceptable', etc., if and only if it is a correct (or acceptable, etc.) codification of logical consequence. The idea that the philosophically primary (but obviously not only) goal of logical theorizing is to provide a formal codification of logical consequence in natural language traces back (at least) to the work of Alfred Tarski. (Cook, 2010, p. 495)

Let us, therefore, assume that the chief subject matter of logic is logical consequence and that logic's canonical application is the analysis of reasoning. One could think that a legitimate way of arguing for pluralism would be to emphasize that there are different ways of accounting for logical consequence, e. g. model-theoretically, proof-theoretically or regarding it as a primitive notion. I have tried to argue for the model-theoretic approach in Benito-Monsalvo (2022) and I believe that the debate is philosophically substantial, but this will not be relevant to our discussion here, since it is not the source of plurality that is interesting for present purposes. We are interested in the thesis that there are different logics which capture different legitimate relations of logical consequence when canonically applied (regardless of whether the logic is presented model-theoretically or proof-theoretically).

Logical localism is one of those pluralist theses, but logical pluralism, understood *à la* Beall and Restall, for instance, is also a position in favour of that kind of plurality. Thus, what I want to do, now, is to start singling out and delimiting the logical localist position. This is a delicate task, because logical pluralism is not an unequivocal thesis and encompasses many positions under the same naming. But I am less interested in the exegetical work than in providing a conceptual map of the available theoretical positions and addressing where localism stands in that map. This is why I will be taking some authors almost like archetypical figures of the different positions. Let me proceed, then, to clarify how I understand logical pluralism, best represented by Beall and Restall, in opposition to logical monism, which has Priest as one of its most popular defenders. This opposition is also interesting because both sides agree on the chief subject matter of logic, namely, logical consequence.

2.1.1 Logical Pluralism

It sounds like a truism, but it is always nice to remember that in order to even conceive logical pluralism we require to know of the existence of different, alternative logic systems. However, it is also always amusing to remember Kant's thesis on Newtonian physics and Aristotelian logic, i. e. syllogistic. Some authors refer now to Hugh McColl as the first proposal of what could be considered a logical pluralist philosophy of logic. It is no coincidence that he was also a pioneer in the development of non-classical logics, including, many-valued, probability, relevant and connexive logics (see Rahman and Redmond (2008)). His pluralism, though, seems to be closer to what I call 'localism' than to Beall and Restall's pluralism, and it would certainly fall under what Cook, in Cook (2010), calls 'relativism'. This is what Rahman and Redmon comment on this respect:

> MacColl's philosophy is a kind of instrumentalism in logic which led him to set the basis of what might be considered to be the first pluralism in logic. The point condensed in the epigraph amounts to the following: it could well be that in some contexts of reasoning the existing argumentation demands a type of logic which is not applicable in others. When constructing a symbolic system for a particular type of logic, the corresponding expressions in use should therefore be taken into careful consideration.
>
> (Rahman and Redmond, 2008, pp. 540–541)

So, we can say that McColl's pluralism is one that claims that the application of logic is relative to 'contexts of reasoning' and, therefore, that there might be different correct logics for different contexts. In this sense, the application of logic is local, despite logic being a theory of reasoning, because this varies from context to context. In other words: the canonical application of logic is the analysis of reasoning, but there are, so to speak, 'sub-canonical applications'.

A second milestone in the history of logical pluralism is Rudolf Carnap and his *principle of tolerance*. In *The Logical Syntax of Language* (1937) Carnap says:

> In logic there are no morals. Everyone is at liberty to build his own logic, i.e. his own language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments.
>
> (Carnap, 1937, §17)

A crucial point that results from this liberty for building our own logic from our own language, is that there is no external logical reality that forces a particular One True Logic. The result is a kind of conventionalism, similar to that of McColl, by which one gets different correct[2] logics by varying the linguistic framework. Cook summarises Carnap's pluralism in a very illuminating way:

> Carnap's view certainly amounts to a form of logical pluralism. [...] But this is a dependent pluralism, resulting from an underlying relativism – that is different logics result from varying the language in question (it is worth noting that Carnap does not advocate pluralism within a framework – different linguistic frameworks might be governed by different logics, but within a particular framework there is a single logic that correctly codifies the (internal) logical consequence relation of that framework). Thus, Carnap's tolerance amounts to a version of logical pluralism, but not a version of SLP [Substantial Logical Pluralism]. (Cook, 2010, pp. 497–498)

What Cook means by 'substantial logical pluralism', is a pluralism such that the language is kept fix, the demarcation of the logical/nonlogical vocabulary is also fixed and, yet, there are (at least) two logical consequence relations that capture two legitimate different senses of 'follows from'. This is, in fact, what Beall and Restall want to achieve. So, let me present the fundamentals of their proposal.

2.1.2 Beall and Restall's Logical Pluralism

As we said before, Beall and Restall adhere to the mainstream tradition of taking logic to be about the consequence relation. About systematically determining what follows from what. The account of logical consequence that Beall and Restall (2000, 2001, 2006) deploy is a generalization of the traditional semantic one, i.e. of Tarski's account of logical consequence. They call it *Generalised Tarski Thesis* (GTT):

- (GTT) An argument is valid$_x$ if and only if, in every case$_x$ in which the premises are true, so is the conclusion.

---

[2] I would say that 'correct' for Carnap means just the logic (or the logics) that better fulfils some pragmatic goal set by the subject.

The key concept in this definition, the one that Beall and Restall exploit in their argumentation, is the concept of *case*. The authors maintain that for GTT to completely define logical consequence, the cases have to be specified. That is, specifying the cases provides the truth conditions for the sentences of a given language. In this sense, to specify a case is to provide an explanation of what it is for a sentence of a given language to be true in that case.

Beall and Restall compare this type of unsettledness with that of vague expressions. For them, 'case' would be like 'bald'. There are determinate cases[3] of bald and not-bald, but there are other cases, given the unsettled nature of language, such that it is indeterminate whether 'bald' applies or not, which leaves room for freedom and for different equally acceptable criteria of baldness. Thus, the authors' conception of logical pluralism is the following:

> *Logical pluralism* is the claim that at least two different instances of GTT provide admissible precisifications of logical consequence. [...] the pluralist endorses at least *two* instances [of GTT], giving rise to two different accounts of deductive logical consequence (for the same language), two different senses of 'follows from'.
>
> (Beall and Restall, 2006, p. 29)

And they go on to add that

> As with unsettledness in general (see §3.1.3), so too with the current topic: the question is ultimately one of utility. Provided that the various GTT accounts of consequence are admissible, there is no sense in asking which is *the correct account*. To the question 'Which account is the *right* account of consequence?' there is no answer -provided, again, that the relevant candidates are admissible. Whether candidates are *admissible* turns on whether they agree with the settled parts of language, on whether they exhibit the features required by the (settled) notion of logical consequence. We hold that the notion settles *some but not all features* of any candidate relation of logical consequence; the unsettled features leave room for plurality.
>
> (Beall and Restall, 2006, p. 29)

So, for instance, one might construe cases as *Tarskian models* which would interpret a language of first-order logic by giving truth conditions to its sentences recursively, in the usual way. This would give rise to *classical logic*[4].

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[3] Meaning 'case' in an informal sense.

[4] It is also possible to specify cases as *worlds*, which is another approach within classical logic concerned with validity as necessary truth preservation.

But, as we said, there are other specifications of cases which give rise to different logics. Beall and Restall (2000, 2006) provide two interesting examples of non-classical logics obtained in such a way: *relevance* and *intuitionistic logics*. Let us illustrate the point with the former.

Relevance logic is obtained by specifying cases as *situations*. To say it briefly, a situation might be understood as a restricted part of a world which can be incomplete[5]. That is, a part of a world that can make true some claims, but where it might not be the case that for every sentence ϕ, either ϕ or $\neg\phi$ are true. So, a situation might be indeterminate with respect to the truth or falsity of some sentences. Now, this has a straightforward consequence; it is easy to see that we already have a disagreement between the classical and the relevant logicians with respect to, at least, one inference. While in classical logic it is valid to infer $\phi \vee \neg\phi$ from α (being $\phi \vee \neg\phi$ logically valid, in classical logic, it can be inferred from no premise at all), that very same inference is not relevantly valid, inasmuch as we can provide a situation in which α is true without neither ϕ nor $\neg\phi$ being true (Beall and Restall, 2006, p. 55).

The way Beall and Restall explain this discrepancy nicely illustrates the sense of their pluralism:

> The virtue of a pluralist account is that we can enjoy the fruits of relevant consequence as a guide to inference without feeling guilty whenever we make an inference which is not relevantly valid. With classical consequence you know you will not make a step from truth to falsehood, assuming, with most philosophers, that *possible* worlds are complete and consistent. With relevant consequence, the strictures are tighter; you know you will not make a step from one that is true in a situation to something not true in it (but which might be true outside it). This is a tighter canon to guide reasoning.
>
> (Beall and Restall, 2000, p. 484)

Hence, Beall and Restall consider that there is not a real rivalry between classical and relevant validity. They can perfectly coexist, since both specifications of cases illuminate different aspects of logical consequence and validity.

Then, it seems as if by construing the notion of case in different ways, we could obtain different logics that systematise distinct levels or features

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[5] It can also be inconsistent, indeed, but we are not going to consider such a situation in our example.

of consequence relations in ordinary reasoning. If logic is about consequence relations, there are various aspects of consequence that require different logics in order to be captured.

An interesting way of getting a better understanding of Beall and Restall's pluralism is to contrast it with Priest's position. As we said above, Priest agrees with Beall and Restall both in the subject matter of logic and in the conceptions of consequence and validity. Nevertheless, he argues in favour of monism. It is important to specify, however, in which sense and with respect to what Priest defends monism. So, Priest does not deny pluralism relative to pure logics. According to him, it is an uncontentious fact that there are many pure logics, each one being a well-defined mathematical structure with a proof-theory, model-theory, etc. (Priest, 2006, p. 195). Neither does he deny theoretical pluralism: that there are different logics that compete for being the most suitable for a given domain (Priest, 2006, p. 196). What Priest denies is that there are a variety of logics that are equally good for its canonical application: the analysis of reasoning. That is, Priest's monism is to the effect that there can be just one correct applied logic in the analysis of reasoning, i.e. in the analysis of what follows from what (*ibidem*)[6].

Thus, the claim that marks his opposition with Beall and Restall is that 'it is only truth-preservation over *all* situations [cases] that is, strictly speaking, validity' (Priest, 2006, p. 202). That is, according to Priest, validity is not something that can be relativized to this or that domain. Logic must work 'come what may' (*ibidem*). Nevertheless, Priest accepts that we might have to reason, say, classically in some domains, despite classical logic not being the correct logic. But this does not imply, according to Priest, that there are different logics for different situations, it means that we can adopt some contingent features of particular domains to recover classical validity enthymematically (Priest, 2006, p. 198).What this means, is that there is, according to Priest, a unique logic having as its set of valid inferences the core of valid inferences that every canonically applied logic shares. So, for instance, if there are situations in which objects are not self-identical, then self-identity is not a logical law. But, in the appropriate domain, the domain of macro-objects, one could use self-identity by appealing to the enthymeme

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[6] In what follows, then, we should have in mind that Priest's monism is restricted to the canonical application of logic. When we speak about 'different domains' or 'all domains' we are not referring to other domains of application on top of the analysis of reasoning, but to the different domains within the canonical application (mathematical reasoning, reasoning about middle sized objects, reasoning about counterfactuals, etc.).

‘all macro-objects are self-identical’ (Priest, 2006, p. 200). Yet, the correct logic would still be the weakest one not containing self-identity.

Now, there is a point in need of clarification here, since Beall and Restall also want to hold that logic works come what may (Beall and Restall, 2001, p. 14), but the sense in which they hold it is rather different. To my mind, the best explanation they provide for supporting together the thesis that logic works come what may and pluralism, is that the universal, ‘every case$_x$’, in (GTT) does not quantify over a fixed domain, as Priest suggests, but that there is a variation in the domain over which the universal quantifies. In this sense, various logics (at least classical, intuitionistic and relevance logics for Beall and Restall) work come what may, because they are applicable for any domain of discourse but, still, each of them is better suited than the others for some job required of deductive validity. That is, each logic gives an incomplete account of what follows from what by specifying the notion of case in different ways, while being applicable to any domain of discourse.

However, the argument for domain variation is not that obvious, so further reasons must be provided in its favour. A positive argument for variation stresses the imprecision of the notion of case. By its own, ‘case’ does not inform us about under which conditions a sentence is true, because cases can be specified in different types of cases. Therefore, logic -or logics- is obtained only after such specifications are provided, since (GTT) does not determine just one such specification.

A second argument, this one negative, highlights the consequences of taking the domain of quantification as fixed: which inferences, if any, would be valid in all cases? This question is difficult to answer, in part because it is not clear how many possible classes of cases there are. The suggestion by Beall and Restall is that the only plausible candidate for being truth-preserving over all cases, in the sense of Priest, is the identity inference, $\alpha \vdash \alpha$ (Beall and Restall, 2000, p. 490). Should we admit, then, that the only valid argument is identity? That there is only one logic whose subject matter is whether α follows from α? And the situation might even be worst, since identity has also been challenged on different grounds as I will illustrate later on.

Another way of characterizing Beall and Restall’s pluralism that might also be helpful for our primary goal of presenting and understanding logical localism, stems from W. V. O. Quine’s *meaning-variance thesis*, which is often summarized with the slogan that ‘change of logic is change of subject’, which opens the sixth chapter in Quine (1970) on deviant logics. The fundamental idea is that the meaning of the logical connectives is determined by the logic. Then, when the non-classical logician proposes an alternative logic system, what she is doing is referring to another thing rather than really disagreeing with the classical logician. They are talking past each other.

This is not exactly the meaning-variance that Beall and Restall defend. In fact, they explicitly say that their pluralism is such that the language is kept fixed. And, obviously, this does not mean that, superficially, the notation is the same, but also that the meaning of the logical symbols is the same.

But, as Hjortland (2013) notices, adopting a distinction made by Susan Haack (1978), there is a sense in which Beall and Restall's pluralism is meaning-variant, namely, in the sense that the meanings of 'valid' and 'follows from' vary across logical theories. This change of meaning occurs because the cases are precisified in different ways depending on the intended utility of the logical system. Then, the pluralism that Beall and Restall are advocating is one which keeps the language fixed, the concept of validity changes in virtue of the different legitimate precisifications of the notion of 'case' and the logics that result from these different precisifications are equally correct with respect to the canonical application, i.e. their application is not relative to domains of reasoning (as McColl, for instance, thought), but global.

Probably due to the fact of being the best and more detailed formulation of logical pluralism, Beall and Restall's proposal has been the most challenged one too. According to Field (2009), for instance, Beall and Restall's pluralism, though interesting, 'falls *far* short of the kind of pluralism that says that advocates of apparently competing *all-purpose* logics don't really disagree' (Field, 2009, p. 346). Cook (2010) points out that, despite Beall and Restall's pluralism not being relative to languages, 'the correct logic is relative to which logical consequence relation in natural language one intends to codify'(Cook, 2010, p. 500). Hjortland (2013) gives convincing arguments to show that Beall and Restall's meaning-variance with respect to 'validity' does entail meaning-variance with respect to connectives, contrary to what they claim, and T. K. Kissel argues that 'negation in intuitionistic logic and negation in relevant logic cannot mean the same thing on any reasonable interpretation of their [Beall and Restall's] account of meaning' (Kissel, 2018, p. 217). On top of these and probably being the most popular among all, there is the *collapse argument* to which we will refer later on, which tries to show that, given Beall and Restall's assumptions on plurality and normativity, their pluralism will collapse to a single logic.

In any case, my aim is not to evaluate Beall and Restall's proposal, but just to have its main properties clarified in order to better understand what localism is against the picture of the most popular pluralist account. The next step, then, is to start developing a more precise notion of what logical localism is.

2.1.3 Characterizing Logical Localism

In this section I want to lay down the most relevant features of localism. Similarly to what happens with pluralism, localism is not a single unequivocal doctrine. There are different versions of localism that have been defended in the literature, but I reckon that there is a core doctrine of localism that many proposals share and that is challenged by the same problem that I want to account for with this thesis.

Let me begin with a general definition of localism. In fact, one of the earliest that can be found in the literature. As I said, McColl's pluralism is already a form of localism but, now, I am looking for more explicit considerations of locality. As far as I know, the first reference that uses the terminology of 'local' and 'global' approaches to logic is Susan Haack (1978). There, she takes logical localism to be a type o logical pluralism, calls it 'local pluralism' and conceives it like this:

> According to *local pluralism*, different logical systems are applicable to (i.e. correct with respect to) different areas of discourse; perhaps classical logic to macroscopic phenomena, and 'quantum logic' to microscopic phenomena, for example, rather as different physical theories may hold for macroscopic phenomena than for microscopic phenomena. The local pluralist relativises the extra-systematic ideas of validity and logical truth, and hence the idea of the correctness of a logical system, to a specific area of discourse; an argument isn't valid, period, but valid-in-*d*. (Haack, 1978, p. 223)

There are many interesting points already in this definition, but the fundamental idea is that, according to the localist thesis, the correct application of logic is not topic-neutral, domain-neutral or irrespective of subject-matter. That is, logical localism is the philosophical thesis stating that different sets of logical principles, forming various alternative logic systems, are required in order to systematically account for correct reasoning in different domains. The variety of localist proposals stems, then, from the alternative ways in which domains can be individuated. Before looking at those alternatives, let me contrast localism with globalism for the sake of making localism more clear and distinct.

Localism against Globalism

The opposite thesis of localism, as I will understand the notions, is not monism but globalism. Globalism is the thesis according to which the application of logic is global, i.e. independent of the subject-matter or the domain of reasoning to which it applies. Therefore, globalism stays within the orthodoxy of logic in that it retains the alleged topic-neutrality of logic and seems to support the traditional idea of the universality of reason.

I agree with some of the intuitions that sustain the idea of universality of reason, but I think it is an oversimplification to infer from that desideratum that, since logic is a theory of correct reasoning and reasoning is universal, then logic has to be applied globally in order for it to be a candidate for being correct. One could push back by arguing that reason can be universal, in the sense of being a human faculty that we can employ irrespective of subject-matter, in any domain of inquiry, despite actual reasoning taking various forms and principles in different domains. To give an analogy, there is something universal to everyone getting a gold medal at the Olympics, namely, they did better and won over their competitors. But, at the same time, winning has many forms and it materializes in different ways, whether you win a gold medal in climbing or in pole vault, for instance.

There is an important empirical argument on the side of the globalist though. We do seem to reason *across* domains, from moral and factual premises to moral conclusions, from mathematical and physical to physical, reasoning about the interplay of macro and micro-objects, and so on and so forth. This is, to my mind, the most important empirical fact that localism has to account for. It is, indeed, the fundamental fact that sustains the problem of mixed inferences. Thus, even if the application of logic is local, we must be able to give an explanation of how those domains might interact.

Moreover, I reckon that there is also important empirical evidence in favour of localism and challenging globalism, namely, the amount of logical systems that are used and are constantly being developed, not just for any application, but for canonical applications, or sub-canonical applications, like formalising reasoning about vague phenomena, reasoning with inconsistent information, reasoning about truth, quantum-mechanical phenomena, etc. Even within the domain of mathematics, that one could regard as a single homogeneous domain, there are proposals for employing different logics in different branches, like paraconsistent, constructive or classical mathematics (see, for instance, Priest (2019); Shapiro (2014a,b)). In fact, there is no reason to suppose that we have reached the ultimate stage of varieties of reasoning and that, therefore, no more new domains of reasoning will appear, which, potentially, could require even new logical systems to be systematised.

Thus, I believe that globalism faces an important challenge too, similar to the *scope problem* that traditional theories of truth have to face, namely, that 'the plausibility of each inflationist's candidate for the [truth] property F differs across different regions of discourse' Pedersen and Wright (2018). Similarly, globalism has to face the empirical fact that the plausibility of any logic system, $\mathcal{L}$, differs across different regions of discourse or domains of reasoning.

Localism as a form of Relativism

Following with the characterization of localism, I believe it is interesting to place localism within the excellent taxonomy of 'pluralist' theories that Cook (2010) provides. Cook's view on relativism is such that,

> One is a relativist about a particular phenomenon X if and only if one thinks that the correct account of X is a function of some distinct set of facts Y. Thus, relativism about X amounts to acceptance of the following schema:
>
> The correct account of X is relative to Y.
>
> (Cook, 2010, pp. 492–493)

In this sense, it is clear that localism is a form of relativism, namely, one holding that the correct account of 'follows from' or 'consequence' or 'valid', is relative to domains (leaving open whether domains are individuated by subject-matter, by the ontological properties of the objects within that domain, or whatnot).

However, localism is not a type of *Substantial Logical Pluralism*, under Cook's categorization, since it is difficult to hold that, if the logic for reasoning about truth is a paraconsistent logic, the logic of evaluative discourse is intuitionistic and the logic for reasoning about middle-size phenomena is classical logic, for instance, those logics' corresponding connectives, say, negations, are going to have the same meaning. Whether one thinks that the meaning of the connectives is determined by the rules of inference or by their truth-conditions or satisfaction conditions, it is reasonable to assume that the meanings will change[7]. Therefore, since Cook's conception of *Substantial Logical Pluralism* implies that the pluralism arises within a fixed language and a fixed interpretation of the logical/non-logical divide, localism cannot be a *Substantial Logical Pluralism*.

This should not be interpreted as localism not being a substantial or interesting thesis. Cook makes this clear, but I guess that the literature has had a tendency to focus more on the type of pluralism claiming that 'there is no genuine debate between advocates of different all-purpose logics' (Field, 2009, p. 344). The reason for this bias might be that the most popular, detailed and best developed account of logical pluralism is Beall and Restall's and they explicitly say that their intended pluralism is not a relativism. The plurality of logics, according to them, are applicable to any domain or subject-matter.

Neither should we conclude that every form of relativism is a localism. A popular relativist view is defended by Achille Varzi in Varzi (2002). According to him, which conclusions are in a logical consequence relation to which premises depends on how one conceives the logical/non-logical divide. For instance, whether or not one regards identity as a logical symbol affects the possible models that one will accept. If one takes it as a logical symbol, she will accept only the models that do justice to the intended interpretation of the predicate. Otherwise, one could accept further models.

Regardless of whether we agree with Varzi or not, it seems clear to me that his relativism is not a form of localism. It is not that the plurality of accounts for the logical consequence relation depends on a domain of reasoning, a subject-matter, some ontological property, etc. His relativism does not have anything to do with domains, but with the specific set of logical constants that one adopts.

To conclude this section on the characterization of logical localism, let me next refer to some localist proposals in the literature. We have already said that McColl's and Carnap's pluralisms can be taken as localist accounts. Now, I will move on to explain how localism has been conceptualized by other authors.

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[7] However, I am going to discuss more this issue once my improvements on juxtaposition have been presented, since I believe that the method might allow for some interpretations compatible with localism without meaning-variance for connectives.

Some Localist proposals in the literature

As I defined it above, logical localism is the thesis stating that different logic systems are required in order to systematically account for correct reasoning in different domains. The main difference between the alternative proposals resides in how one individuates the domains.

In the case of McColl, the domains are contexts of reasoning and his conventionalism seems to leave quite some freedom with respect to what determines a context of reasoning. It might be a particular problem, a topic, etc. There is no specific feature that forces a new context of reasoning, simply some contexts require different tools in order to account for what follows from what in that context. Let me give some other examples of localism now.

Newton da Costa's localism[8]

Newton da Costa's localism is also based on the observation that there are a plurality of logics because different domains ask for different logic systems. Moreover, in da Costa's view, there is a concrete reason for this domain variation, namely, that reasoning about different *kinds* of objects requires different logics. Therefore, domains of reasoning are determined by the ontological properties of the objects belonging to the domain. An example of these kinds of objects and the variation of logics that they require is that of macro-objects versus quantum objects:

> [...] It is clear that for common objects, such as a book or a person, [...] [$\forall x(x = x)$] applies apparently without a single important difficulty. Any person whatever, say A, even though they undergo multiple modifications in the course of their life, remains in a certain sense identical with themself: A = A . That appears even more clearly as concerns abstract objects: for example, the equality $1 = 1$ seems evident and indisputable [...]. However, things are not as simple as naive realism would lead one to believe. In quantum physics, elementary particles, according to all appearances, transgress the principle of identity. Thus Schrödinger affirmed that the relation of identity between particles was devoid of sense: "it is not a problem that depends on our capacity for proving the identity in certain cases and our incapacity for proving it in other cases. It is certain that the issue

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[8] I have not been able to find the relevant material in English and, since my Portuguese and French are not good enough, I will rely on Priest's interpretation of da Costa and in some translations to English made by himself.

> of ‘identity’ is, really and truly, devoid of sense”. It could be that the position of Schrödinger is acceptable only temporarily and that the future will show us that he is mistaken. Nonetheless the fact is that quantum physics shows the possibility of dialectising the idea of identity, and consequently, the very law that corresponds to it.
> (da Costa, 1997, p. 120) as cited from (Priest, 2000, p. 440)

On da Costa’s view, then, the ontological differences between objects enforce different logical properties too. But the account goes even further, as Priest observes:

> Da Costa’s pluralism is more radical than I have so far indicated, though. He envisages not only that objects of different kinds may have different logical properties, but that different logical operators may also need to be used in reasoning about different kinds of objects. Thus, for example, classical negation is appropriate for dealing with platonic objects, and intuitionist negation is appropriate for dealing with mental constructions. (Priest, 2006, p. 198)

Then, da Costa’s localism could imply[9] that there is meaning-variance both at connective level and also with respect to logical consequence and validity. That is, different kinds of objects constitute different domains and the reasoning within those domains varies because the diverse ontological properties of the objects carry over different logical properties. Therefore, different logical connectives and consequence relations are required in each domain in order to provide theories of correct reasoning within those domains.

We will soon see that a counterargument that Priest gives against da Costa’s localism constitutes one of the major challenges that I want to address with the aid of the methods for combining logics. But, there is an issue that Priest does not mention and that to me seems problematic or, at least, dubious. This is the domain individuation just in virtue of the ontological properties of objects. I think there are good reasons, given by D. Edwards in Edwards (2018), for instance, to argue against that criterion. To illustrate, take the statements ‘the number π is irrational’ and ‘the number π is beautiful’. Both statements are about the same object, namely, the mathematical object π. However, while we would certainly assign the first proposition to

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[9] I take the precaution of using ‘could’ because I have not found out whether da Costa defends that there is an extra-systemic notion of validity. If there is one, then it could be argued that the meaning of ‘valid’ does not really change.

the mathematical domain, we will most likely not assign the second one to the domain of mathematics, but to the aesthetical domain. As Edwards suggests,

> it is not what a sentence is about that we should be considering for domain membership, it is rather how the thing the sentence is about is represented, by the use of a predicate to attribute a property.
> (Edwards, 2018, p. 96)

That is, domain membership of a proposition seems to have to do with how the object is represented, i.e. with the predicate, rather than with the object. Another example might come from vague phenomena. It is well known that vagueness has been a fertile field for proposing alternative non-classical logics, but it seems that the objects of which we might predicate vague properties will appear in classical domains too. We might say, for instance, 'Putin is bald', but also 'Putin is the president of Russia'. However, if da Costa's example regarding quantum objects and identity is accepted, that would constitute an example in which the kind of object would be enough to individuate a domain. So, maybe the right criterion for domain individuation has to go beyond da Costa's and Edwards' accounts and make room for both. Domain individuation and the enforcement of a different logic system within a domain could be a matter of kinds of objects and the way objects are represented.

Diderik Batens' localism

In Batens (1990), D. Batens presents a series of arguments against the idea of global paraconsistency, i.e. the idea that the correct logic is paraconsistent and its application is global because inconsistencies are inherent to human reasoning. His view, against what he takes to be a dogmatic attitude towards paraconsistency, is that 'each logic [has] a particular set of domains in which it is adequate' (Batens, 1990, p. 209).

The way in which we should conceive of these domains, following Priest's terminology in Priest (2006), is as 'problem-solving situations'. One of those situations that Batens uses as example is a meta-theoretical situation. According to Batens, despite there being domains in which a paraconsistent logic is required, e. g. inconsistent domains,

> classical descriptions of many logical systems, including paraconsistent ones, are possible, and [...] whenever this is so, paraconsistent descriptions are too poor to be adequate. The same applies to other

> domains: whenever a domain is consistent, a paraconsistent description is incomplete. (Batens, 1990, p. 227)

The reason why paraconsistent logics are too poor for certain situations or domains, is because they lack the expressive strength that classical logic has. For instance, according to Batens, there is a sense of 'rejection' that the classical negations express that cannot be captured by a paraconsistent negation that allows both A and $\neg_p A$ to be true. In this sense, domain individuation seems to be something more pragmatic, having to do with the utility of a logic for a given problem, similarly to McColl's position, than the more substantial criterion of what constitutes a domain that da Costa defends.

Stewart Shapiro's relativism as localism

We have briefly mentioned above S. Shapiro's localism. I think it is worth commenting on it here because it represents an interesting case. So, *prima facie*, one would think that, for most of the possible sound accounts regarding domain individuation, mathematics would be a domain. Whether you characterize domains by the kinds of objects, by the predicates used for representing the objects, by the type of reasoning or problem-solving situations, it seems that mathematics is a good candidate for being a domain. In spite of all that, Shapiro argues for localism *within* mathematics.

As with the previous proposals, there are many details that I am not interested in addressing here. My purpose is to illustrate different existing approaches to localism depending on how one understands domains. So, with Shapiro, the way in which he separates the domains of application of logics is by taking domains as structures, where a structure is a legitimate branch of mathematics. Hence, according to Shapiro, 'logical consequence and validity are relative to structure. That is, one cannot say what the proper logic is until one says which structure is being discussed' (Shapiro, 2014a, p. 321). And, the point is that Shapiro argues for a variety of interesting and applicable mathematical branches, i.e. structures, that require different logics.

For instance, intuitionistic analysis is inconsistent if one has classical logic as its background logical theory. But, there is no reason to dismiss such theory. It is a legitimate branch of mathematics with potential applications and, yet, it requires us to drop some classical principles (most famously, excluded middle) on pain of inconsistency.

That is not the only example that Shapiro points out. Similar situations are found with respect to other intuitionistic theories and also paraconsistent ones. Thus, Shapiro's localism is a localism already within mathematics that, potentially, could be extended if those mathematical structures are applied, say, in some physical theory. Just within mathematics, then, we have legitimate structures some of which require classical logic, others intuitionistic logic and some others paraconsistent logic.

Pedersen and Lynch: from alethic pluralism to localism

Nikolaj J. L. L. Pedersen and Michael P. Lynch argue for a form of localism that Lynch names 'domain-specific logical pluralism' (DLP), in Pedersen (2014) and Lynch (2008), for instance. I group them together because of how similarly motivated they are, namely, they try to connect alethic pluralism with domain-specific logical pluralism. That is, the idea that truth varies across domains (the property of being true or the way propositions are true, for instance) with the idea that this variation of truth forces a variation on logic. Here is how Pedersen frames it:

> features of the truth property of a domain play a crucial role in determining the logic of the domain. In particular, the truth properties of some domains have the feature of being epistemically constrained and go hand in hand with cases that deliver intuitionistic logic, while the truth properties of other domains have the feature of being epistemically unconstrained and go hand in hand with cases that deliver classical logic. In short, alethic pluralism yields logical pluralism.
>
> (Pedersen, 2014, p. 262)

Lynch, similarly, argues that, although there is no direct unavoidable argument from truth pluralism to domain-specific logical pluralism, one can indirectly make the connection like this:

> If there is more than one way to manifest truth, and some of the manifesting properties are epistemically defined properties like superwarrant, and some not, then different domains will admit of different manifestations of the consequence relation. And this means, among other things, that argument forms that are valid in some domains may not be so in others.All this of course, assumes that there is more than one way to play the truth-role. If there is not, then there may still be more than one consequence relation, but this will presumably be motivated by other things than a view about the nature of truth.
>
> (Lynch, 2008, pp. 134–135)

So, according to Pedersen and Lynch, the thing that characterizes a domain and distinguishes it from other domains is how the truth property is manifested within that domain or which truth property a domain has. This, in turn, might be the factor that determines correct reasoning within a domain, making it possible that correct reasoning and, therefore, logic, varies from one domain to another.

Since the argument from alethic pluralism to localism has been one of the most popular in the literature, especially in the literature having to do with theories about truth, we will dive into the details given by Pedersen and Lynch later on in section 2.1.4. Let me finish, now, by adding that this way of individuating domains, by looking at the truth property, could be developed more by linking truth properties with ontological/metaphysical properties, as Pedersen (2014) does, for instance. If we follow this path, we might end up in something not so far from da Costa's idea of domains individuated by kinds of objects. Thus, Pedersen argues that

> if one grants that there is both a correspondence domain and a superwarrant domain [i.e., a domain with an epistemically constrained truth property], one should also grant that there is a domain with respect to which one is committed to metaphysical realism and a domain for which one is not thus committed. But, if one is not committed to metaphysical realism, one must be committed to some other metaphysical view on the entities in the relevant domain. All in all, this amounts to a form of metaphysical pluralism—or at the very least, it seems to be very congenial to a form of metaphysical pluralism.
>
> (Pedersen, 2014, p. 271)

Therefore, domains would be ultimately individuated by the ontological/metaphysical properties, which would then make truth manifest in different ways and, finally, this different truth manifestations would impose different consequence relations.

One can see that there are important localist proposals in the literature. Despite the fact that the philosophy of logic has not been very interested in logical localism (maybe, as I said, because the dominant position has been Beall and Restall's logical pluralism and other authors have tried to challenge it) we can find, already from the beginning of pluralistic proposals about logic, historical and relevant contributions with a localist spirit. Moreover, the literature on theories of truth and, in particular, on alethic pluralism, has clearly had a tendency to favour localist implications with respect to logic, as the natural philosophical position for an alethic pluralist. However, one could be a localist about logic without committing to pluralism about truth.

In the next section, I want to propose a redefinition of the theoretical options and the conceptual map regarding the debate around logical pluralism, broadly understood.

Redefining the conceptual map

What I have presented so far yields a picture of the debate around logical pluralism that allows, I believe, for a new conceptualization. I have tried to frame the localist thesis by contrasting it to globalism and distinguishing it from Beall and Restall's pluralism, which, in turn, I have contrasted with monism.

Thus, the localist thesis states that there is a multiplicity of domains of discourse, with possibly different criteria of correct reasoning, that require adopting different logics. Globalism, on the other hand, is the position defending that the application of logic is global, in the sense that logical laws and valid arguments must be applicable regardless of the content, the subject-matter or the domain of reasoning. Under the assumption that there is a canonical application of logic and that this is the application to reasoning (assumption that I made following Priest (2006)), this means that localism implies, contrary to globalism, that there are sub-canonical applications (since different domains of reasoning require different logical theories).

Now, if we allow pluralism and monism to be theses about the plurality or uniqueness of legitimate logics with respect to a (sub)canonical application, we get a richer conceptual framework. As far as I know, Haack (1978) is the first (and only?) to conceptualize something similar (but only combining localism/globalism with pluralism and with slightly different senses). In this case, we get four theoretical positions:

- *Global Monism*: there is just one correct logic and it is neutral with respect to the domain to which it is applied.
- *Global Pluralism*: there are a variety of logics that are equally correct and their application is global, i.e. independent of the objects of reasoning.
- *Local Monism*: Different domains of discourse require different logics, but there is only one correct logic for each domain.

- *Local Pluralism*: Different domains of discourse require different logics and there might be various equally correct logics for a given domain.

Then, on one hand there is the issue of whether the canonical domain is unique and homogeneous, which would support globalism, or whether there are different canonical or sub-canonical domains. On the other hand, one can argue that given a canonical application, there can only be a unique correct logic, maybe because the properties of the domain of application are such that only allow for one possible formalization and interpretation. The other option is to claim that given a canonical domain, there are still different legitimate ways in which one can capture what follows from what. This might be because of some unsettledness in the concept of consequence within that domain, because there are different legitimate pragmatic ends (e.g. preserving truth or preserving relevant information), or whatnot. So, this makes the localism/globalism and pluralism/monism distinctions orthogonal to each other.

Priest's position seems to correspond clearly to global monism, since he defends both that there is just one correct logic and that it works come what may. That is, independently of the domain of reasoning to which it is applied. Beall and Restall's position, however, does not so obviously belong to a single option. According to Field's interpretation, for instance, Beall and Restall's pluralism, though interesting, 'falls *far* short of the kind of pluralism that says that advocates of apparently competing *all-purpose* logics don't really disagree'(Field, 2009, p. 346). Putting it in our terms, Field claims that Beall and Restall's pluralism is not one in which every equally correct logic applies globally. I do agree with Field in this respect. To my mind, the option that better fits Beall and Restall's account is local pluralism, allowing that some domains to which different logics apply might overlap. Take, for example, the domain of mathematics. On Beall and Restall's view, it makes sense to use both intuitionistic and classical logics within mathematics, while relevant logic has no clear application there (Beall and Restall, 2000, p. 485). In this sense, at least relevant logic would not be global. Nevertheless, their goal seems to be global pluralism: 'we are not *relativists* about logical consequence, or about logic as such. We do not take *logical* consequence to be relative to languages, communities of inquiry, contexts, or anything else' (Beall and Restall, 2006, p. 88).

Notice that the pre-theoretic plausibility of each option does not appear to be the same. Among the great quantity of logical systems that have been developed, it would be enough for one of them to be the best system to capture correct reasoning within a specific domain and not be adequate to capture it within other domain for localism to be true. Globalism, instead, requires that for every possible domain of reasoning, a given logic is the one capturing what follows from what. But, as I said above, we are not in a situation to tell how many other domains of reasoning we are going to get or how varied they can be. We could develop physical theories, biological theories, mathematics, etc. that have different standards of consequence. But, for globalism to be true, every possible such domain would have to fall under the rules of a logic system.

For the pluralism/monism debate, something similar seems to happen. Even if for the majority of the domains a single logic is the best candidate to capture reasoning within that domain, it would be enough with one domain in which there are different legitimate options for pluralism to be true. In any case, the philosophical view that concerns me most for the purposes of the thesis is localism, whether it is local pluralism or local monism. The arguments in favour of it and the challenges that it faces are independent of the other theoretical positions.

2.1.4 Reasons in favour of localism

Before presenting the main challenge to localism, I want to delve a bit more into the reasons in favour of it. I think that we can find reasons coming from the literature on alethic pluralism and also independent reasons for localism.

First, I already mentioned above that the localist thesis has some *prima facie* plausibility stemming from the empirical observation of the practice of logic, specially philosophical logic. New logical systems with canonical applications arise at a stretch. For avoiding or solving a paradox, for formalising reasoning about future contingents, about vague terms, and so on and so forth. Philosophical logic provides a wide variety of canonical applications that need not be all successfully met by a single logic. Of course, universality and homogeneity might be theoretical desiderata worth pursuing, but globalism seems just too dogmatic given the empirical evidence.

This first intuition is what many times is developed into more detailed arguments by the different localist proposals. This is what Shapiro, for instance, does for defending his localism. He identifies cases that involve non-classical modes of reasoning, in his case, intuitionistic and paraconsistent structures, and argue that classical logic is not adequate for those cases.

Batens, does something similar, as we saw, but in his case finding legitimate applications of classical logic, against the view that all reasoning is paraconsistent.

One thing that it is also worth noticing, is that the candidate systems that I have been considering to be sub-canonically applied, although non-classical, are still structural. To clarify,

> *structural properties* are those properties of a logic which concern general features that only depend on considering premises and conclusions as a *manipulable structure of unstructured objects* (for example, reflexivity is a structural property in that it only depends on the premise and conclusion being the same object). *Classical logic* and *many non-classical logics* (for example, intuitionist logic) have a series of noteworthy structural properties: *reflexivity, monotonicity, transitivity, contraction, commutativity* and others. A logic is *substructural* iff it lacks one of the properties in that series. (Zardini, 2018, p. 241)

But it seems a tendency nowadays, within the practice of philosophical logic, to look for substructural solutions to different philosophical problems, like semantic paradoxes. This is, indeed, how E. Zardini, in Zardini (2018), argues against Beall and Restall's pluralism and in favour of what he calls 'regionalism', which is, precisely, another way of referring to our localism. So, in Zardini (2018), we have another very interesting example of how to provide independent reasons for localism, namely, by proposing philosophically significant problems, each of which motivating the rejection of a different structural property.

Let me illustrate the strategy with what might be the most striking case to some people: the property of reflexivity. It is quite common to find arguments in the global monist side to the effect that there is a core of valid inferences that every logic with some interesting application to the analysis of correct reasoning has in common. This core of valid inferences is supposed to be the One True Logic and, among the inferences that some believe to be universally applicable, the most certain one is usually taken to be reflexivity, i.e. $\alpha \vdash \alpha$. However, Zardini (2018) argues that under some epistemic and metaphysical requirements on logical consequence, the circularity that seems to be involved in reflexivity becomes problematic. For instance, one might argue for a tighter connection between validity and transmission, in the sense that a valid argument could provide one with further justification for believing the conclusion. But, in the case of reflexivity, it is difficult to make sense of situations in which the inference $\alpha \vdash \alpha$ provides further justification for α. It seems that the only justification that there is for it is the same as the justification we might have for the premiss. That is, we do not get further justification for α by inferring it from α (Zardini, 2018, p. 242).

The general trend of these strategies for motivating localism, then, is to make the case for the relevance of some domain of reasoning, show that the domain exhibits singular features with respect to what follows from what and, finally, argue that this consequence relation is best captured by this or that logical system.

The second way in which localism has been motivated in the literature is indirect: by appealing to independent reasons in favour of alethic pluralism and later arguing that alethic pluralism provides strong reasons in favour of a localist view of logic.

C. Wright was the first proponent of a form of alethic pluralism based on the idea that truth properties vary across domains and that these properties can be identified by their compliance with a set of platitudes about truth. Thus he put his first insights on the topic:

> The proposal is simply that any predicate that exhibits certain very general features qualifies, just on that account, as a truth predicate. That is quite consistent with acknowledging that there may, perhaps *must* be more to say about the content of any predicate that does have these features. But it is also consistent with acknowledging that there is a prospect of *pluralism* —that the more there is to say may well vary from discourse to discourse. (Wright, 1992, p. 38)

One of the motivations for alethic pluralism comes from the realization that there is no single truth property that naturally fits every domain of discourse. Correspondence theories seem a good candidate when referring to middle-sized objects, for instance, but are not that plausible for moral discourse or mathematics, at least if one has antirealist intuitions about them. In that case, the property of truth has to be epistemically constrained, i.e. 'truth cannot extend beyond our epistemic reach' (Pedersen, 2014, p. 265). This varying plausibility across domains of candidate truth properties is, as we said above, what is known as the scope problem. On the face of it, the pluralist explanation of the scope problem is the following: there is not a single and unique truth property, T, which all true sentences share across domains of discourse. Instead, truth consists in different properties, $T_1, \ldots, T_n$, in such a way that different true sentences from different domains might be true in virtue of having different truth properties from the plurality (cf. Pedersen and Wright (2018)).

Obviously, there are many intricacies when providing the details for a particular pluralist theory of truth, because there are intuitions in favour of the uniqueness of a property of truth that have to be explained. Wright (1992), for instance, argued in favour of a platitude-based strategy, such that every truth property satisfies the set of platitudes while still having some differential features that distinguish them from the other truth properties. Lynch (2008, 2009) offers one of the most popular accounts of alethic pluralism by proposing a functionalist theory of truth. One of the reasons for functionalism is, precisely, to accommodate the seemingly contradictory intuitions of uniqueness and plurality with respect to truth, namely:

> **Truth is One**: There is a single property named by "truth" that all and only true propositions share.
>
> [...]
>
> **Truth is Many**: There is more than one way to be true.
>
> (Lynch, 2008, p. 126)

Inspired by functionalist theories in the philosophy of mind, Lynch proposes to think about propositions being true in terms of having some property that plays the truth-role, which means that the property has, what Lynch calls, the *truish features*. Lynch's pluralism arises from the thought that propositions belonging to different domains may have different properties that play the truth-role. That is, two propositions of different domains may have properties M, for the first proposition, and N, for the second, each of which having the truish features while being different.

In any case, the more interesting point for my purposes, independently of how the pluralist theory is specified, is to note the argumentative path from alethic pluralism to logical localism, by means of which, evidence and reasons for the former might be taken to support also the latter. In Lynch (2008) and Pedersen (2014) we get such an argumentative endeavour. I have already quoted Lynch's view on the general structure of the indirect argument from alethic pluralism to localism (on page 26), which relies on there being epistemically defined properties that manifest truth, like superwarrant.

The truth property that Pedersen and Lynch call 'superwarrant', similar to what Wright calls 'superassertibility', is the epistemically constrained truth property that Pedersen defines as follows:

> A proposition p is superwarranted if and only if believing p is warranted in some state of information I_i and believing p is warranted in any state of information I_j that extends I_i. (*Superwarrant*)
> (Pedersen, 2014, pp. 263)

And Lynch defines (in a possibly different way, depending on the nature of the defeat):

> The proposition that p is superwarranted just when it is warranted without defeat at some stage of inquiry and would remain so at every successive stage of inquiry. A stage of inquiry is a state of information; it is always open to extension, and potentially incomplete. At any particular stage of inquiry, we may have no warrant for a proposition and no warrant for its negation. A *belief is warranted without defeat at a stage of inquiry* as long as any defeater for the belief at a given stage is itself undermined by evidence available at a later stage. In a sentence: *To be superwarranted is to be continually warranted without defeat.* (Lynch, 2008, pp. 124–125)

The epistemic nature of this truth property comes from from the fact that to have such property depends on some state of information and its potential extensions. If a proposition p is superwarranted it is because of the epistemic fact that at some state of information it is warranted and we have evidence undermining any possible defeater for this warrant at later stages of information.

Notice that it is this epistemic nature of the truth property, having to do with stages of information, which allows for incompleteness. Thus, it is very reasonable to conceive of stages of information in such a way that for some stage of information, I, and some proposition p, neither p nor $\neg p$ are warranted. Thus, neither p nor $\neg p$ are superwarranted, so neither of them has the truth property and, hence, $p \vee \neg p$ does not have the truth property, i.e., it is not superwarranted.

Now, it is clear why Pedersen and Lynch have tied the truth property of superwarrant, manifested in epistemically constrained domains, with intuitionistic logic. If there are domains in which $p \vee \neg p$ is not true for every p and every stage of information, then the logic of that domain should not take $p \vee \neg p$ to be valid. Notice, besides, that for this p, if one shows that $\neg\neg(p \vee \neg p)$ is true in I, then we would have that there is a state of information in which $\neg\neg(p \vee \neg p)$ is true while $p \vee \neg p$ is not. Hence, in the logic of that domain $\neg\neg(p \vee \neg p) \nvdash p \vee \neg p$, so more generally, $\neg\neg\alpha \nvdash \alpha$. But, notice that $\neg(p \vee \neg p)$

can be true neither in I nor in any state of information extending I, because we are not allowing for inconsistent states of information. So, $\neg\neg(p \vee \neg p)$ is warranted in I and in any state of information extending I. Therefore, we get that, for that epistemically constrained domain, there is a proposition p and a state of information I such that $\neg\neg(p \vee \neg p)$ is superwarranted, i.e., true, but $p \vee \neg p$ is not.

Therefore, the logic of that epistemically constrained domain should give us $\nvdash \alpha \vee \neg\alpha$ and $\neg\neg\alpha \nvdash \alpha$. This argument, developed by Lynch and Pedersen, strongly suggests that the logic that goes with superwarrant is not classical logic but intuitionistic logic.

But, on the contrary, if we also have the epistemically unconstrained, complete and consistent domain, with correspondence truth property and a strong form of the principle of bivalence (i.e., every sentence is either true or false but not both), then, we will have a domain whose logic is, most likely, classical logic. This would show that the legitimacy of different truth properties for different domains justifies that there are different domains that require different logics. Hence, localism.

We have just considered two kinds of domains with different truth properties but, arguably, one could make a defence of paraconsistency and the virtues of adopting a paraconsistent logic, by a similar strategy. For instance, one could argue that in a domain of discourse like humour, being true might be related to the fact that some people take it as true.With such a low standard for truth, it is very likely that both the proposition that some joke is funny and its negation are both true. So, it seems at least plausible that some interesting logic systems, the ones that are usually invoked as being good candidates to be canonically applied, can be motivated for different domains with a similar strategy, namely, in virtue of being the logic that better fits a given truth property of a domain.

We will see in the next section that this connection of alethic pluralism and localism works for motivating localism, but also carries over to the problems and challenges of these philosophical theories. That is, there is a strong analogy, if not identity, between the problems that are usually attributed to alethic pluralism and localism. While this might itself be problematic, I take it as a possible advantage since I might kill two birds with one stone. Here I am aiming just at localism but, if the other bird falls, it will have been a happy accident.

2.2 Main Challenge: The Problem of Mixed Inferences

We have just argued for the plausibility of logical localism and have given reasons in favour of it. However, there is a crucial empirical fact that would seem to count in favour of globalism and that localism has to account for, namely, that we do reason across domains. Thus, localist theses, intuitive as they might be, have to face an important challenge; a challenge that Priest (2006) raises and that I will summarise as follows:

> **The Problem of Mixed Inferences**: one might defend that there are a variety of domains that require different logics. But there are cases in which one reasons about the interaction of different domains, with premises about different kinds of objects coming from those domains. What kind of logic do we use, then? An underlying logic for both domains? This would give reasons for thinking that there is a logic of global application. Maybe a new logic specific for that domain of interaction? But which one? The intersection of the logics involved in each of the interacting domains might be too weak to be of any utility. Moreover, it should be taken into account that if we start trying to mix the connectives of different logics some of them may collapse. The intuitionistic conditional, for instance, collapses into the classical conditional under the presence of classical negation (Priest, 2006, p. 199).

This, I believe, encompasses all the problems that localism has to answer, specially, for the technical approach of combining logics that I will be taking and that is threatened by the collapse of connectives. The more specific problems that we find by dissecting this main challenge, are the problems of mixed compounds and collapse theorems. Let me clarify that the problem of collapse theorems does not usually appear, even mentioned, in the more philosophical literature[10]. But, since the approach that I will be taking for answering the challenge is that of the methods for combining logics and these methods are threatened by collapse theorems, I include it in characterizing the problem of mixed inferences.

