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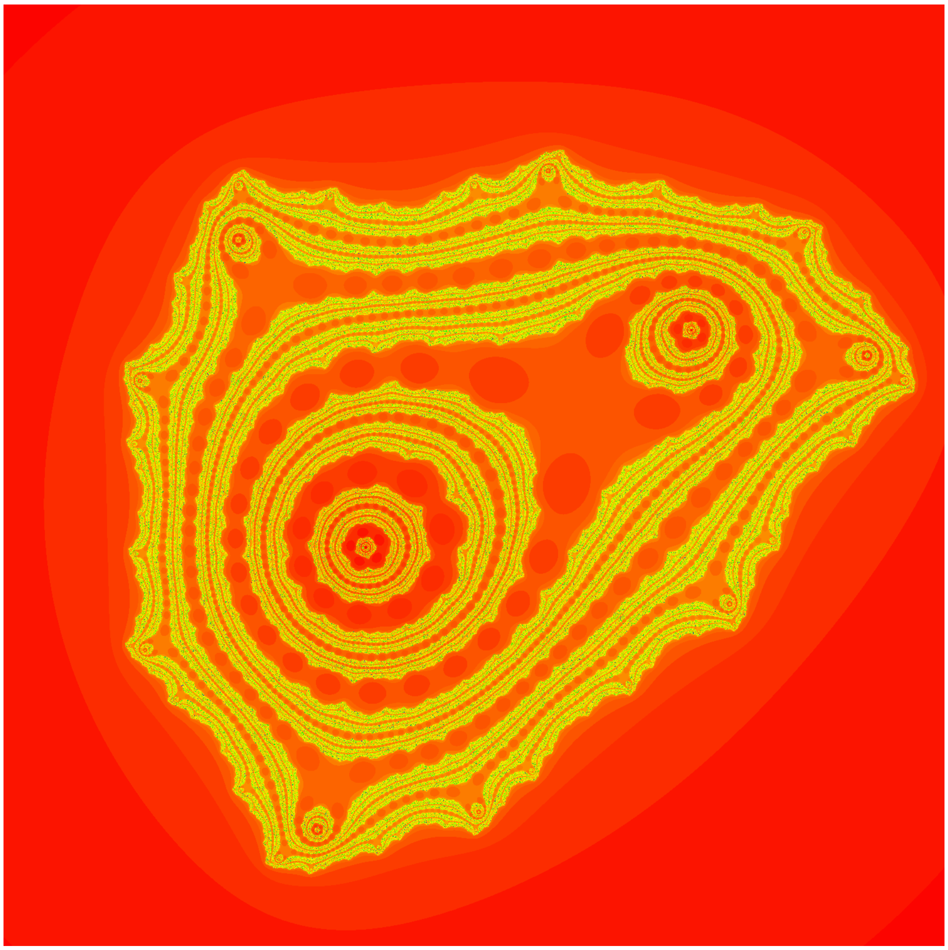

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Departament de Matemàtiques i Informàtica
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Certifico que la següent tesi ha estat realizada per en Dan Alexandru Paraschiv sota la meva codirecció.
Barcelona, abril de 2023.

Xavier Jarque i Ribera

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Jordi Canela Sánchez

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Abstract

Rational iteration is the study of the asymptotic behaviour of the sequences given by the iterates of a rational map on the Riemann sphere. According to Montel's theory on normal families, the phase space (also called the dynamical plane) is divided in two completely invariant sets known as the Fatou set (an open set where the dynamics is tame) and the Julia set (a closed set where the dynamics is chaotic). The main topic of this thesis is the study of the connectivity of the Fatou components for certain families of rational maps. On the one hand, we consider a family of singular perturbations and extend previous results on singular perturbations of Blaschke products. The main result is to show that the dynamical planes for the corresponding maps present Fatou components of arbitrarily large connectivity and determine precisely these connectivities. On the other hand, we consider a different problem related to root-finding algorithms. More precisely, we study the Chebyshev-Halley methods applied to a symmetric family of polynomials of arbitrary degree. The main goal is to show the existence of parameters such that the immediate basins of attraction corresponding to the roots of unity are infinitely connected. Moreover, we also prove that the corresponding dynamical plane contains a connected component of the Julia set, which is a quasiconformal deformation of the Julia set of the map obtained by applying Newton's method.

Keywords: Rational iteration, Fatou and Julia set, connectivity of fatou components, singular perturbations and root finding-algorithms.