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[10] In fact, even the more logically oriented authors, including Priest, despite mentioning the collapse theorems, take them as knockdown arguments, obviating, or ignoring, the fact that there are methods for combining logics designed to avoid the collapse. Hopefully, this dissertation is also useful for closing the gap between those seemingly isolated worlds.

With respect to the problem of mixed compounds, one could argue that it is a sub-problem of mixed inferences. As we present the problems it will be clearer why, but simply noticing that mixed inferences can have as premises or conclusions mixed compounds makes it clear enough[11]. Moreover, it is worth pointing out, as Lynch does, that

> [the problem of mixed inferences] does not arise solely for those who have come to DLP [domain-specific logical pluralism, a.k.a., localism] via truth pluralism. The issue of how to deal with mixed inference and compounds is an issue for any logical pluralist who takes it that distinct logics operate in different domains of discourse.
>
> (Lynch, 2008, p. 137)

The first one who put forward the problem of mixed compounds, as a challenge to alethic pluralism, was T. Williamson in Williamson (1994), where he reviews Wright's *Truth and Objectivity*, though some people also refer to Tappolet (2000) since she raised the same problem in a notorious reply to Beall (2000). The problem of mixed compounds for alethic pluralism goes as follows: a sentence like ‘Alex killed 13 people and killing for fun is wrong’ can well be true. However, it is not obvious how the alethic pluralist could explain the truth of the conjunction. Surely, she can take the first conjunct to be true$_1$, T_1, (maybe in a correspondentist sense) and the second conjunct to be true$_2$, T_2, (maybe a notion of truth grounded in social agreement). But, then, in which sense is the conjunction true? Which is the truth predicate that applies to it? It seems that it can be neither T_1 nor T_2, so maybe there is another truth predicate that applies to the conjunction. But, it is plausible to think that the conjunction will be true in that further way if and only if the conjuncts are true in that very same way too. Why do we need then the other truth predicates T_1 and T_2?

The version of the problem, as applied to logical localism, is analogous. Basically, the challenge is to answer which the logic of a compound proposition should be. This might seem innocuous, but consider a compound proposition like $p_c \vee \neg q_p$, where p_c is a classical proposition and q_p is a paraconsistent proposition. If the logic that governs the domain of p_c is classical

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[11] Another option is to consider mixed compounds as a problem for alethic pluralism only, arguing that the question about the correct logic only makes sense for inferences, not for propositions. In any case, giving a logic for mixed inferences (with mixed compounds) will, most likely, require solving how the semantic value of a mixed compound is determined or explaining which rule is governing the introduction or elimination of the main connective in a mixed compound.

logic and the logic that governs the domain of q_p is Priest's logic of paradox, **LP**, which is the logic of the compound? The answer is not trivial at all, since the way we answer it will affect, say, whether we can use disjunctive syllogism as a valid inference or not (because it is not valid under **LP**). This is the sense in which the problem of mixed compounds is a sub-problem of the problem of mixed inferences, in the versions directed against localism. Responding to the problem of mixed inferences requires having an account of the logic of mixed compounds, to begin with[12].

The problem of mixed inferences was, indeed, first put forward by Tappolet in Tappolet (1997) and it is based on what she takes to be a central platitude about truth: 'truth is what is preserved in valid inferences' (Tappolet, 1997, p. 210). But, then, she proceeds to consider the following valid inference:

$$\frac{\begin{array}{l}\text{Wet cats are funny}\\ \text{This cat is wet}\end{array}}{\text{This cat is funny}}$$

Tappolet's objection, then, goes like this:

> The validity of an inference requires that the truth of the premises necessitates the truth of the conclusion. But how can this inference be valid if we are to suppose with Crispin Wright that two different kinds of truth predicates are involved in these premises? For the conclusion to hold, some unique truth predicate must apply to all three sentences. But what truth predicate is that? And if there is such a truth predicate, why isn't it the only one we need?
>
> (Tappolet, 1997, p. 210)

Thus, Tappolet challenges the truth pluralist with a trilemma: (a) either one denies that mixed inferences are valid, (b) accepts that there is a generic truth property that all domain-specific truth properties have in common (which would make this specific ones redundant) (c) or rejects the standard account of validity as necessary truth preservation.

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[12] Again, one might prefer to reserve the naming 'problem of mixed compounds' to the version affecting alethic pluralism. Then, the version of the problem affecting localism would be just a sub-problem of the problem of mixed inferences. I believe this is just a terminological issue but the reader should keep in mind that the problem of mixed compounds is generally regarded as a problem for alethic pluralism.

Many authors have replied to Tappolet by trying to make the case for a possible satisfactory way out of the trilemma. Especially interesting responses are those of Beall (2000); Cotnoir (2013); Lynch (2008, 2009) and Yu (2017). But, as I said before, I am more interested in solving the problem in its logical localist version. That is the aim of all the combination mechanisms that I will be presenting later on. If, as a side effect, we get an interpretation of the solution that is also satisfactory for alethic pluralism, it would be the icing on the cake.

So, the version of the problem of mixed inferences that is directed against localism goes, roughly, as follows: suppose that there are (at least) two components, within the premises or conclusion of an argument, belonging to different domains whose logics are L_1 and L_2, respectively. Then, which is the criterion of validity for the argument? That is, which is the logic that captures correct reasoning for that mixed domain?

2.2.1 Some attempts to solve the problem of mixed inferences

Although not much, there have been some authors who have attempted to meet the challenge. However, very few of them have been systematic enough in their effort, with the exception of (maybe among others that I am not aware of) Cotnoir (2013); Lynch (2008, 2009); Wrenn (2018) and Yu (2017, 2018). Let me present their accounts and explain why I take them to be unsatisfactory.

Cotnoir's algebraic account

The aim of A. Cotnoir in Cotnoir (2013) is, first and foremost, to solve the problem of mixed inferences in the version directed against alethic pluralism that Tappolet raises. However, after providing an account for that, he goes on to extend the idea to make room for variation in each domain's logic.

So, Cotnoir starts by modelling the idea that each domain has a distinct truth property. In order to do this, he defines semantic values, V, to be n-tuples, n being the number of domains and each element of the tuple being either 1 or 0:

$$V = \{\langle a_1, ..., a_n \rangle |\text{ each } a_i \in \{1, 0\}\}$$

So, having a 1 in the i-th position means that the proposition is in the i-th domain and it is $true_i$. While having a 0 in the i-th position means either that the proposition is not in the i-th domain or that it is $false_i$.

Further, assume that atomic propositions can have at most one truth property, so there can only be a 1 in the tuple representing the semantic value of an atom. He, then goes on to provide a classical account of connectives, in the sense that for each component of the tuple, the negation inverts the values, while conjunction and disjunction are minimum and maximum operations respectively[13]. That is,

- For all proposition A, if $v(A) = \langle a_1, ..., a_n \rangle$ then $v(\neg A) = \langle 1-(a_1), ..., 1-(a_n) \rangle$.
- For all proposition A and B, if $v(A) = \langle a_1, ..., a_n \rangle$ and $v(B) = \langle b_1, ..., b_n \rangle$ then $v(A \wedge B) = \langle min(a_1, b_1), ..., min(a_n, b_n) \rangle$.
- For all proposition A and B, if $v(A) = \langle a_1, ..., a_n \rangle$ and $v(B) = \langle b_1, ..., b_n \rangle$ then $v(A \vee B) = \langle max(a_1, b_1), ..., max(a_n, b_n) \rangle$.

As Yu (2017) rightly notices, these valuation functions already yield unwelcome results. First, notice that if, say, an atomic proposition is true, in the sense specific to its domain, the negation of the proposition will be true in the other domains represented in the tuple. For instance, suppose we are considering a mathematical domain and an ethical domain. Take the proposition A = '1 + 1 = 2'. The value of this atomic proposition will be $v(A) = \langle 1, 0 \rangle$ assuming that the first element of the tuple represents the value in the mathematical domain and the second the value of the ethical domain. It is already questionable that the ethical truth-value of that proposition is false, but it is even worst that the negation of the proposition, i.e., $v(\neg A) = \langle 0, 1 \rangle$, is ethically true. If mathematical propositions are not apt for the ethical truth predicate, why should the negation of a true mathematical sentence be ethically true? And how could one even make sense of the fact that 'one plus one does not equal two' is ethically true?

A second unwelcome result comes from conjunction. Again, consider the mathematical proposition A = '1 + 1 = 2' and the ethical proposition B = 'Killing babies for fun is wrong' with valuations $v(A) = \langle 1, 0 \rangle$ and $v(B) = \langle 0, 1 \rangle$. The conjunction of these propositions, according to Cotnoir's

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[13] Here I am following Yu's characterization of Cotnoir's account, in Yu (2017), since it is a good summary of the proposal.

proposal, is false! It is false in both of the relevant senses, i.e., $v(A \wedge B) = \langle min(1,0), min(0,1)\rangle = \langle 0,0\rangle$. So the valuation making each of the conjuncts true in their relevant domains makes the conjunction false in every sense.

Cotnoir, proposes a solution to the problem with negation, by introducing a third truth-value, $\frac{1}{2}$, and moving to a non-classical logic[14]. The idea is that atomic propositions get at most a 1 in the tuple and for every other place they get the value $\frac{1}{2}$, capturing the idea of being undefined for those domains. With this correction, the problem with negation is alleviated, because if a proposition has value $\frac{1}{2}$ in the i-th place, it will also get that value if we negate the proposition. But, this repair comes at the expense of having renounced to classical logic and, in any case, it does not solve the problem with conjunction. If we have again, A = '1 + 1 = 2' and B = 'Killing for fun is wrong', now with valuations $v(A) = \langle 1, \frac{1}{2}\rangle$ and $v(B) = \langle \frac{1}{2}, 1\rangle$, the conjunction of these propositions is undefined, i.e., $v(A \wedge B) = \langle min(1, \frac{1}{2}), min(\frac{1}{2}, 1)\rangle = \langle \frac{1}{2}, \frac{1}{2}\rangle$. This is equally counterintuitive and unsatisfactory.

On top of these problems with the truth-value functions, there is a further limitation in Cotnoir's account that Yu does not identify, since it is a limitation of his account too, as I will explain in a moment. So, what Cotnoir tries to do is to give a solution to the problem of mixed inferences *for alethic pluralism*. The reason for switching to tuples in order to represent truth-values, is that he wanted to be able to handle different truth-properties, each represented by a different position in the tuple, *within a single logic system*. First, the consequence relation that results, is classical logic (see (Cotnoir, 2013, pp. 570–571)). Then, he introduces a third truth-value and gets a paracomplete and paraconsistent system. And, finally, he introduces Heyting algebras in the tuples and gets intuitionistic logic (see (Cotnoir, 2013, pp. 575–577) for the details). In that way, he claims that localism is accommodated in the broader picture of alethic pluralism. But this is only halfway true. He has attempted to handle different truth-properties with different logics, but with a *single* logic each time! there is no mention of mixed inferences and which the criterion of validity might be when we need to combine different logic systems.

Even a situation as simple as the following does not get an answer: suppose p_c is a classical proposition and q_i a proposition belonging to an epistemically constrained domain. Take the proposition that $\neg p_c \vee q_i$. Do we

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[14] We do not need to enter into many details. Just know that the resulting logic is the intersection of intuitionistic logic and Priest's logic of paradox.

have that $\neg p_c \vee q_i \vdash p_c \to q_i$? We find no criterion in Cotnoir's account and, therefore, no solution to the problem of mixed inferences in its version directed against localism. In fact, Cotnoir says that with his proposal, truth pluralists 'can allow that the logic of *unmixed inferences* can sometimes be domain dependent' (Cotnoir, 2013, p. 577, my emphasis).

Yu's logic for alethic and logical pluralists

We have just seen that A. Yu, in Yu (2017, 2018), rightly identifies some of the unwelcome results in Cotnoir's proposal, so his solution avoids those problems in a quite nice way. Yu's fundamental idea is that there is an isomorphism between domains, truth properties and falsity properties, that behave in the following way:

> there is a one-to-one correspondence between domains, domain-specific truth properties, and domain-specific falsity properties. Pure domains are associated with exactly one subject matter, while impure domains are associated with two or more subject matters. Where domains are either pure or impure, pure domains generate all domains. Each atomic sentence is assigned to exactly one domain. Negations are always assigned to the same domain as the negand, while conjunctions and disjunctions may or may not be assigned to the same domain as each operand, depending on whether or not the operands are assigned to the same domain. Each atomic sentence is then assigned a domain-specific truth value, where the relevant domain is the one it is assigned. The domain-specific truth values of negations, conjunctions, and disjunctions are determined by the domain-specific truth values of the operands. Logical consequence necessarily preserves domain-specific truth. (Yu, 2018, pp. 413–414)

In order to avoid distraction with the details of the proposal,[15] let me give a concrete example to roughly illustrate how it works: suppose we have a proposition about the physical middle-sized domain and a proposition about the ethical domain, say $p_c =$ 'the dog is at home' and $q_i =$ 'torturing is wrong'. Considering that these are pure domains, they can be combined in order to produce the impure domain of 'physical middle-sized and ethical'. Equally,

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[15] To clarify, the details are not important because the proposal is not designed for the problem of mixed inferences 'localism style' but for 'alethic pluralism style'. Otherwise, obviously, the details would be important, as happens with Lynch's and Wrenn's accounts.

the truth-values and falsity-values present the same structure. The physical proposition will take the values T_c or F_c, while the ethical proposition will be either T_i or F_i. However, if we make the conjunction of both propositions to get $p_c \wedge q_i$ that compound proposition will have the truth-values that correspond to its impure domain of 'physical middle-sized and ethical'. Let us call them T_{ci} or F_{ci}.

There are a number of worries with this type of proposal already known in the literature. In Edwards (2008) we find a similar view, which also relies on the intuition that compounds are true in a derivative sense (i.e., 'impurely true', using Yu's terminology) and, for instance, Cotnoir (2009) makes the case against the proliferation of truth-properties that such a view requires. One could argue back, following the literature on the metaphysics of fundamentality, that there is no problem with multiplying the truth properties as long as they are derivative, since a sparse ontology is only relevant at the fundamental level.

In any case, I reckon that Yu's account still has problems. Let me point out a very obvious one. Take p_c to be the previous proposition. By Yu's account, $\neg p_c$ is also from the physical domain and, therefore, $\neg p_c \vee p_c$ too. In fact, since it is a tautology, its truth-value is T_c for any interpretation. By an analogous reasoning $\neg q_i \wedge q_i$, which is a contradiction belonging to the ethical domain, always gets value F_i. Now, make the disjunction of these two propositions to get $(\neg p_c \vee p_c) \vee (\neg q_i \wedge q_i)$. According to Yu, this is a proposition belonging to the impure domain of 'physical middle-sized and ethical' and whose truth-values can only be T_{ci} or F_{ci}. However, this is a very odd result, since there is no doubt that this impure proposition is true in virtue of the same facts as $\neg p_c \vee p_c$ and, therefore, should be true in exactly the same way. In other words, the ethical proposition does not contribute anything to the truth of the impure proposition and, yet, it makes the impure proposition take a different truth property to that of the physical proposition in virtue of which is true.

Be that as it may, this would be a problem for his proposed solution to the problem of mixed inferences as applied to alethic pluralism. That is, it is a way of accommodating a variety of truth values, within *a single consequence relation and criterion of validity*. In Yu (2017) the underlying logic is classical and in Yu (2018), it is extended to allow for a non-classical underlying logic too. But, this is far from being a solution to mixed inferences in which we allow the action of more than one underlying logic within the premises through the conclusion. Yu has just given a way in which a non-classical domain can have a non-classical logic, but has not considered which is the logic of an argument with components governed by different logics. This is the same problem that Cotnoir has. They get non-classical logics for some domains with non-classical truth-values, but they do not offer a method for combining the logics of those domains in mixed inferences.

Lynch's modesty criterion

As we said above, Lynch is aware of the connection between the problem of mixed inferences in its alethic and logical versions and, moreover, notices that it is a problem for localism even if that localism does not come from alethic pluralism. So, once he has solved the problem for alethic pluralists by means of his functionalist account, or so he believes, he moves on to solve the problem for localism. The solution, Lynch claims, has two parts. First, the localist,

> being a pluralist after all, will take it that *within a domain*, what qualifies as the governing logic will be determined by what manifests truth in that domain. (Lynch, 2008, p. 137)

So, for instance, following Lynch's analysis, the logic governing the domain in which superwarrant manifests truth will be intuitionistic logic.

The second part is the more difficult one. Lynch defends that in order to evaluate the validity of a mixed inference we have to apply a criterion of *modesty*. To understand why, consider the next argument[16]:

Nix:

If it is not the case that offensive jokes are funny,
then snow isn't white
Snow is white

Offensive jokes are funny

Which we can formalize as:

$$\begin{array}{l} \neg p_i \to \neg q_c \\ \underline{q_c \qquad\quad} \\ p_i \end{array}$$

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[16] It is a modified version of the argument that Lynch and Wrenn also call 'Nix'. The propositions are different but the structure is the same. I just wanted to use a different exemplar to show the extent of possible instances.

Let us assume that the discourse about humour is an evaluative discourse, for which intuitionistic logic is the most appropriate, and that the discourse about middle-sized objects belongs to a domain for which classical logic is best. Let us further assume that our best theory about humour (maybe, together with our best ethical theory) does not decide upon whether offensive jokes are funny or not and, therefore, neither p_i nor $\neg p_i$ are true in the relevant sense, e.g., they are not superwarranted.

The question is, now, which logic should we use to evaluate the validity of Nix? if we go to the strongest logic, that is, to classical logic, we get the unwelcome result that Nix is a valid argument that does not preserve truth, since we have assumed that p_i is not superwarranted. So, the argument seems to be intuitively invalid. It looks like our only sound option is to use intuitionistic logic as the criterion of validity and, thus, evaluate the argument as invalid.

At the same time, there are other mixed inferences that look intuitively valid. Take for instance,

Mix:

$$\frac{\begin{array}{l}\text{Wet cats are funny}\\ \text{Either snow is white or wet cats are not funny}\end{array}}{\text{Snow is white}}$$

Which we can formalize as:

$$\frac{\begin{array}{l}p_i\\ q_c \vee \neg p_i\end{array}}{q_c}$$

This intuitive validity might be because the argument is a form of disjunctive syllogism, which is a valid argument both in intuitionistic and classical logic. So, Lynch's first option is to propose a modesty criterion stating that the default governing logic of a mixed inference is the weakest of the logics in play. Where, by definition, *a logic L is in play* for an inference when there is a propositional component in the argument which is governed by L. But, after noticing that we might have cases in which the logics in play cannot be ordered, in the sense that one is not an extension of the other, he proceeds to offer what he takes to be the correct modesty criterion, which I will call 'Modest$_1$'[17]:

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[17] The reason for the subindex is that Wrenn will propose another modesty criterion.

> Modest$_1$: where a compound proposition or inference contains propositions coming from distinct domains, the default governing logic is that comprised by the intersection of the domain-specific logics in play. (Lynch, 2008, p. 139)

Hence, a mixed inference is valid, according to Lynch, if all the logics in play for that inference count it as valid. This is a real criterion of validity for mixed inferences meant to solve the problem for localism. The problem is that it is so nice and simple that it has to be incorrect.

I believe it was at the end of my first year as a PhD student when I first came across Lynch's criterion of modesty and what I wrote for my future self was: 'think about possible counterexamples. The logic might be too weak in some cases'. So, after two years of not having followed my orders, I found out C. Wrenn's chapter "A Plea for Immodesty: Alethic Pluralism, Logical Pluralism, and Mixed Inferences" and, I must confess, I wanted to punch my past, present and future selves. As we are going to see in a moment, Wrenn does find counterexamples to Lynch's criterion, but, as it turns out, he also raises a beautiful challenge to any localist proposal and, in doing so, gave me the chance of coming up with something that, hopefully, is more interesting than finding some counterexamples.

Wrenn's challenge is the most systematic and detailed formulation of the problem of mixed inferences for logical localism and my proposed solution is so focused on his formulation that, I reckon, it deserves its own subsection.

2.2.2 Wrenn's plea for immodesty and the ultimate challenge for localism

The modesty criterion that Lynch proposed was the perfect target for anyone wanting to reintroduce the problem of mixed inferences for localism. This is, precisely, what Wrenn (2018) does, by coming up with a series of counterexamples to the modesty criterion and by posing the challenge to the localist in the form of a dilemma.

After introducing Lynch's alethic pluralism and its connection with logical pluralism, he states the problem of mixed inferences for DLP, i.e., for localism:

> given DLP, there are multiple relations of logical consequence that vary from discourse to discourse. So, we need to know *which relation*

> of logical consequence is relevant to evaluating a mixed inference such as Mix. Its premises and conclusion come from different discourses, with different, proprietary relations of logical consequence. DLP does not provide a discourse-neutral relation of logical consequence, and so we seem not to have a relation of logical consequence appropriate for the mixed inference itself. And with no such relation on the table, we have no way of accounting for the inference's validity. What is to be done? (Wrenn, 2018, pp. 393–394)

So, what Wrenn is demanding in this quotation is a criterion of validity on behalf of the localist that is relevant for evaluating a given mixed inference. Of course, the next step on his argumentation is to consider the only criterion that was available, namely, the modesty criterion by Lynch. This criterion, Wrenn will show, is just too modest, since there are intuitively valid mixed inferences that Modest_1 does not account for. Let us see the counterexamples that Wrenn offers:

Inessential Mix:

It's not the case that snow isn't white
Wet cats are funny
―――――――――――――――
Snow is white

The reason why this is a counterexample is that the rule of Double Negation Elimination is not intuitionistically acceptable, and we are assuming intuitionist logic is the logic that governs evaluative discourse, so if we follow Modest_1 we will declare the intuitively valid argument invalid. The name of the argument already hints at the fact that the second premiss does nothing for the validity of the argument. In fact, its whole purpose is to prevent the application of Double Negation Elimination. But, Wrenn goes on and offers another version in which the evaluative components play a more important role.

Essential Mix:

Either it's not the case that snow isn't white or wet cats aren't funny
Wet cats are funny
―――――――――――――――
Snow is white

Which we could formalize as:

$$\begin{array}{l} \neg\neg p_c \vee \neg q_i \\ q_i \\ \hline p_c \end{array}$$

So, while in Inessential Mix, the evaluative sentence was not essential to get the conclusion, in Essential Mix, the evaluative sentence is essential in the sense that, if we deleted the second premise, the argument would be invalid. What Wrenn claims, then, is that these type of cases constitute a clear source of counterexamples for Lynch's modesty criterion. Essential Mix seems an intuitively valid argument and, yet, by Modest$_1$, it is invalid.

However, Wrenn considers the possibility of refining the modesty criterion inspired by the structure of Essential Mix. Notice that we can decompose the argument like this:

1. $\neg\neg p_c \vee \neg q_i$
2. q_i
3. $\neg\neg p_c$ (Disjunctive Syllogism)
4. p_c (Double Negation Elimination)

That is, in both Inessential Mix and Essential Mix, there is a way of getting $\neg\neg p_c$ by the standards of Modest$_1$. But, then, this proposition alone, only belongs to a classical domain, so the application of DNE would be licensed. Thus, Wrenn's refinement aims to incorporate into the modesty criterion this idea of *chains of inference.*

> Modest$_2$: an inference $\Gamma \vdash \varphi$ is valid if there are sentences $\psi_1, \ldots, \psi_n$ such that each one of the arguments $\Gamma \vdash \psi_1$, $\psi_1 \vdash \psi_2$, …, $\psi_{n-1} \vdash \psi_n$, $\psi_n \vdash \varphi$ is valid in the intersection of all the logics that are in play in it.[18]

Thus, under this criterion, the intuitively valid mixed inferences Inessential Mix and Essential Mix get validated. In the case of Inessential Mix, we would first apply the fact that both intuitionistic and classical logics are

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[18] This is an equivalent and simplified version of Wrenn's criterion that my supervisor, José Martínez, suggested me. We worked on these ideas together and presented them at different places.

reflexive, so we get ‘it’s not the case that snow isn’t white’ and, since this is a purely classical proposition, the only logic that is in play now is **CL**. Then, we can apply DNE and get ‘snow is white’. For Essential Mix, since Disjunctive Syllogism is valid in both **IL** and **CL**, we can apply it to get $\neg\neg p_c$ and, again, this being a purely classical proposition, we can eliminate the double negation.

If this was all, it would be good news for localism. There would be a criterion of validity for mixed inferences that fits well with our intuitions on the validity of them. But, of course, Wrenn comes up with the ultimate counterexample that, allegedly, is going to make the alethic pluralist abandon localism for ever.[19]

Disjunctive Mix:

$$\frac{\text{Either it's not the case that snow isn't white or wet cats are funny}}{\text{Either snow is white or wet cats are funny}}$$

Formalized like:

$$\frac{\neg\neg p_c \vee q_i}{p_c \vee q_i}$$

This is an intuitively valid argument that both Modest$_1$ and Modest$_2$ invalidate. The reason with Modest$_2$ is that there is no intermediate step at which we can get that is purely classical and that would allow us to eliminate de double negation and then introduce the disjunction. Therefore, the intuitively valid argument is invalid according to the best criterion that we have.

One could try to modify the criterion and make it stronger, but, then, Wrenn plays the card of Nix and warns that

> if we evaluate inferences according to any logic *stronger than* the logic that applies to any of their components, then we will wind up endorsing certain conclusions that are not true in their home discourses. (Wrenn, 2018, p. 403)

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[19] There is another minor problem with the chains of inferences that Wrenn mentions. It is the fact that if some of the logics in play are not transitive, then we will not be able to build the chains. In this dissertation, though, I will restrict myself to finding solutions for transitive logics.

Being quite pessimistic about the possibility of coming up with a solution, Wrenn raises a challenge for the localist[20] in form of a dilemma:

> **The Dilemma**: either we keep some modest criterion of validity for the mixed inferences, not counting some intuitively valid arguments as valid (e.g., Essential Mix, Disjunctive Mix), or we go beyond modesty (for instance, taking a stronger logic than the one in the intersection), and accept as true some untrue sentences (Nix).

His recommendation is clear: 'alethic pluralists should steer clear of domain-specific logical pluralism' (Wrenn, 2018, p. 388). What I aim to do is to show that there is a possible way out of the dilemma involving a finer analysis of the arguments and the application of some methods for combining logics; specially, the method of juxtaposition and some improvements of it.

Let me recall, though, that the approach of combining logics has its own challenges, like avoiding the collapse of the connectives. In fact, there are more general and relevant things to consider with respect to how the connectives interact. As I see it, the area of combining logics is a natural space in order to look for solutions to the problem of mixed inferences by implementing the technical machinery that has been developed in the field. However, the impression that one gets going over the more philosophical literature is that there has not been interest in, or knowledge of, the methods for combining logics. A clear symptom of this fact is that the people that mention collapse theorems usually take them to be knockdown, unavoidable, objections to combining logics and connectives. Here is what Priest claims, for instance:

> [...]vernacular negation cannot be ambiguous between intuitionist and classical negation. If it were, as I have argued, we could have two formal negations. But it is well known that in the presence of classical negation, many other important intuitionist distinctions collapse. For example, the intuitionist conditional collapses into the classical conditional. (Priest, 2006, p. 199)

The other side of the mirror is equally unspoiled and fruitful, as I see it. In the literature of combining logics there has been an interest in philosophical problems, like the naturalistic fallacy and other interactions among

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[20] The challenge is independent of alethic pluralism, because even if the domains are not individuated by the truth properties and there is some other way of individuating them, we still lack a criterion of validity for mixed inferences that fits our intuitions.

modalities. The collapse theorems were also motivated by reflecting on classical logic and non-classical ones. More concretely, on imagining a dialogue between a classical logician and an intuitionist, willing to cooperate in order to understand what the other party means. However, mixed inferences constitute a new philosophical challenge to combination mechanisms, in the sense that, if logical localism is correct, then there are situations of mixed reasoning that might generate interesting interactions between logic systems that have not been considered yet. I will try to show that this is, in fact, the case and that the philosophical problem of mixed inferences might trigger important developments and research lines for the combinations of logics.

Chapter 3

Mixed Reasoning and Combining Logics

3.1 Interaction Principles

In situations of combined reasoning, such as the ones represented by the cases of mixed inferences, one should expect some interaction between the logical systems that are being combined. If those logical systems aim at capturing the modes of reasoning of given domains, it is reasonable to expect that the combination of the logical systems will capture some interaction of the combined reasoning, otherwise it would not be a case of mixed reasoning in the first place.

One of the most common references when dealing with interaction principles is David Hume's *naturalistic fallacy,* namely, the thesis that from a factual statement one cannot deduce a normative one, that is, that from 'what is' one cannot derive 'what ought to be'. To put it in terms of modern modal logic, Hume's thesis constitutes an objection to interaction principles such as,

$$p \to \bigcirc p$$

stating that, if p is the case, then one ought to p[1].

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[1] One could say that this is a limit case of an interaction principle, since there is no modal operator in the antecedent. We will see in a moment, though, that it falls under the definition of bridge principle, which is a type of interaction principle. In any case, it is reasonable to argue that the lack of a modal operator is irrelevant, since the antecedent is expressing a factual modality and this is what interacts with the deontic one.

Another historical reference on these matters is the ‘ought-implies-can’ thesis, usually attributed to Immanuel Kant. This is the thesis that if something is obligatory then it must be possible, formalized as

$$\bigcirc p \to \Diamond p.$$

And yet another interaction principle that is often times included in epistemic logics comes from Plato’s characterization of knowledge as ‘justified true belief’. Thus, this understanding of what constitutes ‘knowledge’ motivates having an interaction principle capturing that if a subject x knows that p, then p is the case,

$$K_x p \to p.$$

These interaction principles fall under the subcategory of ‘bridge principles’, which were understood as principles linking factualities to norms and, more generally, as principles linking different modalities. Thus, we can define more formally a bridge principle as an axiom schema which has at least one occurrence of an schematic letter under the scope of a modal operator, $\star$, and at least one occurrence outside the scope of $\star$.

One thing to notice is that these bridge principles vary in their analyticity, so while some of them might be highly desired in virtue of their meaning and how the interaction is established, others might be more problematic and in need of some philosophical justification, or even rejected. The usual interdefinition between necessity and possibility, namely, $\Box\alpha \equiv \neg\Diamond\neg\alpha$, could be included among the analytic bridge principles. The bridge principle, previously mentioned, connecting the knowledge of some proposition with the truth of that proposition certainly needs some philosophical justification[2], and an axiom stating that if p is the case then some finite agent x knows that p is clearly bad and rejected by any reasonable epistemic logic.

As we will see later on, some of these bridge principles can be avoided by some of the methods for combining logics. This can be seen as a positive consequence of the combination mechanisms, as one could add, after the combination, the bridge principles that one desires in the combined logic as additional axioms, while avoiding the problematic ones. However, one might take this procedure to be quite *ad hoc* and, therefore, it is an interesting issue to wonder whether one could combine given logical systems in such a

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[2] One could argue that Newton knew some laws of physics, despite those laws not being strictly true.

way that the desired bridge principles arose in the very combination process (Schurz, 1991, p. 46)[3].

But, let me now elaborate and widen more the notion of bridge principle. For now, we have only considered bridge principles to be interactions between variables under the scope of a modal operator and variables outside their scope. However, I reckon, following Carnielli and Coniglio (2007), that we should think of bridge principles as principles establishing, more generally, connections or interactions between *connectives*. But not every interaction between connectives will count as a bridge principle. In a sense, we want these interactions to be "new", meaning that we did not have these in the original logics being combined. Thus, we take bridge principles to be "any interactions (i.e., derivations) among distinct logic operators which are not instances of valid derivations in the individual logics being combined" (Carnielli and Coniglio, 2007, p. 8).

Take, for instance, the following mixed inference:

$$\frac{\begin{array}{l}\text{Wet cats are funny}\\ \text{Fitz Roy isn´t the highest mountain or wet cats are not funny}\end{array}}{\text{Fitz Roy isn´t the highest mountain}}$$

Assume that we think **CL** is the logic of the physical domain and that **IL** is the logic of the domain about humour. So, let us formalize the argument distinguishing the connectives of each logic by the subindexes c and i.

$$\frac{\begin{array}{l}p_i\\ \neg_c q_c \vee_i \neg_i p_i\end{array}}{\neg_c q_c}$$

We can see that there is an interaction between connectives from different logics, but this is not a bridge principle, since it is an instance of a valid derivation in **IL**, namely, Disjunctive Syllogism. However, had we translated the argument like this,

$$\frac{\begin{array}{l}p_i\\ \neg_c q_c \vee_c \neg_i p_i\end{array}}{\neg_c q_c}$$

we would have delivered a bridge principle, since this argument captures an interaction between classical and intuitionistic connectives, but it is no longer an instance of a valid argument in **IL**, nor in **CL**.

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[3] I will show later on that my solution to mixed inferences meets this desideratum.

Another example in order to further illustrate what characterizes bridge principles among interaction principles is given by Carnielli and Coniglio (2007). Consider the combination of the logic of classical conjunction, $\mathcal{L}_\wedge$, and the logic of classical disjunction, $\mathcal{L}_\vee$. In this logic the derivation

$$p \wedge q \vdash (p \wedge q) \vee r$$

is an interaction principle which is not a bridge principle, since it is a substitution instance of

$$p \ \vdash p \vee r$$

that is a valid derivation in $\mathcal{L}_\vee$. However, the distributive law of conjunction over disjunction,

$$p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$$

is a bridge principle of $\mathcal{L}_{\wedge\vee}$, since it is not a substitution instance of any valid derivation either in $\mathcal{L}_\wedge$ or $\mathcal{L}_\vee$. In some sense, then, interaction principles are new derivations just because they involve some vocabulary that we lacked before the combination. But, indeed, the interestingly new interactions come from bridge principles. These really are new derivations that appear in the combination process and that were not present before.

3.1.1 Collapse

Whoever is familiar with the debate around logical pluralism has surely come across the notion of 'collapse', as applied to the pluralist proposals. One of the first collapse arguments, if not the first, appears in Williamson (1988), and other relevant versions can be found in Read (2006), Priest (2006), Keefe (2014) and Stei (2020). Priest´s very well known argument, is directed against Beall and Restall's pluralism:

> Let s be some situation about which we are reasoning; suppose that s is in different classes of situations, say, K_1 and K_2. Should one use the notion of validity appropriate for K_1 or for K_2? We cannot give the answer 'both' here. Take some inference that is valid K_1 but not K_2, $\alpha \vdash \beta$, and suppose that we know (or assume) α holds in s ; are we, or are we not entitled to accept that β does? Either we are or we are not: there can be no pluralism about this. (Priest, 2006, p. 203)

The essence of the argument is that there are two legitimate consequence relations that disagree with respect to a particular argument, i.e. $\alpha \vdash_1 \beta$ but $\alpha \nvdash_2 \beta$, and, at the same time, there is a subject who knows that α holds and that $\alpha \vdash_1 \beta$ but $\alpha \nvdash_2 \beta$. It would seem, then, that the subject is entitled to accept that β holds in s, as Priest defends right after the quoted paragraph. So, the rational thing to do for the agent is to accept it and, therefore, these two logics would collapse to the strongest one.

Nevertheless, this is not the notion of collapse that we are going to focus on in this section; for there is another type of collapse that has received almost no attention in the philosophical literature in spite of being a crucial challenge to some pluralist (meaning 'pluralist' in a relaxed, almost informal, sense) proposals such as localism.

Before going into the details, let us roughly introduce this type of collapse by saying that it does not depend on a particular argument over which two logics disagree. Neither does it involve any assumptions about the normativity of logic in order for it to work. It is rather a technical result, consisting of a number of theorems, known as *collapse theorems*, which show that by freely combining different logic systems, each (possibly) with its own stock of connectives, the logics collapse to one of them, because their different connectives end up behaving as mere notational variants.

Let me notice that I will be following Schechter in the terminology and distinguish, as he does, *collapse* and *weak collapse*. The precise notation and concepts will be presented in the next section. But let us, for the moment, say that a logic, $\mathcal{L}$, with two stocks of logical connectives, *collapses* when for every formula δ, δ', exactly alike except for some or all of their subscripts, $\{\delta\} \vdash \delta'$.

Furthermore, let us say that a logic, $\mathcal{L}$, *weakly collapses* when there is a translation, t, between the set of formulas with connectives from stock 1 and the set of formulas with connectives from stock 2, such that, if $\Gamma \vdash \alpha$, then $t(\Gamma) \vdash t(\alpha)$.

The first collapse results can be traced back to Carnap (1943) and Popper (1948), although the most common references are Harris (1982), in the more philosophically oriented literature, and del Cerro and Herzig (1996) and Gabbay (1996), in the literature revolving around fibring. However, despite their different 'traditions', all these sources share the common feature of dealing uniquely with the case of combining classical and intuitionistic logics.

3.1.2 Collapse Theorems

In his paper ‘What’s So Logical about the “Logical”Axioms?’, J. H. Harris invites us to imagine an intuitionist logician and a classical logician willing to cooperate in order to understand the axioms that the other party deems valid, taking them as *syntactical meaning postulates* of how the other understands the connectives. Assume, then, that both logicians want to entertain a dialogue in a common logic $\mathcal{L}$ over a shared language L.

Harris delivers the following axiom schemata as the ones that both classical and intuitionist logicians would accept[4]:

1. Deduction Property ($DED_{\mathcal{L}}$): $A_1, ..., A_k, A \vdash_{\mathcal{L}} B$ iff $A_1, ..., A_k \vdash_{\mathcal{L}} A \to_x B$ for all L-formulas $A_1, ..., A_k, A$ and B.
2. Modus Ponens ($MP_{\mathcal{L}}$): $\Gamma \vdash_{\mathcal{L}} A$ and $\Gamma \vdash_{\mathcal{L}} A \to_x B$, then $\Gamma \vdash_{\mathcal{L}} B$.
3. $A, B \vdash_{\mathcal{L}} A \wedge_x B$.
4. (a) $\vdash_{\mathcal{L}} A \wedge_x B \to_x A$
    (b) $\vdash_{\mathcal{L}} A \wedge_x B \to_x B$
5. (a) $\vdash_{\mathcal{L}} A \to_x A \vee_x B$
    (b) $\vdash_{\mathcal{L}} B \to_x A \vee_x B$
6. $A \to_x C, B \to_x C \vdash_{\mathcal{L}} A \vee_x B \to_x C$
7. $A \to_x B, A \to_x \neg_x B \vdash_{\mathcal{L}} \neg_x A$
8. $\neg_x A, A \vdash_{\mathcal{L}} B$

On top of these shared axioms, we also both accept that $\Gamma \vdash_{\mathcal{L}} A$ if $A \in \Gamma$ (9). Moreover, the classical logician will also want to include the following axiom:

8_c $\vdash_{\mathcal{L}} \neg_c \neg_c A \to_c A$

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[4] Let us use subscript x for either the intuitionist (i) version or the classical (c) version of the connectives. Also, notice that we will only focus on the propositional part.

However, an interesting point is that we will not need this last axiom, 8_c, in order to prove the collapse theorems. Taken together, these theorems are meant to establish the collapse of the logic, i.e. that for every L-formula A_x, the intuitionistic and the classical versions are interderivable, $\vdash_{\mathcal{L}} A_i \leftrightarrow_x A_c$. Let us now show some of the most interesting collapse theorems.

Theorem 3.1.1. *Assume logic $\mathcal{L}$ satisfies schema 1 - 8 for both $x \in \{i, c\}$. Then for every L-formula A and B and both $x \in \{i, c\}$ we have $\vdash_{\mathcal{L}} A \rightarrow_i B \leftrightarrow_x A \rightarrow_c B$.*

Proof. I prove the right to left direction, which is actually more interesting. The other direction is analogous. We have that $A \rightarrow_c B \vdash_{\mathcal{L}} A \rightarrow_c B$ by (9). Then, $A \rightarrow_c B, A \vdash_{\mathcal{L}} B$ by (1) and $A \rightarrow_c B \vdash_{\mathcal{L}} A \rightarrow_i B$ again by (1). $\square$

Theorem 3.1.2. *Assume logic $\mathcal{L}$ satisfies schema 1 - 8 for both $x \in \{i, c\}$. Then for every L-formula A and both $x \in \{i, c\}$ we have $\vdash_{\mathcal{L}} \neg_i A \leftrightarrow_x \neg_c A$.*

Proof. Again from right to left. $\neg_c A, A \vdash_{\mathcal{L}} B$ by (8) and $\neg_c A \vdash_{\mathcal{L}} A \rightarrow_i B$ by (1). $\neg_c A, A \vdash_{\mathcal{L}} \neg_i B$ by (8) and $\neg_c A \vdash_{\mathcal{L}} A \rightarrow_i \neg_i B$ by (1). Then apply twice $DED_{\mathcal{L}}$ to (7) in order to get $\vdash_{\mathcal{L}} (A \rightarrow_i B) \rightarrow_i ((A \rightarrow_i \neg_i B) \rightarrow_i \neg_i A)$, and, finally, by two applications of $MP_{\mathcal{L}}$ we get $\neg_c A \vdash_{\mathcal{L}} \neg_i A$. $\square$

As one might expect now, the rest of the collapse theorems for $\wedge$ and $\vee$ work in a similar way. Thus, all these theorems lead to the unsettling result that when I state the law of excluded middle using the classical connectives, this is logically equivalent to the intuitionistic version, i.e., $\vdash_{\mathcal{L}} (A \vee_i \neg_i A) \leftrightarrow_x (A \vee_c \neg_c A)$. In fact, one can prove the following stronger theorem which states the collapse of the logic $\mathcal{L}$ over the language L.

Theorem 3.1.3. *Let A_i be any L-formula whose connectives are intuitionistic and let A_c be the corresponding L-formula in its classical version. Assume $\mathcal{L}$ satisfies schema 1 - 8 for both $x \in \{i, c\}$. Then $\vdash_{\mathcal{L}} A_i \leftrightarrow_x A_c$.*

Proof. By induction on the number of connectives (see Harris (1982), Theorem 8).

3.1.3 Collapse: the limiting case of bridge principles

We started the section talking about interaction principles and proceeded to presenting what collapse is. Notice that, indeed, the collapse theorems are nothing more than some results that provide new derivations, i.e., bridge principles, establishing connections between the corresponding logical connectives of each logic. The problem of these theorems, however, is that there is too much interaction between the logical connectives. In fact, depending on the method for combining logics, one can give sufficient conditions for which new interactions provoke the total collapse. In the case of juxtaposition, for instance, one can show that if we added $\alpha \to_1 \beta \dashv\vdash \alpha \to_2 \beta$[5], the logic would collapse (see Schechter (2011), Prop. 7.3.)

As we are going to see later on, one of the most difficult, yet at the same time interesting, aspects of combining logics is to calibrate how much interaction is too much interaction and, also, how little interaction might be too little. We know that the collapse of the connectives is at one extreme of interactions, but there is an analogous problem, first noticed by Béziau in Béziau (2004) and later more deeply analysed in Béziau and Coniglio (2011) at the other extreme, namely, the *anti-collapse problem.* The authors characterize this problem as "the impossibility of obtaining, in the logics obtained by fibring, intended interaction rules which are justified, for instance, by well-known models or sequent rules" (Coniglio, 2007, p. 379). In fact, this is not a problem specific to the combination method of fibring, but also for juxtaposition and, in general, for any combination mechanism that, when combining the logics $\mathcal{L}_1$ and $\mathcal{L}_2$ seeks the logic $\mathcal{L}_{12}$, which is the *minimal conservative extension*[6] of the logics being combined. This means that the combined logic, because of its minimality, will not include among its valid inferences any properly new interactions, that is, bridge principles, nor, *a fortiori*, the intended justified ones.

In Béziau (2004), the anti-collapse problem was illustrated with the case of combining the logic of conjunction, $\mathcal{L}_\wedge$, and the logic of disjunction, $\mathcal{L}_\vee$. We have already seen that some of the desirable bridge principles for this combination are the distributivity laws, both of conjunction over disjunction and vice versa. One can easily check that if we put together the two-valued truth tables for $\wedge$ and $\vee$ we actually get that the distributivity laws are satisfied. Nevertheless, the standard combination mechanisms (like fibring or juxtaposition) will not yield these bridge principles in the combined logic $\mathcal{L}_{\wedge\vee}$, since this logic will be the minimal conservative extension of $\mathcal{L}_\wedge$ and $\mathcal{L}_\vee$.

It might be a matter of disagreement whether the logic of conjunction

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[5] As usual, $\alpha \dashv\vdash \beta$ is an abbreviation for $\alpha \vdash \beta$ and $\beta \vdash \alpha$.

[6] We will give the formal definitions later on.

and disjunction, $\mathcal{L}_{\wedge\vee}$ is distributive or not[7], but there are other applications of combining logics that more obviously require the emergence of bridge principles, for instance, the application that aims at recovering a logic from its fragments. This has been one of the more recent fields of development in the application of combination mechanisms and it has given rise to new methods for combining logics, given how inappropriate the standard mechanisms are in order to go beyond minimality.

One of the few methods developed along those lines is *meta-fibring*, proposed by Coniglio (2007). This method recovers classical logic when combining the logic of (classical) negation with the logic of (classical) conditional. We will see when showing the applications of juxtaposition that this is not possible with it, since we do not get the Principle of Pseudo-Scotus, $\vdash \alpha \rightarrow (\neg\alpha \rightarrow \beta)$, for instance.

But the emergence of some bridge principles can also be problematic, even if they do not lead to collapse, just in virtue of not being well justified or being philosophically faulty. As Carnielli and Coniglio (2007) recall, if one combines two normal modal logics, $\mathcal{L}_1$ and $\mathcal{L}_2$, with the method of *product* of modal logics, each one with its own operators $\Box_i$ and $\Diamond_i$, the following bridge principles pop up at the semantic level:

- $\Box$-commutativity: $\Box_1\Box_2\alpha \leftrightarrow \Box_2\Box_1\alpha$
- $\Diamond$-commutativity: $\Diamond_1\Diamond_2\alpha \leftrightarrow \Diamond_2\Diamond_1\alpha$
- (1, 2) Church-Rosser property: $\Diamond_1\Box_2\alpha \leftrightarrow \Box_2\Diamond_1\alpha$
- (2, 1) Church-Rosser property: $\Diamond_2\Box_1\alpha \leftrightarrow \Box_1\Diamond_2\alpha$

But if the logic $\mathcal{L}_1$ is an alethic modal logic and $\mathcal{L}_2$ is an epistemic logic, we would have,

$$\Diamond K\alpha \leftrightarrow K\Diamond\alpha$$

with 'K´ standing for 'knowledge', which seems highly implausible. Just consider the fact that it is possible that an agent knows that Higgs Boson exists, while the agent not knowing whether it is possible that Higgs Boson exists. This bridge principle clearly has epistemological shortcomings.

One should already see that the issue of bridge principles and how to produce them is a very delicate one. They are not good or bad on their

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[7] See Béziau and Coniglio (2011) for an interesting analysis.

own, but relative to applications and to the logics being combined. Sometimes we might want to go for the minimal conservative extension, but some other times we might seek for some justified and meaningful bridge principles. However, it is not easy to determine a particular method to such end. Meta-fibring, for instance, was useful in order to get distributivity in the logic of conjunction and disjunction, $\mathcal{L}_{\wedge\vee}$, or in order to recover full classical logic from the logic of negation and the logic of conditional. However, if we apply the method of meta-fibring to classical and intuitionistic logics, the resulting mixed logic collapses (see (Coniglio, 2007, p. 380)), since their meta-properties will be preserved, among which is the deduction meta-theorem.

I reckon it is an important contribution of the dissertation trying to show that the problem of mixed inferences turns out to be a fertile area for the emergence of bridge principles. It definitely is a crucial topic in the philosophy of logic and I believe that it forces us to think about the technical side of combining logics in a new way. In fact, the improvements on the combination mechanisms that I will develop later on are motivated by those cases of mixed inferences. By how to account for them considering the logics in play and the philosophical backgrounds of them.

3.1.4 Are there more collapses?

Looking at the general definition of collapse, one can check that it is meant to be applicable to any logic systems, not only to classical and intuitionistic logics. However, as far as I know, there is no other collapse result concerning different logics.

There is a caveat worth keeping in mind when thinking about collapse results, though, namely, that these results are relative to a combination mechanism. Thus, it might happen that for a certain method the collapse occurs proof-theoretically and for certain other method the collapse is semantic.

We have already seen how Harris proved the collapse of **CL** and **IL** axiomatically and Carnap's and Popper's proofs were also given in this way. One could also prove it using natural deduction in the following way. Suppose we have a copy of a natural deduction system for **CL** and a copy of a natural deduction for **IL**. Then, we can prove that $P \to_i Q \dashv\vdash^{ic} P \to_c Q$.

1	$P \to_i Q$	
2	$\quad \mid P$	
3	$\quad \mid Q$	$\to_i$E, 1, 2
4	$P \to_c Q$	$\to_c$I, 2–3

And the other direction goes in a similar way. We can also prove that $\neg_i P \dashv\vdash^{ic} \neg_c P$.

1	$\neg_c P$	
2	$\quad \mid P$	
3	$\quad \mid P \vee_c \neg_i P$	$\vee_c$I, 2
4	$\quad \mid \neg_i P$	Disjunctive Syllogism, 1, 3
5	$\quad \mid \bot_i$	$\neg_i$E, 2, 4
6	$\neg_i P$	$\neg_i$I, 2–5

1	$\neg_i P$	
2	$\quad \mid P$	
3	$\quad \mid P \vee_i \neg_c P$	$\vee_i$I, 2
4	$\quad \mid \neg_c P$	Disjunctive Syllogism, 1, 3
5	$\quad \mid \bot_c$	$\neg_c$E, 2, 4
6	$\neg_c P$	$\neg_c$I, 2–5

However, not with every two logics one will be able to prove the collapse in this way. At least with a natural deduction approach. If one takes natural deduction calculi for $\mathbf{K_3}$ or **LP** (see appendix), one can easily check that the lack of rules for introducing negations and conditionals prevents getting a collapse when combining the natural calculi of each of these logics with the classical natural deduction.

Even so, this does not mean that the problem would be solved for those combinations, at least for two reasons. First, we do not have a systematic method for giving the semantics of those merges and, therefore, this solution would explain very little and would leave a lack of justification for the validity of the mixed inferences involving information formalized by $\mathbf{K_3}$ (**LP**) and **CL**. The reason for this lack of systematicity is that we do not have a guide of which bridge principles are going to appear in this kind of combination process.

And second, related to the last remark, we have already seen that the collapse is just the limit case of bridge principles. We know, moreover, that the collapse is not the only interaction that might be undesirable. Some bridge principles are bad due to the connections they establish (recall the example involving the Church-Rosser property). But putting together the natural deduction calculi in that way will give rise to bridge principles and, probably, to problematic ones. For instance, since we have

$$
\begin{array}{r|lll}
1 & P \vee_c \neg_c P & & \\
\cline{2-2}
2 & & \multicolumn{1}{|l}{P} & \\
3 & & \multicolumn{1}{|l}{P \vee_k \neg_c P} & \vee_k\text{I, } 2 \\
4 & & \multicolumn{1}{|l}{\neg_c P} & \\
5 & & \multicolumn{1}{|l}{\neg_c P \vee_k P} & \vee_k\text{I, } 4 \\
6 & P \vee_k \neg_c P & & \vee_c\text{E, } 1\text{–}5
\end{array}
$$

we get that $P \vee_k \neg_c P$ is a theorem, which seems weird since we do not know why the strong Kleene disjunction would relate in this way a proposition and its classical negation. In fact, we do not have any clue of how to interpret this kind of interactions between the connectives of the different logics. Even principles as easy to get as $P \wedge_p \neg_c P \vdash^{pc} \bot_c$[8] are difficult to account for. We just do not know what justifies that connection between the paraconsistent conjunction and the classical negation and bottom.

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[8] Just eliminate the paraconsistent conjunction (that is what the subindex ‘p’ stands for) and introduce the classical one. We will see later on that, in fact, we have an analogous inference when juxtaposing **IL** and **CL**, but that is, *prima facie*, more plausible given that both logics have Explosion.

On the semantic side the difficulties appear to be even greater. If in the calculus we could sort of throw together all the rules of inference and work with them[9], we do not seem to have the same option here. We cannot simply put the semantic models of the respective logics together and hope for everything to work. It is not only that the logics might collapse but, simply, that we do not know even how to evaluate the formulas, let alone the mixed ones, in that amalgam of models.

Yet the difficulties persist even if one proceeds with a sophisticated combination mechanism such as *algebraic fibring*. Since, even though the fibring of the Hilbert calculi for intuitionistic and classical logics do not collapse, the semantic fibring still collapses (see (Carnielli et.al., 2008, p. 106)). We will have the chance of looking at the system in more detail when presenting different combination mechanisms, but let me very roughly explain now why the semantic fibring of classical and intuitionistic logic collapses.

The way in which algebraic fibring solves the issue of providing the right models for the combination of the logics in play, is by constructing the fibred model out of the models of the given logics, in such a way that the reduct of the fibred structures to each of the signatures is a structure belonging to the class of algebras for that logic. In the case of combining classical and intuitionistic logics, the operations that we are going to have in the fibred structure, $\mathfrak{B}$, are those corresponding to the union of the classical and intuitionistic signatures, $C_c \cup C_i$. Therefore, the reduct of $\mathfrak{B}$ to C_c is a structure, $\mathfrak{B}|_{C_c}$, belonging to the class of Boolean algebras. Equally, the reduct of $\mathfrak{B}$ to C_i is a structure, $\mathfrak{B}|_{C_i}$, such that $\mathfrak{B}|_{C_i} \in \mathbf{HA}$. But, this is possible only if $\mathfrak{B}|_{C_i}$ is also a Boolean algebra, since the carrier set of both reducts coincide. Thus, the intuitionistic connectives in the fibration will behave like classical ones and we will end up having two copies of classical connectives as mere notational variants.

As previously said, as far as I know, there are not further collapse results in the literature involving different logics, but the semantic fibring hints at other possible collapses. For instance, we can see that for any two logics, $\mathcal{L}_1$ and $\mathcal{L}_2$, if the associated class of algebras of $\mathcal{L}_2$ generalizes that of $\mathcal{L}_1$, then the connectives of $\mathcal{L}_2$ should collapse to the connectives of $\mathcal{L}_1$, since the algebras for $\mathcal{L}_2$ that will 'survive' in the fibring process are those that are algebras for $\mathcal{L}_1$ too.

This is exactly what happens when fibring classical and intuitionistic

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[9] In the lucky scenarios in which the logics do not collapse.

logic, but we will be in a similar situation if we try to combine, by fibring, classical logic and any logic whose semantics is given by, say, Kleene algebras (like **K_3** or **LP**), which generalize Boolean algebras. Again, because it seems that the Kleene algebras that will 'survive' the fibring procedure are those which are also Boolean algebras.

However, it would not be correct to describe the collapse phenomenon stating that a weaker logic[10], $\mathcal{L}_2$, behaves as the stronger one, $\mathcal{L}_1$, when combined, for what could we say, then, when neither of them is included in the other? For instance, when combining classical logic with a contra-classical logic, or **IL** and **LP**? In this later scenario, we might have to look not at whether one algebraic semantics generalizes the other, but at the intersection of the classes of algebras. Thus, for the case of combining **IL** and **LP**, we know that both classes generalize Boolean algebras and, so, it could be that the algebras that survive when fibring the logics are the Boolean ones and, so, **IL** and **LP** would collapse to **CL**[11].

Again, these collapse results are not absolute, but relative to a combination mechanism. Nevertheless, note that the phenomenon is quite pervasive, going beyond classical and intuitionistic logics and appearing across different combination methods, both in proof-theoretic and model-theoretic approaches. If this was the whole story regarding collapse theorems, then localism would be a difficult philosophical thesis to swallow. Despite the *prima facie* plausibility it might have, a technical pitfall like this would most certainly be a knockdown. But, of course, this is not the end of the story. Some more sophisticated combination methods have been developed and we now move forward to analyse them.

3.2 Methods for Combining Logics

The ideas and methods for combining logic systems are quite recent and, consequently, the field is relatively fresh and little developed yet. One could argue that the idea of combining logics can be traced back at least to Ramon Llull (1232-1315) and his methods for combining concepts, the *ars combinatoria*, which later influenced Gottfried Leibniz's (1646-1716) *Dissertatio de arte combinatoria* where he pursued a *characteristica universalis*; a universal language. According to Carnielli et.al. (2008), Llull's ideas on combining at-

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[10] 'Weak' in the sense that its set of valid inferences is included in the other logic.

[11] Proving that this is the case will have to wait until future work.

tributes and categories in a symbolic notation and with the aim of relating all forms of knowledge make him the precursor of the modern idea of combining logics.

However, it is not until the 70's that methods of modern logic are developed in order to combine logic systems and, not by chance, just as with interaction principles, the first methods concerned the combination of different modalities by combining modal logics. Among those, the most influential ones were *products of logics*, introduced independently in Segerberg (1973) and Shehtman (1978), *fusion*, by R. Thomason in Thomason (1984) and, probably the most important one for the subsequent developments and generalizations, *fibring*, by D. Gabbay in Gabbay (1996), where the aim was to combine logics having Kripke semantics.

One of the most important steps in the brief history of the field was the generalization of fibring to wider classes of logics, beyond modal ones, by A. Sernadas, C. Sernadas and C. Caleiro, in Sernadas et.al. (1999), with the method of *algebraic fibring*, also known as '*categorial fibring*'. The relevance of this contribution is, at least, twofold. On one hand, it expands the field in a substantial way. From being a topic concerned with the combination of modal logics, to being able to raise the level of abstraction and design a combination mechanism that applies to a wide range of logic systems. On the other hand, their contribution created a school and a research line that continues until today, with many important improvements and expansions of the original ideas of categorial fibring. Among those, it is worth mentioning *modulated fibring* and *cryptofibring* (see Caleiro and Ramos (2004); Sernadas et.al. (2002)), designed to solve the collapsing problem, -since, as I pointed out above, the algebraic fibring of **CL** and **IL** collapses- and to account for "the combination of higher-order, modal, relevance logics or non-truth-functional logics" (Carnielli and Coniglio (2020)).

A close proposal to that of algebraic fibring that also generalizes Gabbay's fibring of modal logics, without using category theory, is *plain fibring*, by M. Coniglio and V. Fernández. In Coniglio and Fernández (2005), the authors generalize the idea of fibring logics with Kripke semantics to fibring logics induced by matrix semantics. In fact, the method that is going to be central to my dissertation and that I am going to use and apply in order to approach the problem of mixed inferences, namely, *juxtaposition* by Schechter (2011), is similar in spirit to plain fibring. The main advantage of juxtaposition is that it is more developed, with more metalogical results and preservation theorems. Moreover, I find it to be more intuitive and clear than plain fibring.

Other methods for combining logics that do not belong to the mainstream of categorial fibring are *ecumenism*, by Prawitz (2015), and *chunk and permeate*, by Brown and Priest (2004)(see, specially, Priest (2014)). There are some reasons why I will not focus on these methods. One of them is that there is only so much time one has for doing a thesis and some things, the ones that do not seem so relevant for certain purposes, have to be left out. Another reason, this one specific of ecumenism, is that the philosophical motivation of Prawitz when developing the method appears to be substantially independent of mine. The philosophical motivation of ecumenism is to give an inferentialist semantics for classical connectives. My motivation, though, is to provide a system that combines logics with the aim of having a localist reading of the mechanism.

This motivation of mine could possibly match one of Priest's applications of chunk and permeate, indeed. Ironic as it might sound, Priest, together with M. B. Brown (Brown and Priest (2004)) developed a strategy for "handling the application of different logics in combination" (Priest, 2014, 333). The problem is that the method of chunk and permeate is not very systematic nor general and, mainly, depends on how one wants to design the mechanism for a specific application. On top of this limitation, there is no metatheoretical result and no preservation theorem that can help us understand more the mechanism for the sake of systematizing it.

Maybe it is because of the monopoly that algebraic fibring has on the field of combining logics, but, the truth is that neither ecumenism, nor chunk and permeate, and not even juxtaposition appear mentioned in the entry "Combining Logics" of the Stanford Encyclopedia. I think this is unfortunate, specially for juxtaposition, which is a very well developed method with so much potential as I hope to show.

Before moving on to the presentation of some of the methods just mentioned, let me refer to two different approaches to the combination of logics in order to clarify which one of those I am focusing on. The approaches are those of *splitting versus splicing logics.* With respect to splitting logics,

> we may think about an analytic procedure that permits us to decompose a given logic into simpler components. [...] A prototypical case of splitting occurs when one succeeds in describing a given logic in terms of simpler components by means of translating the original logic into a collection of simpler, auxiliary logics, using what is called *possible-translations semantics.* (Carnielli et.al., 2008, p. 10)

So, splitting is a top-down analytic approach aiming to decompose a logic into simpler fragments. Splicing, on the other hand, is a "process, by which a bunch of logics is synthesized forming a new logic"(Carnielli et.al., 2008, p. 10). So, it is a bottom-up, synthetic approach, by means of which simple logics are combined in order to obtain a more complex system. This is the approach that we are going to be using. So, every method that I am going to present now, and also juxtaposition, are cases of splicing logics.

3.2.1 Fibring by functions

The method of fibring, also known as 'fibring by functions', was originally proposed by D. Gabbay in Gabbay (1996)[12]. As we said above, this mechanism only applies to logics with Kripke semantics. Still, it is a powerful method for combining such logics. Let me start by giving some definitions:

Definition 3.2.1 (Definition 7 in Coniglio and Fernández (2005)). A *modal signature* is a signature C such that $C^1 = \{\neg, \Box\}$, $C^2 = \{\rightarrow\}$ and $C^k = \emptyset$ in any other case. A *Kripke model* (for modal logics) is a triple $m = \langle W_m, R_m, h_m\rangle$ such that W_m is a nonempty set (the set of *possible-worlds* of m); $R_m \subseteq W_m \times W_m$ (the *accessibility relation* of m); and $h_m : \mathcal{V} \longrightarrow \wp(W_m)$ is a mapping (the m-*valuation*). A *Kripke semantics* is a class Kr of Kripke models.

Let us, then, denote the modal logics by the pair $\mathcal{L} = \langle C_{\mathcal{L}}, Kr\rangle$. Given two logics, $\mathcal{L}_1$ and $\mathcal{L}_2$ we define *the fibred language*, $L(C_{\otimes})$, which is obtained from the *fibred signature*, $C_{\otimes}$, namely:

$$C^1_{\otimes} = \{\neg, \Box_1, \Box_2\};\ C^2_{\otimes} = \{\rightarrow\};\ C^k_{\otimes} = \emptyset \text{ in any other case.}$$

Now, in order to get a fibred logic, we need to perform the fibring of the Kripke models. The fundamental idea is to take fibred Kripke models with distinguished actual worlds and to connect the worlds of one model with the worlds of the other, in such a way that if we are evaluating, say, a formula like $\Box_2\alpha$ in a Kripke model of $\mathcal{L}_1$ in a world from W_{m_1}, we can move to the corresponding world from W_{m_2} in model m_2, in order to check the validity of $\Box_2\alpha$. Thus, a *fibred model of Kr_1 and Kr_2* is a triple (f, g, h) such that:[13]

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[12] In order to present the method of fibring, I will be following Gabbay (1996), but also Carnielli and Coniglio (2020) and Coniglio and Fernández (2005), since these are very clear and concise explanations of the method.

[13] $\uplus$ denotes the disjoint union of sets.

$$f : \biguplus_{m_1 \in Kr_1} W_{m_1} \longrightarrow \biguplus_{m_2 \in Kr_2} W_{m_2};$$

$$g : \biguplus_{m_2 \in Kr_2} W_{m_2} \longrightarrow \biguplus_{m_1 \in Kr_1} W_{m_1};$$

$$h : \mathcal{V} \longrightarrow \wp(W)$$

with $W := (\biguplus_{m_1 \in Kr_1} W_{m_1}) \uplus (\biguplus_{m_2 \in Kr_2} W_{m_2})$ and f and g as transfer mappings: f from the set of worlds of the class of models Kr_1 of $\mathcal{L}_1$ into the class of models Kr_2 of $\mathcal{L}_2$, and g from the set of worlds of the class of models Kr_2 of $\mathcal{L}_2$ into the class of models Kr_1 of $\mathcal{L}_1$.

The *fibred structure* $Kr_1 \otimes Kr_2$ is the class of all the fibred models of Kr_1 and Kr_2. The satisfaction of a formula α by the fibred model (f, g, h) in the world w, denoted $(f, g, h) \vDash_w \alpha$, is defined recursively as usual when the main connective of α is Boolean, i.e., $\neg$ or $\rightarrow$, and when the modal operator and the world correspond to the same logic (that is, in a situation of standard modal logic). The relevant cases are those in which the modal operator and the world of evaluation have different origins. For instance, when evaluating the formula $\Box_1\alpha$ in the fibred model (f, g, h) and the world w_2. The satisfaction clause goes as follows: the model (f, g, h) satisfies $\Box_1\alpha$ in $w_2 \in W_{m_2}$, $(f, g, h) \vDash_{w_2} \Box_1\alpha$ iff $(f, g, h) \vDash_{w'_1} \alpha$ for every $w'_1 \in W_{m_1}$ such that $g(w_2)R_{m_1}w'_1$. The other case, with the subindexes of the box and the world interchanged, works analogously.

Thus, with this notion of satisfaction we characterize logical consequence in the usual way, as preservation of satisfaction from premises to conclusion. The *fibred consequence relation* is, $\vdash_{Kr_1 \otimes Kr_2} \subseteq \wp(L(C_\otimes)) \times L(C_\otimes)$ and the logic $\mathcal{L}_\otimes = \langle C_\otimes, \vdash_{Kr_1 \otimes Kr_2} \rangle$ is the *fibring of* $\mathcal{L}_1$ *and* $\mathcal{L}_2$.

Again, the method of fibring is limited by its applicability to modal logics with Kripke semantics, although, we will soon see that plain fibring is a natural way of extending the method beyond that limit. In any case, fibring is crucial in the history of combining logics, since it inspired the development of new, more general, methods. Among them, categorial fibring has been the one around which almost the whole field of combining logics has orbited.

3.2.2 Categorial (or Algebraic) Fibring

Since categorial fibring was presented in Sernadas et.al. (1999), many developments have emerged around it. So, this brief and rough presentation of the method should not be taken to represent or be faithful to the whole picture. In fact, even the method itself, leaving aside subsequent improvements around it, poses quite a challenge to be summarized in a simple way, since it involves some ideas from category theory. This is why I will be following Schechter (2011), on top of Carnielli et.al. (2008), and his way of presenting categorial fibring; a way that better lends itself to be compared with juxtaposition, which is the main method that I will be using.

Thus, I shall confine myself to the semantics of fibring. Another reason why focusing on the semantics is interesting for us is that, as I pointed out above, the collapse of the algebraic fibring of **CL** and **IL** occurs at the semantic level, not when fibring their Hilbert calculi (see Rasga et.al. (2002)). So, it is interesting for my purposes to show a method for combining logics whose semantics collapse, since this is something that I want to avoid when trying to solve the problem of mixed inferences appealing to a combination mechanism.

Following Carnielli et.al. (2008) the semantic unit of algebraic fibring is an algebra, $\langle B, \Phi \rangle$, understood as a tuple with a set, the carrier set, B, and a family of operations, Φ. But, "in order to ensure the preservation of some properties by fibring, it is convenient to consider enriched algebras, called interpretation structures" (Carnielli et.al., 2008, p. 92). An *interpretation structure* over a signature C, is a tuple $\langle B, \leq, \Phi, \top \rangle$, where $\langle B, \leq, \top \rangle$ is a partial order with a top element $\top$ and $\langle B, \Phi \rangle$ is an algebra over C. In the terminology of Schechter (2011), this interpretation structures are partially ordered unital structures, the 'unital' referring to the fact that the $\top$ is the unique designated value. Moreover, the relation $\leq$ allows us to compare the truth values in B, which makes possible to define two different notions of entailment, although we will only focus on one; global entailment. Given a class of partially ordered unital structures, i.e., a class of interpretation systems, $\mathbb{B}$, we say that Γ *globally entails* α in $\mathbb{B}$ iff for every interpretation structure in $\mathbb{B}$ and valuation, i.e., for every model, if every $\gamma \in \Gamma$ gets value $\top$, then α gets value $\top$.

Now, take an interpretation structure $\mathcal{B} = \langle B, \leq, \Phi, \top \rangle$ over a signature C. Suppose that C' is a subsignature of C, $C' \leq C$. We define the *reduct* of $\mathcal{B}$ to C' as the tuple $\mathcal{B}|_{C'} = \langle B, \leq, \Phi|_{C'}, \top \rangle$, where $\Phi|_{C'}$ is the restriction of Φ to the connectives in C'. Reducts play an important role in algebraic fibring, since when doing the algebraic fibring of two classes of partially ordered unital structures, $\mathbb{B}_1$ and $\mathbb{B}_2$, over the signatures C_1 and C_2, respectively, what we get is a class of partially ordered unital structures and, for each structure in the class, we must have that $\mathcal{B}|_{C_1} \in \mathbb{B}_1$ and $\mathcal{B}|_{C_2} \in \mathbb{B}_2$. That is, the algebraic fibring of $\mathbb{B}_1$ and $\mathbb{B}_2$ is the class of partially ordered unital structures, $\mathbb{B}_{12} = \{\mathcal{B} : \mathcal{B}|_{C_1} \in \mathbb{B}_1 \text{ and } \mathcal{B}|_{C_2} \in \mathbb{B}_2\}$ over the fibred signature $C_1 \cup C_2$.

This way of constructing the fibred semantics has some limitations, though. Limitations that are crucial for the philosophical problem that I am dealing with. This is because, as I pointed out above, algebraic fibring is not well suited for combining the semantics of classical and intuitionistic logics. The reason is the following: classical logic is sound and complete with respect to the class of Boolean algebras, **BA**. Intuitionistic logic is sound and complete with respect to the class of Heyting algebras, **HA**, which generalize Boolean algebras. Then, the algebraic fibring of the semantics of **CL** and **IL** is the class of partially ordered unital structures, let us call it $\mathbb{B}_{ci}$, over the fibred signature $C_c \cup C_i$, with $C_c = \{\neg_c, \wedge_c, \vee_c, \rightarrow_c, \leftrightarrow_c\}$ and $C_i = \{\neg_i, \wedge_i, \vee_i, \rightarrow_i, \leftrightarrow_i\}$, such that $\mathbb{B}_{ci} = \{\mathcal{B} | \mathcal{B}|_{C_c} \in \mathbf{BA} \text{ and } \mathcal{B}|_{C_i} \in \mathbf{HA}\}$. But, if one looks at the definition of reduct, one can easily see that the carrier sets of $\mathcal{B}|_{C_c}$ and $\mathcal{B}|_{C_i}$ coincide. Thus, for every interpretation structure in the fibred class of interpretation structures to be a Boolean algebra and a Heyting algebra for its relevant reducts, every structure has to be a Boolean algebra, i.e., every $\mathcal{B}|_{C_i}$ is a Heyting algebra that is also Boolean. Hence, the intuitionistic connectives will behave exactly like the classical ones and, so, they will be intersubstitutable. This means, according to the definitions given above, that the algebraic fibring of **CL** and **IL** collapses and, *a fortiori*, weakly collapses.

Recall, though, that responding to Wrenn's challenge is an important part of giving a solution to the problem of mixed inferences and that, this challenge, essentially involves being able to give a criterion of validity for mixed inferences with components coming from domains governed by classical and intuitionistic logics. If we are not able to implement a combination mechanism for those logics while avoiding the collapse of the connectives, we will not be able to meet the challenge. But, then, algebraic fibring does not seem a good candidate in order to account for the problem of mixed inferences. We know, however, that there are other options that work better.

3.2.3 Direct Union and Plain Fibring

Direct union and plain fibring are an extension of Gabbay's original notion of fibring to another class of logics, namely, logics characterized by matrix semantics. These methods were proposed in Coniglio and Fernández (2005) and, although they can still be developed more and further metalogical results could be obtained, they are interesting as a continuation of Gabbay's work and also because of the similarities with juxtaposition.

In fact, I should clarify that they are not two totally independent methods. Direct union is the method that one applies when we are in the simple and smooth scenario of combining two logics in "which the domain and designated values of the matrices involved are the same. In such cases, the combined logic can be simply obtained by putting together both matrices" (Coniglio and Fernández, 2005, p. 1596). With plain fibring, however, we can face the more difficult scenario in which the domains and the values designated are different. The idea, in this case, is to build, with appropriate functions, a bigger matrix that encompasses those of the logics being combined. Once we have this general matrix for both logics, we do their direct union. In this sense, direct union is the final stage of plain fibring. Let me begin by laying out some definitions and, then, we will see the methods with a little bit more detail.

DEFINITION 3.2.2 (Definition 5 in Coniglio and Fernández (2005)). Given a signature C, a $C - matrix$ is a pair $M = \langle \mathbf{A}, D \rangle$ where $\mathbf{A} = \langle A, D \rangle$ is an algebra over C and $D \subseteq A$ is the set of *designated values* of M.

DEFINITION 3.2.3 (Definition 6 in Coniglio and Fernández (2005)). Let C be a signature and let $\mathcal{K}$ be a class of C-matrices. The *matrix semantics induced by* $\mathcal{K}$ (denoted by $\vdash_{\mathcal{K}}$) is defined by: $\Gamma \vdash_{\mathcal{K}} \alpha$ iff for every C-matrix $M = \langle \mathbf{A}, D \rangle$ belonging to $\mathcal{K}$ and every valuation v, if $v(\Gamma) \subseteq D$ then $v(\alpha) \in D$.

We consider, first, the simpler case in which the domains and the designated values of matrices are the same. In this case, the direct union consists of putting together both matrices in the following way:

DEFINITION 3.2.4 (Definition 9 in Coniglio and Fernández (2005)). Let $\mathcal{L} = \langle C_i, M_i \rangle$ (with $i \in \{1, 2\}$) be two matrix logics, where each $M_i = \langle \mathbf{A}_i, D_i \rangle$ is a C_i-matrix. Assume that $A_1 = A = A_2$ and $D_1 = D = D_2$. The *direct union of* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ is the logic $\mathcal{L}_1 + \mathcal{L}_2 = \langle C_1 \uplus C_2, \vdash_{M_1+M_2} \rangle$ where $\vdash_{M_1+M_2}$ is the consequence relation defined by the $C_1 \uplus C_2$-matrix $M_1 + M_2 = \langle \mathbf{A}, D \rangle$ such that, if $c \in C_i^k$ and $a_1, ..., a_k \in A$, then $c^{M_1+M_2}(a_1, ..., a_k) = c^{M_i}(a_1, ..., a_k)$ $(i \in \{1, 2\})$.

So, one can see that those types of combinations are pretty straightforward. Since the domains and the designated values are the same, the denotation of a connective from C_i is just the same in the original matrix M_i and in the direct union of matrices, $M_1 + M_2$. That is, we will not get the more difficult case in which the argument of a truth-functional connective is a semantic value for which the truth-function was not defined.

We now consider the more interesting case of combining logics characterized by matrix semantics with different domains. This is the method of plain fibring and the reader will immediately see the similarities with fibring, which is a special case of this more general approach. In Coniglio and Fernández (2005), they treat, first, the case in which there is no restriction to the transfer mappings, i.e., unrestricted plain fibring. However, I directly consider the situation in which we restrict the mappings in a way specified below.

Definition 3.2.5 (Definition 14 in Coniglio and Fernández (2005)). Let $\mathcal{L} = \langle C, M \rangle$ be a matrix logic, where each $M = \langle \mathbf{A}, D \rangle$ is a C-matrix with domain A and set of designated values D. Let A' and D' be two sets such that $D' \subseteq A'$. Suppose, without loss of generality, that $A \cap A' = \emptyset$. Finally, let $f : A' \longrightarrow A$ be a mapping. The C-matrix M_f is defined as follows: its domain is $A \uplus A'$; its set of designated values is $D \uplus D'$ and for $c \in C_i^k$ and $a_1, ..., a_k \in A \uplus A'$, $c^{M_f}(a_1, ..., a_k) = c^M(\overline{a_1}, ..., \overline{a_k})$ where, for every a_j ($j = 1, ..., k$):

- If $a_j \in A$, then $\overline{a_j} = a_j$.
- If $a_j \in A'$, then $\overline{a_j} = f(a_j)$.[14]

Definition 3.2.6 (Definitions 13, 15 in Coniglio and Fernández (2005)). Let $\mathcal{L} = \langle C_i, M_i \rangle$ (with $i = 1, 2$) be two matrix logics, where each $M = \langle \mathbf{A}_i, D_i \rangle$ is a C_i-matrix with domain A_i. The *fibred signature* is given by $C_1 \uplus C_2$ and the *fibred language* is $L(C_1 \uplus C_2)$. A *fibred valuation* is a triple (f, g, v), where $(f, g) \in A_2^{A_1} \times A_1^{A_2}$, such that (f, g) is admissible and $v \in (A_1 \uplus A_2)^{\mathcal{V}}$ ($\mathcal{V}$ being the set of propositional variables). A pair $(f, g) \in A_2^{A_1} \times A_1^{A_2}$ is *admissible* if it satisfies: $f(x) \in D_2$ iff $x \in D_1$, for every $x \in A_1$; and $g(y) \in D_1$ iff $y \in D_2$, for every $y \in A_2$. Given $\phi \in L(C_1 \uplus C_2)$ and a fibred valuation (f, g, v), we define $(f, g, v)(\phi) \in A_1 \uplus A_2$ by recursion on the complexity of ϕ:

- If $\phi \in \mathcal{V}$ then $(f, g, v)(\phi) = v(\phi)$;

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[14] We will see an illustration of this kind of matrices in Example 3.2.1.

- If $\phi = c(\beta_1, ..., \beta_k)$ then $(f, g, v)(\phi) = c(\overline{(f, g, v)}(\beta_1), ..., \overline{(f, g, v)}(\beta_k))$ where for every formula $\beta_j (j = 1, ..., k)$:
    - If $c \in C_i^k$ and $(f, g, v)(\beta_j) \in A_i$ then $\overline{(f, g, v)}(\beta_j) = (f, g, v)(\beta_j)$ for $(i = 1, 2)$;
    - If $c \in C_1^k$ and $(f, g, v)(\beta_j) \in A_2$ then $\overline{(f, g, v)}(\beta_j) = g((f, g, v)(\beta_j))$;
    - If $c \in C_2^k$ and $(f, g, v)(\beta_j) \in A_1$ then $\overline{(f, g, v)}(\beta_j) = f((f, g, v)(\beta_j))$;

We say that a fibred valuation (f, g, v) satisfies ϕ if $(f, g, v)(\phi) \in D_1 \uplus D_2$. The *plain fibred consequence relation* $\vdash_{M_1 \odot M_2} \subseteq \wp(L(C_1 \uplus C_2)) \times L(C_1 \uplus C_2)$ is defined as follows: $\Gamma \vdash_{M_1 \odot M_2} \phi$ if, for every fibred valuation (f, g, v) satisfying simultaneously all the formulas of Γ, we have that (f, g, v) satisfies ϕ. The *plain fibring* of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the pair $\mathcal{L}_1 \odot \mathcal{L}_2 = \langle C_1 \uplus C_2, \vdash_{M_1 \odot M_2} \rangle$.

Now, this is already a much more interesting combination mechanism, because there is a wide variety of logics, or fragments of logics, that we can combine. Moreover, we know by Proposition 10 in Coniglio and Fernández (2005), that the plain fibring of two matrix logics, $\mathcal{L}_1 \odot \mathcal{L}_2$, is a *conservative extension* of both $\mathcal{L}_1$ and $\mathcal{L}_2$. What this means, is that $C_1 \subseteq C_1 \uplus C_2$, $C_2 \subseteq C_1 \uplus C_2$ and $\mathcal{L}_i = (\mathcal{L}_1 \odot \mathcal{L}_2)|_{C_i}$ for $(i = 1, 2)$. That is, when we restrict the plain fibring to each of the signatures, we get the original logics.

Notice that this is an important feature, since it shows already that, if the sets of valid inferences of the logics are different, then the plain fibring of them does not weakly collapse and, therefore, does not collapse either. To see this, think, for instance, about the case of doing the plain fibring of **CL** and **LP**. We know that in **CL** we have *modus ponens* but in **LP** we don't. This means that in **CL**$\odot$**LP** we will have that $p \rightarrow_c q, p \vdash_{\mathbf{CL} \odot \mathbf{LP}} q$, but $p \rightarrow_p q, p \nvdash_{\mathbf{CL} \odot \mathbf{LP}} q$, which is enough to show that **CL**$\odot$**LP** does not weakly collapse, because we have found a valid inference in the plain fibring, whose translation to the inference with corresponding connectives is no longer valid.

Since I want to avoid excessive detail until we plunge into the method of juxtaposition, let me conclude the section by further illustrating the method of plain fibring with an example, given in Coniglio and Fernández (2005) which, I hope, will help to get a better picture of the method.

Example 3.2.1. Take the negation fragment of the paraconsistent matrix logic $\mathbf{P}^1$.[15] Let, then, $\mathcal{L}_1$ be the fragment of $\mathbf{P}^1$ defined over the signature $\{\neg_{\mathbf{P}^1}\}$,

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[15] This is a paraconsistent logic introduced in Sette (1973).

given by the matrix M_1 with domain $A_1 = \{T, T_1, F\}$ and set of designated values $D_1 = \{T, T_1\}$.

$\neg_{\mathbf{P}^1}$	
T	F
T_1	T
F	T

Let also $\mathcal{L}_2$ be the fragment of **CL** defined over the signature $\{\rightarrow_c\}$ given by the matrix M_2 with domain $A_2 = \{1, 0\}$ and set of designated values $D_2 = \{1\}$.

$\rightarrow_c$	1	0
1	1	0
0	1	1

Now, we start fibring the matrices by taking the disjoint unions of the domains and designated values. So, let the domain be $A = \{T, T_1, F, 1, 0\}$ and set of designated values $D = \{T, T_1, 1\}$ and let $(f, g) \in A_2^{A_1} \times A_1^{A_2}$ be the following admissible valuation $f(T) = f(T_1) = 1, f(F) = 0, g(1) = T$ and $g(0) = F$. Then, $(M_1)_g$ and $(M_2)_f$ are given by the following matrices, respectively: [16]

$\neg$	
T	F
T_1	T
1	F
F	T
0	T

$\rightarrow$	T	T_1	1	F	0
T	1	1	1	0	0
T_1	1	1	1	0	0
1	1	1	1	0	0
F	1	1	1	1	1
0	1	1	1	1	1

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[16] These matrices are the extensions of the original matrices with the semantic values of each other. This is possible, because we have the valuations, i.e., the transfer mappings, in order to know how to translate the semantic values of one matrix into the other, so as to be able to determine the semantic matrix of a connective when having as arguments possibly new semantic values that were not in its original definition.

Thus, let $\mathcal{L}$ be the logic over $\{\neg, \rightarrow\}$ characterized by the matrix $M_{(f,g)} = (M_1)_g + (M_2)_f$ given by the two tables above, with $\{T, T_1, 1\}$ as the set of designated values. Notice, though, that there is an additional admissible pair (f, g') such that $g'(1) = T_1$ and $g'(0) = F$. Therefore, the fibred logic $\mathcal{L}_1 \odot \mathcal{L}_2$ is characterized by the set of matrices:

$$M_1 \odot M_2 = \{(M_1)_g + (M_2)_f, (M_1)_{g'} + (M_2)_f\}$$

We can check that, for instance, the mixed formula $(p \rightarrow q) \rightarrow \neg\neg(p \rightarrow q)$ is not valid in $\mathcal{L}_1 \odot \mathcal{L}_2$, since for the pair (f, g') and a valuation of the propositional variables such that $v(p) = v(q) = 1$, the $\overline{(f, g', v)}(p \rightarrow q) = 1$ and, since $g'(1) = T_1$, by the truth-table of negation $\neg(T_1) = T$ and $\neg(T) = F$. But, given that the pair is admissible, $f(F) = 0$ and, therefore, with $p \rightarrow q$ being 1 and $\neg\neg(p \rightarrow q)$ being 0, $(p \rightarrow q) \rightarrow \neg\neg(p \rightarrow q)$ is going to be 0. So, the formula is not valid in $\mathcal{L}_1 \odot \mathcal{L}_2$.

Clearly, plain fibring constitutes a big step forward with respect to Gabbay's fibring and, also, with respect to algebraic fibring (at least concerning the collapse theorems). I, certainly, believe that plain fibring is also a good candidate to successfully be applied in addressing the problem of mixed inferences. However, I will be sticking to juxtaposition since, although maybe being more marginal than any proposal in the fibring tradition, it is more developed than plain fibring.

Modulated fibring and cryptofibring where designed to avoid the collapses but also to be applied to an even larger variety of logic systems, which makes them, definitely, very attractive if the ultimate goal of combination mechanisms is to arrive at a universal methodology for combining any given logic systems, independently of how they are presented, e.g., algebraically, axiomatically, with non-deterministic semantics, etc. The reason for not focusing on them and going with juxtaposition as my preferred method for meeting the challenge, then, is that the cases of mixed inferences that I could potentially address with juxtaposition and the method itself were challenging enough. Moreover, I have been able to come up with ways in which juxtaposition could be improved, in directions that are quite illuminating for thinking about interaction principles and combinations of logics, as I will try to show.

Hence, I am not claiming that juxtaposition, or any of my improvements thereof, are going to be the best solution and the one that encompasses the greater diversity of logics. But, in order to solve any difficult problem it is usually a good idea to divide it to, at some point, conquer it. My modest goal with this dissertation is to offer a solution to one of those divides. Besides, we cannot rule out the funny possibility of there not being a universal method for combining logics and, therefore, of having different combination mechanisms that apply locally, which, in this case, would imply having different methods for different types of logic systems. In that case, my solution would be, at most, one among other correct and necessary solutions.

3.3 Juxtaposition

Let us now focus on the method that, as already announced, is going to be our main object of analysis and inspiration for further developments. This is the method of *juxtaposition*, which was presented by Joshua Schechter in Schechter (2011) as a new method for combining logics, alternative to those of fibring.

The reader will soon find relevant similarities with plain fibring, but I believe that juxtaposition is more elegantly presented and, on top of this, the technical results given by Schechter with respect to preservation results, metalogical theorems, (non)collapse theorems for the system and strengthenings of it, etc., exceed those given for plain fibring. This makes it even more surprising that this work has received so little attention, especially from the people working on the different variations of algebraic fibring.

I will try to follow Schechter's structure in presenting the method and be as precise as possible delivering it. Of course, the reader can always find the full presentation in the original paper.

First, juxtaposition is a method that naturally allows the combination of different propositional logics (for simplicity we will work with two) with a *structural consequence relation*[17], i.e. consequence relations for which **Identity, Weakening**, **Cut** and **Uniform Substitution** obtains (it is not required that consequence relations be **Compact**):

- **Identity**: $\{\alpha\} \vdash \alpha$
- **Weakening**: If $\Gamma \vdash \alpha$ then $\Gamma \cup \Delta \vdash \alpha$
- **Cut**: If $\Gamma \vdash \alpha$ and $\Delta \cup \{\alpha\} \vdash \beta$ then $\Gamma \cup \Delta \vdash \beta$

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[17] In the sense of Łoś and Suszko (1958).

- **Uniform Substitution**: If $\Gamma \vdash \alpha$ then $\Gamma[\beta/p] \vdash \alpha[\beta/p]$ [18]

3.3.1 Syntax

We specify the languages of propositional logics by a signature (i.e. sets of propositional connectives), $C = \{C^n\}_{n \in \mathbb{N}}$, and a set of sentence symbols, P (infinite). The set of sentences generated by C and P, Sent(C, P), is inductively defined as the least set such that:

- If $\alpha \in P$ then $\alpha \in \text{Sent}(C, P)$;
- If $c \in C^n$ and $\alpha^1 ... \alpha^n \in \text{Sent}(C, P)$, then $c\alpha^1 ... \alpha^n \in \text{Sent}(C, P)$.

The juxtaposition of two sets of sentence symbols P_1 and P_2, called P_{12}, is $P_1 \cup P_2$. *The juxtaposition of two signatures* C_1 and C_2, called C_{12}, is $\{C_1^n \cup C_2^n\}_{n \in \mathbb{N}}$, which is the union of each of the sets of connectives of arity n.[19] The set of sentences, Sent(C_{12}, P_{12}), is generated by C_{12} and P_{12} as usual. This allows forming mixed sentences like $\neg_2\neg_1(q_2 \vee_2 \neg_1 p_1)$, for instance, where you can have a connective of C_1 under the scope of a connective of C_2 and vice versa.

3.3.2 Consequence relations

Let us now suppose that C^- is a subsignature of C and P^- is a subset of P. We say that a consequence relation, $\vdash$, *extends* another consequence relation, $\vdash^-$, if and only if for every $\Gamma \subseteq \text{Sent}(C^-, P^-)$ and $\alpha \in \text{Sent}(C^-, P^-)$, if $\Gamma \vdash^- \alpha$ then $\Gamma \vdash \alpha$. We say that $\vdash$ is a *strong conservative extension* of $\vdash^-$, if and only if for every $\Gamma \subseteq \text{Sent}(C^-, P^-)$ and $\alpha \in \text{Sent}(C^-, P^-)$, $\Gamma \vdash^- \alpha$ just in case $\Gamma \vdash \alpha$[20].

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[18] We write $\alpha[\beta/p]$ for indicating the substitution of every instance of p in α by β and $\Gamma[\beta/p]$ for the substitution of every p in γ by β, for every $\gamma \in \Gamma$.

[19] We allow both P_1 and P_2 and C_1 and C_1 to overlap. In the literature on fibring the combination with overlapping signatures is known as "constrained" while the combination with disjoint signatures is referred to as "unconstrained".

[20] Schechter uses 'strong' but some people in the literature refer to this simply as 'conservative extension'. I believe that Schechter's preference for 'strong conservative extension' has to do with making it clear that it is not just for theorems but for inferences.

We define Γ to be *consistent* with respect to $\vdash$ if there is an α such that $\Gamma \nvdash \alpha$. And we say that $\vdash$ is *consistent* if there is an α such that $\nvdash \alpha$. A consequence relation, $\vdash$, is *nontrivial* if there is a nonempty Γ and an α such that $\Gamma \nvdash \alpha$. Let us also say that a consequence relation, $\vdash$, has theorems just in case there is at least one theorem of $\vdash$, and that $\vdash$ is *left-extensional* just in case for every $\alpha, \beta, \delta \in \text{Sent}(C, P)$ and p occurring in δ, $\alpha, \beta, \delta[\alpha/p] \vdash \delta[\beta/p]$.

Going to definitions more specific to juxtaposition, let $\vdash_1$ be a consequence relation for $\text{Sent}(C_1, P_1)$ and $\vdash_2$ a consequence relation for $\text{Sent}(C_2, P_2)$. A *juxtaposed consequence relation* over $\vdash_1$ and $\vdash_2$ is a consequence relation for $\text{Sent}(C_{12}, P_{12})$ that extends both $\vdash_1$ and $\vdash_2$. And *the juxtaposition of the consequence relations* $\vdash_1$ and $\vdash_2$ is the intersection of every juxtaposed consequence relation over $\vdash_1$ and $\vdash_2$. Therefore, *the juxtaposition of the consequence relations* $\vdash_1$ and $\vdash_2$ is the *minimal conservative extension* of $\vdash_1$ and $\vdash_2$[21].

3.3.3 Semantics

The approach to the semantics is broadly algebraic. Let us very briefly recall some general definitions. A *structure* over a signature C is an ordered triple $\mathfrak{B} = \langle B, D, \Phi \rangle$ s.t. B is the set of *semantic values*, D is the (nonempty) set of *designated values* and Φ is the *denotation function* of $\mathfrak{B}$, such that, $\forall c \in C^n$,

$$\Phi(c) : B^n \to B.$$

A *valuation*, V, for a structure, $\mathfrak{B}$, and a set of sentence symbols, P, is a function from P to B. A *model*, $\mathfrak{M} = \langle \mathfrak{B}, V \rangle$, over a signature C and a set of sentence symbols P, based on $\mathfrak{B} = \langle B, D, \Phi \rangle$, is an ordered pair such that $\mathfrak{B}$ is a structure and V is a valuation for $\mathfrak{B}$ and P. Given a class of structures $\mathbb{B}$ over C, we say that $\mathfrak{M}$ is based on $\mathbb{B}$ just in case $\mathfrak{M}$ is based on some element of $\mathbb{B}$.

For any $\alpha \in \text{Sent}(C, P)$, the *value* of α in $\mathfrak{M}$, $||\alpha||^{\mathfrak{M}}$, is recursively defined:

- $||\alpha||^{\mathfrak{M}} = V(\alpha)$ if $\alpha \in P$;

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[21] So, it is a strong conservative extension, as we will see, and it is the minimal among the strong conservative extensions.

- $||c\alpha^1...\alpha^n||^{\mathfrak{M}} = \Phi(c)(||\alpha^1||^{\mathfrak{M}}...||\alpha^n||^{\mathfrak{M}})$ if $c \in C^n$ and $\alpha^1...\alpha^n \in \text{Sent}(C, P)$.

$\mathfrak{M}$ *designates* the sentence α, $\mathfrak{M} \vDash \alpha$, when the value of α in the model belongs to the set of designated values in $\mathfrak{B}$, $||\alpha||^{\mathfrak{M}} \in D$.

α is *valid* in $\mathbb{B}$, $\vDash^{\mathbb{B}} \alpha$, when for every model $\mathfrak{M}$ over C and P based on a class of structures $\mathbb{B}$, $\mathfrak{M} \vDash \alpha$. Lastly, we say that Γ *entails* α in $\mathbb{B}$, $\Gamma \vDash^{\mathbb{B}} \alpha$, when for every model $\mathfrak{M}$ over C and P based on $\mathbb{B}$, if $\mathfrak{M} \vDash \Gamma$ then $\mathfrak{M} \vDash \alpha$.

Having the general overview, let us now move to the semantics specific to juxtaposition. Given two structures $\mathfrak{B}_1$, over C_1, and $\mathfrak{B}_2$, over C_2, *the juxtaposition of the structures* is the ordered pair $\langle\mathfrak{B}_1, \mathfrak{B}_2\rangle$. Notice that, unlike fibring, the juxtaposition does not dissolve the two original structures into one. *The juxtaposition of the classes of structures* $\mathbb{B}_1$ and $\mathbb{B}_2$ is the Cartesian product $\mathbb{B}_1 \times \mathbb{B}_2$.

A *juxtaposed model*, $\mathfrak{M}_{12}$ over C_{12} and P_{12}, based on the juxtaposed structure $\langle\mathfrak{B}_1, \mathfrak{B}_2\rangle$ (or, more generally, based on the class of juxtaposed structures $\mathbb{B}_{12}$) is an ordered quadruple $\langle\mathfrak{B}_1, V_1, \mathfrak{B}_2, V_2\rangle$. So, it is like putting together two models into one.

Now, in order to be able to work with a mixed language and, so, evaluate formulas where one might have to input, say, the value of an intuitionistic fragment into a classical connective and vice versa, it is necessary that each of the two models composing the juxtaposed model evaluates formulas for the entire language. The basic idea for doing this, is that each model treats the sentences from the other logic as if they were additional atoms. So, in a sense, each of the models is not able to interpret the logical structure of the fragments corresponding to the other logic. Let us say, then, that an *i-atom* (in our case $i = 1, 2$) is any element of Sent(C_{12}, P_{12}) that does not have a main connective from C_i. For instance, the subformulas of $\neg_2\neg_1(q_2 \vee_2 \neg_1 p_1)$ that are i-atoms would be:

1-atoms: $\neg_2\neg_1(q_2 \vee_2 \neg_1 p_1)$, $q_2 \vee_2 \neg_1 p_1$, q_2 and p_1.

2-atoms: $\neg_1(q_2 \vee_2 \neg_1 p_1)$, $\neg_1 p_1$, q_2 and p_1.

Once we have the notion of i-atom, we can define the i-value of a formula. For any $\alpha \in$ Sent(C_{12}, P_{12}) the *i-value* of α in $\mathfrak{M}_{12}$, $||\alpha||_i^{\mathfrak{M}_{12}}$, is recursively defined:

- $||\alpha||_i^{\mathfrak{M}_{12}} = V_i(\alpha)$ if α is an i-atom;
- $||c\alpha^1...\alpha^n||_i^{\mathfrak{M}_{12}} = \Phi_i(c)(||\alpha^1||_i^{\mathfrak{M}_{12}}...||\alpha^n||_i^{\mathfrak{M}_{12}})$ if $c \in C_i^n$ and $\alpha^1...\alpha^n \in \text{Sent}(C_{12}, P_{12})$.

We can see that the definition is the standard one, except for the fact that in the first step of the definition we do not only have sentence variables as atomic formulas, but also sentences with main connective from C_i which are i-atoms. Notice, however, that without any further constraints in the possible semantic values a sentence might take, the juxtaposed models could end up being a chaos! and not only that, they would be inconsistent if, for instance, $||\alpha||_1^{\mathfrak{M}_{12}} \in D_1$ but $||\alpha||_2^{\mathfrak{M}_{12}} \notin D_2$.

This is why the following semantic notion is a crucial one for juxtaposition, since it is the one allowing "a sensible definition of designation for juxtaposed models" (Schechter, 2011, p. 567). We say that a model is *coherent* when for every $\alpha \in \text{Sent}(C_{12}, P_{12})$, the 1-value of α is designated just in case its 2-value is designated, $||\alpha||_1^{\mathfrak{M}_{12}} \in D_1$ if and only if $||\alpha||_2^{\mathfrak{M}_{12}} \in D_2$.

Let $\mathfrak{M}_{12}$ be a coherent juxtaposed model over C_{12} and P_{12} based on $\mathbb{B}_{12}$. We say that a juxtaposed model, $\mathfrak{M}_{12}$, *designates* the sentence α, $\mathfrak{M}_{12} \vDash \alpha$, when $||\alpha||_1^{\mathfrak{M}_{12}} \in D_1$, which, given coherence, is equivalent to saying 'when $||\alpha||_2^{\mathfrak{M}_{12}} \in D_2$'. We say that α is *valid* in $\mathbb{B}_{12}$, $\vDash^{\mathbb{B}_{12}} \alpha$, when for every coherent model $\mathfrak{M}_{12}$, $\mathfrak{M}_{12} \vDash \alpha$. And, finally, Γ *entails* α in $\mathbb{B}_{12}$, $\Gamma \vDash^{\mathbb{B}_{12}} \alpha$, when for every coherent model $\mathfrak{M}_{12}$, if $\mathfrak{M}_{12} \vDash \Gamma$ then $\mathfrak{M}_{12} \vDash \alpha$.

Before moving on to some meta-logical results, let me give Schechter's definitions for *strong soundness, strong completeness* and *strong determination*:

Definition 3.3.1 (Strong Soundness, Strong Completeness and Strong Determination). Let $\vdash_{12}$ be a consequence relation for $\text{Sent}(C_{12}, P_{12})$ and $\mathbb{B}_{12}$ is a class of juxtaposed structures over C_1 and C_2. We say that $\vdash_{12}$ is *strongly sound* with respect to $\mathbb{B}_{12}$ just in case for every $\Gamma \subseteq \text{Sent}(C_{12}, P_{12})$ and $\alpha \in \text{Sent}(C_{12}, P_{12})$, if $\Gamma \vdash_{12} \alpha$ then $\Gamma \vDash^{\mathbb{B}_{12}} \alpha$. We say that $\vdash_{12}$ is *strongly complete* with respect to $\mathbb{B}_{12}$ just in case for every $\Gamma \subseteq \text{Sent}(C_{12}, P_{12})$ and $\alpha \in \text{Sent}(C_{12}, P_{12})$, if $\Gamma \vDash^{\mathbb{B}_{12}} \alpha$ then $\Gamma \vdash_{12} \alpha$. We say that $\vdash_{12}$ is *strongly determined* with respect to $\mathbb{B}_{12}$ just in case for every $\Gamma \subseteq \text{Sent}(C_{12}, P_{12})$ and $\alpha \in \text{Sent}(C_{12}, P_{12})$, $\Gamma \vdash_{12} \alpha$ just in case $\Gamma \vDash^{\mathbb{B}_{12}} \alpha$.