Resum en català

La iteració racional és l'estudi del comportament asimptòtic de les seqüències donades pels iterats d'una funció racional sobre l'esfera de Riemann. Segons la teoria de Montel sobre les famílies normals, l'espai de fases (també anomenat pla dinàmic) es divideix en dos conjunts totalment invariants coneguts com a conjunt de Fatou (unió de components oberts on la dinàmica és essencialment senzilla) i el conjunt de Julia (un conjunt tancat on la dinàmica és caòtic). El tema principal d'aquesta tesi és l'estudi de la connectivitat de les components de Fatou per a determinades famílies de funcions racionals. D'una banda, l'autor considera una família de pertorbacions singulars i amplia els resultats anteriors sobre pertorbacions singulars dels productes de Blaschke. El resultat principal és mostrar que els plans dinàmics d'aquestes funcions presenten components de Fatou de connectivitat arbitràriament grans i determinen precisament aquestes connectivitats. D'altra banda, l'autor considera un problema diferent relacionat amb els algorismes de recerca d'arrel. Més precisament, estudia els mètodes Chebyshev-Halley aplicats a una família simètrica de polinomis de grau arbitrari. L'objectiu principal és mostrar l'existència de paràmetres de manera que les conques d'atracció immediates corresponents a les arrels de la unitat tinguin connectivitat infinita. A més, també demostra que el pla dinàmic corresponent conté una component connexa del conjunt de Julia, que és una deformació quasiconforme del conjunt de Julia de la funció obtinguda aplicant el mètode de Newton.

Paraules clau: Iteració racional, conjunt de Fatou i Julia, connectivitat dels components de Fatou, pertorbacions singulars i algorismes de recerca d'arrel.

List of symbols

$\overline{U}$	The closure of the set U.
$\mathbb{R}$	The real line.
$\mathbb{C}$	The complex plane.
$\hat{\mathbb{C}}$	The Riemann sphere.
$\mathbb{D}$	The unit disk.
$\mathbb{D}^*$	The punctured disk.
$\mathbb{S}^1$	The unit circle.
$\mathbb{S}_c$	The circle centered at the origin and of radius $c > 0$.
$N_P(z)$	The Newton map associated to the polynomial $P(z)$.
$\mathcal{J}(f)$	The Julia set of a holomorphic map f.
$\mathcal{F}(f)$	The Fatou set of a holomorphic map f.
$A(w_0)$	The basin of attraction of the attracting fixed point w_0.
$A^*(w_0)$	The immediate basin of the attracting fixed point w_0.
σ_0	The standard complex structure.
μ_0	The Beltrami coefficient of the standard complex structure.
$\mathrm{Fill}(U)$	The minimal simply connected open set which contains the open set U but not $z = \infty$.
$\mathrm{Ext}\,(\gamma)$	The connected component of $\hat{\mathbb{C}} \setminus \gamma$ (where γ is a Jordan curve in $\mathbb{C}$) that contains $z = \infty$.
$\mathrm{Int}\,(\gamma)$	The connected component of $\hat{\mathbb{C}} \setminus \gamma$ (where γ is a Jordan curve in $\mathbb{C}$) that does not contain $z = \infty$.
$A(\gamma_1, \gamma_2)$	The open annulus bounded by Jordan curves γ_1 and γ_2 with $\gamma_1 \subset \mathrm{Int}\,(\gamma_2)$.

Contents

Chapter 1

Introduction

This PhD Thesis belongs to the area of discrete dynamical systems of one complex variable, usually referred to as *complex dynamics* or *holomorphic dynamics*. More precisely, we are interested in studying the asymptotic behaviour of the sequences determined by the iterates of holomorphic functions.

Discrete dynamical systems naturally arise as discretization of continuous dynamical systems associated to differential equations, or as a natural tool to study, locally, invariant objects of continuous dynamical systems such as periodic orbits or graphs (Poincaré maps). In turn, holomorphic (or complex) dynamics come into play as complexifications of real discrete dynamical systems. This complexification or enlargement of the phase portrait allows to have a better understanding of the local dynamics around invariant objects (like fixed points), as well as the global dynamics.

The first one to attempt a rigurous study of the iteration of holomorphic functions was Cayley (see [Cay79]) at the end of the 19th century. He tried to generalize the root-finding algorithm called *Newton's method* to polynomials of complex variable. His attempt was only partially succesfull, as none of the necessary tools were yet developed (including the computer). The French Academy of Sciences announced its major yearly award for 1918 for work related to iteration of functions of complex variables. This motivated French mathematicians to obtain several historical results (like S. Lattès, who constructed the first complex map *chaotic* everywhere). Among them stand out P. Fatou and G. Julia, who independently founded the field of complex dynamics. They used the theory of *normality*, developed by P. Montel at the beginning of the 20th century.