3.3.4 Preservation theorems

We have already mentioned that one of the virtues of juxtaposition is the amount of meta-logical results that Schechter provided. We are going to present the most important ones, just to give the reader an idea of the scope of the mechanism and its force. We start by giving a nice sufficient condition for the existence of coherent nontrivial models.

Proposition 3.3.1 (Existence of Coherent Nontrivial Models (Lemma 5.1 and Proposition 5.2 in Schechter (2011))). *Suppose C_1 and C_2 are disjoint signatures. Suppose $\mathfrak{M}_1 = \langle \mathfrak{B}_1, V_1 \rangle$ is a model over C_1 and P_1 and $\mathfrak{M}_2 = \langle \mathfrak{B}_2, V_2 \rangle$ is a model over C_2 and P_2. Then there is a coherent nontrivial juxtaposed model, $\mathfrak{M}_{12}$, over C_{12} and P_{12} based on $\mathfrak{B}_{12}$, just in case for every $p \in P_1 \cap P_2, \mathfrak{M}_1 \vDash p$ just in case $\mathfrak{M}_2 \vDash p$.*

It is important that the signatures are disjoint in order to prove this proposition. Otherwise it is possible to find structures such that there is no coherent juxtaposed model based on them.

The next important result is the *Preservation of Strong Soundness* (Theorem 5.5 in Schechter (2011)). In order to prove this theorem, Schechter proves two lemmas. The first one establishes that $\vDash^{\mathbb{B}_{12}}$ is a consequence relation, in the sense specified above of having Identity, Weakening, Cut and Uniform Substitution. The second lemma states that, given that $\mathbb{B}_{12}$ is the juxtaposition of $\mathbb{B}_1$ and $\mathbb{B}_2$, if $\Gamma \vDash^{\mathbb{B}_1} \alpha$ or $\Gamma \vDash^{\mathbb{B}_2} \alpha$, then $\Gamma \vDash^{\mathbb{B}_{12}} \alpha$. Applying this two lemmas we get,

Proposition 3.3.2 (Preservation of Strong Soundness). *Suppose $\vdash_{12}$ is the juxtaposition of $\vdash_1$ and $\vdash_2$ and $\mathbb{B}_{12}$ is the juxtaposition of $\mathbb{B}_1$ and $\mathbb{B}_2$. If $\vdash_1$ is strongly sound with respect to $\mathbb{B}_1$ and $\vdash_2$ is strongly sound with respect to $\mathbb{B}_2$, then $\vdash_{12}$ is strongly sound with respect to $\mathbb{B}_{12}$.*

Notice that the proof of this proposition appeals to the fact that $\vdash_{12}$ is *the* juxtaposition of $\vdash_1$ and $\vdash_2$ and, so, by definition, the minimal conservative extension of $\vdash_1$ and $\vdash_2$. This means that there is nothing more in the relation $\vdash_{12}$ than what already belonged either to $\vdash_1$ or $\vdash_2$. That is why it is enough to stablish that if $\Gamma \vdash_1 \alpha$ or $\Gamma \vdash_2 \alpha$, then $\Gamma \vDash^{\mathbb{B}_{12}} \alpha$.

Definition 3.3.2. A consequence relation $\vdash$ *has no mere followers* just in case $\vdash \alpha$ whenever $\Gamma \vdash \alpha$ for every nonempty $\Gamma \subseteq \text{Sent}(C, P)$.

Proposition 3.3.3 (Strong Conservativeness). *Suppose C_1 and C_2 are disjoint signatures. Suppose each of $\vdash_1$ and $\vdash_2$ is consistent and has no mere followers. Suppose $\vdash_{12}$ is the juxtaposition of $\vdash_1$ and $\vdash_2$. Then $\vdash_{12}$ is a strong conservative extension of each of $\vdash_1$ and $\vdash_2$.*

Proposition 3.3.4 (Preservation of Consistency). *Suppose C_1 and C_2 are disjoint signatures. Suppose each of $\vdash_1$ and $\vdash_2$ is consistent and has no mere followers. Suppose $\vdash_{12}$ is the juxtaposition of $\vdash_1$ and $\vdash_2$. Then $\vdash_{12}$ is consistent. If $\Gamma \subseteq$ Sent(C_i, P_i) is consistent with respect to $\vdash_i$, then Γ is consistent with respect to $\vdash_{12}$.*

The preservation of consistency from each consequence relation $\vdash_1$ and $\vdash_2$ to the juxtaposed one, relies on the fact that $\vdash_{12}$ is a strong conservative extension of $\vdash_1$ and $\vdash_2$.

Let us now advance to the more challenging completeness results. Schechter's strategy is based on a modification of the Lindenbaum-Tarski construction. The idea is to provide direct proofs for strong completeness that apply in a variety of cases by finding suitable equivalence relations in order to build the Lindenbaum-Tarski models. Here we focus on the essential results.

Let $\sim$ be an equivalence relation on Sent(C, P). We say that $\sim$ is a *congruence* over C^- whenever:

- For every $c^- \in C^{-n}, \alpha^1, ..., \alpha^n, \beta \in$ Sent(C, P) and $k \in \{1, ..., n\}$, if $\alpha^k \sim \beta$, then $c^-\alpha^1...\alpha^k...\alpha^n \sim c^-\alpha^1...\beta...\alpha^n$.

$\sim$ is *compatible* with $\vdash$ and $\Gamma \subseteq$ Sent(C, P) whenever:

- For every $\alpha, \beta \in$ Sent(C, P), if $\alpha \sim \beta$ then $\Gamma \vdash \alpha$ just in case $\Gamma \vdash \beta$.

We say that $\sim$ is *strongly compatible* with $\vdash$ and $\Gamma \subseteq$ Sent(C, P) just in case $\sim$ is a *compatible* with $\vdash$ and Γ and:

- For every $\alpha, \beta \in$ Sent(C, P), if both $\Gamma \vdash \alpha$ and $\Gamma \vdash \beta$ then $\alpha \sim \beta$.

We say that $\sim$ is *suitable* for C^{-}, $\vdash$ and Γ just in case $\sim$ is a congruence over C^{-} compatible with $\vdash$ and Γ. We say that $\sim$ is *unital suitable* for C^{-}, $\vdash$ and Γ just in case $\sim$ is a congruence over C^{-} strongly compatible with $\vdash$ and Γ. In order to construe the Lindenbaum-Tarski models, Schechter makes use of suitable and unital suitable equivalence relations.

Let us give some additional definitions before moving to the results. Suppose Γ is a nonempty subset of Sent(C_{12}, P_{12}) consistent with respect to $\vdash_{12}$. For each $i \in \{1, 2\}$, suppose $\sim_i^{\Gamma}$ is an equivalence relation on Sent(C_{12}, P_{12}) suitable for C_i, $\vdash_{12}$ and Γ. We define:

- $|\alpha|_i^{\Gamma} = \{\beta \mid \alpha \sim_i^{\Gamma} \beta\}$
- $\mathsf{B}_i^{\Gamma} = \{|\alpha|_i^{\Gamma} \mid \alpha \in \mathrm{Sent}(C_{12},\, P_{12})\}$
- $\mathsf{D}_i^{\Gamma} = \{|\alpha|_i^{\Gamma} \mid \Gamma \vdash_{12} \alpha\}$
- If $c_i \in C_i^n$, $\Phi_i^{\Gamma}(c_i)(|\alpha^1|_i^{\Gamma}, ..., |\alpha^n|_i^{\Gamma}) = |c_i\alpha^1...\alpha^n|_i^{\Gamma}$
- $\mathbf{B}_i^{\Gamma} = \langle \mathsf{B}_i^{\Gamma}, \mathsf{D}_i^{\Gamma}, \Phi_i^{\Gamma} \rangle$
- $\mathbf{B}_{12}^{\Gamma} = \langle \mathbf{B}_1^{\Gamma}, \mathbf{B}_2^{\Gamma} \rangle$
- If α is an i-atom, $\mathsf{V}_i^{\Gamma}(\alpha) = |\alpha|_i^{\Gamma}$
- $\mathbf{M}_{12}^{\Gamma} = \langle \mathbf{B}_1^{\Gamma}, \mathsf{V}_1^{\Gamma}, \mathbf{B}_2^{\Gamma}, \mathsf{V}_2^{\Gamma} \rangle$

Then, $\mathbf{B}_{12}^{\Gamma}$ and $\mathbf{M}_{12}^{\Gamma}$ are the *Lindenbaum-Tarski juxtaposed structure* and the *Lindenbaum-Tarski juxtaposed model* for C_1, C_2, $\vdash_{12}$ and Γ, built with $\sim_1^{\Gamma}$ and $\sim_2^{\Gamma}$.

Now, suppose for every $i \in \{1, 2\}$ and nonempty $\Gamma \subseteq$ Sent(C_{12}, P_{12}) consistent with respect to $\vdash_{12}$, $\sim_i^{\Gamma}$ is an equivalence relation on Sent(C_{12}, P_{12}) suitable for C_i, $\vdash_{12}$ and Γ. Let us define:

$$\mathbb{B}_{12}^{\sim} = \{\mathbf{B}_{12}^{\Gamma} | \Gamma \subseteq \mathrm{Sent}(C_{12},\, P_{12}) \text{ is nonempty and consistent with respect to } \vdash_{12} \text{ and } \mathbf{B}_{12}^{\Gamma} \text{ is built with } \sim_1^{\Gamma} \text{ and } \sim_2^{\Gamma}\}$$

$\mathbb{B}_{12}^{\sim}$ is the *Lindenbaum-Tarski class of juxtaposed structures* for C_1, C_2 and $\vdash_{12}$ built with $\sim_i^{\Gamma}$.

THEOREM 3.3.1 (Strong Completeness). *Suppose* $\vdash_{12}$ *has no mere followers. Suppose for every* $i \in \{1,2\}$ *and nonempty* $\Gamma \subseteq$ *Sent(*C_{12}, P_{12}*) consistent with respect to* $\vdash_{12}$, $\sim_i^\Gamma$ *is an equivalence relation on Sent(*C_{12}, P_{12}*) suitable for* C_i, $\vdash_{12}$ *and* Γ. *Then* $\vdash_{12}$ *is strongly complete with respect to* $\mathbb{B}_{12}^{\sim}$.

THEOREM 3.3.2 (Strong Soundness). *Suppose* $\vdash_{12}$ *is the juxtaposition of* $\vdash_1$ *and* $\vdash_2$. *Suppose for every* $i \in \{1,2\}$ *and nonempty* $\Gamma \subseteq$ *Sent(*C_{12}, P_{12}*) consistent with respect to* $\vdash_{12}$, $\sim_i^\Gamma$ *is an equivalence relation on Sent(*C_{12}, P_{12}*) suitable for* C_i, $\vdash_{12}$ *and* Γ. *Then* $\vdash_{12}$ *is strongly sound with respect to* $\mathbb{B}_{12}^{\sim}$.

PROPOSITION 3.3.5. *Suppose* $\vdash_{12}$ *has no mere followers. Suppose for every* $i \in \{1,2\}$ *and nonempty* $\Gamma \subseteq$ *Sent(*C_{12}, P_{12}*) consistent with respect to* $\vdash_{12}$, $\sim_i^\Gamma$ *is an equivalence relation on Sent(*C_{12}, P_{12}*) suitable for* C_i, $\vdash_{12}$ *and* Γ. *Then if* $\vdash_{12}$ *is consistent, there is a coherent nontrivial juxtaposed model based on* $\mathbb{B}_{12}^{\sim}$.

Summarizing the results:

THEOREM 3.3.3. *Suppose* $\vdash_{12}$ *has no mere followers. Suppose for every* $i \in \{1,2\}$ *and nonempty* $\Gamma \subseteq$ *Sent(*C_{12}, P_{12}*) consistent with respect to* $\vdash_{12}$, $\sim_i^\Gamma$ *is an equivalence relation on Sent(*C_{12}, P_{12}*) suitable for* C_i, $\vdash_{12}$ *and* Γ. *Then:*

1. $\vdash_{12}$ *is strongly complete with respect to* $\mathbb{B}_{12}^{\sim}$.
2. *If* $\vdash_{12}$ *is the juxtaposition of* $\vdash_1$ *and* $\vdash_2$, *then* $\vdash_{12}$ *is strongly sound with respect to* $\mathbb{B}_{12}^{\sim}$.
3. *If* $\vdash_{12}$ *is consistent, then there is a coherent nontrivial juxtaposed model based on* $\mathbb{B}_{12}^{\sim}$.
4. $\mathbb{B}_{12}^{\sim}$ *is a class of juxtaposed unital structures just in case for every* $i \in \{1,2\}$ *and nonempty* $\Gamma \subseteq$ *Sent(*C_{12}, P_{12}*) consistent with respect to* $\vdash_{12}$, $\sim_i^\Gamma$ *is unital suitable for* C_i, $\vdash_{12}$ *and* Γ.

3.3.5 Applying the results to Classical and Intuitionistic Logics

After obtaining this plethora of metalogical results, Schechter himself makes use of the method of juxtaposition in order to apply it to the cases of combining classical and intuitionistic logics, since, as already pointed out, it is for those cases that the collapse theorems have been proved in the literature. Thus, he starts developing the juxtaposition of classical and intuitionistic logics as follows.

Let $P_1 = P_2 = P_{12}$ be a countably infinite set of sentence symbols and for $i = 1, 2$ let C_i be the signature containing these sets of connectives:

- $C_i^1 = \{\neg_i\}$
- $C_i^2 = \{\wedge_i, \vee_i, \rightarrow_i, \leftrightarrow_i\}$

So, C_{12} is the set containing two copies of each of the propositional connectives. Let $\vdash_1^i$ and $\vdash_2^c$ be the intuitionistic and classical consequence relations for Sent(C_1, P_1) and Sent(C_2, P_2) respectively. We say that $\vdash^{ic}$ is the intuitionist-classical juxtaposed consequence relation for Sent(C_{12}, P_{12}). We can also have consequence relations for languages with two copies of classical connectives or two copies of intuitionistic connectives. We call these, $\vdash^{cc}$ and $\vdash^{ii}$, the bi-classical and bi-intuitionist consequence relations respectively.

In the section about interaction principles we advanced some rough definitions of collapse and weak collapse. Let us, now, give some more precise definitions needed in order to study $\vdash^{cc}$, $\vdash^{ii}$ and $\vdash^{ic}$. A consequence relation, $\vdash_{12}$, for Sent(C_{12}, P_{12}) *collapses* just in case for every $\delta, \delta' \in$ Sent(C_{12}, P_{12}) exactly alike except perhaps for some or all of their subscripts, $\{\delta\} \vdash_{12} \delta'$.

Let f be a bijection from Sent(C_1, P_{12}) to Sent(C_2, P_{12}) that maps each sentence $\alpha \in$ Sent(C_1, P_{12}) to the sentence that results from uniformly substituting each connective in α with the corresponding connective from C_2. We say that a consequence relation, $\vdash_{12}$, for Sent(C_1, P_{12}) *weakly collapses* just in case for every $\Gamma \subseteq$ Sent(C_1, P_{12}) and $\alpha \in$ Sent(C_1, P_{12}), $\Gamma \vdash_{12} \alpha$ just in case $f(\Gamma) \vdash_{12} f(\alpha)$.

We consider here the case of intuitionist-classical consequence relation, $\vdash^{ic}$. On one hand, for any nontrivial Boolean algebra, $\langle B, \leq \rangle$, there is a corresponding unital structure, $\langle B, \{1\}, \Phi \rangle$, where B is the set of semantic values, $\{1\}$ is the greatest element of the Boolean algebra given by the order $\leq$, and for every $a, b \in B$:

$$\Phi(\neg)(a) = -a$$

$$\Phi(a,b)(\wedge) = a \sqcap b$$

$$\Phi(a,b)(\vee) = a \sqcup b$$

$$\Phi(a,b)(\rightarrow) = -a \sqcup b$$

$$\Phi(a,b)(\leftrightarrow) = (-a \sqcup b) \sqcap (-b \sqcup a)$$

With $-, \sqcap, \sqcup$ as the complement, infimum and supremum operations in the Boolean algebra. We call these structures "Boolean structures" and the classical consequence relation is strongly determined, i.e., is sound and complete, with respect to the class of Boolean structures. It is consistent, has theorems and it is left-extensional.

On the other hand, for any nontrivial Heyting algebra, $\langle B, \leq \rangle$, there is a corresponding unital structure, $\langle B, \{1\}, \Phi \rangle$, where B is the set of semantic values, $\{1\}$ is the greatest element of the Heyting algebra and for every $a, b \in B$:

$$\Phi(\neg)(a) = a \Rightarrow 0$$

$$\Phi(a,b)(\wedge) = a \sqcap b$$

$$\Phi(a,b)(\vee) = a \sqcup b$$

$$\Phi(a,b)(\rightarrow) = a \Rightarrow b$$

$$\Phi(a,b)(\leftrightarrow) = (a \Rightarrow b) \sqcap (b \Rightarrow a)$$

With $\sqcap, \sqcup, \Rightarrow$ as the infimum, supremum and implication operations in the Heyting algebra and 0 as its least element.We call these structures "Heyting structures" and the intuitionist consequence relation is strongly determined with respect to the class of Heyting structures. It is also consistent, has theorems and it is left-extensional.

By Proposition 3.3.4 $\vdash^{ic}$ is consistent and by Proposition 3.3.3 $\vdash^{ic}$ is a strong conservative extension of $\vdash^{i}$ and $\vdash^{c}$. An interesting point that Schechter just mentions without getting into the details is that the juxtaposed consequence relation $\vdash^{ic}$ can be axiomatized "using a copy of any natural deduction-style axiomatization for intuitionist logic and a copy of any natural deduction-style axiomatization for classical logic, each restricted so that the rules for one stock of connectives cannot be applied within any subderivation used in the application of a metarule governing a connective from the other stock" (Schechter, 2011, p. 595).

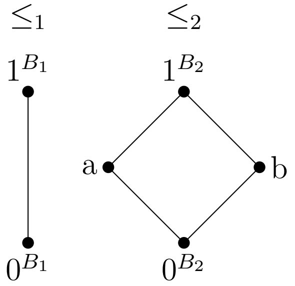
A *Heyting-Boolean structure* is a juxtaposed unital structure $\langle \mathfrak{B}_1, \mathfrak{B}_2 \rangle$ such that $\mathfrak{B}_1$ is a Heyting structure and $\mathfrak{B}_2$ is a Boolean structure. By Proposition (Proposition 6.34 or corollary 6.33 in Schechter (2011)), $\vdash^{ic}$ is strongly determined with respect to the class of Heyting-Boolean structures.

The first non-collapse result that one can easily check is that the intuitionist-classical juxtaposed consequence relation does not collapse. Just notice that since $\vdash^{ic}$ is a strong conservative extension of both $\vdash^{c}$ and $\vdash^{i}$ we will have $\nvdash^{ic} p \vee_1 \neg_1 p$ but $\vdash^{ic} p \vee_2 \neg_2 p$. Moreover, we can prove that no corresponding connectives of C_{12} are interderivable in $\vdash^{cc}$, so neither in the weaker relations $\vdash^{ic}$ and $\vdash^{ii}$.

Proposition 3.3.6 (Proposition 7.1 in Schechter (2011)). *In $\vdash^{cc}$, no pair of corresponding connectives are intersubstitutable. In particular:*

- $\{\neg_1 p\} \nvdash^{cc} \neg_2 p$
- $\{p \vee_1 q\} \nvdash^{cc} p \vee_2 q$
- $\{p \rightarrow_1 q\} \nvdash^{cc} p \rightarrow_2 q$
- $\{p \leftrightarrow_1 q\} \nvdash^{cc} p \leftrightarrow_2 q$
- $\{\neg_2(p \wedge_1 q)\} \nvdash^{cc} \neg_2(p \wedge_2 q)$

Proof. In order to prove this, we have to build a coherent juxtaposed countermodel that invalidates each of the previous entailments. We look, thus, for a model based on bi-Boolean structures. As it is usually done, we represent the Boolean algebras using *Hasse diagrams*. Consider the following Boolean algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$:



Let us choose the following valuations. $V_1(p^1) = V_1(p^2) = V_1(p^3) = 0^{B_1}$, $V_2(p^1) = V_2(p^2) = a$, $V_2(p^3) = b$ and let $V_i(p) = 1^{B_i}$ for every other $p \in P_{12}$. Since $\mathfrak{M}_1 = \langle \mathfrak{B}_1, V_1 \rangle$ and $\mathfrak{M}_2 = \langle \mathfrak{B}_2, V_2 \rangle$ designate the same sentence symbols, we know by Proposition 3.3.1 that there is a coherent juxtaposed model $\mathfrak{M}_{12}$. Since $\vdash^{cc}$ is strongly sound with respect to the class of bi-Boolean structures, we make use of the model, $\mathfrak{M}_{12}$, in order to invalidate the entailments above.

In $\mathfrak{M}_{12}$, $||\neg_1 p^1||_1 = 1^{B_1}$ and $||\neg_2 p^1||_2 = b$, so $\{\neg_1 p^1\} \nvdash^{cc} \neg_2 p^1$. $||p^1 \vee_2 p^3||_2 = 1^{B_2}$ but $||p^1 \vee_1 p^3||_1 = 0^{B_1}$ so $\{p^1 \vee_2 p^3\} \nvdash^{cc} p^1 \vee_1 p^3$ and by symmetry $\{p^1 \vee_1 p^3\} \nvdash^{cc} p^1 \vee_2 p^3$. $||p^1 \rightarrow_1 p^3||_1 = 1^{B_1}$ and $||p^1 \rightarrow_2 p^3||_2 = b$, therefore $\{p^1 \rightarrow_1 p^3\} \nvdash^{cc} p^1 \rightarrow_2 p^3$. $||p^1 \leftrightarrow_1 p^3||_1 = 1^{B_1}$ and $||p^1 \leftrightarrow_2 p^3||_2 = 0^{B_2}$, so $\{p^1 \leftrightarrow_1 p^3\} \nvdash^{cc} p^1 \leftrightarrow_2 p^3$. And the last one, which involves an embedded connective. $||\neg_2(p^1 \wedge_1 p^2)||_2 = -_2||p^1 \wedge_1 p^2||_2$. Since the 1-value of $p^1 \wedge_1 p^2$ is 0^{B_1} and it is a 2-atom, we choose in the construction of the coherent juxtaposed model $V_2(p^1 \wedge_1 p^2) = 0^{B_2}$, therefore $||\neg_2(p^1 \wedge_1 p^2)||_2 = 1^{B_2}$. But, $||\neg_2(p^1 \wedge_2 p^2)||_2 = -_2(a) = b$, so $\{\neg_2(p^1 \wedge_1 p^2)\} \nvdash^{cc} \neg_2(p^1 \wedge_2 p^2)$. □

As already said, since $\vdash^{ii}$ and $\vdash^{ic}$ are weaker relations than $\vdash^{cc}$ the result applies to them as well. Notice that despite $\wedge_1$ and $\wedge_2$ not being intersubstitutable in general, they are intersubstitutable *as main connectives* even in $\vdash^{ii}$. This is because $\{p \wedge_1 q\} \vdash^{ii} p$, $\{p \wedge_1 q\} \vdash^{ii} q$ and $\{p, q\} \vdash^{ii} p \wedge_2 q$, so $\{p \wedge_1 q\} \vdash^{ii} p \wedge_2 q$. Since $\vdash^{ic}$ and $\vdash^{cc}$ are stronger than $\vdash^{ii}$, this also holds for them.

Another interesting result is that $\vdash^{cc}$ is not left-extensional, therefore, $\vdash^{ii}$ and $\vdash^{ic}$ are not left-extensional either.

PROPOSITION 3.3.7. *$\vdash^{cc}$ is not left-extensional.*

PROOF. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be the Boolean structures of the previous proof. Take $V_1(p^1) = V_1(p^2) = 1^{B_1}$, $V_2(p^1) = V_2(p^2) = 1^{B_2}$ and $V_i(p) = 0^{B_i}$ for every other $p \in P_{12}$. Since $\mathfrak{M}_1 = \langle \mathfrak{B}_1, V_1 \rangle$ and $\mathfrak{M}_2 = \langle \mathfrak{B}_2, V_2 \rangle$ designate the same sentence symbols, we know by Proposition 3.3.1 that there is a coherent juxtaposed model $\mathfrak{M}_{12}$. We construe the coherent model in such a way that we can have $||p^2||_2, ||p^1||_2, ||\neg_2\neg_1 p^1||_2 \in D_2$ while $||\neg_2\neg_1 p^2||_2 \notin D_2$. Since $V_1(p^1) = V_1(p^2) = 1^{B_1}$, $||\neg_1 p^1||_1 = ||\neg_1 p^2||_1 = 0^{B_1}$. But both $\neg_1 p^1$ and $\neg_1 p^2$ are 2-atoms, so take $V_2(\neg_1 p^1) = 0^{B_2}$ and $V_2(\neg_1 p^2) = a$ respecting coherence. Now we have that $||\neg_2\neg_1 p^1||_2 = 1^{B_2}$ and $||\neg_2\neg_1 p^2||_2 = b$, so $||p^2||_2, ||p^1||_2, ||\neg_2\neg_1 p^1||_2 \in D_2$ and $||\neg_2\neg_1 p^2||_2 \notin D_2$ as desired. Therefore, $p^1, p^2, \neg_2\neg_1 p^1 \nvdash^{cc} \neg_2\neg_1 p^2$ and, so, $\vdash^{cc}$ is not left-extensional. □

Let us now focus on the case of weak collapse. It is clear from the definition that both $\vdash^{ii}$ and $\vdash^{cc}$ weakly collapse. Notice also that if a consequence relation $\vdash$ collapses, then it weakly collapses. Thus, since we know that the fibring of $\vdash^{i}$ and $\vdash^{c}$ collapses, it does also weakly collapse. However, the juxtaposition of $\vdash^{i}$ and $\vdash^{c}$ does not weakly collapse, because $\vdash^{ic}$ is a strong conservative extension of both $\vdash^{i}$ and $\vdash^{c}$. That is, we will have, for instance, $\neg_c\neg_c p \vdash^{ic} p$ but $\neg_i\neg_i p \nvdash^{ic} p$.

As I said before, there is a wealth of results that Schechter proved in his paper and here we have just seen some of the most relevant ones, but Schechter's paper contains other interesting results. My next step, though, is to present some new, further applications of juxtaposition.

3.3.6 Further applications of juxtaposition

Since Schechter's seminal paper on juxtaposition, no further developments have been made on it (I believe that it has not even been applied in the literature). In this section, we apply the method of juxtaposition in order to obtain further results. Some of these have already been explored in the literature (see Béziau and Coniglio (2011) and Coniglio (2007)), but not under the method of juxtaposition. We start with a result that I have already mentioned above.

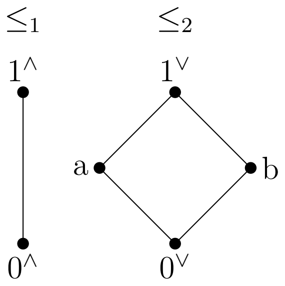
Proposition 3.3.8. *The juxtaposition of the logic of conjunction, $\mathcal{L}_\wedge$, and the logic of disjunction, $\mathcal{L}_\vee$, is not distributive. Furthermore, $\mathcal{L}_{\wedge\vee}$ is not the logic of lattices since absorption does not hold.*

Proof. Let $P_1 = P_2 = P_{12}$ be a countably infinite set of sentence symbols and let C_1 and C_2 be the signatures containing these sets of connectives:

- $C_1^2 = \{\wedge\}$
- $C_2^2 = \{\vee\}$

Let, then, $\vdash^\wedge$ and $\vdash^\vee$ be the consequence relations for $\text{Sent}(C_1, P_1)$ and $\text{Sent}(C_2, P_2)$ respectively. Thus, $\vdash^{\wedge\vee}$ is the juxtaposed consequence relation for the set of sentences $\text{Sent}(C_{12}, P_{12})$.

In order to prove this, let us make use again of the algebras we employed in the proof of Proposition 3.3.6, since, as Boolean algebras, they are both join and meet-semilattices.



We have to build coherent juxtaposed models in which nor distributivity neither absorption hold. That is:

- Distributivity of $\wedge$ over $\vee$: $p \wedge (q \vee r) \nvdash^{\wedge\vee} (p \wedge q) \vee (p \wedge r)$
- Distributivity of $\vee$ over $\wedge$: $p \vee (q \wedge r) \nvdash^{\wedge\vee} (p \vee q) \wedge (p \vee r)$
- Absorption-1: $p \vee (p \wedge q) \nvdash^{\wedge\vee} p$
- Absorption-2: $p \wedge (p \vee q) \nvdash^{\wedge\vee} p$

Let us take the following valuation $V_1(p) = 1^\wedge$, $V_1(q) = V_1(r) = 0^\wedge$, $V_2(p) = 1^\vee$, $V_2(q) = a$ and $V_2(r) = b$. We know by Proposition 3.3.1 that there is a coherent juxtaposed model $\mathfrak{M}_{12}$. In this model, $||p \wedge (q \vee r)||_1 = 1^\wedge \sqcap V_1(q \vee r)$. Since $||q \vee r||_2 = 1^\vee$, $V_1(q \vee r) = 1^\wedge$, therefore, $||p \wedge (q \vee r)||_1 = 1^\wedge$. But, in this very same model, $||(p \wedge q) \vee (p \wedge r)||_2 = ||p \wedge q||_2 \sqcup ||p \wedge r||_2$. Since $||p \wedge q||_1 = ||p \wedge r||_1 = 1^\wedge \sqcap 0^\wedge = 0^\wedge$, we can take $V_2((p \wedge q)) = V_2((p \wedge r)) = 0^\vee$ and we get that the premise is designated in the model while the conclusion is not. Then, the splicing of the logic of conjunction and the logic of disjunction by juxtaposition does not prove distributivity of $\wedge$ over $\vee$.

Let us now show that absorption-1 does not hold either. Take $V_1(p) = V_1(q) = 0^\wedge$ and $V_2(p) = a$. In this model, $||p \vee (p \wedge q)||_2 = a \sqcup V_2(p \wedge q)$. Since $||p \wedge q||_1 = 0^\wedge$, take $V_2(p \wedge q) = b$. Thus, we get $||p \vee (p \wedge q)||_2 = 1^\vee$, while the conclusion is not designated. So, the juxtaposition of the logics of a meet-semilattice and a join-semilattice does not result in the logic of a lattice.

□

The reader can easily check by playing around with the valuations that there are juxtaposed countermodels for distributivity of $\vee$ over $\wedge$ and absorption-2 as well.

The next results have to do with recovering a logic from its fragments. More specifically, we deal with the case of classical logic.

Proposition 3.3.9. *The juxtaposition of the logic of (classical) negation, $\mathcal{L}_{\neg}$, and the logic of (classical) conditional, $\mathcal{L}_{\rightarrow}$, does not recover classical logic.*

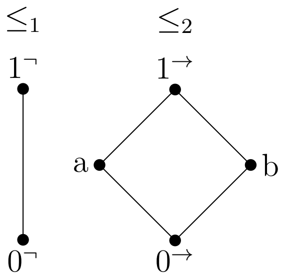
Proof. We prove this proposition by showing that for the juxtaposed consequence relation, $\vdash^{\neg\rightarrow}$, despite *Ex Contradictione Quodlibet* holding as a rule, i.e. $p, \neg p \vdash^{\neg\rightarrow} q$, we do not get the Principle of Pseudo-Scotus, $\vdash^{\neg\rightarrow} p \rightarrow (\neg p \rightarrow q)$, which is valid in **CL**.

Let $P_1 = P_2 = P_{12}$ be a countably infinite set of sentence symbols and let C_1 and C_2 be the signatures containing these sets of connectives:

- $C_1^1 = \{\neg\}$
- $C_2^2 = \{\rightarrow\}$

Let, then, $\vdash^{\neg}$ and $\vdash^{\rightarrow}$ be the consequence relations for Sent(C_1, P_1) and Sent(C_2, P_2) respectively. Thus, $\vdash^{\neg\rightarrow}$ is the juxtaposed consequence relation for the set of sentences Sent(C_{12}, P_{12}).

We make use, again, of the previous Boolean algebras:



In order to prove $\nvdash^{\neg\rightarrow} p \rightarrow (\neg p \rightarrow q)$ we look for a coherent juxtaposed countermodel based on the above pair of structures. Thus, let us take a valuation such that $V_1(p) = 1^{\neg}$, $V_1(q) = 0^{\neg}$, $V_2(p) = 1^{\rightarrow}$ and $V_2(q) = 0^{\rightarrow}$. Since $||\neg p||_1 = 0^{\neg}$, respecting coherence we take $V_2(\neg p) = a$ in order to have $||p \rightarrow (\neg p \rightarrow q)||_2 = -1^{\rightarrow} \sqcup (-a \sqcup 0^{\rightarrow}) = b$, which is not designated, therefore, $\nvdash^{\neg\rightarrow} p \rightarrow (\neg p \rightarrow q)$ as desired. However, we do have $p, \neg p \vdash^{\neg\rightarrow} q$ as announced. Just notice that since juxtaposition is a strong conservative extension and $p, \neg p \vdash^{\neg} q$ holds, we will also have that inference in $\vdash^{\neg\rightarrow}$. Moreover, there is no juxtaposed model satisfying both p and $\neg p$, so the argument is trivially valid. □

Next we present a similar result involving negation, disjunction and the Law of Excluded Middle.

Proposition 3.3.10. *The juxtaposition of the logic of (classical) negation, $\mathcal{L}_\neg$, and the logic of (classical) disjunction $\mathcal{L}_\vee$, does not recover classical logic.*

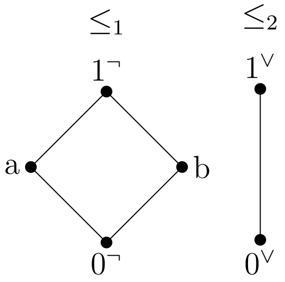
Proof. We prove this proposition by showing that for the juxtaposed consequence relation, $\vdash^{\neg\vee}$, the Law of Excluded Middle, $\vdash^{\neg\vee} p \vee \neg p$, does not hold.

Let $P_1 = P_2 = P_{12}$ be a countably infinite set of sentence symbols and let C_1 and C_2 be the signatures containing these sets of connectives:

- $C_1^1 = \{\neg\}$
- $C_2^2 = \{\vee\}$

Let, then, $\vdash^\neg$ and $\vdash^\vee$ be the consequence relations for Sent(C_1, P_1) and Sent(C_2, P_2) respectively. Thus, $\vdash^{\neg\vee}$ is the juxtaposed consequence relation for the set of sentences Sent(C_{12}, P_{12}).

Take the following structures:



Now, for the valuations $V_1(p) = a$, $V_2(p) = 0^\vee$, we get $||\neg p||_1 = b$, so choose $V_2(\neg p) = 0^\vee$. With these, we have that $||p \vee \neg p||_2 = 0^\vee$ as we wanted. ☐

A curious case is that of juxtaposing the logic of negation, $\mathcal{L}_\neg$, and the logic of conjunction, $\mathcal{L}_\wedge$. One might think, at first, that this logic should not have the principle of explosion, since that looks like a bridge principle and juxtaposition is not supposed to contain those. However, there are some interactions that are unavoidable. We have already seen, for instance, that in $\vdash^{ii}$ the two conjunctions are intersubstitutable as main connectives.

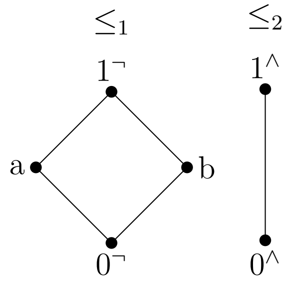
PROPOSITION 3.3.11. *The juxtaposition of the logic of (classical) negation, $\mathcal{L}_{\neg}$, and the logic of (classical) conjunction $\mathcal{L}_{\wedge}$, validates de Principle of Explosion, $p \wedge \neg p \vdash^{\neg\wedge} q$.*

PROOF. In order to invalidate explosion we would need $||p \wedge \neg p||_x \in D_x$ for $x = 1$ or 2, that is, both $||p||_x \in D_x$ and $||\neg p||_x \in D_x$. But this is not possible in the juxtaposed models given coherence. Suppose $V_1(p) = 1^{\neg}$, then $||\neg p||_1 = 0^{\neg}$. Given coherence, $V_2(\neg p) \notin D_2$. Take this value to be $x_2 \neq 1^{\wedge}$. Then, $||p \wedge \neg p||_2 = min(1^{\wedge}, x_2) = x_2 \notin D_2$. ☐

Notice, however, that the juxtaposed consequence relation $\vdash^{\neg\wedge}$ does not recover classical logic either.

PROPOSITION 3.3.12. *The juxtaposition of the logic of (classical) negation, $\mathcal{L}_{\neg}$, and the logic of (classical) conjunction $\mathcal{L}_{\wedge}$, does not validate $\vdash^{\neg\wedge} \neg(p \wedge \neg p)$.*

PROOF. Take, for instance, the following structures,



with valuations $V_1(p) = 1^{\neg}$, $V_2(p) = 1^{\wedge}$. Then, $||\neg p||_1 = 0^{\neg}$, so $V_2(\neg p) = 0^{\wedge}$. Therefore, $||p \wedge \neg p||_2 = 0^{\wedge}$, so choose $V_1(p \wedge \neg p) = a$. Hence, $||\neg(p \wedge \neg p)||_1 = b$. ☐

For the next result we go back to combining "full" logics, i.e. splicing two logical systems in order to get a new combined one. In this case, we are going to apply the method of juxtaposition for combining classical logic, **CL**, and Priest's logic of paradox, **LP**. Of course, the fact that we can do this is crucial for localism and, hopefully, for solving the problem of mixed inferences.

We consider, then, the case of paraconsistent-classical consequence relation, $\vdash^{pc}$.[22] On the classical side, the things remain the same as above,

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[22] I assume here some knowledge of **LP**.

when combining classical and intuitionist logics. On the paraconsistent side, in particular for the case of Priest´s *Logic of Paradox*, **LP**, we need some definitions in order to roughly present its semantics [23].

We say that a subset X of the set of semantic values A of an algebra $\mathfrak{A}$ is *contradictory* if $\exists a \in A : a \in X$ and $\neg a \in X$. A *filter* of $\mathfrak{A}$ is an $X \subseteq A$ such that $\forall a, b \in A, a \wedge b \in X$ iff $a \in X$ and $b \in X$. Let X be a filter of $\mathfrak{A}$. We say that X is *prime* in $\mathfrak{A}$ if, $\forall a, b \in A, a \vee b \in X$ iff $a \in X$ or $b \in X$.

Kleene algebras are the relevant algebraic structures for a semantic presentation of **LP**. A Kleene algebra is a *bounded distributive lattice* $(A, \wedge, \vee, 1, 0)$ with a unary operation, $\neg$, which is a *de Morgan involution* (i.e., $\neg$ satisfies $\neg(x \wedge y) = \neg x \vee \neg y$ and $\neg\neg x = x$) that additionally satisfies $x \wedge \neg x \leq y \vee \neg y$.

Theorem 3.3.4 (Algebraic Completeness Theorem for LP (Theorem 2.3 in Pynko (1995))). *Let $\mathfrak{A}$ be a Kleene lattice and F a proper contradictory filter of $\mathfrak{A}$. Then the matrix $\mathcal{M} = \langle \mathfrak{A}, F \rangle$ is strongly determined for the logic of paradox **LP**.*

Similarly to Boolean algebras and the two-element Boolean algebra, the variety of Kleene algebras is generated by the three-element chain Kleene algebra, $\mathfrak{K}_3$. Thus, **LP** is semantically defined by the logical matrix $\mathcal{M}_{LP} = \langle \mathfrak{K}_3, D_{LP} \rangle$, where $\mathfrak{K}_3 = \langle K_3, \wedge, \vee, \neg \rangle$ is Kleene's three-element chain algebra with $K_3 = \{1^p, a, 0^p\}$ as its set of semantic values and $D_{LP} = \{1^p, a\}$ its set of designated values. The truth functions for the connectives are those of Strong Kleene (see Appendix B), and the conditional, $p \to q$, can be expressed as $\neg p \vee q$.

Hence, we have that **LP** is strongly determined with respect to the class of Kleene algebras with a proper contradictory filter. We also know that $\vdash^p$ is consistent (because $\exists \alpha \in \mathrm{Sent}(C_p,\ P)$: $\nvdash^p \alpha$), has theorems but that it is not left-extensional. To see this, notice simply that $p, q, \neg r \vdash^p \neg r$, but $p, q, \neg r[p/r] \nvdash^p \neg r[q/r]$ because, with the three-element chain, $\mathfrak{K}_3$, for instance, for $V(p) = a, V(q) = 1^p$ the premises are designated and the conclusion is not. But this is unproblematic, since the Kleene matrices for **LP** are not unital and, therefore, the juxtaposed structures will not be unital either. [24]

By Proposition 3.3.4 $\vdash^{pc}$ is consistent and by Proposition 3.3.3 $\vdash^{pc}$ is a strong conservative extension of $\vdash^p$ and $\vdash^c$. The juxtaposed consequence

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[23] My main reference here is Pynko (1995).

[24] Recall that left-extensionality is a necessary condition for the preservation of strong unital determination, but not for strong determination.

relation $\vdash^{pc}$ can be axiomatized using a copy of any natural deduction-style axiomatization for the logic of paradox and a copy of any natural deduction-style axiomatization for classical logic, each restricted so that the rules for one stock of connectives cannot be applied within any subderivation used in the application of a metarule governing a connective from the other stock.

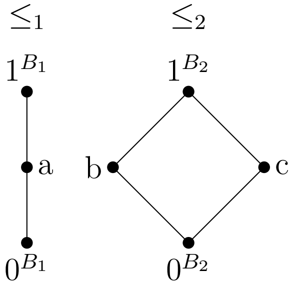
A *Kleene-Boolean structure*[25] is a juxtaposed structure $\langle \mathfrak{B}_1, \mathfrak{B}_2 \rangle$ such that $\mathfrak{B}_1$ is a Kleene structure and $\mathfrak{B}_2$ is a Boolean structure. By Proposition (Proposition 6.34 or corollary 6.33 in Schechter (2011)), $\vdash^{pc}$ is strongly determined with respect to the class of Kleene-Boolean structures. Let us now present some results about $\vdash^{pc}$.

Proposition 3.3.13. *In $\vdash^{pc}$, no pair of corresponding connectives are intersubstitutable in every context. In particular:*

- $\{\neg_1 p\} \nvdash^{pc} \neg_2 p$
- $\{p \vee_1 q\} \nvdash^{pc} p \vee_2 q$
- $\{p \rightarrow_1 q\} \nvdash^{pc} p \rightarrow_2 q$
- $\{p \leftrightarrow_1 q\} \nvdash^{pc} p \leftrightarrow_2 q$
- $\{\neg_2(p \wedge_1 q)\} \nvdash^{pc} \neg_2(p \wedge_2 q)$

Therefore, $\vdash^{pc}$ does not collapse.

Proof. We have to build a coherent juxtaposed countermodel that invalidates each of the previous entailments. We look, thus, for a model based on Kleene-Boolean structures. Consider the following Kleene and Boolean algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$:



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[25] Let me clarify that, when dealing with **LP**, "Kleene structure" should be understood as Kleene algebra with a proper contradictory filter.

Notice that, since the connectives of **LP** behave classically for the top and bottom semantic values of the Kleene algebra, we can take the valuations we took in Proposition 3.3.6 and by Proposition 3.3.1[26] we know that there exists a coherent juxtaposed countermodel for each of the entailments. □

Proposition 3.3.14. $\vdash^{pc}$ *does not weakly collapse.*

Proof. $\vdash^{pc}$ is a strong conservative extension of both $\vdash^{p}$ and $\vdash^{c}$, so while we have $p \rightarrow_c q, p \vdash^{pc} q$, we do not have $p \rightarrow_p q, p \vdash^{pc} q$ and so, the juxtaposed consequence relation does not weakly collapse. □

Proposition 3.3.15. $\vdash^{pc}$ *is not left-extensional.*

Proof. Take $p, q, r \wedge_c \neg_p r \vdash^{pc} r \wedge_c \neg_p r$. We want to show that $p, q, (r \wedge_c \neg_p r)[p/r] \nvdash^{pc} (r \wedge_c \neg_p r)[q/r]$. Just consider $V_p(p) = a$, i.e. the intermediate value, which is designated in the three-element chain Kleene algebra, $V_p(q) = 1^p$, $V_c(p) = 1^c$, $V_c(q) = 1^c$. With these, $||p \wedge_c \neg_p p||_c = 1^c$, but $||q \wedge_c \neg_p q||_c = 0^c$. Thus, we get a coherent juxtaposed model which makes the argument invalid. □

Notice that there are many more logical systems that are suitable to be combined in the same way as we have done with **CL**, **IL** and **LP**. This opens up the possibility of creating new interesting logical systems, by splicing simpler ones, that could potentially fit a variety of applications. One could argue, for instance, that a reasonable way of dealing with an inconsistent theory is to separate the inconsistencies from the consistent information, deal with it by means of a paraconsistent strategy and then try to work with the refined whole theory again[27]. Well, if we build, e.g., the juxtaposition of **CL** and **LP** as we have just done, we are in a sweet spot in order to proceed in that way.

But, the main application that concerns us, in this case, is the application of the methods for combining logics to solving the problem of mixed inferences. This is what I want to do now and, to such end, we will need to go beyond the juxtaposition of logics.

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[26] This proposition is in no need of modification for the case of **LP**, since it is proved for structures with an arbitrary number of designated values, not just for unital structures.

[27] This is, in fact, an application that Priest (2014) considers for his method of Chunk and Permeate.

Chapter 4

Towards a Solution to the Problem of Mixed Inferences

4.1 A reply to Wrenn: Partial Solution

As we already mentioned when presenting the different versions in the literature of the problem of mixed inferences, Chase Wrenn's challenge is the most and best developed, and the one that goes deeper into the details of the problem. Recall that the challenge is posed by Wrenn as a dilemma: either we keep some modest criterion of validity for the mixed inferences (e.g. the logic of the mixed inference is the one in the intersection of the logics in play), not counting some intuitively valid arguments as valid (Mix, Essential Mix, Disjunctive Mix), or we go beyond modesty (e.g. taking a stronger logic than the one in the intersection), accepting as true some untrue sentences (Nix).

What I want to do, now, is to start developing a possible way out of the dilemma, by offering a solution consisting of making a finer analysis of the arguments and using the method of juxtaposition as the criterion for deciding the validity of arguments. This, on top of being another novel application of the method of juxtaposition, is going to be a crucial clue for developing new methods that go beyond juxtaposition in order to solve the problem of mixed inferences while still avoiding the collapse.

Since we are going to work, for the moment, with classical and intuitionistic logic, we avoid repeating what we said about the juxtaposition of classical and intuitionistic language, consequence relation and semantics. The only difference is that we distinguish, now, the sets of sentence symbols. So, $P_c \neq P_i$ and $P_c \cup P_i = P_{ic}$.

Let us start with the easiest case, namely, Mix, to see how it is done. First, we formalize the argument in the juxtaposed language and, then, we apply the method of juxtaposition as the validity criterion.

1 **Mix**:

Wet cats are funny
Either snow is white or wet cats are not funny
Snow is white

Recall the assumption that the discourse about humour is an evaluative discourse in which reasoning is best captured by intuitionistic logic, while discourse about middle-sized objects, such as snow, is a discourse in which reasoning is best captured by classical logic. Under this assumption, let us formalize the argument by translating it into our juxtaposed language.

$$\begin{array}{l} p_i \\ q_c \vee_x \neg_i p_i \\ \hline q_c \\ {\scriptstyle (x\,=\,i,\,c)} \end{array}$$

There is a first difficulty right away. It seems natural to translate the negation as the intuitionistic one, since it is being applied to an intuitionistic proposition[1], i.e., ‘wet cats are funny’. However, which disjunction should we use in order to formalize the argument? Since it is a mixed sentence with one disjunct from the classical domain and another disjunct from the intuitionistic, it is not obvious whether the disjunction should be classical or intuitionistic. Remember, though, that Mix is among the arguments that are intuitively valid and, as we will see, the choice of the disjunction is not innocuous in this respect.

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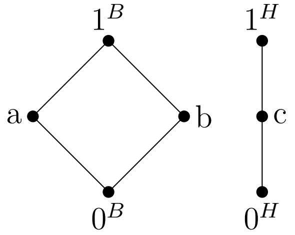
[1] This is not a strict rule and the context usually plays a major role in determining which the correct interpretation is, as it also happens with a single stock of connectives when, for instance, the use of an ‘if’ in a sentence might be ambiguous and further context might be needed in order to translate it as a conditional or a biconditional. Thus, there might be cases in which what the natural expression is trying to convey is, say, a classical negation applied to an intuitionistic proposition. We will discuss more these issues of translation later on.

Indeed, if we translate the disjunction as the classical disjunction, the argument is invalid. An easy way to see this is noticing that, since the disjunction is classical, $\neg_i p_i$ is a c-atom (classical atom) and so, the logical form of the argument is this: $p_i, q_c \vee_c r \vdash^{ic} q_c$. '$r$' is not a totally independent variable because if $||p_i||_x \in D_x$, then $||\neg_i p_i||_x = ||r||_x \notin D_x$, but this constraint is not enough to make the argument valid. Let us offer a juxtaposed countermodel to show it.

PROPOSITION 4.1.1. *$p_i, q_c \vee_c \neg_i p_i \vdash_{ic} q_c$ is invalid.*[2]

PROOF. We build a coherent juxtaposed countermodel that invalidates the argument. We look, then, for a model based on a Heyting-Boolean structure. Consider the following algebras:



We want to find valuations such that $||p_i||_x \in D_x$, $||q_c \vee_c \neg_i p_i||_x \in D_x$ and $||q_c||_x \notin D_x$[3]. Since $||p_i||_x \in D_x$, $V_i(p_i) = 1^H$and $V_c(p_i) = 1^B$, so $||\neg_i p_i||_i = 0^H$. Given coherence, $V_c(\neg_i p_i) \notin D_c$, so take $V_c(\neg_i p_i) = b$. Now we can choose $V_c(q_c) = a$. With these valuations we get $||p_i||_c = 1^B$, $||q_c \vee_c \neg_i p_i||_c = a \sqcup_c b = 1^B$ and $||q_c||_c = a$. So, we have built a coherent juxtaposed model in which $||p_i||_x \in D_x$, $||q_c \vee_c \neg_i p_i||_x \in D_x$ and $||q_c||_x \notin D_x$ as desired. ☐

But this cannot be the end of the story with Mix, otherwise juxtaposition would not be even a partial solution to the problem of mixed inferences because it would not be able to explain the validity of the easiest of the cases that Lynch and Wrenn consider. In fact, we have a way of explaining the intuitive validity of Mix, and this requires that we translate the disjunction as the intuitionistic one. That way, the logical form of the argument is that of

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[2] This could already be problematic for juxtaposition. In fact, I will try to correct this when improving the method.

[3] Of course, the valuations that we are going to take have to respect coherence, in order to comply with the sufficient condition for the existence of coherent nontrivial juxtaposed models (Proposition 3.3.1).

an intuitionistic disjunctive syllogism and, so, the intuitively valid argument will be validated by juxtaposition.

Proposition 4.1.2. *$p_i, q_c \vee_i \neg_i p_i \vdash_{ic} q_c$ is valid.*

Proof. We want to show that for every coherent juxtaposed model that designates the premises, the conclusion will also be designated. So, suppose that $||p_i||_x \in D_x$ and $||q_c \vee_i \neg_i p_i||_x \in D_x$. For $||p_i||_x \in D_x$, $V_i(p_i) = 1^H$, so $||\neg_i p_i||_i = 0^H$. Now, $||q_c \vee_i \neg_i p_i||_i \in D_i$, so $||q_c \vee_i \neg_i p_i||_i = 1^H$, but having $||\neg_i p_i||_i = 0^H$ the disjunction is designated only if $||q_c||_i \in D_i$. Thus, if the premises are designated the conclusion has to be designated too. Then, the argument is validated by juxtaposition. □

Let us consider, now, the mixed inference dubbed Essential Mix:

2 **Essential Mix**:

Either it's not the case that snow isn't white or
wet cats aren't funny
Wet cats are funny

Snow is white

We formalize the argument as:

$$\begin{array}{l} \neg_c \neg_c p_c \vee_x \neg_i q_i \\ \underline{q_i \qquad\qquad} \\ p_c \end{array}$$

Here, again, we have a similar situation. If we translate the disjunction as the classical disjunction, it turns out that the argument is invalid. The previous structures together with valuations $V_c(p_c) = a$, $V_i(q_i) = 1^H$, $V_c(\neg_i p_i) = b$ yield a coherent juxtaposed countermodel. However, we can explain the intuitive validity of the argument if we translate the disjunction as the intuitionistic one.

Proposition 4.1.3. *$\neg_c \neg_c p_c \vee_i \neg_i q_i$, $q_i \vdash^{ic} p_c$ is valid.*

Proof. Let us prove the validity of the argument, in this case using the juxtaposed natural deduction calculus, in the way that Schechter describes it.

1	$\neg_c\neg_c p_c \vee_i \neg_i q_i$	
2	q_i	
3	$\mid\ \neg_c\neg_c p_c$	
4	$\mid\ \neg_c\neg_c p_c$	Identity, 3
5	$\mid\ \neg_i q_i$	
6	$\mid\ \bot_i$	$\neg_i$E, 2, 5
7	$\mid\ \neg_c\neg_c p_c$	$\bot_i$E, 6
8	$\neg_c\neg_c p_c$	$\vee_i$E, 1–8
9	p_c	$\neg_c\neg_c$E, 8

So, we have a derivation of the conclusion from the premises showing that the argument is valid. ☐

Remember that Wrenn's improvement on Lynch's modesty criterion was able to explain the intuitive validity of these arguments we have just considered too. However, Wrenn comes up with the argument 'Disjunctive Mix', which is allegedly the knockdown mixed inference against localist proposals. He claims that, if the proposal is modest enough, which needs to be in order to invalidate arguments like 'Nix', then it will not be able to account for the validity of 'Disjunctive Mix'. We will show, now, that this is not in fact the case.

3 **Disjunctive Mix**:

Either it's not the case that snow isn't white
or wet cats are funny
Either snow is white or wet cats are funny

We formalize the argument as:

$$\frac{\neg_c\neg_c p_c \vee_x q_i}{p_c \vee_x q_i}$$

$(x = i, c)$

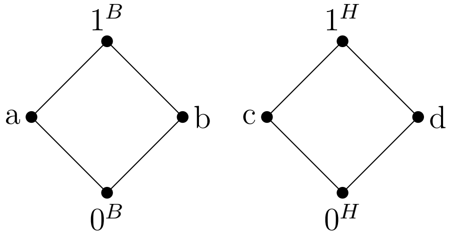
Notice that in this mixed inference the translation that would preserve the logical form of a valid argument is the one with a classical disjunction. Under this translation $||\neg_c\neg_c p_c||_c = ||p_c||_c$ and, so, $||\neg_c\neg_c p_c \vee_c q_i||_c = ||p_c \vee_c q_i||_c$, which makes the argument valid, contrary to what Wrenn predicts (given that we are able to explain the intuitive invalidity of 'Nix', as we will see below).

Nevertheless, let us see that the argument would not be valid if we translated it as:

$$\frac{\neg_c\neg_c p_c \vee_i q_i}{p_c \vee_i q_i}$$

Proposition 4.1.4. $\neg_c\neg_c p_c \vee_i q_i \vdash^{ic} p_c \vee_i q_i$ *is invalid.*

Proof. We build a coherent juxtaposed countermodel that invalidates the argument. We look, then, for a model based on a Heyting-Boolean structure. Consider, for instance, the following algebras:



We want valuations such that $||\neg_c\neg_c p_c \vee_i q_i||_i \in D_i$ and $||p_c \vee_i q_i||_i \notin D_i$. Take the following valuation, $V_i(q_i) = d$, $V_i(p_c) = 0^H$, $V_i(\neg_c\neg_c p_c) = c$, $V_c(p_c) = 0^B$ and $V_c(q_i) = a$. It is easy to check that those structures together with these valuations give a coherent juxtaposed countermodel for the argument. □

Up to this point we have shown that juxtaposition is not too modest with respect to accounting for the intuitive validity of the mixed inferences that Wrenn poses as a challenge to localism. Now, we have to see that juxtaposition is not too immodest. That is, we have to be able to explain the intuitive invalidity of Nix.

4 **Nix**:

$$\frac{\begin{array}{l}\text{If it is not the case that offensive jokes are funny,}\\ \text{then grass is not green}\\ \text{Grass is green}\end{array}}{\text{Offensive jokes are funny}}$$

We formalize the argument as:

$$\frac{\begin{array}{l}\neg_i p_i \rightarrow_x \neg_c q_c\\ q_c\end{array}}{p_i}$$
$$(x = i, c)$$

There is an important difference to notice in this case. We cannot simply find a translation in which the argument is (in)valid, as we did in the previous cases. Now, both translations of the conditional have to be invalidated by juxtaposition, otherwise juxtaposition would leave open a way of validating an argument that seems to be intuitively invalid[4]. Let us see how to find countermodels for each case.

Proposition 4.1.5. $\neg_i p_i \rightarrow_c \neg_c q_c,\ q_c \vdash^{ic} p_i$ *is invalid.*

Proof. We build a coherent juxtaposed countermodel based on the following Heyting-Boolean structure:

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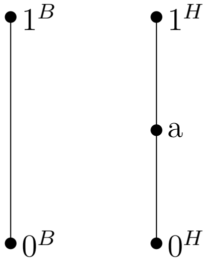
[4] This asymmetry between what is required for validating an argument (having a translation for which the argument is valid) and invalidating an argument (that no translation makes it valid) is not a bizarre feature of juxtaposition. Take the typical argument concluding Socrates' mortality, which, obviously, is intuitively valid. If we were to translate it to propositional logic, the resulting argument would not be valid, but we can translate it to first-order predicate logic in order to account for its validity. On the contrary, if we have an argument in natural language which is intuitively invalid (say, '2 = 2, therefore, there is a McDonald's on the dark side of the moon') we would expect that any reasonable formalization to the language of a legitimate logic system would invalidate the argument. So, the problem that juxtaposition might have with this asymmetry is not because of the asymmetry *per se*, but because the other possible translations also seem intuitively valid and juxtaposition does not account for it.



Choose $V_c(q_c) = 1^B$, $V_i(p_i) = a$. Then, $||\neg_c q_c||_c = 0^B$ and $||\neg_i p_i||_i = 0^H$. Given coherence, since $||\neg_i p_i||_i \notin D_i$, $V_c(\neg_i p_i) \notin D_c$, so $V_c(\neg_i p_i) = 0^B$. Therefore, $||\neg_i p_i \rightarrow_c \neg_c q_c||_c = 1^B$ $(0^B \rightarrow_c 0^B)$ and the argument is not valid, because $\neg_i p_i \rightarrow_c \neg_c q_c \in D_x$, $q_c \in D_x$ but $p_i \notin D_x$. ☐

PROPOSITION 4.1.6. *$\neg_i p_i \rightarrow_i \neg_c q_c$, $q_c \vdash^{ic} p_i$ is invalid.*

PROOF. Consider the same Heyting-Boolean structure and, in order to build the model, take $V_c(q_c) = 1^B$, $V_i(p_i) = a$. Then, $||\neg_c q_c||_c = 0^B$, so $\notin D_c$. Given coherence, $V_i(\neg_c q_c) \notin D_i$, so choose $V_i(\neg_c q_c) = a$. Since, $V_i(p_i) = a$, $||\neg_i p_i||_i = 0^H$. So, $||\neg_i p_i \rightarrow_i \neg_c q_c||_i = 1^H$ $(0^H \Rightarrow a)$[5]. Therefore, for the valuation $V_i(p_i) = a$, $V_c(q_c) = 1^B$ and $V_i(\neg_c q_c) = a$, the premises are designated while the conclusion is not, making the argument not valid as desired. ☐

This concludes the application of the method of juxtaposition to the mixed inferences that Wrenn presents against localist proposals. Let me summarize the results:

- **Mix**: $q_c \vee_i \neg_i p_i$, $p_i \vdash^{ic} q_c$ and $q_c \vee_c \neg_i p_i$, $p_i \nvdash^{ic} q_c$
- **Essential Mix**: $\neg_c \neg_c p_c \vee_i \neg_i q_i$, $q_i \vdash^{ic} p_c$ and $\neg_c \neg_c p_c \vee_c \neg_i q_i$, $q_i \nvdash^{ic} p_c$
- **Disjunctive Mix**: $\neg_c \neg_c p_c \vee_c q_i \vdash^{ic} p_c \vee_c q_i$ and $\neg_c \neg_c p_c \vee_i q_i \nvdash^{ic} p_c \vee_i q_i$
- **Nix**: $\neg_i p_i \rightarrow_x \neg_c q_c$, $q_c \nvdash^{ic} p_i$ $(x = i, c)$

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[5] Recall that $a \Rightarrow b$ is the relative pseudo-complement of a with respect to b, which is the greatest element x such that $a \wedge x \leq b$. Since $\neg a = a \Rightarrow \bot$ the value of $\neg a$ is the greatest x such that $a \wedge x = \bot$.

Remark 4.1.1. Let me briefly comment on these results. First, I reckon that the results make it clear that Wrenn's claim against localism is, at least, too hasty. The application of the method of juxtaposition has allowed us to account for the (in)validity of the mixed inferences respecting our intuitions about them. This should be more than enough to avoid the direct conclusion that the problem of mixed inferences rules localism out. Even if it is not the perfect and definitive solution, juxtaposition is a first stepping stone towards a more satisfactory explanation of how we might combine different logics and systematise what follows from what in a mixed discourse, while adhering to a localist philosophy of logic.

There is a good reason for juxtaposition to be in a good level of (im)modesty: the juxtaposition of two consequence relations is the minimal conservative extension of each of them. Therefore, the juxtaposed consequence relation will contain every inference that was already valid in each logic for its own stock of connectives and will not create new interactions or inferences for those stocks of connectives. As we saw when presenting juxtaposition, this is what guarantees that the combined consequence relation does not collapse.

However, recall that this minimality and conservativeness was also the reason for blocking the appearance of bridge principles, such as the distributivity of conjunction over disjunction when juxtaposing the logics $\mathcal{L}_\wedge$ and $\mathcal{L}_\vee$. Now, I believe that we are in a situation in which juxtaposition, despite being a first step towards a solution, falls short of being a conclusive answer, precisely due to the incapacity for allowing interesting bridge principles. Which bridge principles are those? Luckily enough we already have them at hand.

Consider again the mixed inferences that were supposed to be intuitively valid according to Wrenn. Take the case of Essential Mix, for instance, and try to think about the verdict given by juxtaposition (i.e., $\neg_c\neg_c p_c \vee_i \neg_i q_i$, $q_i \vdash^{ic} p_c$ and $\neg_c\neg_c p_c \vee_c \neg_i q_i$, $q_i \nvdash^{ic} p_c$) from a localist point of view. That is, from the point of view of a person (most likely a philosopher) that claims that the logic of evaluative discourse is intuitionistic logic and the logic of middle-sized objects is classical logic. Knowing that the set of valid inferences of **IL** is included in the set of valid inferences of **CL**, that the semantics of **IL** (Heyting algebras, Kripkean possible world semantics,...) generalizes that of **CL** (Boolean algebras, truth-conditional two-valued semantics, ...) and that the notion of 'construction', which is at the center of **IL** semantics, sets a higher epistemological standard than that of 'truth', which is crucial to **CL** semantics, how can we possibly justify that our method validates $\neg_c\neg_c p_c \vee_i \neg_i q_i$, $q_i \vdash^{ic} p_c$ but invalidates the argument when changing the intuitionistic disjunction by the classical one?

Look at it from this perspective: if we had a construction for q_i we would know that the value of $\neg_i q_i$ is the bottom element in the Heyting algebra, and so that if we had a construction for $\neg_c\neg_c p_c \vee_i \neg_i q_i$ it must be because we had a construction for $\neg_c\neg_c p_c$. Now, if one is a localist and its intuitionistic semantics is telling you that you have a construction for the proposition q_i how can you not assign to $\neg_i q_i$ the bottom value of your classical semantics? And similarly, if when having the intuitionistic disjunction, the disjunction is designated because we have a construction for the disjunct $\neg_c\neg_c p_c$, how can it be that changing the disjunction to the classical one allows you to have $\neg_c\neg_c p_c$ not designated? It is an awkward situation to say the least.

The fact is that $\neg_c\neg_c p_c \vee_c \neg_i q_i$, $q_i \vdash^{ic} p_c$ is a bridge principle, i.e. a new interaction principle between connectives of **IL** and **CL** that seems to be intuitively valid on the face of the examples that we have been considering, together with the philosophy of the logics in play in the mixed inference and a localist standpoint. And an identical reasoning works for the intuitive validity of $q_c \vee_c \neg_i p_i$, $p_i \vdash^{ic} q_c$.

The case for $\neg_c\neg_c p_c \vee_i q_i \vdash^{ic} p_c \vee_i q_i$ is not so clear, though. Since, the intuitionistic standards for designation are higher than those of classical logic, it is not so obvious that the classical fact that a proposition and its double negation have the same semantic value should be preserved once we switch a classical connective into an intuitionistic one. Moreover, given that the set of valid inferences of **IL** is included in that of **CL**, perhaps it is not so awkward that an inference that was valid with a **CL** connective turns out to be invalid when changed by its corresponding intuitionistic one. Nevertheless, I reckon that there are legitimate localist positions that could make a case for the validity of $\neg_c\neg_c p_c \vee_i q_i \vdash^{ic} p_c \vee_i q_i$ too.

Later on, I will try to present a modified method of combination in order to be able to account for it as well. For the moment, let us try to make room for the more obviously valid bridge principles of Mix and Essential Mix.

4.2 Improving on the method of juxtaposition: coordinating logics for mixed inferences

Let me very informally suggest the motivation behind this improvement with an analogy. Imagine you are trekking in an alpine terrain. You carry a map of the area and a compass in case you get lost. There are situations in which, only with the map, one could find the way back home. Suppose you are heading towards a shelter to spend the night over there and suddenly you stop seeing the milestones that mark the trail. You look around trying to look for something to orient yourself. Far in the distance you recognise the shape of a summit and look for it in the map. Your shelter is at the foot of the mountain, so you know which direction you must take. Now, there are also situations in which just the compass will do, for instance, if you know that the shelter is to the south-west of where you are, you might be able to find your way out. But there are other situations in which, neither the compass nor the map alone will be enough. That is, situations in which you need to use the map and the compass in combination because each of them can have more applications when used coordinated with the other. When used in coordination, you can calculate the exact course you must follow, even if you do not know much about the geographical features surrounding you.

Following the analogy, juxtaposition is like having the map and the compass in your backpack but using them as if they where independent instruments. You have, say, intuitionistic and classical logic, you have every valid inference for their respective languages, but you do not allow properly new interactions. However, there seem to be situations, in our case, contexts in which we reason across domains employing mixed inferences, that require new interactions. That is, the logics in combination can potentially generate more valid inferences than only by themselves, just like the map and the compass are useful in more situations when they are used in combination than by their own. In fact, the very name of 'juxtaposition' suggests this situation of having two or more things together but not interacting. That is why, even if it is a modification of juxtaposition, I will call the improved method '*coordination*' (**C**) in order to make clear that the combination we are looking for allows for the emergence of bridge principles.

The goal, then, is to modify the combination mechanism in such a way that we can account for those mixed inferences by going beyond juxtaposition and allowing the emergence of bridge principles, while still avoiding the collapse and invalidating arguments like Nix. As one might already imagine, I am targeting a tricky spot here, because it is quite possible that, in that space between the juxtaposition and the collapse, despite finding the desired bridge principles, one also gets undesired and unjustified ones. That is something to keep in mind and avoid.

I believe that, from the semantic point of view, the problem of juxtaposition is that we have too much freedom when looking for counter models. More specifically, in the method of juxtaposition, this freedom is located within the notion of 'coherence', in the sense that the mere 'designated/non-designated' criterion for 'translating' semantic values from one logic to another seems to be too loose. If we look at the counter models for Mix and Ess.Mix, what happens is that we can assign a non-designated-non-bottom Boolean value to the intuitionist formula that had value 0^H in the Heyting algebra and, thus, get the disjunction designated while no disjunct is. In particular, the one appearing in the conclusion.

Then, the way of making 'coherence' a strong enough criterion involves restricting more the possible valuations, in such a way that we get more 'structure preservation' from the semantic values of a logic to the other (in our case, from intuitionistic to classical logic) and hinder the construction of countermodels. Let us see how to achieve this.

4.2.1 Coordination (C)

We start defining a new method for combining logics that I will call 'Coordination'. Of course the method does not come from scratch, but from a modification of a semantic criterion for *the* juxtaposition. This is an important point because it allows us to benefit from some of the results obtained by Schechter. Coordination is a special case of juxtaposition, as the reader will shortly appreciate. In fact, with respect to the syntax, we have nothing new to add to the case of juxtaposition. So we move to consequence relations[6].

1 **Consequence relations**

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[6] Since the motivation for coordination has come from bridge principles involving intuitionistic and classical logic, we leave aside any pretension of generality for the moment. That is, I will be presenting coordination as a special way of juxtaposing intuitionistic and classical logics.

We still require the coordinated consequence relations to be *structural* in the sense of having Identity, Weakening, Cut and Uniform Substitution. *The coordinated consequence relation*, $\vdash^{ic}_{C}$, is a juxtaposed consequence relation that extends $\vdash^{i}$ and $\vdash^{c}$. Minimality characterizes the juxtaposed consequence relation among juxtaposed consequence relations in general. In the case of the coordinated consequence relation, it will be a property of its models what will characterize it among other extensions[7].

2 Semantics of Coordination

With respect to the semantics the approach is fairly similar too. *The coordination of the structures* $\mathfrak{B}_i$, over C_i, and $\mathfrak{B}_c$, over C_c, is the juxtaposition of the structures, $\langle \mathfrak{B}_i, \mathfrak{B}_c \rangle$, and *the coordination of the classes of structures* $\mathbb{B}_i$ and $\mathbb{B}_c$ is the Cartesian product $\mathbb{B}_i \times \mathbb{B}_c$, which is the juxtaposition of the classes of structures.

A *coordinated model,* $\mathfrak{M}^{C}_{ic} = \langle \mathfrak{B}_i, V_i, \mathfrak{B}_c, V_c \rangle$ over C_{ic} and P_{ic}, based on the coordinated structure $\langle \mathfrak{B}_i, \mathfrak{B}_c \rangle$ (or, more generally, based on the class of coordinated structures $\mathbb{B}_{ic}$) is a (coherent) juxtaposed model satisfying the following property:

- **Coordination**: A model is coordinated when for every $\alpha \in \text{Sent}(C_{ic}, P_{ic})$,
  1. $||\alpha||_i = \top_i$ iff $||\alpha||_c = \top_c$[8]
  2. If $||\alpha||_i = \bot_i$, then $||\alpha||_c = \bot_c$

The rationale for the second clause of coordination should already be quite straightforward after the diagnosis of the problems of juxtaposition and the justification for the bridge principles. As a localist, I should take seriously my intuitionistic semantic standards, that is why I make room in

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[7] Schechter uses the symbol $\vdash$ as a neutral way of referring to the juxtaposed consequence relation, since he can appeal to the notion of minimality in order to characterize it regardless of the semantics or the calculus. Since my characterization of the coordinated consequence relation (and subsequent consequence relations) is going to be mainly semantic, I will be using $\vDash$ most of the times to refer to the (semantically defined) consequence relation of the systems and $\vdash$ as the consequence relation of the natural deduction calculus.

[8] It is obvious that this condition follows from coherence. We could have chosen the more general criterion of $||\alpha||_i \in D_i$ iff $||\alpha||_c \in D_c$, but since in intuitionistic and classical logics we only have one designated value we stick to this simpler version of coherence.

my logical repertoire for intuitionism. Then, if the intuitionistic semantics of my coordinated system is evaluating a sentence with the bottom value of the Heyting algebra, it means that it is being interpreted as having a construction for its absurdity. This is because if $||\alpha||_i = \bot_i$, then $||\neg_i\alpha||_i = \top_i$, but, precisely, the intuitionistic negation is a shorthand for $\alpha \Rightarrow \bot_i$. So, $||\neg_i\alpha||_i = \top_i$, means that we have a way of turning a construction for α into absurdity. Clearly, the epistemological standard is higher than that of falsity in classical logic, that is why, if our system evaluates $||\alpha||_i = \bot_i$ we should not be able to give α any other Boolean value than $\bot_c$. In other words, if I have reasons for $||\alpha||_i = \bot_i$, I have even more reasons for $||\alpha||_c = \bot_c$.

3 A Natural Deduction calculus for Coordination

One thing that does not have much weight on Schechter's presentation of juxtaposition is the deductive calculus. One of the reasons for this is that the main meta-logical results only require the semantic machinery plus the fact that juxtaposition is the *minimal* conservative extension of the logics being combined. But since we are no longer looking for minimality, but searching for fresh interaction principles, it will be useful to have a calculus for coordination. On top of this, it is always nice to have a possibly sharper way of proving validity claims, other than the semantic procedure.

Let me, then, modify Schechter's natural deduction calculus for the juxtaposition of **CL** and **IL**. Recall that this natural deduction consisted of a copy of a natural deduction for **CL** and another copy for **IL**, each one restricted so that the rules for one stock of connectives cannot be applied within subderivations of the rules governing the other stock. For instance, one cannot apply $\neg_i$E within $\neg_c$I. Since we want to go beyond the minimality that results from that calculus and allow new interactions, we have to either introduce new rules or relax Schechter's restriction. In the case of coordination, it is enough to relax the restriction.

The idea behind the relaxation of Schechter's criterion is the following: the semantic property of coordination has reduced the possible ways of generating countermodels in the sense that, given that the intuitionistic value of a formula is $\bot_i$, the classical value of that formula can only be $\bot_c$. So, the new semantic criterion reduces the uncertainty of the classical semantic values given the intuitionistic ones, so to speak. This asymmetry is also reflected in how we relax the calculus, as one would expect. In the coordinated natural deduction for **CL** and **IL** we have a copy of a natural deduction calculus for **CL** and a copy of a natural deduction calculus for **IL** each one restricted in different ways. Like juxtaposition, we can only use intuitionistic rules within intuitionistic subderivations but, unlike juxtaposition, we can use intuitionistic rules within classical subderivations with a caveat. Those subderivations, i.e., $\neg_c$I, $\rightarrow_c$I and $\vee_c$E, have to be closed in certain specific ways if intuitionistic rules have been used within them. The informal way of putting it is that we need to get an intuitionistic contradiction in order to be able to close, because that is what guarantees that, from the semantic perspective, the assumption opening the subderivation gets value $\bot_i$ given that the premisses are $\top_i$ and, therefore, the assumption will also be $\bot_c$ given coordination.

More formally now, we consider each classical rule opening subderivations and specify how to apply intuitionistic rules within them. Consider, first, the rule of classical negation introduction:

$$(\neg_c\mathrm{I}) \qquad \begin{array}{|l} \quad \begin{array}{|l} \alpha \\ \hline \Pi \\ \bot_c \end{array} \\ \neg_c\alpha \end{array}$$

If no application of an intuitionistic rule occurred within $\neg_c$I, the rule is just the standard classical rule. If there is an application of an intuitionistic rule within $\neg_c$I, then the subderivation has to be closed like this:

$$(\neg_c\mathrm{I}) \qquad \begin{array}{|l} \quad \begin{array}{|l} \alpha \\ \hline \Pi \\ \bot_i \end{array} \\ \neg_c\alpha \end{array}$$

In fact, we require that the only rules that can be applied in the derivation Π are intuitionistic rules, together with repetitions of formulas derived from the premisses of the argument[9]. In all of the examples of mixed inferences that we are going to analyse the way of getting $\bot_i$ is by deriving an

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[9] Let me point out that I know that these rules are not going to be complete for the

intuitionistic contradiction and applying $\neg_i$E. However, it is also possible to get $\bot_i$ if it is contained in the assumption α. For instance, if $\alpha = p \wedge_i \bot_i$ we can eliminate the conjunction to get $\bot_i$ and, then, close and conclude $\neg_c \alpha$.

A second classical rule that opens subderivations is $\rightarrow_c$ –introduction:

$$(\rightarrow_c\mathrm{I}) \quad \begin{array}{|l} \quad \begin{array}{|l} \alpha \\ \hline \Pi \\ \beta \end{array} \\ \alpha \rightarrow_c \beta \end{array}$$

Here, again, if there is no application of intuitionistic rules within $\rightarrow_c$I, the rule is the standard one. If only intuitionistic rules (or repetitions) are applied within the classical subderivation, then we need to modify the rule in order to close the subderivation:

$$(\rightarrow_c\mathrm{I}) \quad \begin{array}{|l} \quad \begin{array}{|l} \alpha \\ \hline \Pi \\ \bot_i \\ \beta \end{array} \\ \alpha \rightarrow_c \beta \end{array}$$

---

semantics that I have developed. There are cases in which classical rules should be allowed in Π in order to get a derivation of a semantically valid argument. However, due to time limitations I have not been able to get a completeness proof and proving the soundness for the more general rules has been more subtle than what I expected. That is why I have opted for weaker rules, which are enough to account for the cases of mixed inferences that I consider in this dissertation and which are more easily proved to be sound.

Finally, the last rule that we have to consider is $\vee_c$−elimination:

$$
\begin{array}{r|l}
(\vee_c\text{E}) & \alpha \vee_c \beta \\ \cline{2-2}
& \begin{array}{|l} \alpha \\ \hline \Pi_1 \\ \delta \end{array} \\
& \begin{array}{|l} \beta \\ \hline \Pi_2 \\ \delta \end{array} \\
& \delta
\end{array}
$$

Again, we modify the rule in order to allow for the application of intuitionistic rules in the following way:

$$
\begin{array}{r|l}
(\vee_c\text{E}) & \alpha \vee_c \beta \\ \cline{2-2}
& \begin{array}{|l} \alpha \\ \hline \Pi_1 \\ \bot_i \\ \delta \end{array} \\
& \begin{array}{|l} \beta \\ \hline \Pi_2 \\ \delta \end{array} \\
& \delta
\end{array}
$$

It should be noted that since this rule opens two subderivations, one for each disjunct, it is enough that one of them finishes with $\bot_i$ and δ, but both could end like that. Moreover, if just one of the disjuncts ends with $\bot_i$ and δ, we allow both classical and intuitionistic rules in the other subderivation (Π_2 in the presentation of the rule). We will see the details in the soundness proof. Let me now present some meta-logical results for coordination.

4 Preservation theorems for Coordination

Now that we know the combination mechanism, let us offer some meta-theoretical results. Some of them are direct consequences of those given by Schechter and some other's require more or less significant modifications to be proved. We begin by proving the existence of coordinated nontrivial models. First, we define the semantic notion of a coordination of two models just as Schechter defines a juxtaposition of two models.

Suppose $\mathfrak{M}_i = \langle \mathfrak{B}_i, V_i \rangle$ is a model over C_i and P_i and $\mathfrak{M}_c = \langle \mathfrak{B}_c, V_c \rangle$ is a model over C_c and P_c. A *coordination of the models* $\mathfrak{M}_i$ and $\mathfrak{M}_c$ is a coordinated model $\langle \mathfrak{B}_i, V_i^+, \mathfrak{B}_c, V_c^+ \rangle$ over C_i, C_c and P_{ic} such that:

- If $p \in P_i, V_i^+(p) = V_i(p)$ and
- If $p \in P_c, V_c^+(p) = V_c(p)$

Notice that if $\mathfrak{M}_{ic}^C$ is a coordination of $\mathfrak{M}_i$ and $\mathfrak{M}_c$, for any $\alpha \in \text{Sent}(C_x$, $P_x)$ (for $x = i, c$), $||\alpha||_x^{\mathfrak{M}_{ic}^C} = ||\alpha||^{\mathfrak{M}_x}$. Therefore, $\mathfrak{M}_{ic}^C \vDash \alpha$ just in case $\mathfrak{M}_x \vDash \alpha$. We now provide a necessary and sufficient condition for the existence of coordinated models.

Proposition 4.2.1 (Existence of Coordinated Nontrivial Models). *Suppose C_i and C_c are disjoint signatures. Suppose $\mathfrak{M}_i = \langle \mathfrak{B}_i, V_i \rangle$ is a model over C_i and P_i and $\mathfrak{M}_c = \langle \mathfrak{B}_c, V_c \rangle$ is a model over C_c and P_c, satisfying the condition that if the cardinality of the set of semantic values of $\mathfrak{B}_i$ is two, $|\mathcal{B}_i| = 2$, then $|\mathcal{B}_c| = 2$*[10]*. Then there is a coordinated nontrivial model, $\mathfrak{M}_{ic}^C$, over C_{ic} and P_{ic} based on $\mathfrak{B}_{ic}$, just in case for every $p \in P_i \cap P_c, \mathfrak{M}_i \vDash p$ just in case $\mathfrak{M}_c \vDash p$ and if $||p||^{\mathfrak{M}_i} = \bot_i$, then $||p||^{\mathfrak{M}_c} = \bot_c$.*

Proof. Suppose there is some $p \in P_i \cap P_c$ such that, either $\mathfrak{M}_i \vDash p$ and $\mathfrak{M}_c \nvDash p$ or $\mathfrak{M}_i \nvDash p$ and $\mathfrak{M}_c \vDash p$ or $||p||^{\mathfrak{M}_i} = \bot_i$ but $||p||^{\mathfrak{M}_c} \neq \bot_c$, then there is no coordination of $\mathfrak{M}_i$ and $\mathfrak{M}_c$.

Now, suppose that for every $p \in P_i \cap P_c, \mathfrak{M}_i \vDash p$ just in case $\mathfrak{M}_c \vDash p$ and if $||p||^{\mathfrak{M}_i} = \bot_i$, then $||p||^{\mathfrak{M}_c} = \bot_c$. We show that there is a weak coordination of $\mathfrak{M}_i$ and $\mathfrak{M}_c$.

---

[10] This is because, if we allowed $|\mathcal{B}_c| > 2$ when $|\mathcal{B}_i| = 2$, if $\alpha \in \text{Sent}(C_c, P_c)$ and $||\alpha||^{\mathfrak{M}_c}$ is non-designated but not-bottom, when doing the coordination, we would have to evaluate this intuitionistic atom as $\bot_i$, since it is the only non-designated value in the two-element Boolean algebra. But, then, we would not get a coordinated model, since we would have $||\alpha||_i^{\mathfrak{M}_{ic}^C} = \bot_i$ and $||\alpha||_c^{\mathfrak{M}_{ic}^C} = b_c \neq \bot_c$, where b_c is a non-designated non-bottom value.

Let $\top_x$ be the top value of B_x, $\bot_x$ be the bottom value of B_x and let b_x be an element of $B_x - \{\top_x\}$. We inductively define, $[\ ]_x$, the function from Sent(C_{ic}, P_{ic}) to B_x such that:

- If $p \in P_x, [p]_x = V_x(p)$;
- If $p \in P_i - P_c, [p]_c = \top_c$ if $V_i(p) = \top_i$, $[p]_c = \bot_c$ if $V_i(p) = \bot_i$ and $[p]_c = b_c$ otherwise;
- If $p \in P_c - P_i, [p]_i = \top_i$ if $V_c(p) = \top_c$, $[p]_i = b_i$ if $V_c(p) = \bot_c$ and $[p]_i = b_i(\neq \bot_i)$ otherwise;
- If $c \in C^n_x, [c\alpha^1...\alpha^n]_x = \Phi_x(c)([\alpha^1]_x...[\alpha^n]_x)$;
- If $c \in C^n_i, [c\alpha^1...\alpha^n]_c = \top_c$ if $\Phi_i(c)([\alpha^1]_i...[\alpha^n]_i) = \top_i$, $[c\alpha^1...\alpha^n]_c = \bot_c$ if $\Phi_i(c)([\alpha^1]_i...[\alpha^n]_i) = \bot_i$ and $[c\alpha^1...\alpha^n]_c = b_c$ otherwise;
- If $c \in C^n_c, [c\alpha^1...\alpha^n]_i = \top_i$ if $\Phi_c(c)([\alpha^1]_c...[\alpha^n]_c) = \top_c$, $[c\alpha^1...\alpha^n]_i = b_i$ if $\Phi_c(c)([\alpha^1]_c...[\alpha^n]_c) = \bot_c$ and $[c\alpha^1...\alpha^n]_i = b_i(\neq \bot_i)$ otherwise;

If α is an x-atom, let $V^+_x(\alpha) = [\alpha]_x$. Let $\mathfrak{M}_{ic} = \langle \mathfrak{B}_i, V^+_i, \mathfrak{B}_c, V^+_c \rangle$. We show that for every $\alpha \in$ Sent(C_{ic}, P_{ic}), $||\alpha||^{\mathfrak{M}_{ic}}_i = \top_i$ iff $||\alpha||^{\mathfrak{M}_{ic}}_c = \top_c$ and if $||\alpha||^{\mathfrak{M}_{ic}}_i = \bot_i$, then $||\alpha||^{\mathfrak{M}_{ic}}_c = \bot_c$. That is, $[\alpha]_i = \top_i$ iff $[\alpha]_c = \top_c$ and if $[\alpha]_i = \bot_i$ then $[\alpha]_c = \bot_c$.

If $p \in P_i \cap P_c$, $[p]_i = V_i(p)$. So, $[p]_i = \top_i$ iff $V_i(p) = \top_i$ iff $V_c(p) = \top_c$ iff $[p]_c = \top_c$, and $[p]_i = \bot_i$ iff $V_i(p) = \bot_i$. But, if $V_i(p) = \bot_i$, then $V_c(p) = \bot_c$ and, so, $[p]_c = \bot_c$.

If $p \in P_i - P_c$, $[p]_i = \top_i$ iff $V_i(p) = \top_i$ iff $[p]_c = \top_c$. Also, $[p]_i = \bot_i$ iff $V_i(p) = \bot_i$ and if $V_i(p) = \bot_i$, then $[p]_c = \bot_c$.

If $p \in P_c - P_i, [p]_c = \top_c$ iff $V_c(p) = \top_c$ iff $[p]_i = \top_i$. Also, if $[p]_i = \bot_i$ then $V_c(p) = \bot_c = [p]_c$[11].

If $c \in C^n_i$ and $\alpha^1...\alpha^n \in$ Sent(C_{ic}, P_{ic}), $[c\alpha^1...\alpha^n]_i = \top_i$ iff $\Phi_i(c)([\alpha^1]_i...[\alpha^n]_i) = \top_i$ iff $[c\alpha^1...\alpha^n]_c = \top_c$[12]. Also, $[c\alpha^1...\alpha^n]_i = \bot_i$ iff $\Phi_i(c)([\alpha^1]_i...[\alpha^n]_i) = \bot_i$ and $[c\alpha^1...\alpha^n]_c = \bot_c$ if $\Phi_i(c)([\alpha^1]_i...[\alpha^n]_i) = \bot_i$.

---

[11] Notice that there is no other option. If $[p]_i = \bot_i$ then it can only be that $V_c(p) = \bot_c$. The reasoning is by contraposition. If it had any other non-designated classical value then p could not have the intuitionistic bottom value.

[12] Since C_i and C_c are disjoint, the only reason for having $[c\alpha^1...\alpha^n]_c = \top_c$ is because $\Phi_i(c)([\alpha^1]_i...[\alpha^n]_i) = \top_i$. Again, reasoning with the contrapositive might be helpful.

If $c \in C^n_c$ and $\alpha^1 ... \alpha^n \in \text{Sent}(C_{ic}, P_{ic})$, $[c\alpha^1 ... \alpha^n]_c = \top_c$ iff $\Phi_c(c)([\alpha^1]_c ... [\alpha^n]_c) = \top_c$ iff $[c\alpha^1 ... \alpha^n]_i = \top_i$. If $[c\alpha^1 ... \alpha^n]_i = \bot_i$ then $\Phi_c(c)([\alpha^1]_c ... [\alpha^n]_c) = [c\alpha^1 ... \alpha^n]_c = \bot_c$.

Since we are doing the coordination of intuitionistic and classical models, it is obvious that this coordinated model will be nontrivial (because there are $\alpha_1, \alpha_2 \in \text{Sent}(C_{ic}, P_{ic})$ such that $\mathfrak{M}^C_{ic} \vDash \alpha_1$ and $\mathfrak{M}^C_{ic} \nvDash \alpha_2$). □

Now, we proceed with the proof of strong soundness of the coordinated consequence relation (axiomatized by the coordinated natural deduction calculus), $\vdash^{ic}_C$, with respect to the class of coordinated Heyting-Boolean structures, $\mathbb{B}_{ic}$. Before proving the main result we need two auxiliary lemmas (similar to Lemmas 5.3 and 5.4 in Schechter (2011)).

Lemma 4.2.1. *Suppose $\mathbb{B}_{ic}$ is the class of coordinated structures over C_i and C_c. Then, $\vDash^{\mathbb{B}_{ic}}_C$ is a consequence relation for Sent(C_{ic}, P_{ic}).*

Proof. We have to show that $\vDash^{\mathbb{B}_{ic}}_C$ satisfies Identity, Weakening, Cut and Uniform Substitution. For the first three the proof is basically that of Schechter in Lemma 5.3. We focus, then, on Uniform Substitution, i.e., if $\Gamma \vDash^{\mathbb{B}_{ic}}_C \alpha$ then $\Gamma[\beta/p] \vDash^{\mathbb{B}_{ic}}_C \alpha[\beta/p]$, to extend and clarify what Schechter does.

We prove its contrapositive. So, suppose $\Gamma[\beta/p] \nvDash^{\mathbb{B}_{ic}}_C \alpha[\beta/p]$. This means that there is a coordinated model, $\mathfrak{M}^C_{ic}$, such that $\mathfrak{M}^C_{ic} \vDash \Gamma[\beta/p]$ and $\mathfrak{M}^C_{ic} \nvDash \alpha[\beta/p]$. With the help of this model, we build another coordinated model, $\mathfrak{M}'^C_{ic} = \langle \mathfrak{B}_i, V'_i, \mathfrak{B}_c, V'_c \rangle$, by letting $V'_x(\delta) = ||\delta[\beta/p]||^{\mathfrak{M}^C_{ic}}_x$ whenever δ is an x-atom. We show that $||\delta||^{\mathfrak{M}'^C_{ic}}_x = ||\delta[\beta/p]||^{\mathfrak{M}^C_{ic}}_x$ for every $\delta \in \text{Sent}(C_{ic}, P_{ic})$, by induction on the complexity of the formulas:

- Base case: we have defined $V'_x(\delta)$ in such a way that if δ is an x-atom the equality holds.

- Inductive Hypothesis (IH): Assume that the equality holds for all formulas less complex than α. We show that the equality holds for any possible α.

  Assume that $\alpha = \neg_x \gamma$. By IH, we know that $||\gamma||^{\mathfrak{M}'^C_{ic}}_x = ||\gamma[\beta/p]||^{\mathfrak{M}^C_{ic}}_x$. So, clearly $||\neg_x \gamma||^{\mathfrak{M}'^C_{ic}}_x = ||\neg_x \gamma[\beta/p]||^{\mathfrak{M}^C_{ic}}_x$.

  Assume now that $\alpha = \gamma \vee_x \omega$. By IH, we know that $||\gamma||^{\mathfrak{M}'^C_{ic}}_x = ||\gamma[\beta/p]||^{\mathfrak{M}^C_{ic}}_x$ and $||\omega||^{\mathfrak{M}'^C_{ic}}_x = ||\omega[\beta/p]||^{\mathfrak{M}^C_{ic}}_x$. Again, it is clear that the same operation, namely, $\vee_x$, over the same semantic values will yield the same semantic value. So, $||\gamma \vee_x \omega||_x^{\mathfrak{M}'^C_{ic}} = ||(\gamma \vee_x \omega)[\beta/p]||_x^{\mathfrak{M}^C_{ic}}$. One can easily check that the same holds for $\alpha = \gamma \wedge_x \omega$ and $\alpha = \gamma \rightarrow_x \omega$.

Therefore, $\mathfrak{M}'^C_{ic}$ is a coordinated model such that $\mathfrak{M}'^C_{ic} \vDash \Gamma$ and $\mathfrak{M}'^C_{ic} \nvDash \alpha$. ☐

Lemma 4.2.2. *Suppose $\mathbb{B}_{ic}$ is the coordination of $\mathbb{B}_i$ and $\mathbb{B}_c$. If $\Gamma \vDash^{\mathbb{B}_i} \alpha$ or $\Gamma \vDash^{\mathbb{B}_c} \alpha$, then $\Gamma \vDash^{\mathbb{B}_{ic}}_C \alpha$.*

Proof. Similar to the proof of Schechter for Lemma 5.4. The idea is that if $\Gamma \vDash^{\mathbb{B}_x} \alpha$, with $\Gamma \cup \alpha \subseteq \text{Sent}(C_x, P_x)$, then we take a coordinated model $\mathfrak{M}^C_{ic} = \langle \mathfrak{B}_i, V_i, \mathfrak{B}_c, V_c \rangle$ such that $\mathfrak{M}^C_{ic} \vDash \Gamma$ and $\mathfrak{M}_x|_{P_x} = \langle \mathfrak{B}_x, V_x|_{P_x} \rangle$ is the restriction of $\mathfrak{M}_x$ to P_x, making $||\beta||^{\mathfrak{M}_x|_{P_x}} = ||\beta||_x^{\mathfrak{M}^C_{ic}}$ for every $\beta \in \text{Sent}(C_x, P_x)$. With this and knowing that $\mathfrak{M}_x|_{P_x}$ is based on $\mathbb{B}_x$, we get that $\mathfrak{M}^C_{ic} \vDash \alpha$. So, $\Gamma \vDash^{\mathbb{B}_{ic}}_C \alpha$. ☐

Now we have the ingredients to prove the main result.

Theorem 4.2.1 (Strong Soundness). *The coordinated natural deduction calculus is strongly sound with respect to the class of Heyting-Boolean structures. That is,*

$$\Gamma \vdash^{ic}_C \alpha \Rightarrow \Gamma \vDash^{\mathbb{B}_{ic}}_C \alpha.$$

Proof. By the previous lemmas, we know that $\vDash^{\mathbb{B}_{ic}}_C$ is a consequence relation and that if $\Gamma \vDash^{\mathbb{B}_i} \alpha$ or $\Gamma \vDash^{\mathbb{B}_c} \alpha$, then $\Gamma \vDash^{\mathbb{B}_{ic}}_C \alpha$. We also know that the juxtaposed natural deduction of intuitionistic and classical logics is the minimal conservative extension of them and that **IL** and **CL** are strongly sound w.r.t. the classes of Heyting and Boolean algebras respectively. From there, we get that, for the juxtaposed natural deduction derivation, for any Γ and α, if $\Gamma \vdash^{ic} \alpha$ then $\Gamma \vDash^{\mathbb{B}_{ic}} \alpha$. Therefore, we know that $\Gamma \vdash^{ic} \alpha \Rightarrow \Gamma \vdash^{ic}_C \alpha$ (because we have all the previous rules and some of them have fewer restrictions), we also know that $\Gamma \vdash^{ic} \alpha \Rightarrow \Gamma \vDash^{\mathbb{B}_{ic}} \alpha$ (from Schechter's proof of Soundness) and, finally, also that $\Gamma \vDash^{\mathbb{B}_{ic}} \alpha \Rightarrow \Gamma \vDash^{\mathbb{B}_{ic}}_C \alpha$ (because every coordinated model is a juxtaposed model but not vice versa). Thus, now we just need to consider the rules that we have changed, i.e., relaxed, for giving the coordinated natural deduction calculus of intuitionistic and classical logics, and check that they are strongly sound with respect to the coordinated semantics.

The proof is, as usual, by induction on the length of the derivation. Let us start with the base case, namely, proofs of size $k = 1$:

- Base case ($k = 1$): if $\Gamma \vdash^{ic}_{C} \alpha$ and the length of the derivation is $k = 1$, then $\alpha \in \Gamma$. Since $\alpha \in \Gamma$, for every coordinated model, $\mathfrak{M}^{C}_{ic}$, if $\mathfrak{M}^{C}_{ic} \vDash \Gamma$, then $\mathfrak{M}^{C}_{ic} \vDash \alpha$. Therefore, $\Gamma \vDash^{\mathbb{B}_{ic}}_{C} \alpha$.
- Inductive Hypothesis (IH): assume that if $\Gamma \vdash^{ic}_{C} \alpha$ and the length of the derivation is $\leq k$, then $\Gamma \vDash^{\mathbb{B}_{ic}}_{C} \alpha$.

We have to show that this also holds for the new rules with derivations of length $k + 1$. Let me start with $\neg_c$I:

$$\begin{array}{l|ll}
(\neg_c\mathrm{I}) & \gamma_1 & \\
 & \vdots & \\
 & \gamma_j & \\
 & \Pi_1 & \\ \cline{2-2}
 & & \multicolumn{1}{|l}{\alpha} \\ \cline{3-3}
 & & \multicolumn{1}{|l}{\Pi_2} \\
k & & \multicolumn{1}{|l}{\bot_i} \\
k+1 & \neg_c\alpha &
\end{array}$$

Recall that, within Π_2 only **IL** rules are allowed or repetitions of ϕ such that ($\Gamma \vdash^{ic}_{C} \phi$)$\in \Pi_1$[13]. The derivation ends with an application of $\neg_c$I in line $k + 1$, but previously, we get to $\bot_i$ from undischarged assumptions $\Gamma \cup \{\alpha\}$, in $\leq k$ lines. Therefore, $\Gamma \cup \{\alpha\} \vdash^{ic}_{C} \bot_i$ and, by IH, $\Gamma \cup \{\alpha\} \vDash^{\mathbb{B}_{ic}}_{C} \bot_i$.[14]

Now, we want to show that for every model $\mathfrak{M}^{C}_{ic}$ designating every formula in Γ, $|| \bigwedge_x(\Gamma)||_x^{\mathfrak{M}^{C}_{ic}} = \top_x$, the value of α has to be bottom, $||\alpha||_i^{\mathfrak{M}^{C}_{ic}} = \bot_i$, and, also, that within Π_1 we could have used rules both from **CL** and **IL**

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[13] Of course, this includes any of the $\gamma \in \Gamma$ and, in fact, in most of the cases in which I will apply the rules, what is going to be repeated within Π_2 is one of the premisses in Γ.

[14] Notice that with $\Gamma \cup \{\alpha\} \vdash^{ic}_{C} \bot_i \Rightarrow \Gamma \cup \{\alpha\} \vDash^{\mathbb{B}_{ic}}_{C} \bot_i$ we are in the same scenario as in a standard intuitionistic negation introduction rule, since in Π_2 we have only applied **IL** rules. So, the reasoning for showing that $||\alpha||_i = \bot_i$ should be the same.

natural deductions. So, notice that for whatever rule we applied in Π_1, if $||\bigwedge_x(\Gamma)||_x^{\mathfrak{M}^C_{ic}} = \top_x$ and $\Gamma \vDash_C^{\mathbb{B}_{ic}} \phi$ (because $(\Gamma \vdash_C^{ic} \phi) \in \Pi_1$ in $\leq k$ lines), then $||\alpha||_x^{\mathfrak{M}^C_{ic}} = \top_x$[15]. So, if some formula ϕ was used together with α for obtaining $\bot_i$, that formula takes the value $\top_i$ in the Heyting algebra for the models that make $||\bigwedge_i(\Gamma)||_i^{\mathfrak{M}^C_{ic}} = \top_i$.