Julia and Fatou divided the dynamical plane of a holomorphic map f based on whether the family of iterations $\{f^n\}_{n\geq 0}$ is normal in some neighbourhood of the point (we now call this the *Fatou set*) or not (the *Julia set*). The Fatou set is open, while the Julia set is closed. Both sets are completely invariant. For rational maps, the Julia set is either the Riemann sphere, or it has empty interior. Therefore, we usually study the topology and geometry in the dynamical plane using connected components of the Fatou set, which are called *Fatou components*. A cornerstone result is the Classification Theorem for periodic Fatou components of rational maps (due to the work of Fatou and and Sullivan, compare Theorem 2.3.4). Thanks to the No Wandering Domain Theorem proven by Sullivan (see [Sul85], compare Theorem 2.3.3), we know that any Fatou component of a rational map is eventually mapped onto such a periodic Fatou component. All together concludes that any Fatou domain of a rational mao eventually is mapped to a periodic Fatou component, which is either the attracting basin of an attracting cycle, the attracting basin of a parabolic cycle, or a periodic rotation domain (which can be simply connected, a Siegel disk, or doubly connected, a Herman ring). Consequently, a natural question is to know how many of these tame behaviours can share the same dynamical plane (*phase portrait*). To solve this question, the cornerstone tool is the control of the orbits of the critical points, that is, the points $c \in \hat{\mathbb{C}}$ such that $f'(c) = 0$. In fact the image of a critical point $v = f(c)$ is called a *critical value* and notice that f is a local homeomorphism at every point which is not a singular value (alternatively, the *singular set* is the set of points at which not all branches of the inverse map are well defined). It is proven that every attracting or parabolic cycle has at least one critical value lying on its immediate basin of attraction. For the case of rotation domains, the situation is more delicate since the result is that at least one critical orbit should accumulate on the boundary of the Fatou component (two boundary components in the case of Herman rings) but, a priori, the same critical orbit might share many different rotation domains. Due to fundamental results of Fatou and Shishikura, we know that in fact the number of critical points (finite for a rational map) bound the number of periodic cycles of Fatou components.

We consider the basic example of family of non-trivial maps, the *quadratic family*

$$Q_c(z) \coloneqq z^2 + c,$$

where $z \in \hat{\mathbb{C}},\ \ c \in \mathbb{C}$. The polynomials of degree 2 have 2 simple critical points. The point $z = \infty$ is always a *superattracting fixed point* (this holds true for all polynomials). The other critical point is said to be a *free critical point.* However, the quadratic family is precisely the set of monic polynomials of degree 2 for which the free critical point is $z = 0$. Its image, the coresponding *critical value*, is $z = c$. Thererefore, the corresponding *singular set* is given by $S(Q_c) = \{\infty, c\}$. Since this set has dimension 1, we say that the family is *unicritical.*

A well-known result (proven independently by both Fatou and Julia) related to the quadratic family is the following dichotomy. It offers a characterization of the Julia set based on whether the orbit of the free critical value $z = c$ is bounded, or not (see Figure 1.1). We have a first example that understanding the dynamics of the singular set of a map is sufficient for understanding the dynamical plane.

Theorem 1.0.1 (The Fundamental Dichotomy Theorem)**.** *Let $Q_c(z) = z^2 + c$, where $c \in \mathbb{C}$. If $|Q_c^i(0)| \leq 2$ for any $i \in \mathbb{N}$ (in particular, the orbit of the critical value $z = c$ is bounded), then the Julia set $\mathcal{J}(Q_c)$ is connected. Otherwise, the Julia set is totally disconnected (it is homeomorphic to a Cantor set) and the orbit of $z = c$ is unbounded.*

The *connected locus* of the quadratic family is

$$\mathcal{M} \coloneqq \{c \in \mathbb{C} \,|\, \mathcal{J}(Q_c) \text{ is connected}\}.$$

It is known as the *Mandelbrot set* (see Figure 1.2), one of the most famous fractals. According to the previous arguments, if c is a parameter for which the map has an attracting periodic cycle of a certain period, then $c \in \mathcal{M}$. The Implicit Function Theorem implies that in fact $c \in \operatorname{Int}(\mathcal{M})$. Components of the interior of $\mathcal{M}$ for which the associated polynomials have an attracting periodic cycle are called *hyperbolic components.* Parameters in the same hyperbolic component present attracting periodic cycles of the same period. This is closely related to the concept of $\mathcal{J}-$*stability.* Meanwhile, parameters lying on the boundary of the Mandelbrot set have in any neighbourhood some parameters for which the Julia set is a Cantor set, and other parameters for which the Julia set is connected. Moreover, if a parameter lies on the boundary