Since from assuming α only **IL** rules were used in Π_2 and these are sound with respect to the class of Heyting algebras, where for any Δ and ω, $\Delta \vDash^{\mathbb{HA}} \omega$ iff $||\bigwedge(\Delta)||^{\mathbb{HA}} \leq ||\omega||^{\mathbb{HA}}$, then, $\Gamma \cup \{\alpha\} \vDash_C^{\mathbb{B}_{ic}} \bot_i$ iff $||\bigwedge_i(\Gamma) \wedge_i \alpha||_i^{\mathfrak{M}^C_{ic}} \leq ||\bot_i||_i^{\mathfrak{M}^C_{ic}}$ for every $\mathfrak{M}^C_{ic}$. So, take any coordinated model $\mathfrak{M}_j$ such that $||\bigwedge_i(\Gamma)||_i^{\mathfrak{M}_j} = \top_i$. Given that for every $\mathfrak{M}^C_{ic}$, $||\bigwedge_i(\Gamma) \wedge_i \alpha||_i^{\mathfrak{M}^C_{ic}} = \bot_i$, for $\mathfrak{M}_j$ in particular, $||\bigwedge_i(\Gamma) \wedge_i \alpha||_i^{\mathfrak{M}_j} = \bot_i$. That is, $||\alpha||_i^{\mathfrak{M}_j} = \bot_i$ and, by coordination, $||\alpha||_c^{\mathfrak{M}_j} = \bot_c$. Therefore, for every $\mathfrak{M}^C_{ic}$, if $||\bigwedge_i(\Gamma)||_i^{\mathfrak{M}^C_{ic}} = \top_i$, then $||\neg_c\alpha||_c^{\mathfrak{M}^C_{ic}} = \top_c$. Hence, $\Gamma \vDash_C^{\mathbb{B}_{ic}} \neg_c\alpha$ as desired.

The next rule we have to consider is $\rightarrow_c$I. If no intuitionistic rule is applied within the subderivation, we know that the rule is sound because the natural deduction calculus for **CL** is strongly sound with respect to the class of Boolean algebras. When intuitionistic rules are applied within $\rightarrow_c$I, then the rule is as follows and it is sound with respect to the coordinated semantics:

$$
\begin{array}{r|ll}
(\rightarrow_c\mathrm{I}) & \multicolumn{2}{l}{\gamma_1}\\
& \multicolumn{2}{l}{\vdots}\\
& \multicolumn{2}{l}{\gamma_j}\\
& \multicolumn{2}{l}{\Pi_1}\\ \cline{2-2}
& & \multicolumn{1}{|l}{\alpha}\\ \cline{3-3}
& & \multicolumn{1}{|l}{\Pi_2}\\
& & \multicolumn{1}{|l}{\bot_i}\\
k & & \multicolumn{1}{|l}{\delta}\\
k+1 & \multicolumn{2}{l}{\alpha \rightarrow_c \delta}
\end{array}
$$

---

[15] This shows that whatever we used by repetition after assuming α (any ϕ such that $(\Gamma \vdash_C^{ic} \phi) \in \Pi_1$) has value $\top_x$ in the coordinated models that make $||\bigwedge_x(\Gamma)||_x^{\mathfrak{M}^C_{ic}} = \top_x$.

Just as with $\neg_c$I, notice that within Π_2 only **IL** rules are allowed or repetitions of ϕ such that $(\Gamma \vdash^{ic}_{C} \phi) \in \Pi_1$. In this case, the derivation ends with an application of $\rightarrow_c$I in line $k+1$, but previously, we get to $\bot_i$ and δ from undischarged assumptions $\Gamma \cup \{\alpha\}$, in $\leq k$ lines. Therefore, $\Gamma \cup \{\alpha\} \vdash^{ic}_{C} \bot_i$ and, by IH, $\Gamma \cup \{\alpha\} \vDash^{\mathbb{B}_{ic}}_{C} \bot_i$.

Now, notice that what we need to do is exactly what we did in the previous case, which is to prove that for every model $\mathfrak{M}^C_{ic}$ designating every formula in Γ, $|| \bigwedge_x(\Gamma)||^{\mathfrak{M}^C_{ic}}_x = \top_x$, the value of α has to be bottom, $||\alpha||^{\mathfrak{M}^C_{ic}}_i = \bot_i$. So, by the same argument as before, which relied on the fact that $\Gamma \cup \{\alpha\} \vdash^{ic}_{C} \bot_i \Rightarrow \Gamma \cup \{\alpha\} \vDash^{\mathbb{B}_{ic}}_{C} \bot_i$, we know that for every $\mathfrak{M}^C_{ic}$, if $|| \bigwedge_i(\Gamma)||^{\mathfrak{M}^C_{ic}}_i = \top_i$, then $||\alpha||^{\mathfrak{M}^C_{ic}}_i = \bot_i$ and, by coordination, $||\alpha||^{\mathfrak{M}^C_{ic}}_c = \bot_c$. Given the truth conditions of the classical conditional, if in every coordinated model in which Γ is designated, the value of α is $\bot_c$, then for every coordinated model, if $\mathfrak{M}^C_{ic} \vDash \Gamma$ then $\mathfrak{M}^C_{ic} \vDash \alpha \rightarrow_c \delta$. Hence, $\Gamma \vDash^{\mathbb{B}_{ic}}_{C} \alpha \rightarrow_c \delta$ as desired.

We conclude by showing the soundness of the rule of $\vee_c$E when **IL** rules are applied within the subderivations:

$$\begin{array}{r|l} (\vee_c\text{E}) & \gamma_1 \\ & \vdots \\ & \gamma_j \\ & \Pi_1 \\ & \alpha \vee_c \beta \\ \hline & \begin{array}{|l} \alpha \\ \hline \Pi_2 \\ \bot_i \\ \delta \end{array} \\ & \begin{array}{|l} \beta \\ \hline \Pi_3 \\ \delta \end{array} \\ k+1 & \delta \end{array}$$

Like in the previous cases, within Π_2 only **IL** rules are allowed or repetitions of ϕ such that $(\Gamma \vdash^{ic}_{C} \phi) \in \Pi_1$. The derivation ends with an application of $\vee_c$E in line $k+1$, but previously, we get to $\bot_i$ and δ from undischarged assumptions $\Gamma \cup \{\alpha\}$, in $\leq k$, and we also get δ from undischarged assumptions $\Gamma \cup \{\beta\}$, in $\leq k$. Therefore, $\Gamma \cup \{\alpha\} \vdash^{ic}_{WC} \bot_i$ and $\Gamma \cup \{\beta\} \vdash^{ic}_{WC} \delta$ and, by IH, $\Gamma \cup \{\alpha\} \vDash^{\mathbb{B}_{ic}}_{C} \bot_i$ and $\Gamma \cup \{\beta\} \vDash^{\mathbb{B}_{ic}}_{C} \delta$. And, again, notice that the same argument that we used above can be applied here to conclude that for every $\mathfrak{M}^{C}_{ic}$, if $||\bigwedge_i(\Gamma)||^{\mathfrak{M}^{C}_{ic}}_{i} = \top_i$, then $||\alpha||^{\mathfrak{M}^{C}_{ic}}_{i} = \bot_i$ and, by coordination, $||\alpha||^{\mathfrak{M}^{C}_{ic}}_{c} = \bot_c$.

Now, we also have that $\Gamma \cup \{\beta\} \vDash^{\mathbb{B}_{ic}}_{C} \delta$, from which we know that every coordinated model that satisfies $\Gamma \cup \{\beta\}$ also satisfies δ. Thus, we can conclude that every model satisfying $\Gamma \cup \{\alpha \vee_c \beta\}$ is a model satisfying $\Gamma \cup \{\beta\}$ and, therefore, also δ[16]. Hence, for every coordinated model, if $\mathfrak{M}^{C}_{ic} \vDash \Gamma \cup \{\alpha \vee_c \beta\}$

---

[16] Notice that here the semantic clause of coordination is crucial, as in the previous rules. This is what guarantees that, since we concluded that the intuitionistic semantic value of

then $\mathfrak{M}_{ic}^{C} \vDash \delta$, so, $\Gamma \cup \{\alpha \vee_c \beta\} \vDash_{C}^{\mathbb{B}_{ic}} \delta$, as desired.

This concludes the strong soundness proof for weak coordination.

□

4.2.2 Applying coordination to mixed inferences: a better reply to Wrenn

One of the motivations for developing coordination was to allow for the emergence of bridge principles when combining intuitionistic and classical logics, especially, those cases of bridge principles belonging to Mix and Essential Mix. If we are capable of doing this while assuming a localist stand towards logic, then we are in a good position for meeting Wrenn's challenge against localism and, in general, for giving a localist account of mixed inferences. Let us show that we can, in fact, account for the validity of Mix and Essential Mix when they are formalized as bridge principles.

1 **Mix**:

Wet cats are funny
Either snow is white or wet cats are not funny

Snow is white

Let us formalize the argument as a bridge principle by translating it into our coordinated language and recall the assumption that the discourse about humour is an evaluative discourse in which reasoning is best captured by intuitionistic logic, while discourse about middle-sized objects, such as snow, is a discourse in which reasoning is best captured by classical logic.

$$\begin{array}{l} p_i \\ q_c \vee_c \neg_i p_i \\ \hline q_c \end{array}$$

I showed in Proposition 4.1.1 that this argument was invalidated by juxtaposition. Now I will show that coordination makes the argument valid, as desired.

Proposition 4.2.2. *$p_i, q_c \vee_c \neg_i p_i \vDash_{C}^{ic} q_c$ is valid.*

α has to be $\bot_i$, its classical semantic value, under the scope of $\vee_c$, has to be $\bot_c$.

Proof. We want to check that for every coordinated model based on $\mathbb{B}_{ic}$, if the premises are designated, then the conclusion is designated too. Suppose then, that $||p_i||_x \in D_x$ and $||q_c \vee_c \neg_i p_i||_x \in D_x$. $||p_i||_x \in D_x$ just in case $||p_i||_i = \top_i$. If $||p_i||_i = \top_i$, then $||\neg_i p_i||_i = \bot_i$ and $V_c(\neg_i p_i) = \bot_c$ (by the second condition of coordination, i.e. if $||\alpha||_i = \bot_i$ then $||\alpha||_c = \bot_c$). Thus, in order for $q_c \vee_c \neg_i p_i$ to be designated, $||q_c||_c = \top_c$. Therefore, $||q_c||_c \in D_c$ and, so, the argument is validated by coordination. ☐

2 **Essential Mix**:

Either it's not the case that snow isn't white or
wet cats aren't funny
Wet cats are funny
———
Snow is white

We formalize the argument as:

$$\begin{array}{l} \neg_c \neg_c p_c \vee_c \neg_i q_i \\ q_i \\ \hline p_c \end{array}$$

This argument too was invalidated by juxtaposition. Let us see that we can now account for its validity applying coordination.

Proposition 4.2.3. $\neg_c \neg_c p_c \vee_c \neg_i q_i,\ q_i \vDash^{ic}_{C} p_c$ *is valid.*

Proof. We prove it, this time, using our coordinated natural deduction, since we know that it is strongly sound with respect to the class of Heyting-Boolean structures.

$$\begin{array}{llll}
1 & \multicolumn{2}{l}{\neg_c\neg_c p_c \vee_c \neg_i q_i} & \\
2 & \multicolumn{2}{l}{\underline{q_i}} & \\
3 & c & \underline{\neg_c\neg_c p_c} & \\
4 & & \neg_c\neg_c p_c & \text{Identity, 3} \\
5 & c & \underline{\neg_i q_i} & \\
6 & & \bot_i & \neg_i\text{E, 2, 5} \\
7 & & \neg_c\neg_c p_c & \bot_i\text{E, 6} \\
8 & \multicolumn{2}{l}{\neg_c\neg_c p_c} & \vee_c\text{E, 1–7} \\
9 & \multicolumn{2}{l}{p_c} & \neg_c\neg_c\text{E, 7}
\end{array}$$

$\square$

Hence, coordination is less modest than juxtaposition, but it is not too immodest yet, since we can still invalidate the intuitively invalid arguments of Nix (in its two versions)[17]. In fact, we already have the countermodels because the ones that we built for the proofs of propositions 4.1.5 and 4.1.6 are coherent juxtaposed countermodels that also are coordinated countermodels.

Recall that we also have to consider a second version of Disjunctive Mix, namely, $\neg_c\neg_c p_c \vee_i q_i \vDash^{ic}_{C} p_c \vee_i q_i$. We have briefly argued above about why the argument could be regarded as invalid, but we have left room for other interpretations. I will still suspend my judgement on that matter and just say that the argument continues to be invalid under the analysis by coordination. Again, the countermodel that we offered for the case of juxtaposition is a coordinated countermodel too.

Therefore, this is the summary of the results obtained by applying the method of coordination to the mixed inferences proposed by Wrenn as counterexamples to localism:

- **Mix**: $q_c \vee_i \neg_i p_i, p_i \vDash^{ic}_{C} q_c$ and $q_c \vee_c \neg_i p_i, p_i \vDash^{ic}_{C} q_c$
- **Essential Mix**: $\neg_c\neg_c p_c \vee_i \neg_i q_i, q_i \vDash^{ic}_{C} p_c$ and $\neg_c\neg_c p_c \vee_c \neg_i q_i, q_i \vDash^{ic}_{C} p_c$

---

[17] Of course, notice that the coordinated consequence relation does neither collapse nor weakly collapse. Simply consider that $\vDash^{ic}_{C} p \vee_c \neg_c p$ but $\nvDash^{ic}_{C} p \vee_i \neg_i p$.

- **Disjunctive Mix**: $\neg_c\neg_c p_c \vee_c q_i \vDash^{ic}_C p_c \vee_c q_i$ and $\neg_c\neg_c p_c \vee_i q_i \nvDash^{ic}_C p_c \vee_i q_i$
- **Nix**: $\neg_i p_i \rightarrow_x \neg_c q_c, q_c \nvDash^{ic}_C p_i$ $(x = i, c)$

I believe that these results are already quite exciting for a number of reasons. The first one is that they address better the specific challenge by Wrenn, because we are able to account for more versions of the arguments that should be intuitively valid. That is, by allowing the emergence of bridge principles we have more chances of being able to account for potential intuitively valid mixed inferences. On the flip side, we also have a higher risk of allowing for bridge principles that appear to be intuitively invalid. That is, since we are aiming at an unknown space of interactions, the risk of allowing for unjustified and unwanted interactions is quite high. This might be a reason for justifying a certain level of specificity in our coordinated systems, in the sense that the method for combining **IL** and **CL** is quite likely to be different from a method for combining **IL** and **LP**, if we are looking for the emergence of meaningful interactions[18]. Thus, aiming for a more general combination mechanism that is so prone to counterexamples is, probably, a bad idea.

Another reason is that the motivation that has guided the semantic modifications resulting in coordination has been quite substantive. It was not just a technical modification for its own sake, which can in fact be very interesting already, but a technical modification resulting from a critical localist analysis of the results obtained by juxtaposition and of the method itself. That is, I have tried to give reasons, appealing to philosophical justifications, for how the Heyting and Boolean values should relate to each other, for which type of mixed inferences should be reasonable to validate and for how the intuitionistic and classical connectives should interact in order to allow for the bridge principles that better represent, formally, those mixed inferences.

The last more general reason is that it seems that the philosophical problem of mixed inferences can help developing the more technical area of combinations of logics in an innovative way. I will focus more on this point when presenting the concluding remarks, but let me just say, now, that we have

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[18] In fact, it could be even more specific than this. It could be that the nature of epistemic constraints is such that the combination of the ethical domain with a physical domain allows for more bridge principles than the combination of the mathematical domain with the physical domain (or vice versa), even if one believes that reasoning within the ethical and the mathematical domains is best captured by intuitionistic logic. I will explore a bit more this idea later on.

aimed at a space of possible results that, as far as I know, has not been aimed at in the literature when combining logic systems. We are starting to see that we can go beyond minimality in a controlled way, looking for certain bridge principles without ending up in the collapse or weak collapse of the combined consequence relation.

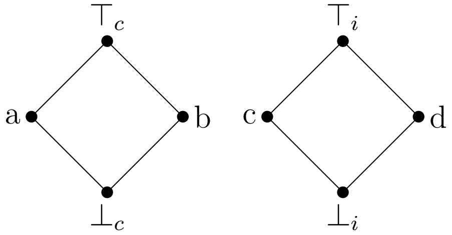
In order to better appreciate the method of coordination, as well as its potential virtues and shortcomings, let me offer some other results regarding bridge principles that the method (in)validates.

4.2.3 Applying coordination: more bridge principles

Let me end the section on coordination providing a catalogue of results with respect to bridge principles. The aim of the section is twofold. On one hand, to give more details of how the method works, semantically and syntactically, by means of examples, and, on the other hand, to analyse where, if anywhere, the shortcomings of the method are. This is important because, as it happened with juxtaposition, looking at what the method invalidates might give us hints for possible further modifications.

- $p \vee_c q \not\models^{ic}_{C} p \vee_i q$

Proof.



Take this Heyting-Boolean structure and the following valuation in order to get the coordinated countermodel: $V_i(p) = V_i(q) = c$, $V_c(p) = a$ and $V_c(q) = b$. $\square$

- $p \to_i q \not\models^{ic}_{C} p \to_c q$

The countermodel we have just given is a countermodel for this one too. Notice that these two examples have symmetrical versions and their countermodels will be the same but with the subindexes changed in the valuations.

- $\neg_i p \vDash^{ic}_{C} \neg_c p$

Proof. Suppose that $||\neg_i p||_i = \top_i$, then $||p||_i = \bot_i$[19]. By coordination, if $||p||_i = \bot_i$, then $||p||_c = \bot_c$. Therefore, $||\neg_c p||_c = \top_c$ and $||\neg_c p||_i = \top_i$ again by coordination (more concretely, by the clause corresponding to coherence).

We can also prove it with our coordinated natural deduction:

$$\begin{array}{l|l|l l}
1 & \neg_i p & & \\
\hline
2 & c & p & \\
3 & & \bot_i & \neg_i\text{E, 1, 2} \\
4 & \neg_c p & & \neg_c\text{I, 2–3}
\end{array}$$

$\square$

However, we do not have it in the other direction:

- $\neg_c p \nvDash^{ic}_{C} \neg_i p$

Proof. Take the pair of structures we had above, for instance, and the valuations $V_c(p) = \bot_c$ and $V_i(p) = c$. This gives as a coordinated countermodel.

From the point of view of the calculus, notice that we cannot derive $\neg_i p$ from $\neg_c p$ because we cannot apply a classical rule, namely, $\neg_c$E, within an intuitionistic subderivation, i.e., $\neg_i$I. $\square$

In fact, the iteration of negations yields interesting results:

- $\neg_i \neg_c p \vDash^{ic}_{C} \neg_c \neg_c p$

Proof. It follows from $\neg_i p \vDash^{ic}_{C} \neg_c p$ and uniform substitution. $\square$

- $\neg_i \neg_c p \vDash^{ic}_{C} p$

Proof. Immediate from the previous proof. $\square$

- $\neg_i \neg_c p \vDash^{ic}_{C} \neg_i \neg_i p$

---

[19] Notice that for any Heyting algebra the only value for which its negation is $\top_i$ is $\bot_i$. Since, $\neg_i \alpha = \alpha \Rightarrow \bot_i$, if the value of the relative pseudo-complement of α with respect to $\bot_i$ is $\top_i$, that means that $\top_i$ is the greatest x such that $\alpha \wedge x \leq \bot_i$. So the only possible value α can have for the inequality $\alpha \wedge \top_i \leq \bot_i$ to hold is $\bot_i$.

Proof. Suppose that $||\neg_i\neg_c p||_i \in D_i$, that is $||\neg_i\neg_c p||_i = \top_i$. Then, $||\neg_c p||_i = \bot_i$, so $||\neg_c p||_c = \bot_c$, which means that $||p||_c = \top_c$. Therefore, by the first clause of coordination, we know that $||p||_i = \top_i$. Then, $||\neg_i p||_i = \bot_i$ and $||\neg_i\neg_i p||_i = \top_i$. So, the argument is validated by coordination.

We prove it with the natural deduction too. Notice that this derivation taken up to line 4 is a proof for the previous inference.

$$\begin{array}{r|lll}
1 & \multicolumn{2}{l}{\underline{\neg_i\neg_c p}} & \\
2 & c & \multicolumn{1}{|l}{\underline{\neg_c p}} & \\
3 & & \multicolumn{1}{|l}{\bot_i} & \neg_i\text{E, 1, 2} \\
4 & \multicolumn{2}{l}{\neg_c\neg_c p} & \neg_c\text{I, 2–3} \\
5 & \multicolumn{2}{l}{p} & \neg_c\neg_c\text{E, 4} \\
6 & i & \multicolumn{1}{|l}{\underline{\neg_i p}} & \\
7 & & \multicolumn{1}{|l}{\bot_i} & \neg_i\text{E, 5, 6} \\
8 & \multicolumn{2}{l}{\neg_i\neg_i p} & \neg_i\text{I, 6–7}
\end{array}$$

□

- $\neg_c\neg_i p \not\vDash^{ic}_{C} \neg_c\neg_c p$

Proof. Take the structures used above and valuations $V_i(p) = c$, $V_c(p) = b$ and $V_c(\neg_i p) = \bot_c$. With these, $||\neg_c\neg_i p||_c = \top_c$ but $||\neg_c\neg_c p||_c = b$, so the argument is invalidated by coordination. □

- $\neg_c\neg_i p \not\vDash^{ic}_{C} \neg_i\neg_i p$

Proof. The previous coordinated countermodel is a countermodel for this inference too. From the natural deduction perspective, we will not be able to do the derivation because we are not allowed to apply classical rules within intuitionistic subderivations. □

I reckon that the inferences that could potentially be more problematic are $\neg_i\neg_c p \vDash^{ic}_{C} \neg_c\neg_c p$ and $\neg_i\neg_c p \vDash^{ic}_{C} p$. But, indeed, since they follow from the fact that $\neg_i p \vDash^{ic}_{C} \neg_c p$, the philosophical justification for these should be the same. Moreover, one could argue that the intuitionistic negation of a classical negation is a negation of a classical proposition and, being a localist, this affects how the intuitionistic part behaves. This is because, since $\neg_c p$ takes the classical bottom value, we are forced from the classical part of our system to designate p.

Let me offer some more examples:

- $\nvDash^{ic}_{C} \neg_i p \rightarrow_i \neg_c p$

Proof. Again, take the structures we have been considering in these examples and the valuations $V_i(p) = d$, $V_c(p) = b$ and $V_i(\neg_c p) = d$. With these, since $||\neg_i p||_i = c$, $||\neg_i p \rightarrow_i \neg_c p||_i = (c \Rightarrow d)$, which is the greatest x such that $c \wedge x \leq d$, i.e. $||\neg_i p \rightarrow_i \neg_c p||_i = d$. Thus, we have given a coordinated countermodel. □

One might wonder what is wrong with this natural deduction derivation compared to the one establishing that $\neg_i p \vDash^{ic}_{C} \neg_c p$. Well, notice that, in this case, we would require an application of a classical rule within an intuitionistic subderivation.

$$\begin{array}{r|l|l|l l}
1 & i & \neg_i p & & \\ \cline{3-3}
2 & & c & p & \\ \cline{4-4}
3 & & & \bot_i & \neg_i\text{E, 1, 2} \\
4 & & \neg_c p & & !!\neg_c\text{I, 2–3} \\
5 & \multicolumn{3}{l}{\neg_i p \rightarrow_i \neg_c p} & \rightarrow_i\text{I, 1–4}
\end{array}$$

But, then, it could seem like the argument with the classical conditional instead of the intuitionistic one would be a tautology. However, this is not the case:

- $\nvDash^{ic}_{C} \neg_i p \rightarrow_c \neg_c p$

Proof. The construction of the countermodel is analogous to the previous case. The more interesting point is to see why the natural strategy to prove it in the coordinated natural deduction fails.

$$\begin{array}{r|llll}
1 & c \mid \neg_i p & & & \\
2 & \quad \mid & c \mid p & & \\
3 & \quad \mid & \quad \mid \bot_i & & \neg_i\mathrm{E},\ 1,\ 2 \\
4 & \quad \mid & \neg_c p & & \neg_c\mathrm{I},\ 2\text{–}3 \\
5 & \neg_i p \to_c \neg_c p & & & !!{\to_c}\mathrm{I},\ 1\text{–}4
\end{array}$$

□

The problem with this derivation is that it does not respect the restrictions that we have established in order to apply **IL** rules within **CL** subderivations. On one hand, the rules allow the application of **IL** rules within **CL** subderivations but, in that case, every rule applied within the **CL** subderivation must be intuitionistic. However, in this derivation both $\neg_i$E and $\neg_c$I are applied within $\to_c$I. On the other hand, if **IL** rules are applied within $\to_c$I, the only way of closing is with $\bot_i$ and δ in the outermost subderivation (in this case the one having $\neg_i p$ as an assumption). But we do not get that in this subderivation and, therefore, the rule $\to_c$I is badly applied.

Notice, though, that once p is forced to have a semantic value that reduces the possible translations between structures (i.e. $||p||_i = \top_i$ iff $||p||_c = \top_c$ or if $||p||_i = \bot_i$ then $||p||_c = \bot_c$) and, therefore, the flexibility for getting countermodels, the argument might be valid. For instance, we can put p in the premises to get a valid argument:

- $p \vDash^{ic}_{C} \neg_i p \to_c \neg_c p$

Proof. Suppose $||p||_i = \top_i$. Then, $||\neg_i p||_i = \bot_i$, so $||\neg_i p||_c = \bot_c$. This is enough to see that the conclusion will be designated too, i.e. $||\neg_i p \to_c \neg_c p||_c = \top_c$.

And with the natural deduction,

$$\begin{array}{lll}
1 & \mid p & \\
2 & \mid c \mid \neg_i p & \\
3 & \mid \quad \mid \bot_i & \neg_i\mathrm{E},\ 1,\ 2 \\
4 & \mid \quad \mid \neg_c p & \bot_i\mathrm{E},\ 3 \\
5 & \mid \neg_i p \rightarrow_c \neg_c p & \rightarrow_c\mathrm{I},\ 1\text{–}4
\end{array}$$

$\square$

- $\neg_i p, q, \neg_i r, (p \vee_i r) \vee_c (q \rightarrow_i s) \vDash^{ic}_{C} s$

Proof. Suppose that $||\neg_i p||_x = ||q||_x = ||\neg_i r||_x = ||(p \vee_i r) \vee_c (q \rightarrow_i s)||_x = \top_x$. Since, $||\neg_i p||_i = ||\neg_i r||_i = \top_i$, then $||p||_i = ||r||_i = \bot_i$ and, so, $||p \vee_i r||_i = \bot_i$. By coordination, $||p \vee_i r||_c = \bot_c$ whenever $||p \vee_i r||_i = \bot_i$. Therefore, $||(p \vee_i r) \vee_c (q \rightarrow_i s)||_x = \top_x$ iff $||q \rightarrow_i s||_x = \top_x$ iff $||s||_x = \top_x$ (because we have supposed $||q||_x = \top_x$).

And by the coordinated natural deduction,

1	$\neg_i p$	
2	q	
3	$\neg_i r$	
4	$(p \vee_i r) \vee_c (q \rightarrow_i s)$	
5	$\neg_i p \wedge_i \neg_i r$	$\wedge_i$I, 1, 3
6	$\neg_i (p \vee_i r)$	De Morgan, 5
7	$c \mid p \vee_i r$	
8	$\quad \mid \bot_i$	$\neg_i$E, 6, 7
9	$\quad \mid q \rightarrow_i s$	$\bot_i$E, 8
10	$c \mid q \rightarrow_i s$	
11	$\quad \mid q \rightarrow_i s$	Identity, 10
12	$q \rightarrow_i s$	$\vee_c$E, 4, 7–11
13	s	$\rightarrow_i$E, 2, 12

□

I will end up with a couple more cases to make sure that different types of inferences are available to the reader in order to enhance the comprehension of the method.

- $p, q \vDash^{ic}_{C} (p \rightarrow_i \neg_i q) \rightarrow_c \neg_c q$

Proof.

$$\begin{array}{lll}
1 & \vert\; p & \\
2 & \vert\; \underline{q} & \\
3 & \vert\; c \;\vert\; \underline{p \to_i \neg_i q} & \\
4 & \vert\; \;\vert\; \neg_i q & \to_i\text{E, 1, 3} \\
5 & \vert\; \;\vert\; \bot_i & \neg_i\text{E, 2, 4} \\
6 & \vert\; \;\vert\; \neg_c q & \bot_i\text{E, 5} \\
7 & \vert\; (p \to_i \neg_i q) \to_c \neg_c q & \to_c\text{I, 3–6}
\end{array}$$

$\square$

Let me conclude with an argument which I believe might be quite suggestive, since it is the 'mirror image' of a version of Mix. Consider the following argument:

$$\begin{array}{l}
\text{Snow is white} \\
\underline{\text{Either wet cats are funny or snow isn't white}} \\
\text{Wet cats are funny}
\end{array}$$

So, we formalize the argument like this:

$$\begin{array}{l}
p_c \\
\underline{q_i \vee_x \neg_c p_c} \\
q_i \\
{\scriptstyle (x = i, c)}
\end{array}$$

In this case, the disjunctive syllogism is occurring with the classical part of the logic while in the original Mix, we had an intuitionistic disjunctive syllogism. In this second version, if we translate the argument with a classical disjunction, the argument is clearly valid (already in juxtaposition and *a fortiori* in coordination), since it is an instance of a classically valid argument. However, if we translate the argument with an intuitionistic disjunction, we can give a coordinated countermodel to it. So,

- $p_c, q_i \vee_i \neg_c p_c \nvDash^{ic}_{C} q_i$

Proof. Take the pair of structures that we have been using in this section and valuations $V_c(p_c) = \top_c$, $V_i(q_i) = c$. $||\neg_c p_c||_c = \bot_c$, so by coordination, $||\neg_c p_c||_i \notin D_i$, so take $V_i(\neg_c p_c) = d$. Then, $||q_i \vee_i \neg_c p_c||_i = \top_i$, so $||p_c||_x \in D_x$, $||q_i \vee_i \neg_c p_c||_x \in D_x$, while $||q_i||_x \notin D_x$. □

And, from the natural deduction side, it is quite easy to see that we cannot make the derivation because that requires the application of a classical rule within an intuitionistic subderivation.

However, the important question is whether our combined logic system of classical and intuitionistic logic should make this inference valid or not. So, is the argument intuitively valid? Or are there philosophical and localist reasons that could be provided in favour of the validity of the argument? Or, another way of putting it, should our combined system aim for this kind of bridge principles too? Is coordination still too modest?

To my mind, it is not obvious that it is a bridge principle for which there is good enough justification. In fact, some of the reasons that we gave for the clause 'if $||\alpha||_i = \bot_i$ then $||\alpha||_c = \bot_c$' and so, for letting the emergence of bridge principles like $q_c \vee_c \neg_i p_i,\ p_i \vDash^{ic}_{C} q_c$, would speak against accepting this new bridge principle. The intuitionistic epistemic standard is higher than the classical, intuitionistic logic is weaker than (i.e. it is included in) classical logic, having a proof for the absurdity of a proposition seems to imply the falsity of that proposition, etc. So, there might be localists who will find these reasons appealing enough as to stick with coordination and avoid going beyond it by letting more suspicious bridge principles emerge.

Yet, there might be some other localists who find the bridge principle appealing for other reasons. To start with, it is true that the epistemic standard is higher for intuitionism than for classical logic, but that might have less to do with the top and bottom values of the algebras and more with the structural features of the intermediate ones. That is, with the fact that not for every semantic value, x, in a Heyting algebra, $\mathfrak{A}$, $\neg\neg x = x$ and $\neg x \vee x = \top$. In fact, despite the different epistemic standards and semantic conceptions of what it takes for a proposition to get the top value, the localist defending coordination is already accepting that $||\alpha||_i = \top_i$ iff $||\alpha||_c = \top_c$, so why not accept also that $||\alpha||_i = \bot_i$ iff $||\alpha||_c = \bot_c$? This is enough for bridge principles like $p_c, q_i \vee_i \neg_c p_c \vDash^{ic}_{C} q_i$ to emerge, indeed.

In the following section I will explore this new strengthening of juxtaposition that I call '*strong coordination*'. I hope it is clear enough that I am not using 'strong' or the lack of an adjective as a value judgement.For the moment, I am exploring the different possibilities and analysing how to make them technically viable while assessing their possible philosophical consequences. I reckon it is a complex and delicate matter how to weigh the technical and philosophical virtues and shortcomings of each of them, but hopefully the analysis will shed some light on this new issue before us.

4.2.4 Strong Coordination (SC)

Let me introduce a further strengthening of the juxtaposition, which is also a strengthening of coordination. From the syntactic point of view, strong coordination is a consequence relation and, like coordination, it is a particular case of juxtaposed consequence relation. That is, the strongly coordinated consequence relation, $\vdash^{ic}_{SC}$, is a juxtaposed consequence relation that extends $\vdash^{i}$, $\vdash^{c}$, $\vdash^{ic}$ and $\vdash^{ic}_{C}$. Since we cannot appeal to minimality, as it is done with the juxtaposed consequence relation, in order to discriminate $\vdash^{ic}_{SC}$ from other juxtaposed consequence relations, we will characterize it by means of its semantic and syntactic properties.

1 **Semantics of Strong Coordination**

With respect to the semantics the approach is almost the same. *The strong coordination of the structures* $\mathfrak{B}_i$, over C_i, and $\mathfrak{B}_c$, over C_c, is the juxtaposition of the structures, $\langle \mathfrak{B}_i, \mathfrak{B}_c \rangle$, and *the strong coordination of the classes of structures* $\mathbb{B}_i$ and $\mathbb{B}_c$ is the Cartesian product $\mathbb{B}_i \times \mathbb{B}_c$, which is the juxtaposition of the classes of structures.

A *strongly coordinated model,* $\mathfrak{M}^{SC}_{ic} = \langle \mathfrak{B}_i, V_i, \mathfrak{B}_c, V_c \rangle$ over C_{ic} and P_{ic}, based on the strongly coordinated structure $\langle \mathfrak{B}_i, \mathfrak{B}_c \rangle$ (or, more generally, based on the class of strongly coordinated structures $\mathbb{B}_{ic}$) is a coherent juxtaposed model satisfying the following property:

- **Strong Coordination**: A model is strongly coordinated when for every $\alpha \in \text{Sent}(C_{ic}, P_{ic})$,
  1. $||\alpha||_i = \top_i$ iff $||\alpha||_c = \top_c$
  2. $||\alpha||_i = \bot_i$ iff $||\alpha||_c = \bot_c$

Again, part of the motivation for the strengthening of the second clause has come from the fact that bridge principles like $p_c, q_i \vee_i \neg_c p_c \vdash q_i$ were invalidated by coordination. We have seen some reasons for this argument to be invalid and for the clause of coordination, but I have also pointed out some reasons that might justify strong coordination. In any case, after presenting the method and some of its results, we will come back to its adequacy as a localist solution to mixed inferences and to its justification.

2 A Natural Deduction calculus for Strong Coordination

As one might expect, just as with coordination the natural deduction calculus reflected the asymmetry of the semantic clauses, now the strongly coordinated natural deduction calculus has to be modified in such a way that we get the symmetry of the semantic clauses of strong coordination in the calculus. Therefore, what we need to do is just to relax the intuitionistic rules opening subderivations in the same way that we relaxed the classical ones. That is, on top of the coordinated natural deduction calculus, we allow the application of classical rules within $\neg_i$I, $\rightarrow_i$I and $\vee_i$E with the same caveats that had their analogous classical rules.

Let us consider each intuitionistic rule opening subderivations and specify, just in case, how to apply classical rules within them. Consider, first, the rule of intuitionistic negation introduction:

$$(\neg_i\mathrm{I}) \qquad \begin{array}{|l} \quad \begin{array}{|l} \alpha \\ \hline \Pi \\ \bot_i \end{array} \\ \neg_i\alpha \end{array}$$

If no application of a classical rule occurred within $\neg_i$I, the rule is just the standard intuitionistic rule. If there is an application of a classical rule within $\neg_i$I, then the subderivation has to be closed like this:

$$(\neg_i\mathrm{I}) \qquad \begin{array}{|l} \quad \begin{array}{|l} \alpha \\ \hline \Pi \\ \bot_c \end{array} \\ \neg_i\alpha \end{array}$$

Analogously, here we require that the only rules that can be applied in the derivation Π are classical rules, together with repetitions of formulas derived from the premisses of the argument.

For $\to_i$I and $\vee_i$E, let me jump directly to the relaxed versions of the standard rules. The details for applying the rules mirror exactly those of the coordinated natural deduction calculus.

$$(\to_i\mathrm{I}) \quad \begin{array}{|l} \begin{array}{|l} \alpha \\ \hline \Pi \\ \bot_c \\ \beta \end{array} \\ \alpha \to_i \beta \end{array}$$

$$(\vee_i\mathrm{E}) \quad \begin{array}{|l} \alpha \vee_i \beta \\ \hline \quad \begin{array}{|l} \alpha \\ \hline \Pi_1 \\ \bot_c \\ \delta \end{array} \\ \quad \begin{array}{|l} \beta \\ \hline \Pi_2 \\ \delta \end{array} \\ \delta \end{array}$$

3 Preservation theorems for Strong Coordination

Just as we did in the case of coordination, let me present now some meta-theoretical results concerning strong coordination. We start, as we did before, proving the existence of strongly coordinated nontrivial models. This time, we skip the definition of a *strong coordination of the models* $\mathfrak{M}_i$ and $\mathfrak{M}_c$, since it remains the same as before.

Proposition 4.2.4 (Existence of Strongly Coordinated Nontrivial Models). *Suppose C_i and C_c are disjoint signatures. Suppose $\mathfrak{M}_i = \langle \mathfrak{B}_i, V_i \rangle$ is a model over C_i and P_i and $\mathfrak{M}_c = \langle \mathfrak{B}_c, V_c \rangle$ is a model over C_c and P_c, satisfying the condition that $|\mathcal{B}_i| = 2$ just in case $|\mathcal{B}_c| = 2$*[20]*. Then there is a strongly coordinated nontrivial model, $\mathfrak{M}_{ic}^{SC}$, over C_{ic} and P_{ic} based on $\mathfrak{B}_{ic}$, just in case for every $p \in P_i \cap P_c, \mathfrak{M}_i \vDash p$ just in case $\mathfrak{M}_c \vDash p$ and $||p||^{\mathfrak{M}_i} = \bot_i$ iff $||p||^{\mathfrak{M}_c} = \bot_c$.*

Proof. Suppose there is some $p \in P_i \cap P_c$ such that, either $\mathfrak{M}_i \vDash p$ and $\mathfrak{M}_c \nvDash p$ or $\mathfrak{M}_i \nvDash p$ and $\mathfrak{M}_c \vDash p$ or $||p||^{\mathfrak{M}_i} = \bot_i$ and $||p||^{\mathfrak{M}_c} \neq \bot_c$ or $||p||^{\mathfrak{M}_c} = \bot_c$ and $||p||^{\mathfrak{M}_i} \neq \bot_i$, then there is no strong coordination of $\mathfrak{M}_i$ and $\mathfrak{M}_c$.

Now, suppose that for every $p \in P_i \cap P_c, \mathfrak{M}_i \vDash p$ just in case $\mathfrak{M}_c \vDash p$ and $||p||^{\mathfrak{M}_i} = \bot_i$ just in case $||p||^{\mathfrak{M}_c} = \bot_c$. We show that there is a strong coordination of $\mathfrak{M}_i$ and $\mathfrak{M}_c$.

Let $\top_x$ be the top value of B_x, $\bot_x$ be the bottom value of B_x and let b_x be an element of $B_x - \{\top_x, \bot_x\}$. We inductively define, $[\ ]_x$, the function from Sent(C_{ic}, P_{ic}) to B_x such that:

- If $p \in P_x, [p]_x = V_x(p)$;
- If $p \in P_i - P_c, [p]_c = \top_c$ if $V_i(p) = \top_i$, $[p]_c = \bot_c$ if $V_i(p) = \bot_i$ and $[p]_c = b_c$ otherwise;
- If $p \in P_c - P_i, [p]_i = \top_i$ if $V_c(p) = \top_c$, $[p]_i = \bot_i$ if $V_c(p) = \bot_c$ and $[p]_i = b_i$ otherwise;
- If $c \in C_x^n, [c\alpha^1...\alpha^n]_x = \Phi_x(c)([\alpha^1]_x...[\alpha^n]_x)$;
- If $c \in C_i^n, [c\alpha^1...\alpha^n]_c = \top_c$ if $\Phi_i(c)([\alpha^1]_i...[\alpha^n]_i) = \top_i$, $[c\alpha^1...\alpha^n]_c = \bot_c$ if $\Phi_i(c)([\alpha^1]_i...[\alpha^n]_i) = \bot_i$ and $[c\alpha^1...\alpha^n]_c = b_c$ otherwise;
- If $c \in C_c^n, [c\alpha^1...\alpha^n]_i = \top_i$ if $\Phi_c(c)([\alpha^1]_c...[\alpha^n]_c) = \top_c$, $[c\alpha^1...\alpha^n]_i = \bot_i$ if $\Phi_c(c)([\alpha^1]_c...[\alpha^n]_c) = \bot_c$ and $[c\alpha^1...\alpha^n]_i = b_i$ otherwise;

If α is an x-atom, let $V_x^+(\alpha) = [\alpha]_x$. Let $\mathfrak{M}_{ic} = \langle \mathfrak{B}_i, V_i^+, \mathfrak{B}_c, V_c^+ \rangle$. We show that for every $\alpha \in$ Sent(C_{ic}, P_{ic}), $||\alpha||_i^{\mathfrak{M}_{ic}} = \top_i$ iff $||\alpha||_c^{\mathfrak{M}_{ic}} = \top_c$ and $||\alpha||_i^{\mathfrak{M}_{ic}} = \bot_i$ iff $||\alpha||_c^{\mathfrak{M}_{ic}} = \bot_c$. That is, $[\alpha]_i = \top_i$ iff $[\alpha]_c = \top_c$ and $[\alpha]_i = \bot_i$ iff $[\alpha]_c = \bot_c$.

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[20] The reason is analogous to that given for the existence of coordinated nontrivial models.

If $p \in P_i \cap P_c$, $[p]_i = V_i(p)$. So, $[p]_i = \top_i$ iff $V_i(p) = \top_i$ iff $V_c(p) = \top_c$ iff $[p]_c = \top_c$, and $[p]_i = \bot_i$ iff $V_i(p) = \bot_i$ iff $V_c(p) = \bot_c$ iff $[p]_c = \bot_c$.

If $p \in P_i - P_c$, $[p]_i = \top_i$ iff $V_i(p) = \top_i$ iff $[p]_c = \top_c$. Also, $[p]_i = \bot_i$ iff $V_i(p) = \bot_i$ iff $[p]_c = \bot_c$.

If $p \in P_c - P_i$, $[p]_c = \top_c$ iff $V_c(p) = \top_c$ iff $[p]_i = \top_i$. Also, $[p]_i = \bot_i$ iff $V_c(p) = \bot_c = [p]_c$.

If $c \in C_i^n$ and $\alpha^1 ... \alpha^n \in \text{Sent}(C_{ic}, P_{ic})$, $[c\alpha^1 ... \alpha^n]_i = \top_i$ iff $\Phi_i(c)([\alpha^1]_i ... [\alpha^n]_i) = \top_i$ iff $[c\alpha^1 ... \alpha^n]_c = \top_c$. Also, $[c\alpha^1 ... \alpha^n]_i = \bot_i$ iff $\Phi_i(c)([\alpha^1]_i ... [\alpha^n]_i) = \bot_i$ iff $[c\alpha^1 ... \alpha^n]_c = \bot_c$.

If $c \in C_c^n$ and $\alpha^1 ... \alpha^n \in \text{Sent}(C_{ic}, P_{ic})$, $[c\alpha^1 ... \alpha^n]_c = \top_c$ iff $\Phi_c(c)([\alpha^1]_c ... [\alpha^n]_c) = \top_c$ iff $[c\alpha^1 ... \alpha^n]_i = \top_i$. Also, $[c\alpha^1 ... \alpha^n]_c = \bot_c$ iff $\Phi_c(c)([\alpha^1]_c ... [\alpha^n]_c) = \bot_c$ iff $[c\alpha^1 ... \alpha^n]_i = \bot_i$.

Since we are doing the strong coordination of intuitionistic and classical models, it is obvious that this strongly coordinated model will be nontrivial (because there are $\alpha_1, \alpha_2 \in \text{Sent}(C_{ic}, P_{ic})$ such that $\mathfrak{M}_{ic}^{SC} \vDash \alpha_1$ and $\mathfrak{M}_{ic}^{SC} \nvDash \alpha_2$). ☐

In order to prove the strong soundness of the strongly coordinated natural deduction calculus, we rely on the same lemmas that we did for coordination, appropriately adapted for strong coordination.

Lemma 4.2.3. *Suppose $\mathbb{B}_{ic}$ is the class of strongly coordinated structures over C_i and C_c. Then, $\vDash_{SC}^{\mathbb{B}_{ic}}$ is a consequence relation for Sent(C_{ic}, P_{ic}).*

Lemma 4.2.4. *Suppose $\mathbb{B}_{ic}$ is the strong coordination of $\mathbb{B}_i$ and $\mathbb{B}_c$. If $\Gamma \vDash^{\mathbb{B}_i} \alpha$ or $\Gamma \vDash^{\mathbb{B}_c} \alpha$, then $\Gamma \vDash_{SC}^{\mathbb{B}_{ic}} \alpha$.*

The proofs for these lemmas are almost identical to the ones given for coordination. Since we have just relaxed three more rules compared to the natural deduction calculus that we had for coordination, it is enough to prove that, for the inferences that those new rules allow in the calculus, we have designation preservation in the semantics.

Theorem 4.2.2 (Strong Soundness). *The strongly coordinated natural deduction calculus is strongly sound with respect to the class of Heyting-Boolean structures. That is,*

$$\Gamma \vdash_{SC}^{ic} \alpha \Rightarrow \Gamma \vDash_{SC}^{\mathbb{B}_{ic}} \alpha.$$

Proof. Immediate from the proof of Strong Soundness for coordination. It is enough to change the subindexes '*c*' and '*i*'. ☐

4.2.5 Applying strong coordination: new interactions and bridge principles

Let me now apply the method of strong coordination in order to see how far does it go, in terms of allowing for new interactions, with respect to juxtaposition and coordination. We start with a result having to do with the intersubstitutability of logical connectives, so that we start grasping what is new and how the corresponding connectives interact. Moreover, this result allows us to affirm that $\vDash^{ic}_{SC}$ does not collapse nor weakly collapse.

PROPOSITION 4.2.5. *In* $\vDash^{ic}_{SC}$*, no pair of corresponding connectives are intersubstitutable in every context. In particular:*

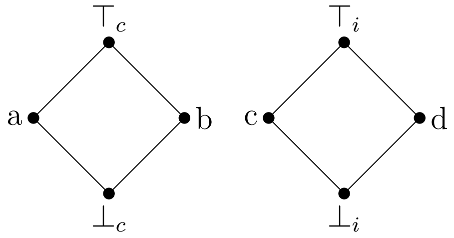
- $p \vee_i q \nvDash^{ic}_{SC} p \vee_c q$
- $p \rightarrow_i q \nvDash^{ic}_{SC} p \rightarrow_c q$
- $p \leftrightarrow_i q \nvDash^{ic}_{SC} p \leftrightarrow_c q$
- $\neg_c(p \wedge_i q) \nvDash^{ic}_{SC} \neg_c(p \wedge_c q)$
- $\neg_c p \rightarrow_i \neg_c p \nvDash^{ic}_{SC} \neg_i p \rightarrow_i \neg_c p$

However, notice that $\neg_x$ *and* $\wedge_x$ *are intersubstitutable as main connectives:*

- $\neg_c p \vDash^{ic}_{SC} \neg_i p$
- $\neg_i p \vDash^{ic}_{SC} \neg_c p$
- $p \wedge_c q \vDash^{ic}_{SC} p \wedge_i q$
- $p \wedge_i q \vDash^{ic}_{SC} p \wedge_c q$

PROOF. In order to prove it, we look for a strongly coordinated countermodel for the first set of invalidities and then we prove that the inferences in the last set hold for every strongly coordinated model.

Let me use again, the following pair of structures in order to build the countermodel:



- $p \vee_i q \not\models^{ic}_{SC} p \vee_c q$

Take $V_i(p) = c$, $V_i(q) = d$ and $V_c(p) = V_c(q) = a$.

- $p \rightarrow_i q \not\models^{ic}_{SC} p \rightarrow_c q$

Take $V_i(p) = V_i(q) = c$, $V_c(p) = a$ and $V_c(q) = b$.

- $p \leftrightarrow_i q \not\models^{ic}_{SC} p \leftrightarrow_c q$

The same valuation as before will do.

- $\neg_c(p \wedge_i q) \not\models^{ic}_{SC} \neg_c(p \wedge_c q)$

Take $V_i(p) = c$, $V_i(q) = d$ and $V_c(p) = V_c(q) = a$. With these, $||p \wedge_i q||_i = \bot_i$, so, by strong coordination, $||p \wedge_i q||_c = \bot_c$. Therefore, $||\neg_c(p \wedge_i q)||_c = \top_c$, but $||\neg_c(p \wedge_c q)||_c = b$.

- $\neg_c p \rightarrow_i \neg_c p \not\models^{ic}_{SC} \neg_i p \rightarrow_i \neg_c p$

The premise is a tautology, so simply take a valuation that makes the conclusion non-designated, e.g. $V_i(p) = c$ and $V_i(\neg_c p) = c$.

Thus, we know that $\models^{ic}_{SC}$ does not collapse nor weakly collapse. But, as we said:

- $\neg_c p \models^{ic}_{SC} \neg_i p$
- $\neg_i p \models^{ic}_{SC} \neg_c p$
- $p \wedge_c q \models^{ic}_{SC} p \wedge_i q$
- $p \wedge_i q \models^{ic}_{SC} p \wedge_c q$

Remember that the last two were already valid for juxtaposition. For the first two, notice that the premiss, in both cases, is only $\top_x$ iff $||p||_x = \bot_x$. But, given strong coordination, this means that the conclusion will also be $\top_x$ whenever the premiss is. □

Now, we show that the alternative version of Mix is valid for strong coordination. In the case of coordination we showed that $p_c, q_i \vee_i \neg_c p_c \nvDash^{ic}_{C} q_i$. Let us prove that, for strong coordination, $p_c, q_i \vee_i \neg_c p_c \vDash^{ic}_{SC} q_i$.