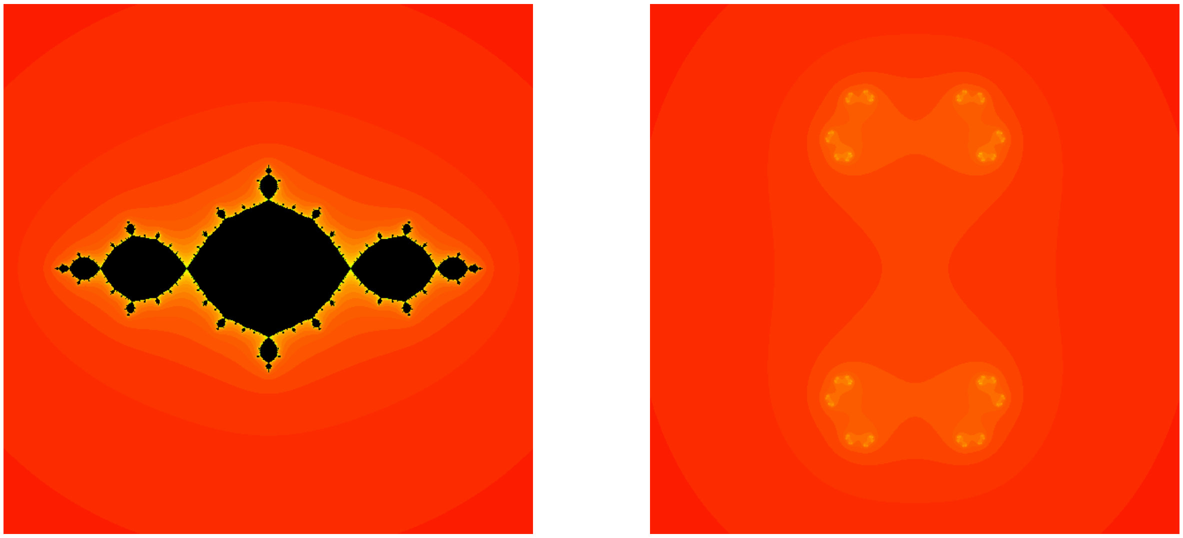


Figure 1.1: The left figure illustrates the dynamical plane of $Q_c = z^2 + c$ for $c = -1$, which is called the *basilica*. In the right figure we can see the dynamical plane of $Q_c = z^2 + c$ for $c = 1$, where the Julia set is totally disconnected and homeomorphic to the Cantor set.

of 2 hyperbolic components, any neighbourhood contains parameters with dynamical planes containing attracting cycles of different periods. The boundary of the Mandelbrot set forms the *bifurcation locus* of the quadratic family.

The natural extension of polynomials is the set of maps holomorphic over the Riemann sphere, that is, the *rational* maps. Each of these maps can be described as a quotient:

$$R(z) := \frac{p(z)}{q(z)},$$

where p and q are poynomials. The *degree* of the rational map R is given by:

$$d := \max\{\deg(p), \deg(q)\},$$

and it concides with the topological degree; every point $z \in \hat{\mathbb{C}}$ has precisely d preimages (counting multiplicity) under iteration by a rational map of degree d. Furthermore, every rational map has precisely $2d - 2$ critical points (counting multiplicity).

The extension of the quadratic maps in the rational maps space are, of course, the *quadratic rational maps*. While the space Rat_2 of quadratic rational maps defined over the Riemann sphere is a $5-$dimensional manifold, the moduli space of holomorphic conjugacy classes of quadratic rational maps has dimension 2 (the same as the singular set). Roughly speaking, we say that quadratic rational maps have 2 degrees of freedom. Since we are only able to visualize $1-$dimensional complex spaces, we study $1-$dimensional *slices*. This approach makes clear the analogy between quadratic rational maps and cubic polynomials, a topic much larger than the one of quadratic maps. Thus, increasing the degree of the map is a primary way of obtaining more complicated dynamical systems.

Rational maps are known to present richer dynamics than polynomials (Lattès' example in 1918, of a Julia set which is the entire Riemann sphere, was a non-polynomial rational map). For a start, $z = \infty$ is not in general a superattracting fixed point, but it might present

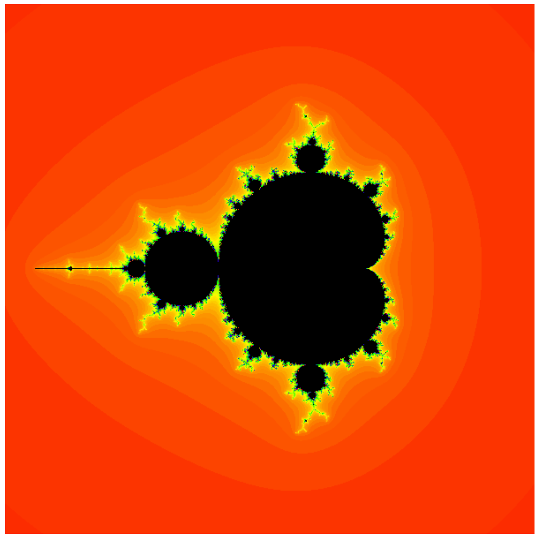


Figure 1.2: The Mandelbrot set.

any type of dynamical behaviour. Herman rings, a type of periodic Fatou component, are proven to not exist for polynomial maps (they were first observed in dynamical planes of rational maps by Herman in [Her79]).

A major role in this resurgence of complex dynamics has been played by the development of the technique called *quasiconformal surgery*. It was the works of Sullivan, Douady, and Hubbard in the 1980s which established the current framework of holomorphic dynamics. They proved fundamental results like No Wandering Domains Theorem (see [Sul85], see also Theorem 2.3.3) and the Straightening Theorem (see [DH85], see also Theorem 2.6.4). Quasiconformal surgery uses the theory of *quasiconformal maps*, homeomorphisms with specific analytic and geometric properties. First used by Sullivan in 1983 to prove the No Wandering Domain Theorem, quasiconformal surgery was later used by Douady and Hubbard in 1985 to prove the Straightening Theorem. Using *polynomial-like mappings*, this theorem explains why deformations of the Julia sets of polynomials can be seen in dynamical planes of holomorphic maps.

Another way of increasing the complexity of a dynamical system is by studying particular families of rational maps called *singular perturbations*. Informally, a family is called a singular perturbation if it is defined by a *base* family (called the *unperturbed family* and for which we have a deep understanding of the dynamical plane) plus a *local* perturbation, that is, a perturbation which has a significant effect on the orbits of points in some part(s) of the dynamical plane, but a very small dynamical relevance on other regions.

Singular perturbations of rational maps were introduced by McMullen in [McM88]. He proposed the study of the family

$$Q_{n,d,\lambda}(z) = z^n + \frac{\lambda}{z^d}, \tag{1.1}$$

where $n, d \geq 2$ and $\lambda \in \mathbb{C}$, $|\lambda|$ small. Observe that in (1.1) the unperturbed map is the *simplest* possible: z^n. He considered the case $n = 2$ and $d = 3$ and he proved that if $|\lambda|$ is small enough then the Julia set is a Cantor sets of quasicircles (the result actually holds for n and d satisfying $1/n + 1/d < 1$). Later, Devaney, Look, and Uminsky (see [DLU05]) considered (1.1) as a λ-family of rational maps and they extended McMullen's result by proving the Escape Trichotomy. More specifically, they showed that if all critical points belong to the basin of attraction of infinity then the Julia set is a Cantor set, a Sierpinski carpet, or a Cantor set of quasicircles (McMullen's case). Other families of maps that were used as *base* for singular perturbations are the unicritical polynomials (see [BDGR08, GMR13]) and finite generalized Blaschke products of degree 4 (see [Can17, Can18]).

The *connectivity* of a domain $D \subset \hat{\mathbb{C}}$ is defined as the number of connected components of its boundary. It is known that periodic Fatou components have connectivity 1, 2, or ∞. Indeed, Siegel disks have connectivity 1, Herman rings have connectivity 2, and immediate basins of attraction might have connectivity 1 or ∞. Preperiodic Fatou components can have finite connectivity greater than 2. The first such example, with connectivities 3 and 5, was presented in [Bea91]. Moreover, for any given $n \in \mathbb{N}$, there are examples of rational maps with Fatou components of connectivity n. These examples can either be obtained by quasiconformal surgery (see [BKL91]) or by giving explicit families of rational maps (see [QG04] and [Ste93]). However, the degree of the rational maps obtained in all previous examples grows rapidly with n. The first example of rational map whose dynamical plane contains Fatou components of arbitrarily large finite connectivities was presented in [Can17] (see also [Can18]) by using singular perturbations. Finally, the main tool that we use to compute connectivities of Fatou components is a particular version of the Riemann-Hurwitz formula, and its corollaries (see Section 2.4).

Another important topic in rational dynamics is the study of asymptotic behaviour of iteration of maps obtained using numerical methods. Numerical methods have been extensively used to give accurate approximations of the solutions of systems of non-linear equations. Those equations or systems of equations correspond to a wide source of scientific models from biology to engineering and from economics to social sciences, and so their solutions are a cornerstone of applied mathematics. One of the most used families of numerical methods are the so called root-finding algorithms; that is, iterative methods which asymptotically converge to the zeros (or some of the zeros) of the non linear equation, say $g(z) = 0$.

The universal and most studied root-finding algorithm is known as *Newton's method*. If g is holomorphic, the Newton's method applied to g is the iterative root finding algorithm defined as follows

$$z_{n+1} = z_n - \frac{g(z_n)}{g'(z_n)}, \quad z_0 \in \mathbb{C}.$$

It is well known that if $z_0 \in \mathbb{C}$ is chosen close enough to one of the solutions of the equation $g(z) = 0$, say α, then the sequence $\{z_n = g^n(z_0)\}_{n \geq 0}$ converges to α when n tends to ∞. Moreover, the speed of (local) convergence is generically quadratic. It was Cayley (see [Cay79]) the first to consider Newton's method as a (holomorphic) dynamical system, that is studying the convergence of these sequences for all possible seeds $z_0 \in \mathbb{C}$ at once, under the assumption that g was a degree 2 or 3 polynomial. This was known as *Cayley's problem*.

Root-finding algorithms are a natural topic for complex dynamics. In particular, maps obtained by applying Newton's method to polynomials are a much studied topic (see [Shi09], [HSS01], [Tan97]). The key dynamical property of the map N_P obtained by applying Newton's method to the polynomial P is the following: if $z \in \mathbb{C}$ is a simple root of P, then it is a superattracting fixed point of N_P. This makes clear why iterating N_P is a reasonable way of approximating the roots of P. However, since the Julia set is not empty, we know that all not all seeds will converge to a root of the polynomial. For a numerical method, approaching components of the Julia set, is a problem to be avoided. A priori, the immediate basin of attraction can be multiply connected, hence infinitely connected. However, a result due to Shishikura ([Shi09]) states that the Julia set for a map obtained by applying Newton's method to a polynomial is connected, and therefore, all Fatou components are simply connected. An extension of this theorem to the transcendental case can be found in [BFJK14].

The problems studied in this thesis are related to connectivities of Fatou components for 2 families of rational maps. The first family of maps is a family of singular perturbations, where we encounter Fatou components of finite arbitrarily large connectivity in one dynamical plane. The second family of maps is a family derived from root-finding algorithms, where we encounter infinitely connected immediate basins of attraction. The results regarding these two families were published in the papers [CJP22] and [Par23].