- $p_c, q_i \vee_i \neg_c p_c \vDash^{ic}_{SC} q_i$

Proof. From the semantic perspective, it is easy to see that for the valuations that designate the premisses $||\neg_c p||_c = \bot_c$, so, by strong coordination, $||\neg_c p||_i = \bot_i$. Then, for $||q_i \vee_i \neg_c p_c||_x \in D_x$, $||q_i||_i = \top_i$.

Let us see also how to make the derivation in the strongly coordinated natural deduction calculus:

1	$q_i \vee_i \neg_c p_c$			
2	p_c			
3	i	q_i		
4		q_i	Identity, 3	
5	i	$\neg_c p_c$		
6		$\bot_c$	$\neg_c$E, 2, 5	
7		q_i	$\bot_c$E, 6	
8	q_i		$\vee_i$E, 1–7	

□

For coordination we also had that $\neg_c\neg_i p \nvDash^{ic}_{C} \neg_i\neg_i p$ and $\neg_c\neg_i p \nvDash^{ic}_{C} \neg_c\neg_c p$. In the case of strong coordination, though, we have that the former holds while the latter is still invalid.

- $\neg_c\neg_i p \vDash^{ic}_{SC} \neg_i\neg_i p$

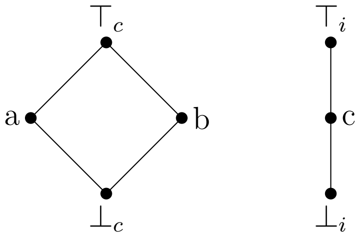
Proof. It follows from $\neg_c p \vDash^{ic}_{SC} \neg_i p$ and uniform substitution, but I will give the direct proofs too. Suppose that $||\neg_c\neg_i p||_c = \top_c$, then $||\neg_i p||_c = \bot_c$. By strong coordination, $||\neg_i p||_i = \bot_i$, therefore $||\neg_i\neg_i p||_i = \top_i$.

1		$\neg_c\neg_i p$	
2	i	$\neg_i p$	
3		$\bot_c$	$\neg_c$E, 1, 2
4	$\neg_i\neg_i p$		$\neg_i$I, 2–3

□

- $\neg_c\neg_i p \nvDash^{ic}_{SC} \neg_c\neg_c p$

Proof. We build a strongly coordinated countermodel based on the following Boolean-Heyting structure:



Suppose that $||\neg_c\neg_i p||_c = \top_c$, then $||\neg_i p||_c = \bot_c$. By strong coordination, $||\neg_i p||_i = \bot_i$, so take $V_i(p) = c$. Then, we can chose $V_c(p) = a$ so that $||\neg_c\neg_c p||_c = a$.

Let me also show what goes wrong in the strongly coordinated natural deduction derivation:

1	$\neg_c \neg_i p$			
2	c	$\neg_c p$		
3		i	p	
4			$\bot_c$	!!$\neg_c$E, 2, 3
5		$\neg_i p$		$\neg_i$I, 3–4
6		$\bot_c$		$\neg_c$E, 1, 5
7	$\neg_c \neg_c p$			!!$\neg_c$I, 2–6

□

Notice that there are some problems with this derivation. On the one hand, we have applied both classical and intuitionistic rules within $\neg_c$I, which our rules do not allow. And, on the other hand, having applied **IL** rules within $\neg_c$I, the rule is not closed with $\bot_i$ as it should.

Finally, let me just mention that, although strong coordination introduces more bridge principles than coordination, arguments like Nix that appear to be intuitively invalid are still invalidated by strong coordination. Simply notice that the countermodel that we gave for Nix when applying juxtaposition is also a strongly coordinated countermodel. The same goes for the second version of Disjunctive Mix, but the intuitions with respect to its validity are not so clear as I explained above. Then, if we include the new version of Mix among the mixed inferences that Wrenn considers to be challenging for localism, the verdict by applying strong coordination is this:

- **Mix**: $q_c \vee_i \neg_i p_i,\, p_i \vDash^{ic}_{SC} q_c$; $q_c \vee_c \neg_i p_i,\, p_i \vDash^{ic}_{SC} q_c$ and $q_i \vee_i \neg_c p_c, p_c \vDash^{ic}_{SC} q_i$
- **Essential Mix**: $\neg_c \neg_c p_c \vee_i \neg_i q_i,\, q_i \vDash^{ic}_{SC} p_c$ and $\neg_c \neg_c p_c \vee_c \neg_i q_i,\, q_i \vDash^{ic}_{SC} p_c$
- **Disjunctive Mix**: $\neg_c \neg_c p_c \vee_c q_i \vDash^{ic}_{SC} p_c \vee_c q_i$ and $\neg_c \neg_c p_c \vee_i q_i \nvDash^{ic}_{SC} p_c \vee_i q_i$
- **Nix**: $\neg_i p_i \to_x \neg_c q_c,\, q_c \nvDash^{ic}_{SC} p_i$ $(x = i, c)$

4.2.6 The localism of strong coordination and beyond

The results obtained by applying strong coordination to mixed inferences look quite promising. We are able to go beyond coordination without allowing the emergence of invalid bridge principles like Nix, or of bridge principles that provoke the collapse of $\vDash^{ic}_{SC}$. Moreover, we have given some reasons for the rationale behind strong coordination and for the type of mixed inferences that allows, i.e. $q_i \vee_i \neg_c p_c, p_c \vDash^{ic}_{SC} q_i$, on top of those allowed by coordination.

Nevertheless, I believe that strong coordination leaves us in an unsatisfactory middle ground. On the one hand, we could still argue that it is philosophically less plausible than coordination. Even taking into account the reasons that we have given for it, there is still something odd in forcing the bottom value in the Heyting algebra whenever we get the Boolean bottom value for a formula, if one is a localist about classical and intuitionistic logics. It seems that if the classical side tells me about an intuitionistic proposition (say, 'wet cats are funny', 'Sophie's choice is morally wrong' or 'every even natural number greater than 2 is the sum of two prime numbers') that it is false, in the classical sense, I should not interpret this as there being a construction for its negation, since this goes, precisely, against the reasons for preferring intuitionistic logic over classical logic in those domains.

It is true that we already have a connection between the top values of the Heyting and Boolean algebras, as we said, and that this could be used to argue that the different epistemological standards do not impede the possible connection between the bottom values. But having the tops connected is almost mandatory if we want a sensible definition of satisfaction and of consequence as designation preservation. However, connecting the bottoms is just a way of going beyond juxtaposition and coordination, so it might be easier that the philosophical reasons against it outweigh the benefits of allowing for some more mixed inferences.

On the other hand, strong coordination does not constitute a great breakthrough with respect to the kind of bridge principles that it allows compared with coordination. I do not mean that it is not an important improvement, but one might expect even more capacity for validating bridge principles, even the doubtful version of Disjunctive Mix, once we have started to pay the price of entering into more delicate philosophical terrains.

One possible way of upping the ante and trying to go further in the way our combined system captures the localist spirit is by dropping uniform substitution. In the case of strong coordination, maybe it was not a sound philosophical approach to impose that whenever $||\alpha||_c = \bot_c$ then $||\alpha||_i = \bot_i$, for any $\alpha \in \text{Sent}(C_{ic}, P_{ic})$. But, being a localist and defending that classical logic has its correct domain of application, I might have good philosophical justification for imposing in my coordinated semantics that if my classical semantics evaluates a classical proposition as false, since **CL** is the 'authority' in that domain, the intuitionistic part defers to it and evaluates that proposition as the classical side dictates. That is, if one believes that classical logic is the correct logic for certain propositions (or inferences constituted by certain propositions), one might want that, whenever $||\alpha||_c = \bot_c$, then $||\alpha||_i = \bot_i$, for any α with main connective $c \in C_c^n$ or $\alpha \in P_c - P_i$.

In the next section, I will develop further this idea by trying to impose even more structure preservation between the semantic value translations. The way of preserving more structure while still avoiding the collapse is, precisely, to drop uniform substitution. But this seems to match the localist spirit maybe even more than the previous methods.

4.2.7 (Weak/Strong) Coordination with Embedding

The guiding idea for this new method is that, not only do we have different connectives for each logical system (in our case intuitionistic and classical) being combined, but also these connectives can behave in different ways depending on whether they act upon an intuitionistic or a classical proposition. To be more precise, what I want to do is to let the intuitionistic connectives treat the classical terms as 'stable' terms. That is, I want intuitionistic connectives to behave like classical ones when applied to propositions that are not epistemologically constrained.

For intuitionistic logic, it is said[21] that a formula α is *stable* iff $\neg\neg\alpha \Rightarrow \alpha$[22]. Since the idea is to introduce some criterion of classicality in order to be able to allow for a different behaviour of the intuitionistic connectives when applied to classical formulas, we will have to syntactically characterize which formulas count as stable. Equally, we need a semantic criterion of stability. Luckily enough, this is something already known in the literature. Semantically, stable formulas are the ones taking regular elements as semantic values in the Heyting algebras. We say that an element $a \in HA$ is *regular* iff $\neg\neg a = a$. That is, for formulas which take regular values we have that $||\neg\neg\alpha||^{\mathfrak{HA}} = ||\alpha||^{\mathfrak{HA}}$.

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[21] See, for instance (Dummett, 1977, p. 24).

[22] As far as as I know, a more general definition of stability is that a stable formula with respect to an operation, $*$, is a formula for which $*\alpha \Leftrightarrow \alpha$ holds. Since in intuitionistic logic we already have that $\alpha \Rightarrow \neg\neg\alpha$, we define stable formulas just with the left-to-right direction.

Moreover, we know that the set of all regular elements of a Heyting algebra, denoted $\mathfrak{HA}_{\neg\neg}$, is a Boolean algebra (though it is not in general a sublattice of $\mathfrak{HA}$[23]) (see, for instance (Johnstone, 1982, p. 10)). We also know that the only regular elements that will appear in every Heyting algebra are $\top^{\mathfrak{HA}}$ and $\bot^{\mathfrak{HA}}$. So, in every Heyting algebra, we will have at least the two-element Boolean algebra.

With these elements in mind, let us now present the details of the method. In fact, I should say, in plural, 'the methods'. I will cover them within the same section but, in reality, the semantic standards for the models that we consider vary. However, the main idea of implementing a concept of stability remains constant. That is why I group them together. Also, because as we will see, the process for getting at those methods has been like that of making a sculpture with a hammer and chisel. We start with a method that adds a layer to strong coordination, namely, *strong coordination with embedding* (henceforth **SCE**), and then proceed to refine and chisel the parts that can be made more fine-tuned with certain localist readings of the combination mechanisms.

Thus, **SCE** is meant to extend the previous methods while avoiding the collapse and invalidating intuitively invalid arguments like Nix. As we announced, these methods will not have uniform substitution, but I will try to justify this and make room for restricted forms of substitutions or 'safe' substitutions.

Syntax and Consequence Relation of SCE

Before going to the crucial semantic clauses, let me briefly highlight a couple of important syntactic notions as well as the type of relation that **SCE** is. First, in order to have a syntactic criterion of stability, let us say that a formula, α, is *stable* just in case, either for every propositional variable, p, occurring in α, $p \in P_c$, or $\alpha = c\alpha^1 ... \alpha^n$ with $c \in C_c^n$. When presenting the semantics, in a moment, it will become clearer why we adopt this syntactic criterion, the reason being that those types of formulas are the ones which are going to have regular values in the Heyting algebras.

With respect to what type of consequence relation **SCE**, $\vDash^{ic}_{SCE}$, is, we will see that it is not *structural*, in the sense that we used in the previous cases, because it will not have uniform substitution. An easy way of seeing this is

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[23] That is, it might be the case that $\exists a, b \in \mathfrak{HA}_{\neg\neg} : a \wedge b \notin \mathfrak{HA}_{\neg\neg}$ or $a \vee b \notin \mathfrak{HA}_{\neg\neg}$.

noticing that we will have $\neg_i\neg_i p_c \vDash^{ic}_{SCE} p_c$ and yet $\neg_i\neg_i p_i \nvDash^{ic}_{SCE} p_i$. Lastly, from the semantic clauses it will be clear that **SCE**, $\vDash^{ic}_{SCE}$, extends strong coordination, $\vDash^{ic}_{SC}$, and, therefore, also $\vDash^{ic}_{C}$, $\vDash^{ic}$, $\vDash^{i}$ and $\vDash^{c}$.

Semantics of SCE

*The **SCE** of the structures* $\mathfrak{B}_i$, over C_i, and $\mathfrak{B}_c$, over C_c, is the juxtaposition of the structures, $\langle \mathfrak{B}_i, \mathfrak{B}_c\rangle$, and *the **SCE** of the classes of structures* $\mathbb{B}_i$ and $\mathbb{B}_c$ is the Cartesian product $\mathbb{B}_i \times \mathbb{B}_c$, which is the juxtaposition of the classes of structures.

A ***SCE**-model,* $\mathfrak{M}^{SCE}_{ic} = \langle \mathfrak{B}_i, V_i, \mathfrak{B}_c, V_c\rangle$ over C_{ic} and P_{ic}, based on the **SCE**-structure $\langle \mathfrak{B}_i, \mathfrak{B}_c\rangle$ (or, more generally, based on the class of **SCE**-structures $\mathbb{B}_{ic}$) is a (coherent) juxtaposed model satisfying the following property:

- **Strong Coordination with Embedding**: A model is a **SCE**-model when for every $\alpha \in \text{Sent}(C_{ic},\ P_{ic})$,
    1. $||\alpha||_i = \top_i$ iff $||\alpha||_c = \top_c$
    2. $||\alpha||_i = \bot_i$ iff $||\alpha||_c = \bot_c$
    3. If $||\alpha||_c \notin D_c$ then $||\alpha||_i = \bot_i$, for every α such that $\alpha = c\alpha^1...\alpha^n$ with $c \in C^n_c$ or $\alpha \in P_c$.

Notice that the loss of uniform substitution comes from the fact that we have introduced a new constraint in the possible strongly coordinated models, in order to have more structure preservation from the semantic values of the Boolean algebra to those of the Heyting algebra, but this new constraint only affects classical sentences. With this we will secure that stable sentences exhibit a special behaviour compared to the non-stable ones, namely, they will behave classically, since their range of values will be the two-element Boolean algebra[24]. This is the localist intuition that we are trying to represent in our models. If the sentences that the intuitionistic part of the model

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[24] To see this, simply look at the definition of stable sentence and evaluate its possible values in the Heyting algebra. If the formula has a classical main connective or is a classical propositional variable it is immediate. If it has an intuitionistic main connective but all its propositional variables are classical, then the intuitionistic truth-value functions will have as domain and range only $\top^{\mathfrak{H}\mathfrak{A}}$ and $\bot^{\mathfrak{H}\mathfrak{A}}$.

has to evaluate have that trace of classicality, then they can only take regular values. This is perfectly sound with the intuitionistic intuitions that the localist might have for sentences belonging to other domains.

We will dig deeper into these ideas after having applied the method to the mixed inferences, which, I believe, will be of some help to better see its scope. But, before, let me explore what possible modification of the calculus could match this new semantic clause.

A Natural Deduction calculus for SCE

As we have just seen, **SCE** extends strong coordination by adding a further restriction to the models: on top of having strong coordination we have a clause sending every non-designated Boolean value to the bottom value of any Heyting algebra, which allows us to embed the two-element Boolean algebra in any Heyting algebra. Therefore, the natural deduction calculus of **SCE** needs to have the rules of the strongly coordinated natural deduction calculus (which, we know, is strongly sound w.r.t the **SCE** semantics, since $\vDash^{\mathbb{B}_{ic}}_{SC} \subseteq \vDash^{\mathbb{B}_{ic}}_{SCE}$) plus something extra in order to extend it. That something extra is going to be reflecting the idea that has motivated **SCE**, namely, that intuitionistic connectives behave classically when applied to certain formulas. More concretely, to those formulas having a trace of classicality that we called stable formulas.

Thus, the **SCE** natural deduction calculus consists of the rules of the strongly coordinated natural deduction calculus plus the following rule that I will call 'intuitionistic double-negation elimination for stable formulas'[25]

$$(\neg_i\neg_i\mathrm{E}_s) \quad \begin{array}{|l} \neg_i\neg_i\alpha_s \\ \hline \alpha_s \end{array}$$

In order to prove that the **SCE** natural deduction calculus is strongly sound with respect to $\vDash^{\mathbb{B}_{ic}}_{SCE}$, then, we just need to prove that $\neg_i\neg_i\mathrm{E}_s$ is a sound rule. But this is immediate since, for any α_s, its only possible semantic values in any Heyting algebra will be the top and the bottom, as clarified in footnote 24. So, for any **SCE**-model designating $\neg_i\neg_i\alpha_s$, $\mathfrak{M}^{SCE}_{ic} \vDash \neg_i\neg_i\alpha_s$, we will also have $\mathfrak{M}^{SCE}_{ic} \vDash \alpha_s$.

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[25] The 's' in the subindexes stands for stable and recall that I defined stable formula as those α s.t. either for every propositional variable, p, occurring in α, $p \in P_c$, or $\alpha = c\alpha^1 ... \alpha^n$ with $c \in C^n_c$.

Syntax and Consequence Relation of Coordination with Embedding (CE)

With respect to the syntax there is nothing new to add for **CE**. As a consequence relation, though, the **CE** consequence relation, $\vDash^{ic}_{CE}$, adds something interesting into the picture. For, despite extending the coordinated consequence relation, $\vDash^{ic}_{C}$, the juxtaposed consequence relation, $\vDash^{ic}$, and, so, also $\vDash^{i}$ and $\vDash^{c}$, and being included in $\vDash^{ic}_{SCE}$, it 'deviates' from the strongly coordinated consequence relation, $\vDash^{ic}_{SC}$. That is, there are inferences $\Gamma \vDash \alpha$ and $\Delta \vDash \beta$ such that $\Gamma \vDash \alpha \subseteq \vDash^{ic}_{CE}$ but $\Delta \vDash \beta \not\subseteq \vDash^{ic}_{CE}$, and $\Gamma \vDash \alpha \not\subseteq \vDash^{ic}_{SC}$ but $\Delta \vDash \beta \subseteq \vDash^{ic}_{SC}$. We will see why in a moment and give examples of this deviation when applying the method.

Semantics of CE

With respect to structures and classes of structures **CE** remains the same. Again, the novelty comes with the models. A ***CE**-model,* $\mathfrak{M}^{CE}_{ic} = \langle \mathfrak{B}_i, V_i, \mathfrak{B}_c, V_c \rangle$ over C_{ic} and P_{ic}, based on the **CE**-structure $\langle \mathfrak{B}_i, \mathfrak{B}_c \rangle$ (or, more generally, based on the class of **CE**-structures $\mathbb{B}_{ic}$) is a (coherent) juxtaposed model satisfying the following property:

- **Coordination with Embedding**: A model is a **CE**-model when for every $\alpha \in \mathrm{Sent}(C_{ic}, P_{ic})$,
  1. $||\alpha||_i = \top_i$ iff $||\alpha||_c = \top_c$
  2. If $||\alpha||_i = \bot_i$ then $||\alpha||_c = \bot_c$
  3. If $||\alpha||_c \notin D_c$ then $||\alpha||_i = \bot_i$, for every α such that $\alpha = c\alpha^1 ... \alpha^n$ with $c \in C^n_c$ or $\alpha \in P_c$.

The loss of uniform substitution responds to the same reason as for **SCE**. We have a semantic clause that imposes different restrictions on the possible values a sentence might take, depending on its syntactic features.

The difference with respect to **SCE** is subtle but important. **SCE** added a new layer to the semantic clauses of strong coordination and we have already seen how one could doubt the requirement of having the intuitionistic bottom whenever we got the classical bottom. This was problematic because it applied also to intuitionistic sentences that might be epistemologically constrained according to our localism. Now, however, we are sending to the intuitionistic bottom only sentences that can be 'safely' sent there, namely, the sentences that are stable and should take regular values in the Heyting algebra. And the way of making sure that the Boolean values are sent to regular values of the Heyting algebra is by sending the designated ones to $\top^{\mathfrak{HA}}$ and the non-designated ones to $\bot^{\mathfrak{HA}}$.

When applying the methods we will clearly see that **CE** and strong coordination deviate in the sense explained above, but I reckon it is already quite obvious that they do. On the one hand, we have gained some flexibility for building countermodels going from strong coordination to **CE**, since now it is possible to have $||\alpha||_c = \bot_c$ and $||\alpha||_i \notin D_i \neq \bot_i$ for non-stable sentences. On the other hand, we have also imposed further restrictions that reduce the possible valuations and so make it harder to build countermodels. This is the case for stable sentences, which for **CE** can only take the top and bottom values in the Heyting algebra. Thus, we have relaxed one clause but added a new one and this will result in our method allowing for some new bridge principles while loosing some others, as we will shortly see.

Syntax and Consequence Relation of Weak Coordination with Embedding (WCE)

The syntax of **WCE** remains the same. As a consequence relation **WCE**, $\vDash^{ic}_{WCE}$, extends the juxtaposed consequence relation, $\vDash^{ic}$, deviates from coordination, $\vDash^{ic}_{C}$, and strong coordination, $\vDash^{ic}_{SC}$, and is included in **CE**, $\vDash^{ic}_{CE}$. The semantics and the application to mixed inferences will show why and which inferences are in or out.

Semantics of WCE

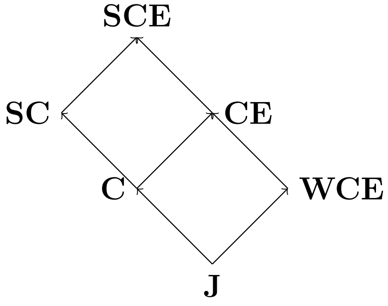
We continue chiselling the models in order to obtain a new, more refined, way of implementing a localist standpoint. We say that a ***WCE**-model,* $\mathfrak{M}^{WCE}_{ic} = \langle \mathfrak{B}_i, V_i, \mathfrak{B}_c, V_c \rangle$ over C_{ic} and P_{ic}, based on the **WCE**-structure $\langle \mathfrak{B}_i, \mathfrak{B}_c \rangle$ (or, more generally, based on the class of **WCE**-structures $\mathbb{B}_{ic}$) is a (coherent) juxtaposed model satisfying the following property:

- **Weak Coordination with Embedding**: A model is a **WCE**-model when for every $\alpha \in \text{Sent}(C_{ic}, P_{ic})$,
  1. $||\alpha||_i = \top_i$ iff $||\alpha||_c = \top_c$
  2. If $||\alpha||_i = \bot_i$ then $||\alpha||_c = \bot_c$, for every α such that $\alpha = c\alpha^1 ... \alpha^n$ with $c \in C^n_i$ or $\alpha \in P_i$.

3. If $||\alpha||_c \notin D_c$ then $||\alpha||_i = \bot_i$, for every α such that $\alpha = c\alpha^1...\alpha^n$ with $c \in C_c^n$ or $\alpha \in P_c$.

What we are trying to capture in this case is that, in the same way that classical logic has its domain and imposes some restrictions for the possible semantic values that the sentences of that domain might get, the intuitionistic part could also be restricted to its own domain and only have the 'authority' of imposing the bottom value of the Boolean algebra for the sentences of its own domain that get the bottom value in the Heyting algebra.

I guess it is not so straightforward which the correct intuitions about this clause should be, but I would say that the application of the method and the analysis of the bridge principles that it leaves out will help to clarify the situation and the possible localist readings of the method. The mixed inferences involving negations and iterations of negations are particularly interesting as we will see. Moreover, it can be helpful to keep in mind how each system relates to the others by inclusion, which is expressed in the following graph:



4.2.8 Applying (W/S)CE to mixed reasoning

Let us now move on to some examples of mixed inferences in order to apply the methods that we have just presented and analyse their results. I will be considering the mixed inferences in turn and applying the three methods to each of them. Of course, since **WCE** is extended by the other two consequence relations, if an inference is valid according to **WCE** it will also be valid for the other cases. Equally, given that **WCE** extends juxtaposition, every valid inference of juxtaposition will be valid according to **WCE** too. Let me start, then, with the mixed inferences proposed by Wrenn as challenges to localism.

- $q_c \vee_c \neg_i p_i,\ p_i \vDash^{ic}_{WCE} q_c$

Proof. Suppose that $||q_c \vee_c \neg_i p_i||_x = ||p_i||_x = \top_x$. Since, $||p_i||_i = \top_i$, then $||\neg_i p_i||_i = \bot_i$ and, given **WCE**, $||\neg_i p_i||_c = \bot_c$. So, in order for $||q_c \vee_c \neg_i p_i||_c = \top_c$, $||q_c||_c = \top_c$.[26] □

- $q_i \vee_i \neg_c p_c, p_c \vDash^{ic}_{WCE} q_i$

Proof. Suppose that $||q_i \vee_i \neg_c p_c||_x = ||p_c||_x = \top_x$. Since, $||p_c||_c = \top_c$, then $||\neg_c p_c||_c = \bot_c$ and, given **WCE**, $||\neg_c p_c||_i = \bot_i$. So, in order for $||q_i \vee_i \neg_c p_c||_i = \top_i$, $||q_i||_i = \top_i$. □

- $\neg_c \neg_c p_c \vee_c \neg_i q_i,\ q_i \vDash^{ic}_{WCE} p_c$

Proof. Suppose that $||\neg_c \neg_c p_c \vee_c \neg_i q_i||_x = ||q_i||_x = \top_x$. Since, $||q_i||_i = \top_i$, then $||\neg_i q_i||_i = \bot_i$ and, given **WCE**, $||\neg_i q_i||_c = \bot_c$. So, in order for $||\neg_c \neg_c p_c \vee_c \neg_i q_i||_c = \top_c$, $||\neg_c \neg_c p_c||_c = \top_c$. Therefore, $||p_c||_c = \top_c$. □

Now, we evaluate the second version of Disjunctive Mix that we had not been able to accommodate in the previous methods. In this case, we can validate the mixed inference by **WCE**.

- $\neg_c \neg_c p_c \vee_i q_i \vDash^{ic}_{WCE} p_c \vee_i q_i$

Proof. Suppose that $||\neg_c \neg_c p_c \vee_i q_i||_x = \top_x$. This can only be either because $||q_i||_x = \top_x$ or $||\neg_c \neg_c p_c||_x = \top_x$[27]. If it is because $||q_i||_i = \top_i$, then $||p_c \vee_i q_i||_i = \top_i$. If it is because $||\neg_c \neg_c p_c||_i = \top_i$, then $||\neg_c \neg_c p_c||_c = ||p_c||_c = \top_c$, so $||p_c||_i = \top_i$ and $||p_c \vee_i q_i||_i = \top_i$. □

However, we can still invalidate the intuitively invalid argument Nix, in its two versions. We show that **SCE** invalidates Nix and, therefore, **WCE** and **CE** invalidate it too.

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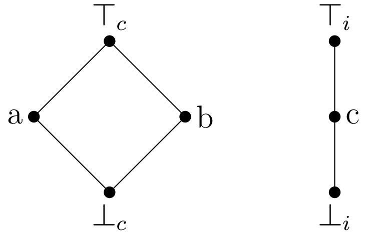
[26] Notice that the way of extending the intuitionistic (classical) valuations for the new intuitionistic (classical) atoms in order to get **(W/S)CE**-models is the same as for the previous methods. The only difference is that the restrictions on the valuations have changed because the amount of structure preservation allowed by the semantic clauses is different.

[27] Notice that with **WCE**-models we no longer can have that the premiss is designated while the disjunct is not, since $\neg_c \neg_c p_c$ is a classical formula and it can only take the values top or bottom in the Heyting algebra.

- $\neg_i p_i \rightarrow_i \neg_c q_c, q_c \nvDash^{ic}_{SCE} p_i$

Proof. We are going to build a **SCE**-countermodel. Take the following **SCE** Boolean-Heyting structure,



together with the valuations $V_x(q_c) = \top_x$, $V_i(p_i) = c$, $V_c(p_i) = a$. Since $||q_c||_c = \top_c$, $||\neg_c q_c||_c = \bot_c$ and $||\neg_c q_c||_i = \bot_i$. So, with $||p_i||_i = c$, $||\neg_i p_i||_i = \bot_i$ and, therefore, $||\neg_i p_i \rightarrow_i \neg_c q_c||_i = \top_i$[28], $||q_c||_i = \top_i$ but $||p_i||_i = c$. ☐

- $\neg_i p_i \rightarrow_c \neg_c q_c, q_c \nvDash^{ic}_{SCE} p_i$

Proof. The same **SCE**-model is a countermodel for this version of Nix too. ☐

Therefore, by applying **(W/S)CE** to the inferences presented in Wrenn (2018), we get these results;

Proposition 4.2.6.

- **Mix**: $q_c \vee_i \neg_i p_i$, $p_i \vDash^{ic}_{(W/S)CE} q_c$; $q_c \vee_c \neg_i p_i$, $p_i \vDash^{ic}_{(W/S)CE} q_c$ and $q_i \vee_i \neg_c p_c, p_c \vDash^{ic}_{(W/S)CE} q_i$
- **Essential Mix**: $\neg_c \neg_c p_c \vee_i \neg_i q_i$, $q_i \vDash^{ic}_{(W/S)CE} p_c$ and $\neg_c \neg_c p_c \vee_c \neg_i q_i$, $q_i \vDash^{ic}_{(W/S)CE} p_c$
- **Disjunctive Mix**: $\neg_c \neg_c p_c \vee_c q_i \vDash^{ic}_{(W/S)CE} p_c \vee_c q_i$ and $\neg_c \neg_c p_c \vee_i q_i \vDash^{ic}_{(W/S)CE} p_c \vee_i q_i$
- **Nix**: $\neg_i p_i \rightarrow_x \neg_c q_c$, $q_c \nvDash^{ic}_{(W/S)CE} p_i$ $(x = i, c)$

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[28] Because it is the greatest x, s.t. $\bot_i \wedge x \leq \bot_i$.

We continue now with other bridge principles and evaluate their validity according to **(W/S)CE**. The cases in which the results diverge are specially interesting for assessing their virtues and shortcomings from a technical, but also philosophical, point of view.

- $\neg_i p_i \vDash^{ic}_{WCE} \neg_c p_i$

Proof. Suppose $||\neg_i p_i||_i = \top_i$. For this, $||p_i||_i = \bot_i$ and, by **WCE**, $||p_i||_c = \bot_c$. Therefore, $||\neg_c p_i||_c = \top_c$. ☐

- $\neg_c p_c \vDash^{ic}_{WCE} \neg_i p_c$

Proof. Analogous. ☐

However, for **WCE** things change when the negation of the premiss does not match the domain of the propositional variable to which it applies.

- $\neg_i p_c \nvDash^{ic}_{WCE} \neg_c p_c$

Proof. Take the Boolean-Heyting structure that we considered above and the valuation $V_c(p_c) = a$. Then, $||p_c||_i = \bot_i$, so $||\neg_i p_c||_i = \top_i$ while $||\neg_c p_c||_c = b$. ☐

Notice, that this already shows that **WCE** and coordination deviate, since for coordination we had $\neg_i p \vDash^{ic}_{C} \neg_c p$, for any p regardless of its domain.

- $\neg_c p_i \nvDash^{ic}_{WCE} \neg_i p_i$

Proof. Take the same Boolean-Heyting structure and the valuation $V_i(p_i) = c$. Then, choose $||p_i||_c = \bot_c$, so $||\neg_c p_i||_c = \top_c$ while $||\neg_i p_i||_i = \bot_i$. ☐

But, for **CE**:

- $\neg_i p_c \vDash^{ic}_{CE} \neg_c p_c$

Proof. This already follows from the fact that the argument is valid in coordination and **CE** extends it, but I offer the direct proof in any case. Suppose $||\neg_i p_c||_i = \top_i$. For this, $||p_c||_i = \bot_i$ and, by **CE**, $||p_c||_c = \bot_c$. Therefore, $||\neg_c p_c||_c = \top_c$. ☐

And still:

- $\neg_c p_i \not\models^{ic}_{CE} \neg_i p_i$

Proof. Same countermodel as for **WCE**. □

These results might appear to be difficult to interpret because we do not have clear intuitions about these type of mixed inferences with mixed languages. But, let me try to suggest a possible reading of these last two results of **CE** and why they might make sense for a localism of classical and intuitionistic domains.

First, with respect to the inference $\neg_i p_c \models^{ic}_{CE} \neg_c p_c$, the sense in which the premiss is supposed to be true is that we suppose that there is a construction for the absurdity of p_c. Imagine, for instance, that we have a box A with three candies and a box B with two. Let p_c be the proposition 'there are more candies in B than in A'. Certainly, we do not need a mathematical proof to know that it is not the case that p_c, i.e., $\neg_c p_c$, humans knew this well before Pythagoras. But, it also seems to be the case that if we have a proof for the absurdity of two being bigger than three, then we also know that it is not the case that p_c.

With the inference $\neg_c p_i \not\models^{ic}_{CE} \neg_i p_i$ things seem to be different, though. If one is a localist but at the same time a defender of intuitionism for its domain, then it is reasonable that the assumption of not being the case that p_i does not imply having a construction for the absurdity of p_i, because the standard for truth is higher in that domain than what the classical negation in the premiss is expressing. Recall that for **CE** we still have $\neg_c p_c \models^{ic}_{CE} \neg_i p_c$, because in this case the propositional variable is classical. The localist reading is that, given that the domain of p_c is not epistemologically constrained, the truth of the premiss in a classical semantics sense is enough justification for the intuitionistic negation of that classical proposition to be true. That is, being a localist, one should have no problem accepting that, within the classical domain, the standards of classical logic are the correct ones, even with intuitionistic connectives.

Thus, maybe **SCE** can be challenged on those lines, as being too immodest. Because, on top of validating $\neg_c p_c \models^{ic}_{SCE} \neg_i p_c$, $\neg_i p_i \models^{ic}_{SCE} \neg_c p_i$ and $\neg_i p_c \models^{ic}_{SCE} \neg_c p_c$, we also get,

- $\neg_c p_i \models^{ic}_{SCE} \neg_i p_i$

Proof. It follows directly from the fact that the argument is valid within strong coordination and **SCE** extends strong coordination. Let me also give the direct proof. Suppose that $||\neg_c p_i||_c = \top_c$. For this, $||p_i||_c = \bot_c$, therefore, $||p_i||_i = \bot_i$ and, so, $||\neg_i p_i||_i = \top_i$. □

Let us see, now, what happens when we iterate negations. For **WCE** we already have that,

- $\neg_i \neg_i p_c \vDash^{ic}_{WCE} p_c$

Proof. Suppose that $||\neg_i \neg_i p_c||_i = \top_i$. For this, $||\neg_i p_c||_i = \bot_i$. Since p_c is a classical proposition, it can only take the values $\top_i$ or $\bot_i$ in the Heyting algebra. Therefore, for $||\neg_i p_c||_i = \bot_i$, $||p_c||_i = \top_i$ □

However, we still have $\neg_i \neg_i p_i \nvDash^{ic}_{WCE} p_i$. In fact, we have $\neg_i \neg_i p_i \nvDash^{ic}_{SCE} p_i$, so not even **SCE** collapses intuitionistic logic into classical logic.

- $\neg_c \neg_i p_c \vDash^{ic}_{WCE} \neg_c \neg_c p_c$

Proof. Suppose that $||\neg_c \neg_i p_c||_c = \top_c$. For this, $||\neg_i p_c||_c = \bot_c$. Since p_c is a classical proposition, it can only take the values $\top_i$ or $\bot_i$ in the Heyting algebra. Therefore, for $||\neg_i p_c||_i \notin D_i$, $||p_c||_i = \top_i$, so $||p_c||_c = \top_c$ and $||\neg_c \neg_c p_c||_c = \top_c$. □

The rationale for this inference is similar to the previous one. Since the embedded intuitionistic negation is being applied to a classical proposition, its behaviour should be classical, that is why the truth of the classical negation of $\neg_i p_c$ implies the truth of p_c. Nevertheless, **WCE** invalidates other combinations:

- $\neg_c \neg_i p_i \nvDash^{ic}_{WCE} \neg_c \neg_c p_i$

Proof. Take the Boolean-Heyting structure that we considered above and the valuation $V_i(p_i) = c$. Then, choose $||p_i||_c = b$. With this, $||\neg_c p_i||_c = a$ and $||\neg_i p_i||_i = \bot_i$. Hence, $||\neg_i p_i||_c = \bot_c$ and, so, $||\neg_c \neg_i p_i||_c$
$= \top_c$ while $||\neg_c \neg_c p_i||_c = b$. □

- $\neg_i \neg_c p_c \nvDash^{ic}_{WCE} \neg_c \neg_c p_c$

Proof. Take again the same Boolean-Heyting structure and the valuation $V_c(p_c) = a$. Then, $||\neg_c p_c||_c = b$, so choose $V_i(\neg_c p_c) = \bot_i$. With this, $||\neg_i \neg_c p_c||_i = \top_i$ while $||\neg_c \neg_c p_c||_c = a$. □

- $\neg_i \neg_c p_i \not\models^{ic}_{WCE} \neg_c \neg_c p_i$

Proof. Take again the same Boolean-Heyting structure and the valuation $V_i(p_i) = c$. Then, choose $||p_i||_c = b$. With this, $||\neg_c p_i||_c = a$ and, therefore, $||\neg_c p_i||_i = \bot_i$. Hence, $||\neg_i \neg_c p_i||_i = \top_i$ while $||\neg_c \neg_c p_i||_c = b$. ☐

For **CE** and **SCE** we will also get $\neg_c \neg_i p_i \not\models^{ic}_{(S)CE} \neg_c \neg_c p_i$. In fact, the same countermodel that we used for **WCE** is also a countermodel for **(S)CE**. However, for these consequence relations, we will get the other cases. We show it for **CE**, which implies that they also hold for **SCE**.

- $\neg_i \neg_c p_c \models^{ic}_{CE} \neg_c \neg_c p_c$

Proof. Suppose that $||\neg_i \neg_c p_c||_i = \top_i$. Then, $||\neg_c p_c||_i = \bot_i$ so, by **CE**, $||\neg_c p_c||_c = \bot_c$. Therefore, $||\neg_c \neg_c p_c||_c = \top_c$. ☐

- $\neg_i \neg_c p_i \models^{ic}_{CE} \neg_c \neg_c p_i$

Proof. Suppose that $||\neg_i \neg_c p_i||_i = \top_i$. Then, $||\neg_c p_i||_i = \bot_i$ so, by **CE**, $||\neg_c p_i||_c = \bot_c$. Therefore, $||\neg_c \neg_c p_i||_c = \top_c$. ☐

The justification for these latter inferences is the same as for $\neg_i p_c \models^{ic}_{(S)CE} \neg_c p_c$, since what we have now under the scopes of $\neg_i$ and $\neg_c$ is $\neg_c p_c$ and $\neg_c p_i$, i.e. stable sentences in both cases. Then, having a construction for the absurdity of those sentences should be enough for the truth, in the classical semantic sense, of those sentences.

I will dedicate some more space later on to discussing the significance of these results and the methods themselves. As I said before, it is not easy to get clear intuitions on these different combinations of logics, even when we restrict the combination, or coordination, to just classical and intuitionistic logic. But let me end this section with a localist philosophical interpretation of these **(W/S)CE** relations and with a novel result that applies to them.

I started motivating these different methods appealing to the notion of 'stability'. The key idea behind this is that I wanted a method capturing the localist intuition that one might have along the following lines: 'classical logic is just right for its domain of application, but there are some domains (say, humour, ethics, mathematics...) where it goes too far and forces us to accept undesirable conclusions. But, as long as we are in a domain adequate to classical logic, intuitionistic connectives should agree with what classical logic delivers.' Then, when we are dealing with classical propositions, we should expect to get all the deductive strength of classical logic, even with the intuitionistic connectives.

However, if we are in a mixed domain, with propositions from the classical and intuitionistic domains, then we get **IL** + **CL** (for their own languages) + interaction principles, some of which might be bridge principles. And, finally, if the domain is purely intuitionistic, we have **IL** and also a classical reading of the classical connectives when these are applied to intuitionistic propositions. For instance, even for the case of juxtaposition, we get that $\nvdash^{ic} p_i \vee_i \neg_i p_i$ but $\vdash^{ic} p_i \vee_c \neg_c p_i$. The first one is usually interpreted like 'either there is a construction of p or there is a construction of not-p' which is intuitively invalid, at least if one is an intuitionist about certain epistemically constrained domains. But the second inference could be read as 'either there is a construction of p or there is not a construction of p', which expresses something different to $p_i \vee_i \neg_i p_i$, but also to $p_c \vee_c \neg_c p_c$, and which seems intuitively valid.

Hence, when we are in a classical domain, it is as if the whole combined system collapsed to **CL**, but this is something sought and controlled, because the semantic clauses I gave made us drop uniform substitution. In fact, these results suggest that there is another possible type of collapse, on top of collapse and weak collapse. Let me give the definition:

Definition 4.2.1. Let f be the bijection from $\text{Sent}(C_1, P_1)$ to $\text{Sent}(C_2, P_{12})$ that maps each sentence $\alpha \in \text{Sent}(C_1, P_1)$ to the sentence that results from uniformly substituting each connective in α with the corresponding connective from C_2. A consequence relation, $\vdash^{12}$, for $\text{Sent}(C_{12}, P_{12})$, *partially collapses* just in case for all $\Gamma \subseteq \text{Sent}(C_1, P_1)$ and $\alpha \in \text{Sent}(C_1, P_1)$, $\Gamma \vdash^{12} \alpha$ just in case $f(\Gamma) \vdash^{12} f(\alpha)$.

From the definition, it is clear that if $\vdash^{12}$ weakly collapses, then it partially collapses[29]. So, both $\vdash^{cc}$ and $\vdash^{ii}$ partially collapse. It should also be clear that $\vdash^{ic}_{WCE}$, $\vdash^{ic}_{CE}$ and $\vdash^{ic}_{SCE}$ partially collapse. Just notice that with every propositional variable being from P_c, the intuitionistic connectives will take as arguments only $\top_i$ and $\bot_i$. Then, intuitionistic connectives will behave exactly like the classical truth-value functions over the two-element Boolean algebra, for which classical logic is strongly determined. This is, exactly,

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[29] Just notice that, in this definition, the sentences in $\Gamma \cup \{\alpha\}$ are from $\text{Sent}(C_1, P_1)$ and in the definition of weak collapse are from $\text{Sent}(C_1, P_{12})$.

the sense in which the **(W/S)CE**-consequence relations partially collapse for intuitionistic and classical logics. If in a classically valid inference, for Sent(C_c, P_c), we substitute every classical connective for its corresponding intuitionistic connective, then this new argument is also valid.

This result together with the localist interpretations of **SCE**, **CE** and **WCE** that I have suggested, open up the space of possibilities for a more radical and, maybe, more interesting system for certain localist conceptions. Let me conclude the section with a tentative approximation to the idea. Notice that we have obtained a controlled partial-collapse result for the new systems that drop uniform substitution. But, then, we could wonder, do we need the classical connectives any more? In a sense, the only difference between the classical and intuitionistic connectives was that they were operations defined over different classes of algebras but, from a more abstract point of view, they were essentially the same[30]. Now, we have developed a way in which the same connective, defined over a unique class of algebras, can express the two behaviours in virtue of the propositional variables that occur under its scope. When the propositional variables are classical we force the operations to have as their domain and range the $\top_i$ and $\bot_i$ of any Heyting algebra. That is, we have embedded the two-element Boolean algebra in every Heyting algebra and restricted the operations to that algebra when the propositions are classical. Let me illustrate the idea by applying it with the semantic criterion of **SCE**, for instance, to the mixed inferences considered by Wrenn.

What changes with respect to the previous presentation of **SCE** is just the language. Instead of having two stocks of connectives we will only have one, $C = \{\neg, \wedge, \vee, \rightarrow\}$, defined in a standard way as operations in a Heyting algebra, such that for any elements $a, b \in \mathrm{HA}$:

$$\Phi(\neg)(a) = a \Rightarrow \bot$$

$$\Phi(a, b)(\wedge) = a \sqcap b$$

$$\Phi(a, b)(\vee) = a \sqcup b$$

$$\Phi(a, b)(\rightarrow) = a \Rightarrow b$$

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[30] Notice, for instance, that despite $p \rightarrow_i q$ not being in general equivalent to $\neg_i p \vee_i q$, they are equivalent when p and q take regular values.

With $\sqcap, \sqcup, \Rightarrow$ as the infimum, supremum and implication operations in the Heyting algebra. Then, we translate the mixed inferences as follows:

- **Mix**: $q_c \vee \neg p_i,\, p_i \vDash_{SCE} q_c$
- **Essential Mix**: $\neg\neg p_c \vee \neg q_i,\, q_i \vDash_{SCE} p_c$
- **Disjunctive Mix**: $\neg\neg p_c \vee q_i \vDash_{SCE} p_c \vee q_i$
- **Nix**: $\neg p_i \to \neg q_c,\, q_c \nvDash_{SCE} p_i$

Let me prove that those consequence claims hold for the more interesting cases of Essential Mix, Disjunctive Mix and Nix.

- $\neg\neg p_c \vee \neg q_i,\, q_i \vDash_{SCE} p_c$

Proof. Suppose that $||\neg\neg p_c \vee \neg q_i||_x = ||q_i||_x = \top_x$. Since, $||q_i||_i = \top_i$, then $||\neg q_i||_i = \bot_i$. So, in order for $||\neg\neg p_c \vee \neg q_i||_i = \top_i$, $||\neg\neg p_c||_i = \top_i$. Since, p_c is a classical propositional variable, if $||p_c||_c = \top_c$, then $||p_c||_i = \top_i$ and if $||p_c||_c \neq \top_c$, then $||p_c||_i = \bot_i$. Therefore, in order to have $||\neg\neg p_c||_i = \top_i$, we need $||p_c||_i = \top_i$. □

- $\neg\neg p_c \vee q_i \vDash_{SCE} p_c \vee q_i$

Proof. Suppose that $||\neg\neg p_c \vee q_i||_i = \top_i$. This can only be either because $||q_i||_i = \top_i$ or $||\neg\neg p_c||_i = \top_i$[31]. If it is because $||q_i||_i = \top_i$, then $||p_c \vee_i q_i||_i = \top_i$. If it is because $||\neg\neg p_c||_i = \top_i$, then $||\neg\neg p_c||_i = ||p_c||_i = \top_i$, so $||p_c||_i = \top_i$ and $||p_c \vee_i q_i||_i = \top_i$. □

However, we can still invalidate the intuitively invalid argument Nix because, in this case, the double negation of the antecedent that we get by contraposition affects an intuitionistic propositional variable.

- $\neg p_i \to \neg q_c,\, q_c \nvDash_{SCE} p_i$

Proof. We are going to build a **SCE**-countermodel. Take the following **SCE** Boolean-Heyting structure,

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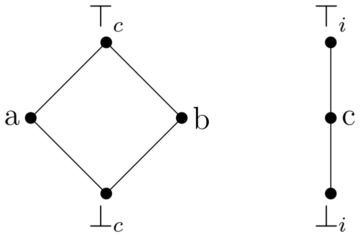
[31] Again, because $\neg\neg p_c$ is a stable formula and it can only take the values top or bottom in the Heyting algebra.



together with the valuations $V_x(q_c) = \top_x$, $V_i(p_i) = c$, $V_c(p_i) = a$. Since $||q_c||_i = \top_i$, $||\neg q_c||_i = \bot_i$. So, with $||p_i||_i = c$, $||\neg p_i||_i = \bot_i$ and, therefore, $||\neg p_i \to \neg q_c||_i = \top_i$, $||q_c||_i = \top_i$ but $||p_i||_i = c$. $\square$

Hence, we get a solution to the dilemma raised by Wrenn in which a single stock of connectives is used, namely, intuitionistic connectives, but that is able to capture classical reasoning since those intuitionistic connectives can behave classically when having regular values under their scope. This versatility of the connectives for showing different behaviours in a systematic way is due to the semantic clauses translating the semantic values between Boolean and Heyting structures, which force the rejection of uniform substitution.[32]

One could argue that losing uniform substitution is a major cost compared with the technical and philosophical improvements that these embedded systems offer. However, I believe that there are good reasons to think that it is natural that the combined system most faithful to a localist philosophy of logic does not have uniform substitution. One of the motivations for localism is to have the propositions classified and separated in virtue of the domains to which they belong. Thus, if we could substitute any sentence for any propositional variable, we would be dissolving the localism, because we would have that an inference is valid with, say, classical propositional variables and that it remains valid when intuitionistic formulas are replaced for

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[32] Maybe it is possible to dispense with the Boolean structures altogether, since in this case their only role is to assign a semantic value to the classical propositional variables, which later on is used to extend the intuitionistic valuations complying with the semantic clauses of the given combination mechanism. But, if the aim is to embed the two-element Boolean algebra in any Heyting algebra and to restrict the possible semantic values of the classical propositional variables to those two, i.e. $\top_i$ and $\bot_i$, maybe we can put the constraint directly in the valuations for the intuitionistic models, in such a way that the range of the valuation function is $\top_i$ and $\bot_i$,when it takes as argument any of the new classical propositional variables with which we have extended the original intuitionistic language.

the classical variables, even if the argument is invalid in intuitionistic logic, which is supposed to be the logic of the intuitionistic variables.

Lastly, let me point out that this combined system with a single stock of connectives suggests another interesting philosophical interpretation. As we advanced in section 2.1.3, the idea that localism, as a form of relativism, goes hand in hand with meaning-variance at connective level is a reasonable and widely accepted thesis. However, our last system challenges this common idea. If one interprets that the meaning of the connective is given by the abstract operations that each connective performs in the algebraic structures (in our case, Heyting and Boolean structures), then the operations are just the same and, therefore, the meanings are the same. What changes are the objects, i.e., the elements of the domains with a certain order, that we allow to fall under those operations. Of course, this might be the case just because of the fact that Heyting algebras generalize Boolean ones. In a situation in which the operations are defined over classes of algebras which are not included one in the other meaning-variance of the corresponding connectives might be harder to resist. For instance, it might be harder to argue that the meaning of a paraconsistent negation, which does not validate explosion, is the same as the meaning of an intuitionistic negation, which does not make excluded middle tautological.

But, maybe the important thought behind these ideas is that meaning-variance is a matter of degree and not a dichotomous issue. We might find quite intuitive that the meaning of negation is the same, or very similar, when it has all of its Boolean algebraic properties and when it lacks one of them, for instance, the property of being an involution with period two ($\neg\neg p \equiv p$). But if we continue removing algebraic properties to the point where $\neg$ is not even order reversing or antimonotone (if $x \leq y$, then $\neg y \leq \neg x$) it gets progressively harder to defend that the meaning of negations is just the same.

4.2.9 Responding to the challenge

The problem of mixed inferences and collapse theorems, as challenges to logical localism, have been considered as knockdown arguments. The challenge of mixed inferences was formulated in its most precise and systematic form by Wrenn (2018) and summarized as a dilemma: keep some modest criterion of validity for the mixed inferences, not counting some intuitively valid arguments as valid, or go beyond modesty and accept as true some untrue sentences.

I reckon that the methods for combining logics have been, mostly, unknown to many who have developed the challenges and to those who, in addition, have invoked collapse theorems as a further limitation to any possible solution. As we have seen, even the existing methods like juxtaposition, which were not designed with the purpose of answering the problem of mixed inferences, might be used to give reasonably good solutions to the problem and, more concretely, to the dilemma raised by Wrenn. That is, even without going beyond minimality in the combination process, we get a combined system which is modest enough to invalidate arguments like Nix (and not collapse), but which goes beyond the modesty that results from the criterion of taking the intersection of the logics in play.

However, the problem of mixed inferences also shows that there might be some bridge principles that, depending on the logics being combined and their philosophical interpretations, are reasonable and justified. Therefore, our best combination mechanism should allow for the emergence of this kind of bridge principles whereas juxtaposition, just as well the other combination mechanisms in the literature (to my knowledge), simply are not appropriate for this task, because they were not developed with these problems in mind. In fact, notice that the problem of mixed inferences, and, more specifically, the desirable bridge principles that we have found, constitute a type of *anti-collapse* problem for juxtaposition and for any combination mechanism that seeks the minimal conservative extension of the logics being combined.

Thus, I have developed different ways of combining intuitionistic and classical logics, in order to analyse alternative mechanisms that go beyond the minimality of juxtaposition. In fact, these methods allow the emergence of different sets of bridge principles and, therefore, are alternative solutions to the problem of mixed inferences. One positive and exciting feature of these methods is that we do not get the desired bridge principles just by adding them as new *ad hoc* axioms, as happened with some of the methods for combining modal logics. Instead, we fulfil Schurz (1991)'s desideratum of obtaining the desired bridge principles, while avoiding the problematic ones, in the very combination process, just in virtue of the semantic clauses that constrain the possible models for each method (and, also, in virtue of the relaxed natural deduction rules in the case of coordination, strong coordination and strong coordination with embedding).

However, since every strengthening of juxtaposition that we have done stands as an alternative solution to the problem of mixed inferences, it should be possible to weigh which one of them provides a more satisfactory answer. But, as I have been pointing out through the presentation of the methods, it appears to be quite tricky to get our heads around the different intuitions. So, while each one of them, like juxtaposition, is able to validate some version of the intuitively valid mixed inferences (Mix, Essential Mix, Disjunctive Mix) and invalidate both versions of the intuitively invalid one (Nix) -avoiding, moreover, the collapse- each of them allows for different sets of bridge principles. Thus, I believe that the philosophical analysis of the methods can be done at two levels, at least, namely: the level of the semantic clauses and the level of the bridge principles that each one allows. That is, we can look at the philosophical justification of the amount of structure preservation that the semantic clauses allow, between the intuitionistic and classical semantic values, and we can look, on top of that, to the consequences of those clauses, i.e., the bridge principles that each method allows, and their philosophical justification.

If one focuses on the semantic clauses, there is one feature that distinguishes strong coordination among the other systems: it is the only one whose semantic clauses are symmetric. That is, the same amount of information/structure is transferred between the classical and intuitionistic models with respect to the values of sentences. This, I believe, is a weird feature given the relationship between classical and intuitionistic logic, i.e., given that the class of Boolean algebras is included in the class of Heyting algebras. It could be that the bridge principles that these clauses allow to emerge justify the symmetry, but given that the system **SCE** extends strong coordination and that **CE** has very similar bridge principles[33], it seems difficult to justify that symmetry against its implausibility.

There is another obvious feature of the semantic clauses that singles out some systems. It is the feature that the systems with Embedding have, i.e., **SCE**, **CE** and **WCE**, namely, that some clauses do not affect every sentence in the combined language, but just some sentences with concrete syntactic features (more concretely, syntactic features that aim to capture their classical or intuitionistic origin). This is, in fact, the reason why the systems drop uniform substitution. As I have just commented in the previous section, these systems might be the ones that better fulfil the localists desiderata. The reason being that the validity of an inference, involving the connectives

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[33] **CE** lacks $\neg_c p_i \vDash^{ic} \neg_i p_i$, but as I said above, there are some reasons for challenging that inference.

of a logic, depends on the type of formulas that occur in the inference, i.e., on whether they belong to the classical or the intuitionistic domains. So, if one takes seriously the idea that some principles of reasoning are local, one might seek a combined system in which one cannot 'blindly' substitute a variable for any type of formula.

With respect to the bridge principles that each system allows to emerge the analysis is even more subtle, but let me comment on some distinctive cases. First, I take it that having the two versions of Disjunctive Mix is a point in favour of **SCE**, **CE** and **WCE**. This is because, even if the disjunction is intuitionistic, it is reasonable that the intuitionistic part of the system should defer to the classical part when the semantic value of a classical formula is at issue. Therefore, we should expect that the intuitionistic part of our systems agrees with the classical semantic fact that a classical formula and its classical double negation have the same semantic value. This is something that we have achieved by imposing more structure preservation in the semantic clauses and dropping uniform substitution.

Second, in order to compare the systems dropping uniform substitution, I have pointed out some inferences that could be crucial for evaluating their plausibility. Among these, I reckon that the most insightful ones are $\neg_i p_c \vDash^{ic}_{SCE/CE} \neg_c p_c$ and $\neg_c p_i \vDash^{ic}_{SCE} \neg_i p_i$. As I have tried to show above, there are good reasons for thinking that the proof for the absurdity of a classical proposition should imply that the proposition is not the case (recall the example with boxes and candies) and, so, for considering $\neg_i p_c \vDash^{ic}_{SCE/CE} \neg_c p_c$ a justified and desirable bridge principle. This would be an advantage for preferring both **SCE** and **CE** and a problem for **WCE**, which invalidates the argument.

On the other hand, I have cast doubt on the plausibility of $\neg_c p_i \vDash^{ic}_{SCE} \neg_i p_i$. But, now, I would like to offer an alternative point of view[34]. I argued above that not being the case that there is a construction of p_i should not imply that there is a construction of $\neg_i p_i$, but this relied on an implicit assumption about the nature of the epistemic constraint, namely, that there might be eternal gaps of knowledge[35]. If the nature of the epistemically constrained domain is such that there might be those eternal gaps, then not being the case that there is a construction of p_i does not imply that

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[34] Thanks to my supervisor, Elia Zardini, for pointing out this view.

[35] Eternal gaps of knowledge meaning that for certain proposition, p, there is, in principle, no proof of it nor of its negation.

there is, even in principle, a construction of $\neg_i p_i$, since p_i could be one of those propositions in the knowledge gap. However, there might be other epistemically constrained domains such that there are no knowledge gaps. One could argue that the ethical domain is like that and that, for every ethical proposition it is impossible that both the proposition and its negation lack a proof in principle. So if a proposition lacks a proof, then its negation has it. In that case, then, the fact that it is not the case that there is, in principle, a proof of p_i does imply that there is, in principle, a proof of $\neg_i p_i$. Therefore, one would consider that the combined system that better captures the justified bridge principles is **SCE**.

However, the most interesting upshot is not that **SCE** might be the best system to solve the problem of mixed inferences. I reckon that it is more striking to realize that our combination mechanisms permit us to extract more fine-grained information about the domains by analysing, among other things, the bridge principles that they allow. Notice that we have been putting on a par any epistemically constrained domain, under the assumption that whether it was an evaluative domain or the domain of mathematics or whatnot, their logic was best captured by intuitionistic logic. But intuitionistic logic alone was not telling us anything about the nature of the epistemic constraint with regard to possible eternal gaps of knowledge. The issue only arose when combining those possible epistemically constrained domains with a classical domain. That is, when wondering about the justification for meaningful interactions between the classical and intuitionistic connectives.

This suggests that whether a combination mechanism is correct or not is a more nuanced question than just looking at the logics that are in play, since the justification of the bridge principles also has to do with our philosophical conceptions of the domains and their properties.

Before concluding this chapter, let me notice that the methods that I have developed, as well as juxtaposition, have a quite straightforward interpretation as possible answers to the mixed compounds/inferences versions of the challenges directed against alethic pluralism.

4.2.10 Mixed compounds and inferences for alethic pluralism (killing the second bird)

I take it that the problem of mixed inferences for alethic pluralism is solved if my systems for combining logics are good candidates in order to meet Wrenn's challenge. More concretely, although the combination mechanisms are new and aim at bridge principles that had not been considered previously, the semantics are basically standard algebraic semantics together with functions that restrict the possible ways of extending the valuations to the expanded language. So, as long as Tappolet regards Heyting algebras as appropriate semantics for the intuitionistic consequence relation, then my mixed semantics should satisfy Tappolet's desideratum of keeping a standard notion of consequence.

With respect to mixed compounds, since we are working in a language with two stocks of connectives -one for each logic being combined-, the question about the sense in which a mixed compound is true has to be more nuanced. However, the procedure for calculating the truth-values is clear in every method, since the connectives are still truth-functional in spite of there being two copies of each. Therefore, technically we have no difficulties; we have a systematic way of determining the semantic value of every sentence in the language. What is nuanced is the semantic/philosophical interpretation of the procedure, so let me propose the reading that I find most plausible.

I will begin by giving the general picture of the mechanisms. Let's assume that we are in the more straightforward situation in which no connectives are shared and the sets of propositional variables are disjoint. That is, we have a set of classical propositional atoms and a set of intuitionistic atoms. The first thing to notice is that, despite the fact that we distinguish between classical and intuitionistic formulas, all of them receive both classical and intuitionistic semantic values. This might sound as contrary to alethic pluralism at first glance. But, I reckon that there is a natural way of explaining this for juxtaposition and the extensions: each formula gets its semantic value (namely, classical (intuitionistic) formulas, classical (intuitionistic) semantic values) from the class of algebras that better models the truth-property of the domain to which the formula belongs. That is, the fundamental semantic value of a classical formula is a value of a structure belonging to the class of Boolean algebras, which are supposed to better model the correspondence truth-property. Thus, the classical models are somehow the 'authority' for determining the semantic value of the classical formulas.

But, as I said, classical formulas also get a semantic value from a (potentially non-Boolean) structure belonging to the class of Heyting algebras. However, this semantic value is given as if the classical formulas were intuitionistic atoms, with no interpretation of the connectives occurring in them. This intuitionistic valuation is done in virtue of the classical semantic value that the classical formula gets in its classical model. For juxtaposition the only restriction was coherence and, as I have shown, other mechanisms are obtained by further restricting the possible valuations. What seems clear, though, is that the intuitionistic value of the classical formula is less fundamental, or more derivative, than its classical semantic value. That is, the intuionistic models are not the authority for determining the value of the classical formula but, on the contrary, they defer to the classical models and expand the valuation function following (more or less restrictively) what the classical model dictates. Likewise, models based on the class of Heyting algebras are the authority for determining the semantic value of intuitionistic formulas and those formulas get the classical value as if they were further atoms that are added to the classical language. Thus, the classical valuation for those intuitionistic formulas will have to defer to the interpretation that the intuitionistic models yielded.

So, the fundamental semantic value of a formula α such that $\alpha = c\alpha^1 ... \alpha^n$ with $c \in C^n$ or $\alpha \in P$ is that given by the semantics of the logic $\mathcal{L}$ that is the logic of C and P, while the derivative semantic value allows to determine the semantics of mixed formulas. That is, when α occurs under the scope of a connective, c_i, that is not from C, the derivative semantic value of α is what c_i is going to take as an argument to give a semantic value belonging to the semantics of its logic. Let me focus on some cases concerning classical-intuitionistic compounds in order to illustrate the intended interpretation. In order to simplify the explanation, let us assume that we are working with the method of coordination. The idea behind the interpretation that I want to suggest, is that we should make a second-order reading of the truth-property that corresponds to a mixed formula: e.g., consider again the inference $\vdash^{ic}_{C} p_i \vee_c \neg_c p_i$. Since the intuitionistic propositional variable occurs under the scope of a classical connective (the disjunct on the left under the scope of $\vee_c$ and the propositional variable on the right disjunct under the scope of $\vee_c$ and $\neg_c$) we have to read the correspondence truth-property as a second-order property of the intuitionistic atoms which have, say, 'having a construction/proof' as their fundamental truth-property. Thus, the interpretation of the mixed compound would be 'either it is a fact that there is a construction of p_i or it is not a fact that there is a construction of p_i' and the fundamental semantic value in every coordinated model of this mixed compound, with a classical main connective, is the top value of a Boolean algebra.

More generally, an interesting question is how do we understand that the truth-value of $p_c \vee_c q_i$ is $\top_c$ (modelling correspondence truth) when the value of p_c is $\bot_c$ and the value of q_i is $\top_i$ (modelling 'constructive truth') and, derivatively, $\top_c$. The answer is, again, to interpret the mixed formula with a second-order reading of correspondence: 'either it is a fact that p_c or it is a fact that there is a construction of q_i (that is, it corresponds to reality that there is a construction of q_i)'. Given the semantics of coordination, we know that if the value of q_i is $\top_i$, then, from coherence, its semantic value in the classical model will be $\top_c$ and, therefore, the classical disjunction will also be $\top_c$.

Similarly, when the main connective is intuitionistic, we should make a second-order reading of the intuitionistic truth-property of having a construction. For instance, take $p_c \wedge_i q_i$. We should read this sentence as 'there is a proof that it is a fact that p_c and there is a proof of q_i'. Clearly, I am assuming that what constitutes a proof is different for classical and intuitionistic propositions. In a sense, what constitutes a proof of the classical proposition is that according to the classical model of the world that proposition is the case. So, maybe, in order to make it clear that a proof of that kind is not a proof at the object level, but more a deference or an acceptance of a testimony from the classical model, we could interpret $p_c \wedge_i q_i$ as 'there is a proof that according to the classical world 'p_c' and there is a proof of q_i'.

Notice that this solution does not add further truth-properties on top of those of the propositional variables that occur in each sentence. If we are doing the coordination of classical and intuitionistic logic, for any mixed formula, the only truth-properties that we need in order to determine the truth-value of any mixed compound are correspondence and an epistemically constrained truth-property (like having a construction or coherence, for instance). Moreover, we do not have the level of indeterminacy that worried Tappolet, with respect to the relevant truth-value of mixed compounds. In our system (let me continue assuming coordination) each formula has a main connective that is either from C_c or C_i. So, each formula has a determinate fundamental truth-property and a determinate derivative truth-property. This is because each connective is an operation over a determinate class of algebras.

Tappolet could respond, similarly to what she said responding to Beall (2000) in Tappolet (2000), that the property of being designated on a coordinated model is playing the role of a generic truth-property and that, therefore, the only property that we need is the generic one because the others are redundant. I believe, though, that this objection is not right. To begin with, Tappolet should explain what the connection is between that generic truth-property and the technical concept of being designated in a structure (in our case, in a coordinated structure). Correspondence truth-property and constructions have been modelled by the semantics of classical and intuitionistic logics, respectively. But, the fact that these semantics are the optimal ones for modelling those truth-properties is a substantive philosophical claim, involving theories about negation as complementation, consequence as truth-preservation, verificationist theories of meaning, etc. However, Tappolet is asking to abandon all that, because one can avoid the talk about specific-truth properties by simply using the notion of designation. That is, according to Tappolet, the specific truth-properties are redundant because designation encompasses them.

Nevertheless, the contrary seems to be more plausible, as Cotnoir (2009) also notices: the generic truth-property modelled by designation is redundant and much less fine-grained and informative than the specific truth-properties. On one hand, being designated is dependent on having a specific truth-property. Not only in the order of explanation, but in the ontological order: a formula is designated in virtue of having the truth-property T_x, but it does not have the specific property T_x in virtue of being designated. On the other hand, in order to capture the semantics of a specific-domain, not only truth is important, but also falsity and other intermediate semantic values. An epistemically constrained truth-property can be adequately modelled by an intuitionistic semantics that explains why for certain proposition we might not have justification for it nor for its negation. But, if we just take designation we lose this kind of nuances that the various alternative semantics might exhibit. We have no order or degree of designation and this makes the concept very coarse for modelling something as subtle as truth for mixed compounds.

Thus, since we have a systematic way of mixing semantic values and there is no ambiguity in our systems between which the fundamental and derivative truth-properties are for each sentence, it is more reasonable to keep the non-dependent and more fine-grained properties and regard designation as a technical concept with no modelling purpose. Moreover, I have given a reasonable interpretation of the mixed sentences that only appeals to the specific-truth properties and not to designation. Of course, a more systematic criterion for interpreting any candidate mixed compound would be desirable. But I reckon that my developments in the dissertation are a good starting point.

Chapter 5

Final Remarks

The main goal of this dissertation was to account for the problem of mixed inferences by applying and improving on the method of juxtaposition for combining logics. I have mainly focused on the version of the problem presented by Wrenn and directed against logical localism, but the possible solutions that I have proposed go beyond the specifics of Wrenn's challenge.

The philosophical literature dealing with the problem of mixed inferences, and related problems for alethic and logical pluralisms, had not considered combination mechanisms as useful technical tools that could be applied in order to handle the philosophical challenges. Similarly, the literature on combining logics had been interested in philosophical issues but not in the problem of mixed inferences. However, I reckon that one of the most interesting conclusions of my dissertation is that both areas of inquiry highly benefit from the interplay of the different analyses and results of each other. The philosophical side can obviously take advantage of the methods for combining logics, because these can systematically account for what follows from what in arguments involving multiple logics. But, also, the methods for combining logics benefit from the philosophical discussion on mixed inferences, since that discussion can yield justified and desirable mixed inferences that pose an anti-collapse problem for certain combination mechanisms, as I have shown.

In fact, this has been the dialectics that has guided my different improvements on the method of juxtaposition. That is, I have been modifying the semantics of the method in order to capture the mixed inferences that seemed to be reasonable and justified in virtue of the philosophical analyses of the logics being combined and their connections (in my case, the meaningful connections between classical and intuitionistic logics). Nevertheless, I am convinced that the potential of the inquiry that I have started here extends much further and that I have only started scratching the surface of it. Thus, I would like to conclude by looking at various promising research lines both on the philosophical and on the technical sides.

Starting with the philosophical aspects, one of the tasks that I want to carry out is to develop a localist theory of logic inspired by some of the coordinated systems. That is, every localist theory that exists in the literature has been proposed without having in mind the methods for combining logics. Thus, it is to be expected that a localist theory that stems from one of the methods, or some of them, will be better equipped against the typical challenges to localism. The biggest difficulty for developing such a localist theory is that we do not know *a priori* which are the logics that will be regarded as the best candidates for capturing reasoning in different domains. It could be that there are few correct logics and similar enough as to be captured by a single combination mechanism. But it is also possible that there are more than a few and so varied that different combination mechanisms are needed.

Regardless of whether there are few or many correct logics, it will be important to improve on the criteria that I have been using for comparing coordination methods by introducing more systematicity. Of course, the comparison will depend on the logics that we are combining, since, as I argued above, it is very likely that for different logics the bridge principles that we want to get will differ. However, even restricting to classical and intuitionistic logics, I reckon that a more systematic way of evaluating each method should be developed. When comparing the methods, I have focused primarily on the negations and the bridge principles involving negations because the semantic clauses that I have considered for going beyond juxtaposition had the most obvious consequences for those bridge principles. However, I guess that a systematic criterion for comparing combination methods should focus on the other connectives and the potential bridge principles concerning them as well. For instance, a bridge principle involving disjunction, like $p_i \vee_i q_i \vdash p_i \vee_c q_i$ seems quite reasonable[1], since having a proof either for p_i or for q_i could arguably imply that either p_i or q_i is the case.

With respect to the technical aspects left for future work, the most obvious one is to fine-tune the natural deduction calculi that I have proposed for

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[1] Notice that we get a version of this principle in **SCE**, **CE** and **WCE** when the propositional variables are classical.

coordination, strong coordination and strong coordination with embedding, in order to get the calculi that could be proved to be complete. I believe that the completeness proofs for coordination and strong coordination should be fairly similar to Schechter's completeness proof for juxtaposition, since his proof is in general for any juxtaposed consequence relation and coordination and strong coordination are juxtaposed consequence relations. So, one should check that the Lindenbaum-Tarski models resulting from the construction meet the additional semantic restrictions of each method, on top of coherence. Moreover, I would like to get some further preservation results for the coordinated methods and analyse which additional interactions or bridge principles lead to collapse.

Closely related to this last remark is the task of trying to combine **CL** and **IL** (and hopefully other logics too) in a way that we can have shared connectives[2]. One could argue that the fundamental differences between these logics arise because of the negation and the conditional, so, maybe, we could study which combination mechanisms yield meaningful interactions when the language consists of, say, two negations and two conditionals (one for each logic) while the other connectives are shared. The idea is that we would seek the least redundant language possible while still being able to express the relevant differences between the systems and capture the meaningful interactions. In fact, the interactions that I have analysed in the case of **SCE**, for instance, suggest another possibility which, interestingly enough, could be a good candidate for being a strongly determined semantics for Prawitz's ecumenical system. In **SCE**, we have that $\neg_x$ and $\wedge_x$ are intersubstitutable as main connectives and moreover, $\neg_x \neg_x \phi_s \vDash^{ic}_{SCE} \phi_s$ whenever ϕ_s is stable. So, maybe we could modify the language in such a way that we share the negation and the conjunction while still having two versions, classical and intuitionistic, of the other connectives. This is very similar to the language that Prawitz chooses for ecumenism. As I said above, ecumenism's motivation was quite different from the motivation I had for combining logics. But if some of the semantics that I have developed here are adequate for Prawitz's combined calculus that would be another interesting application of coordinated methods.

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[2] In fact, what I tentatively explored at the end of Chapter 4 was the limit case of having just one stock of connectives. So, in a sense, all the connectives were shared despite the fact that they were defined generally in the class of Heyting algebras, because we were able to restrict the operations to the two-element Boolean algebra when classical propositions were involved.

Moreover, when thinking about ways of going beyond juxtaposition and inducing the emergence of desired bridge principles, one of the possibilities that came up in discussion with my supervisors was to restrict the Heyting algebras that I was considering for the intuitionistic semantics. A reasonable and promising option is to consider just linearly ordered Heyting algebras. Dummett (1959) showed that this logic, the logic of all finite chains, is the logic that extends intuitionistic propositional logic with the axiom $(p \to q) \vee (q \to p)$, also known as *Gödel-Dummett logic*. So, we would not be combining **CL** and **IL**, exactly, but something a little stronger than **IL**. Of course, one would have to justify which the domain of application of that logic is and, perhaps also, how one should interpret the logic or why it is the correct model for preserving certain truth-property. In any case, if we combined Gödel-Dummett logic with classical logic using **SCE**, for instance, we would get the bridge principle involving disjunction that I just referred to above, $p_i \vee_i q_i \vdash p_i \vee_c q_i$. A quick way of seeing this is noticing that for every finite chain, the supremum of two elements is the top only if one of the elements is the top. So every **SCE**-model satisfying $p_i \vee_i q_i$ is a model such that either $||p_i||_i = \top_i$ or $||q_i||_i = \top_i$ and by coherence, then, every **SCE**-model satisfying $p_i \vee_i q_i$ is a model in which $||p_i||_c = \top_c$ or $||q_i||_c = \top_c$. Hence, every **SCE**-model satisfying $p_i \vee_i q_i$ is a model satisfying $p_i \vee_c q_i$. Thus, we can see that this restriction on the class of Heyting algebras could be a way of inducing new and desirable bridge principles. I reckon this is an idea worth exploring.

In a similar vein, another quite straightforward task for future work is to develop coordinated methods for combining logics other than **CL** and **IL**. In Chapter 3, I already did the juxtaposition of **CL** and **LP**, so the way of going beyond their juxtaposition would be to study which type of bridge principles are justified and desirable for the coordinated systems, so as to be able to modify the semantic clauses accordingly. A reasonable guess, for instance, is that if a classical formula gets the top value in the classical models, then it should get the top value of the Kleene algebra, not any designated value. This would give us bridge principles like $q_p \vee_c \neg_p p_c, p_c \vdash^{cp} q_p$, which is *prima facie* plausible even for a defender of paraconsistency who accepts that propositions belonging to the classical domain are not susceptible of being designated if their negation is designated.

Thus, there is a wide range of logics that could be combined and each combination will probably require different mechanisms depending on the bridge principles that seem reasonable for the various combinations. Moreover, notice that we are not limited to combining just pairs of logics. The methods that I have obtained by modifying juxtaposition, and juxtaposition itself, allow the combination of more than two logics. Suppose, though, that we have three logics $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ and that we want to go beyond minimality when combining them. It could be that the bridge principles that are reasonable when interacting $\mathcal{L}_1$ and $\mathcal{L}_2$ are different from those of $\mathcal{L}_1$ and $\mathcal{L}_3$ or, in fact, any other combination. This is an interesting situation to further investigate, since we would have that the semantic clauses relating the algebraic values of the sentences of each logic vary. That is, we could have that the semantic clauses between $\mathcal{L}_1$ and $\mathcal{L}_2$ are those of, say, **SCE** while those of $\mathcal{L}_1$ and $\mathcal{L}_3$ correspond to, say, **WCE**.

Finally, my starting point has been juxtaposition and every system that I have developed has been an extension of it. That is, I have built up on top of the semantic clause of coherence in order to add more constraints on the possible models for the systems. However, it is possible and, certainly, worth investigating that the rejection of coherence could result in other interesting systems. Maybe in systems that allow for even more accurate localist interpretations. For the case of combining **CL** and **IL**, one of the heuristics that I tried to put forward in order to justify and compare the different mechanisms was that the standard of truth for intuitionism (i.e., constructive proof) is higher than the classical standard of truth (i.e., correspondence). This seemed to justify that when we evaluate a formula as bottom in the Heyting algebra, we should evaluate that same formula as bottom in the Boolean algebra. However, the other direction seemed to be more problematic, at least in general for every formula of the combined language. Thus, we could maybe follow this idea even further and analyse what would happen if we have only one direction of coherence. That is, that if a formula is evaluated as top in the Heyting algebra, then it has to be evaluated as top in the Boolean algebra, but not the other way around. In fact, we could be even more meticulous and restrict it to just intuitionistic formulas.

The upshot is that there are a number of plausible mechanisms that can be still developed. Some of them might be relevant for the problem of mixed inferences, but it is possible that other philosophical problems require different mechanisms to address them. As I said, this dissertation is a first step in the direction of exploiting the logical tools that the combination methods offer. But, I reckon that the step is important since it covers a central philosophical problem involving two of the most salient logics in the history of the field.

Appendices

A Natural Deduction Calculi for IL, CL, K$_3$ and LP

1 Natural Deduction Calculus for IL

$(\bot_i\mathrm{E})$
$$\begin{array}{|l} \bot_i \\ \hline \alpha \end{array}$$

$(\neg_i\mathrm{E})$
$$\begin{array}{|l} \alpha \\ \neg_i\alpha \\ \hline \bot_i \end{array}$$

$(\neg_i\mathrm{I})$
$$\begin{array}{|l} \quad\begin{array}{|l} \alpha \\ \hline \vdots \\ \bot_i \end{array} \\ \neg_i\alpha \end{array}$$

$(\wedge_i\mathrm{E1})$
$$\begin{array}{|l} \alpha \wedge_i \beta \\ \hline \alpha \end{array}$$

$(\wedge_i\mathrm{E2})$
$$\begin{array}{|l} \alpha \wedge_i \beta \\ \hline \beta \end{array}$$

$(\wedge_i\mathrm{I})$
$$\begin{array}{|l} \alpha \\ \beta \\ \hline \alpha \wedge_i \beta \end{array}$$

$(\vee_i\mathrm{E})$
$$\begin{array}{|l} \alpha \vee_i \beta \\ \hline \quad\begin{array}{|l} \alpha \\ \hline \vdots \\ \delta \end{array} \\ \quad\begin{array}{|l} \beta \\ \hline \vdots \\ \delta \end{array} \\ \delta \end{array}$$

$(\vee_i\mathrm{I1})$
$$\begin{array}{|l} \alpha \\ \hline \alpha \vee_i \beta \end{array}$$

$(\vee_i\mathrm{I2})$
$$\begin{array}{|l} \beta \\ \hline \alpha \vee_i \beta \end{array}$$

$$(\to_i\mathrm{E})\quad \begin{array}{|l} \alpha \to_i \beta \\ \alpha \\ \hline \beta \end{array} \qquad\qquad (\to_i\mathrm{I})\quad \begin{array}{|l} \begin{array}{|l} \alpha \\ \hline \vdots \\ \beta \end{array} \\ \alpha \to_i \beta \end{array}$$

2 Natural Deduction Calculus for CL

$$(\bot_c\mathrm{E})\quad \begin{array}{|l} \bot_c \\ \hline \alpha \end{array} \qquad\qquad (\neg_c\mathrm{E})\quad \begin{array}{|l} \alpha \\ \neg_c\alpha \\ \hline \bot_i \end{array}$$

$$(\neg_c\neg_c\mathrm{E})\quad \begin{array}{|l} \neg_c\neg_c\alpha \\ \hline \alpha \end{array} \qquad\qquad (\neg_c\mathrm{I})\quad \begin{array}{|l} \begin{array}{|l} \alpha \\ \hline \vdots \\ \bot_c \end{array} \\ \neg_c\alpha \end{array}$$

$$(\wedge_c\mathrm{E1})\quad \begin{array}{|l} \alpha \wedge_c \beta \\ \hline \alpha \end{array} \qquad (\wedge_c\mathrm{E2})\quad \begin{array}{|l} \alpha \wedge_c \beta \\ \hline \beta \end{array} \qquad (\wedge_c\mathrm{I})\quad \begin{array}{|l} \alpha \\ \beta \\ \hline \alpha \wedge_c \beta \end{array}$$

$$(\vee_c\text{I1})\quad \begin{array}{|l} \alpha \\ \hline \alpha \vee_c \beta \end{array}$$

$$(\vee_c\text{I2})\quad \begin{array}{|l} \beta \\ \hline \alpha \vee_c \beta \end{array}$$

$$(\vee_c\text{E})\quad \begin{array}{|l} \alpha \vee_c \beta \\ \hline \quad \begin{array}{|l} \alpha \\ \hline \vdots \\ \delta \end{array} \\ \quad \begin{array}{|l} \beta \\ \hline \vdots \\ \delta \end{array} \\ \delta \end{array}$$

$$(\rightarrow_c\text{E})\quad \begin{array}{|l} \alpha \rightarrow_c \beta \\ \alpha \\ \hline \beta \end{array}$$

$$(\rightarrow_c\text{I})\quad \begin{array}{|l} \quad \begin{array}{|l} \alpha \\ \hline \vdots \\ \beta \end{array} \\ \alpha \rightarrow_c \beta \end{array}$$

3 Natural Deduction Calculus for $\mathbf{K}_3$

$$(\wedge_k\text{E1})\quad \begin{array}{|l} \alpha \wedge_k \beta \\ \hline \alpha \end{array} \qquad (\wedge_k\text{E2})\quad \begin{array}{|l} \alpha \wedge_k \beta \\ \hline \beta \end{array} \qquad (\wedge_k\text{I})\quad \begin{array}{|l} \alpha \\ \beta \\ \hline \alpha \wedge_k \beta \end{array}$$

$(\vee_k\mathrm{I}1)$
$$\begin{array}{|l} \alpha \\ \hline \alpha \vee_k \beta \end{array}$$

$(\vee_k\mathrm{I}2)$
$$\begin{array}{|l} \beta \\ \hline \alpha \vee_k \beta \end{array}$$

$(\vee_k\mathrm{E})$
$$\begin{array}{|l} \alpha \vee_k \beta \\ \hline \quad \begin{array}{|l} \alpha \\ \hline \vdots \\ \delta \end{array} \\ \quad \begin{array}{|l} \beta \\ \hline \vdots \\ \delta \end{array} \\ \delta \end{array}$$

$$\begin{array}{|l} \neg_k(\alpha \wedge_k \beta) \\ \hline \neg_k\alpha \vee_k \neg_k\beta \end{array} \qquad \begin{array}{|l} \neg_k\alpha \vee_k \neg_k\beta \\ \hline \neg_k(\alpha \wedge_k \beta) \end{array}$$

$$\begin{array}{|l} \neg_k(\alpha \vee_k \beta) \\ \hline \neg_k\alpha \wedge_k \neg_k\beta \end{array} \qquad \begin{array}{|l} \neg_k\alpha \wedge_k \neg_k\beta \\ \hline \neg_k(\alpha \vee_k \beta) \end{array}$$

$$\begin{array}{|l} \alpha \\ \hline \neg_k\neg_k\alpha \end{array} \qquad \begin{array}{|l} \neg_k\neg_k\alpha \\ \hline \alpha \end{array}$$

$$\begin{array}{|l} \alpha \\ \neg_k\alpha \\ \hline \beta \end{array}$$

4 Natural Deduction Calculus for LP

$(\wedge_p\text{E1})$
$$\begin{array}{|l} \alpha \wedge_p \beta \\ \hline \alpha \end{array}$$

$(\wedge_p\text{E2})$
$$\begin{array}{|l} \alpha \wedge_p \beta \\ \hline \beta \end{array}$$

$(\wedge_p\text{I})$
$$\begin{array}{|l} \alpha \\ \beta \\ \hline \alpha \wedge_p \beta \end{array}$$

$(\vee_p\text{I1})$
$$\begin{array}{|l} \alpha \\ \hline \alpha \vee_p \beta \end{array}$$

$(\vee_p\text{I2})$
$$\begin{array}{|l} \beta \\ \hline \alpha \vee_p \beta \end{array}$$

$(\vee_p\text{E})$
$$\begin{array}{|l} \alpha \vee_p \beta \\ \hline \quad \begin{array}{|l} \alpha \\ \hline \vdots \\ \delta \end{array} \\ \quad \begin{array}{|l} \beta \\ \hline \vdots \\ \delta \end{array} \\ \delta \end{array}$$

$$\begin{array}{|l} \neg_p(\alpha \wedge_p \beta) \\ \hline \neg_p\alpha \vee_p \neg_p\beta \end{array}$$

$$\begin{array}{|l} \neg_p\alpha \vee_p \neg_p\beta \\ \hline \neg_p(\alpha \wedge_p \beta) \end{array}$$

$$\begin{array}{|l} \neg_p(\alpha \vee_p \beta) \\ \hline \neg_p\alpha \wedge_p \neg_p\beta \end{array} \qquad \begin{array}{|l} \neg_p\alpha \wedge_p \neg_p\beta \\ \hline \neg_p(\alpha \vee_p \beta) \end{array}$$

$$\begin{array}{|l} \alpha \\ \hline \neg_p\neg_p\alpha \end{array} \qquad \begin{array}{|l} \neg_p\neg_p\alpha \\ \hline \alpha \end{array}$$

$$\begin{array}{|l} \\ \hline \alpha \vee_p \neg_p\alpha \end{array}$$

B Strong Kleene (K_3) connectives

$\neg_k$	
1	0
1/2	1/2
0	1

$\vee_k$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\wedge_k$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$\rightarrow_k$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	1/2
0	1	1	1

Bibliography

Barrio, E. A., F. Pailos, and D. Szmuc, 2018, “Substructural logics, pluralism and collapse”, *Synthese*. DOI: 10.1007/s11229-018-01963-3

Batens, D. , 1990 “Against Global Paraconsistency”, *Studies in Soviet Thought* 39: 209–229.

Beall, J. C., 2000 , “On Mixed Inferences and Pluralism about Truth Predicates”, *The Philosophical Quarterly* 50: 380–382.

Beall, J. C., and G. Restall, 2000 , “Logical pluralism”, *Australasian Journal of Philosophy* 78 (4): 475–493. DOI: 10.1080/00048400012349751

Beall, J. C., and G. Restall, 2001, “Defending logical pluralism”, in Woods, J., and B. Brown (eds.), *Logical Consequence: Rival Approaches. Proceedings of the 1999 Conference of the Society of Exact Philosophy*, Stanmore: Hermes: 1–22.

Beall, J. C., and G. Restall, 2006, *Logical Pluralism*, Oxford University Press, Oxford. DOI: 10.1093/acprof:oso/9780199288403.001.0001

Benito-Monsalvo, C., 2022, “Local Applications of Logics via Model-Theoretic Interpretations”, *Logic and Logical Philosophy* Online:1–22.DOI: 10.12775/LLP.2022.023

Béziau, J.-Y., 2004, “A paradox in the combination of logics”, in W. A. Carnielli, F. M. Dionísio and P. Mateus (eds), *Proceedings of CombLog’04*, Lisbon: 87–92.

Béziau, J.-Y. and ,M. Coniglio, 2011, “To distribute or not to distribute?” *Logic Journal of IGPL* 19(4): 566–583. DOI: 10.1093/jigpal/jzp084

Brown, M. B. and G. Priest, 2004, “Chunk and Permeate, a Paraconsistent Inference Strategy, Part I: the Infinitesimal Calculus”, *Journal of Philosophical Logic* 33: 379–388.

Caleiro, C., and J. Ramos, 2004, “Cryptofibring”, in W. Carnielli, M. Dionísio, and P. Mateus (eds), *Proceedings of CombLog’04, Workshop on Combination of*

Logics: Theory and Applications, Lisboa: 87–92.

Caleiro, C., W. Carnielli, J. Rasga, and C. Sernadas, 2005, "Fibring of Logics as a Universal Construction", in D., M. Gabbay and F. Guenthner (eds),*Handbook of Philosophical Logic*, volume 13, Kluwer Academic Publishers.

Carnap, R., 1937, *The Logical Syntax of Language*, London: Kegan Paul.

Carnap, R., 1943, *The Formalization of Logic*, Cambridge, MA: Harvard University Press.

Carnielli. W, and M. Coniglio, 2007, "Bridge principles and combined reasoning", in T. Müller and A. Newen (eds), *Logik, Begriffe, Prinzipien des Handelns (Logic, Concepts, Principles of Action)*, Mentis Verlag, Paderborn: 32–48.

Carnielli. W, M. Coniglio, D. Gabbay, P. Gouveia, and C. Sernadas, 2008, *Analysis and Synthesis of Logics*,Volume 35 in the *Applied Logic* Series, Dordrecht: Springer.

Carnielli. W, and M. Coniglio, 2020, "Combining Logics" *The Stanford Encyclopedia of Philosophy*,Edward N. Zalta (ed). URL = <https://plato.stanford.edu/archives/fall2020/entries/logic-combining/>.

Coniglio, M. and V. Fernández, 2005, "Plain fibring and direct union of logics with matrix semantics", in B. Prasad (ed), *Proceedings of the 2nd Indian International Conference on Artificial Intelligence*:1590–1608.

Coniglio, M. , 2007, "Recovering a logic from its fragments by meta-fibring", *Logica Universalis* 1(2): 1–39. DOI: 10.1007/s11787-007-0019-6

Cook, R. T., 2010, "Let a Thousand Flowers Bloom: A Tour of Logical Pluralism", *Philosophy Compass* 5/6: 492–504. DOI: 10.1111/j.1747-9991.2010.00286.x

Cotnoir, A. J., 2009, "Generic truth and mixed conjunctions: some alternatives", *Analysis* 69 (3): 473–479.

Cotnoir, A. J., 2013, "Validity for Strong Pluralists", *Philosophy and Phenomenological Research* 83 (3): 563–579. DOI: 10.1111/j.1933-1592.2011.00544.x

Cozzo, C., 1994, *Meaning and Argument*, Almqvist & Wiksell, Stockholm.

da Costa, N. C. A., 1997, *Logique Classique et Non Classique: Essai sur les Fondements de la Logique,*, Masson, Paris.

del Cerro, L. F. and A. Herzig, 1996, "Combining classical and intuitionistic logic, or: Intuitionistic implication as a conditional", in F. Baader and K. Schulz (eds), *Frontiers of Combining Systems: Proceedings of the 1st International Workshop, Munich (Germany)*, Applied Logic, Dordrecht: Kluwer Academic

Publishers: 93–102.

Dummett, M., 1959, "A propositional calculus with denumerable matrix", *The Journal of Symbolic Logic* 24: 97–106.

Dummett, M., 1977, *Elements of Intuitionism*, Clarendon Press, Oxford.

Edwards, D., 2008, "How to solve the problem of mixed conjunctions", *Analysis* 68: 143–149.

Edwards, D., 2018, "The Metaphysics of Domains", in J. Wyatt, N. J. L. L. Pedersen and N. Kellen (eds.), *Pluralisms in Truth and Logic*, Palgrave Macmillan. DOI: 10.1007/978-3-319-98346-2

Field, H., 2009, "Pluralism in Logic", *The Review of Symbolic Logic* 2 (2): 342–359. DOI: 10.1017/S1755020309090182

Gabbay, D., 1996, "Fibred semantics and the weaving of logics: Part 1", *The Journal of Symbolic Logic* 61 (4): 1057–1120.

Haack, S., 1978, *Philosophy of Logics*, Cambridge University Press, Cambridge. DOI: 10.1017/CBO9780511812866.002

Harris, J. H., 1982, "What's so logical about the 'logical axioms'?", *Studia Logica* 41: 159–171.

Hjortland, O. T., 2013, "Logical Pluralism, Meaning-Variance, and Verbal Disputes", *Australasian Journal of Philosophy* 91 (2): 355–373. DOI: 10.1080/00048402.2011.648945

Hjortland, O. T., 2017, "Anti-exceptionalism about logic", *Philosophical Studies* 174: 631–658. DOI: 10.1007/s11098-016-0701-8

Johnstone, P. T., 1982, *Stone Spaces*, Cambridge University Press.

Keefe, R., 2014, "What Logical Pluralism Cannot Be", *Synthese* 191: 1375–1390.

Kissel, T. K., 2018, "Connective Meaning in Beall and Restall's Logical Pluralism", in J. Wyatt, N. J. L. L. Pedersen and N. Kellen (eds.), *Pluralisms in Truth and Logic*, Palgrave Macmillan. DOI: 10.1007/978-3-319-98346-2

Lynch,M. P., 2008, "Alethic Pluralism, Logical Consequence andnthe Universality of Reason", *Midwest Studies in Philosophy, XXXII*: 122–140.

Lynch,M. P., 2009, *Truth as One and Many*, Oxford University Press, Oxford. DOI: 10.1093/acprof:oso/9780199218738.003.0005

Łoś, J. and R. Suszko, 1958, "Remarks on sentential logics", *Indagationes Mathematicae* 20: 177–183.

MacFarlane, J. G. , 2000, *What Does it Mean to Say that Logic is Formal?*, PhD Thesis, University of Pittsburgh.

Pedersen, N. J. L. , 2014, "Pluralism x 3: Truth, Logic, Metaphysics", *Erkenntnis* 79: 259–277.DOI: 10.1007/s10670-013-9476-x

Pedersen, N. J. L. and C. Wright, 2018, "Pluralist Theories of Truth", *The Stanford Encyclopedia of Philosophy* URL = <https://plato.stanford.edu/archives/win2018/entries/truth-pluralist/>.

Popper, K., 1948, "On the theory of deduction, part II: The definitions of classical and intuitionistic negation", *Indagationes Mathematicae* 10: 111–120.

Popper, K. R., 1979, *Truth, rationality and the growth of scientific knowledge*, Frankfurt am Main: Klostermann.

Prawitz, D., 1965, *Natural Deduction*, Almqvist & Wiksell, Stockholm.

Prawitz, D., 2015, "Classical versus intuitionistic logic", *Why is this a Proof?, Festschrift for Luiz Carlos Pereira* 27: 15–32.

Priest, G. , 2000, "Reviewed work(s): Logiques Classiques Et Non Classiques: Essai Sur Les Fondements De La Logique by N. C. A. da Costa", *Studia Logica* 64 (3): 435–443.

Priest, G. , 2003, "On Alternative Geometries, Arithmetics, and Logics; A Tribute to Łukasiewicz", *Studia Logica* 74: 441–468.

Priest, G. , 2006, *Doubt Truth to be a Liar*, Oxford University Press, Oxford. DOI: 10.1093/0199263280.001.0001

Priest, G., 2014, "Logical Pluralism: Another Application for Chunk and Permeate", *Erkenntnis* 79: 331–338.

Priest, G. , 2019, "A note on mathematical pluralism and logical pluralism", *Synthese*. DOI: 10.1007/s11229-019-02292-9

Prior, A. N., 1960, "The Runabout Inference-Ticket", *Analysis* 21 (2): 38–39. DOI: 10.1093/analys/21.2.38

Pynko, A., 1995, "On Priest's logic of paradox", *Journal of Applied Non- Classical Logics* 5 (2): 219–225.DOI: 10.1080/11663081.1995.10510856

Rahman, S., and J. Redmond, 2008, "Hugh MacColl and the birth of logical pluralism", in D. Gabbay and J. Woods (eds.), *The Handbook of the History of Logic* (Volume 4: British Logic in the Nineteenth Century), Amsterdam: Elsevier: 533–604.

Rasga, J., A. Sernadas, C. Sernadas, L. Viganò, 2002, "Fibring labelled deduction systems", *Journal of Logic and Computation* 12 (3): 443–473.

Read, S., 2006, "Monism: The One True Logic", in D. DeVidi and T. Kenyon (eds), *A Logical Approach to Philosophy: Essays in Honour of Graham Solomon*, Dordrecht: Springer: 193–209.

Quine, W. V., 1970, *The Philosophy of Logic*, Prentice-hall.

Schechter, J., 2011, "Juxtaposition: a new way to combine logics", *The Review of Symbolic Logic* 4: 560–606.

Schurz, G., 1991, "How far can Hume's is-ought-thesis be generalized? An investigation in alethic-deontic modal predicate logic", *Journal of Philosophical Logic* 20: 37–95.

Segerberg, K., 1973, "Two-dimensional modal logic", *Journal of Philosophical Logic* 2 (1): 77–96.

Sernadas, A., C. Sernadas, and C. Caleiro, 1999, "Fibring of logics as a categorical construction", *Journal of Logic and Computation* 9 (2): 149–179.

Sernadas, C., J. Rasga, and W. Carnielli, 2002, "Modulated fibring and the collapsing problem", *The Journal of Symbolic Logic* 67(4): 1541–1569.

Sette, A. M., 1973, "On the propositional calculus P^1", *Mathematica Japonicae* 18: 173–180.

Shapiro, S., 2014, "Structures and Logics: A Case for (a) Relativism", *Erkenntnis*,79: 309–329. DOI: 10.1007/s10670-013-9480-1

Shapiro, S., 2014, *Varieties of Logic*, Oxford University Press, Oxford. DOI: 10.1093/acprof:oso/9780199696529.001.0001

Shehtman, V., 1978, "Two-dimensional modal logic", *Akademiya Nauk SSSR. Matematicheskie Zametki* 23 (5): 759–772.

Stei, E., 2020, "Rivalry, normativity, and the collapse of logical pluralism", *Inquiry*63 (3-4):411–432.

Tappolet, C., 1997, "Mixed inferences: A problem for pluralism about truth predicates´´, *Analysis* 57: 209–210.

Tappolet, C., 2000, "Truth pluralism and many-valued logics: A reply to Beall´´, *Philosophical Quarterly* 50: 382–385.

Thomason, R., 1984, "Combinations of tense and modality", in D., M. Gabbay and F. Guenthner (eds)*Handbook of Philosophical Logic* volume 2, D. Reidel, Dordrecht: 135–165.

van Benthem, J., 2008, “Logical dynamics meets logical pluralism?”, *The Australasian Journal of Logic* 6: 182–209. DOI: 10.26686/ajl.v6i0.1801

Varzi, A., 2002, “On Logical Relativity”, *Philosophical Issues* 10: 197–219. DOI: 10.1111/j.1758-2237.2002.tb00067.x

Williamson, T., 1988, “Equivocation and Existence”, *Proceedings of the Aristotelian Society* 88 (1): 109–128.DOI: 10.1093/aristotelian/88.1.109

Williamson, T., 1994, “A critical study of ‘Truth and Objectivity’ ”, *International Journal of Philosophical Studies* 30: 130–144.

Wittgenstein, L., 1922, *Tractatus logico-philosophicus*, Routledge & Kegan Paul, London.

Wrenn C. B., 2018, “A Plea for Immodesty: Alethic Pluralism, Logical Pluralism, and Mixed Inferences”, in J. Wyatt, N. J. L. L. Pedersen and N. Kellen (eds.), *Pluralisms in Truth and Logic*, Palgrave Macmillan. DOI: 10.1007/978-3-319-98346-2

Wright, C., 1992, *Truth and Objectivity*, Harvard University Press, Cambridge.

Wright, C., 1998, “Truth: A Traditional Debate Reviewed”, *Canadian Journal of Philosophy*: 31–74.

Yu A. D., 2017, “Logic for Alethic Pluralists”, *The Journal of Philosophy* 114 (6): 277–302. DOI: 10.5840/jphil2017114621

Yu A. D., 2018, “Logic for Alethic, Logical, and Ontological Pluralists”, in J. Wyatt, N. J. L. L. Pedersen and N. Kellen (eds.), *Pluralisms in Truth and Logic*, Palgrave Macmillan. DOI: 10.1007/978-3-319-98346-2

Zardini, E., 2018, “Generalised Tarski’s Thesis Hits Substructure”, in J. Wyatt, N. J. L. L. Pedersen and N. Kellen (eds.), *Pluralisms in Truth and Logic*, Palgrave Macmillan. DOI: 10.1007/978-3-319-98346-2