Singular perturbations

In this thesis we consider the family of degree $n+d+1$ rational maps given by

$$\mathcal{S}_{n,d,\lambda}(z) = \frac{z^n(z-a)}{Q(z)} + \frac{\lambda}{z^d}, \tag{1.2}$$

$\lambda \in \mathbb{C}^*$, $d \geq 2$, where $n \geq 2$, $a \in \mathbb{C}^*$, and Q is a polynomial of degree at most n with $Q(0)Q(a) \neq 0$. For $\lambda \neq 0$ the point $z = 0$ is a pole of degree d. We impose upon this family of maps 4 conditions, which we denote by (a), (b), (c), and (d), that we will describe later, in Chapter 3. The first 3 conditions arise from the need for the Julia set of the *base* family to be relatively simple, in our particular case a quasicircle. We must point out that our chosen family of maps is the maximal family to satisfy all the conditions. The last condition is the same as in the McMullen family.

Our goal is to extend the results in [Can17, Can18] to a wider family of singular perturbations and to study which connectivities are attainable for this family. We assume that all critical points iterate towards $z = \infty$, and therefore, the Fatou set is the basin of attraction of the superattracting fixed point $z = \infty$. The immediate basin of attraction of $z = \infty$ has precisely one preimage other than itself, which contains $z = 0$, and it is called the *trap door* T_λ. We prove that there exists a preimage of T_λ which is an annulus that contains $n+d$ critical points, and we denote it by A_λ. There exists another critical point, ν_λ, which is crucial in order to increase the connectivities beyond 2. For $\lambda = 0$, ν_λ is a critical point of the unperturbed map, that lies in the immediate basin of attraction of the superattracting fixed point $z = 0$. For $\lambda \neq 0$, $|\lambda|$ small enough, if ν_λ belongs to a preimage U_ν of A_λ, then the Fatou component U_ν is triply connected. The configuration of the dynamical plane (the condition of assuming all critical points iterate towards $z = \infty$ is in fact equivalent to assuming that ν_λ iterates towards $z = \infty$) allows for the family to be viewed as unicritical for $|\lambda|$ small enough. Moreover, if U_ν surrounds $z = 0$, then we can find sequences of iterated preimages of U_ν which increase the (finite) connectivity with every iteration.

Theorem A. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d) and let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. Assume also that $\nu_\lambda \in \mathcal{U}_\nu$, where $\mathcal{U}_\nu$ is an iterated preimage of A_λ which surrounds $z = 0$. Let k be the minimal number of iterations needed by the free critical point ν_λ to be mapped into Fill(A_λ). Let U be a Fatou component of connectivity $\kappa > 2$. Then, there exist $i, j, \ell \in \mathbb{N}$ such that $\kappa = (n+1)^i d^j n^\ell + 2$ and $\ell \leq jk$.*

In other words, Theorem A is telling us all *potential* connectivities $\kappa > 2$ for a Fatou component of a map in $\mathcal{S}_{n,d,\lambda}$ for $|\lambda| < \mathcal{C}$ (see Chapter 3 for the definition of this constant), but it is not claiming the existence of a Fatou component of each (i, j, ℓ)-connectivity. The next result complements Theorem A and it gives the connectivities that are certainly achieved for any parameter λ as long as $|\lambda|$ is sufficiently small and ν_λ satisfies certain dynamical conditions.

Theorem B. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d) and let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. Assume also that $\nu_\lambda \in \mathcal{U}_\nu$, where $\mathcal{U}_\nu$ is an iterated preimage of A_λ which surrounds $z = 0$. Let $k \geq 1$ be the minimal number of iterations needed by the free critical point ν_λ to be mapped into Fill(A_λ). For any given $i, j, \ell \in \mathbb{N}$ such that $\ell \leq j(k-1)$, there exists a Fatou component U of connectivity $\kappa = (n+1)^i d^j n^\ell + 2$.*

In Theorem A and Theorem B the achievable connectivities depend on the minimal number of iterations $k > 0$ needed by the free critical point ν_λ to be mapped into $\text{Fill}(A_\lambda)$. However, choosing λ appropriately we can make this k as big as desired. Therefore, for any ℓ and j we can find λ so that the inequality $\ell \leq j(k-1)$ is satisfied. From this, we obtain Theorem C.

Theorem C. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d) and let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. For any given $i, l \geq 0$ and $j > 0$, there exists a parameter λ such that $\mathcal{S}_{n,d,\lambda}(z)$ has a Fatou component of connectivity $\kappa = (n+1)^i d^j n^\ell + 2$, and a Fatou component of connectivity $\kappa = (n+1)^i + 2$.*

A dynamical view of the Chebyshev-Halley family of methods

Newton's method is the most well-known iterative root-finding algorithm. Many authors have studied alternative iterative methods having, for instance, a better local speed of convergence. Two of the best known root-finding algorithms are *Chebyshev's method* and *Halley's method* (see [ABP04]). They are included in the *Chebyshev-Halley family* of root-finding algorithms, which was introduced in [CTV13] (see also [Ama16]). For a holomorphic map g, the Chebyshev-Halley family of root finding algorithms is given by the iterative procedure

$$z_{n+1} = z_n - \left(1 + \frac{1}{2}\frac{L_g(z_n)}{1 - \alpha L_g(z_n)}\right)\frac{g(z_n)}{g'(z_n)}, \tag{1.3}$$

where $\alpha \in \mathbb{C}$ and $L_g(z) = \frac{g(z)g"(z)}{(g'(z))^2}$. For $\alpha = 0$ we have Chebyshev's method and for $\alpha = \frac{1}{2}$ Halley's method. As α tends to ∞, the Chebyshev-Halley algorithms tend to Newton's method.

Previously, Campos, Canela, and Vindel have studied the Chebyshev-Halley family applied to $f_{n,c}(z) = z^n + c$, $c \in \mathbb{C}^*$ (see [CCV18], [CCV20]). The maps obtained by applying the Chebyshev-Halley family to $f_{n,c}$ are all conjugated to the map obtained by applying the Chebyshev-Halley family to $f_n(z) \coloneqq f_{n,-1}(z) = z^n - 1$. By applying the Chebyshev-Halley method to $f_n(z)$ we obtain the map:

$$O_{n,\alpha}(z) \coloneqq \frac{(1-2\alpha)(n-1) + (2 - 4\alpha - 4n + 6\alpha n - 2\alpha n^2)z^n + (n-1)(1 - 2\alpha - 2n + 2\alpha n)z^{2n}}{2nz^{n-1}(\alpha(1-n) + (-\alpha - n + \alpha n)z^n)}.$$

The map $O_{n,\alpha}$ has degree $2n$ and it has $4n-2$ critical points, counting multiplicity. The point $z = 0$ is a critical point of multiplicity $n-2$, which is mapped to the fixed point $z = \infty$. The n-th roots of the unity are superattracting fixed points of local degree 3 and, therefore, they have multiplicity 2 as critical points. This leaves n free critical points. This family is symmetric with respect to rotation by the n-th root of the unity. The symmetry ties the orbits of the n free critical points, so the family $O_{n,\alpha}$ has only one degree of freedom.

In [CCV20], the authors studied in detail the topology of the immediate basins of attraction of the fixed points of $O_{n,\alpha}(z)$ given by the $n-$th root of unity, that is, the zeros of $f_n(z)$. In what follows we refer to these basins as

$$A_{n,\alpha}(\xi) \coloneqq A_{O_{n,\alpha}}(\xi) \left[A^*_{n,\alpha}(\xi) \coloneqq A^*_{O_{n,\alpha}}(\xi)\right],$$

where $\xi^n = 1$. For one particular case, the immediate basins of attraction are infinitely connected (see Figure 4.1). We study the Julia set of $O_{n,\alpha}$ for this particular case and relate it to the Julia set of the map obtained by applying Newton's method to f_n. Our main goal is to show that the unbounded connected component of the Julia set of the Chebyshev-Halley maps applied to $z^n - c$ (for large enough α) is homeomorphic to the Julia set of the map obtained by applying Newton's method to $z^n - 1$.

Theorem D. *Fix $n \geq 2$ and assume that $A^*_{n,\alpha}(1)$ is infinitely connected for some $\alpha \in \mathbb{C}$. Then, there exists an invariant Julia component Π (which contains $z = \infty$) which is a quasiconformal copy of the Julia set of N_{f_n}, where N_{f_n} is the map obtained by applying Newton's method to the polynomial $f_n(z) = z^n - 1$.*

We conclude by proving that there exist parameters such that the hypothesis of Theorem D holds.

Theorem E. *Let $n \geq 2$. Then, there exists $\alpha_0 > 0$ large enough such that for $\alpha > \alpha_0$, $\alpha \in \mathbb{R}$, $A^*_{n,\alpha}(1)$ is infinitely connected. Moreover, for $n = 2$, the statement is true for any $\alpha \in \mathbb{C}$ such that $|\alpha| > \alpha_0$.*

Structure of the thesis

The thesis is structured as follows. In Chapter 2 we include the background necessary to understand the work in Chapters 3 and 4. We start with basics of rational iteration, local theory, and fundamental results about Fatou components.We continue with a few results related to the Riemann-Hurwitz formula, the main tool in computing connectivities of Fatou components. This is followed by a section on quasiconformal surgery, where we introduce quasiconformal mappings and their properties, almost complex structures and their pullbacks, and the Measurable Riemann Mapping Theorem (also known as the Integrability Theorem), which connects the previous concepts. We conclude the chapter with a section on polynomial-like mappings, a tool that will be used in Chapter 3.

In Chapter 3 we introduce the families of maps $S_{n,a,Q}$ and $\mathcal{S}_{n,d,\lambda}$. We motivate the choice of these families and the conditions applied. Then, using quasiconformal surgery, we show the conjugation between maps in the family $\mathcal{S}_{n,d,\lambda}$ and a specific Blaschke product. This is followed by an exhaustive description of the topology of Fatou components in the dynamical plane, which allows us to explicitly compute all the achievable connectivities in one dynamical plane (Theorem A). Then, we show that in any dynamical plane satisfying the necessary conditions, almost all connectivities described in Theorem A are achieved (Theorem B). Afterwards, we describe the variation of the dynamical plane with respect to the parameter λ. Finally, we prove Theorem C, which shows that for any possible connectivity described in Theorem A, there exists a parameter λ such that the corresponding dynamical plane contains a Fatou component of the desired connectivity.

In Chapter 4 we introduce the family $O_{n,\alpha}$, which is a family of root-finding algorithms applied to $f_n(z) = z^n - 1$. We highlight several necessary symmetries and properties of the maps $O_{n,\alpha}$. Using surgery and properties of Newton maps, we prove that the unbounded component of the Julia set for the map $O_{n,\alpha}$ is a quasiconformal copy of the Julia set of a specific Newton map (Theorem D). Finally, we prove Theorem E, which shows that there exist parameters for which the hypothesis in Theorem D holds. This is done by separating the cases $n = 2$ and $n \geq 3$. For $n = 2$ and $\lambda > 0$, the map is conjugated to a Blaschke product. We prove that, for any $\lambda \in \mathbb{C}$ of modulus large enough, the hypothesis in Theorem D holds. For $n \geq 3$, the map is not conjugated to a Blaschke product anymore. We still prove that, for $\lambda \in \mathbb{R}$ large enough, the hypothesis in Theorem D is true.

Chapter 2

Preliminaries

In this chapter we present the main tools that are common to chapters 3, and 4. We introduce basic holomorphic dynamics tools and results (for a more detailed introduction to the topic, see [Bea91], [BF14], [CG93], and [Mil06]).

2.1 Iteration of rational maps

We study discrete holomorphic dynamics, that is, we study the dynamical system obtained from the iteration of holomorphic maps. In particular, we are interested in the asymptotic behaviour of the iterates of the map. We focus on iteration of rational maps.

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. Then, R is a rational map of the form

$$R(z) = \frac{p(z)}{q(z)},$$

where p and q are polynomials, not both being the zero polynomial (it also holds that any rational map is a holomorphic map over $\hat{\mathbb{C}}$). Let $d = \max\{\deg(p), \deg(q)\}$. Then, we say that the rational map R has degree d. The degree of R coincides with its topological degree, that is, every point $z \in \hat{\mathbb{C}}$ has precisely d preimages under R, counting multiplicity. In other words, R is a d-fold branched covering of the Riemann sphere. We also denote by

$$R^n := \underbrace{R \circ R \circ \cdots \circ R}_{\text{n times}},$$

the n-th iterate of R.

We start by giving some definitions necessary for the study of any dynamical system.

Definition 2.1.1. Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and $z_0 \in \hat{\mathbb{C}}$. Then, the set

$$\mathcal{O}_R(z_0) = \bigcup_{k \geq 0} R^k(z_0)$$

is said to be the *(forward) orbit* of z_0.

Points with finite orbit play a fundamental role in dynamical systems.

Definition 2.1.2. Let $U, V \subset \hat{\mathbb{C}}$. Let $f : U \to V$ be a function and let $z_0 \in U$ such that $f^p(z_0) = z_0$, $f^k(z_0) \neq z_0$, $k \in \{1, 2, \ldots, p-1\}$. Then, z_0 is said to be a *periodic point of period* p of f. If $p = 1$, then z_0 is called a *fixed point* of f. A point z_0 with finite orbit is called a *preperiodic point*. A preperiodic point z_0 such that $f^k(z_0) \neq z_0$, $\forall k \geq 1$, is said to be *strictly preperiodic*.

Periodic points are key for understanding discrete dynamical systems. Since periodic points of period p of f are in fact fixed points of f^p, then we present here only the classification for fixed points. The same notation can be extended to periodic points.

Definition 2.1.3. Let $R : U \to \hat{\mathbb{C}}$ be a holomorphic map. Let z_0 be a fixed point of f, and let $\lambda := f'(z_0)$ be the *multiplier* of z_0. Then, z_0 is called:

1. *attracting*, if $|\lambda| < 1$ (in particular, if $\lambda = 0$, z_0 is *superattracting*);
2. *neutral*, if $|\lambda| = 1$;
3. *repelling*, if $|\lambda| > 1$.

Moreover, if z_0 is a neutral fixed point, then it is called:

1. *parabolic*, if $\lambda = e^{\frac{2\pi i p}{q}}$, with $\frac{p}{q} \in \mathbb{Q}$;
2. *irrational*, if $\lambda = e^{2\pi i \alpha}$, with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Another important feature when studying dynamical systems are conjugacies since conjugated maps present the same dynamics.

Definition 2.1.4. Let $R, Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be rational maps. We say that R and Q are *conformally* (respectively *topologically conjugated*) if there exists a conformal map (respectively a homeomorphism) $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $R = h \circ Q \circ h^{-1}$.

Several properties of conjugacies follow immediately from the definition. Two topologically conjugated rational maps have the same degree. Moreover, if the conjugacy is also conformal, the multipliers of the corresponding fixed points of the two maps coincide.

Lemma 2.1.5. *Let $R, Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be two conformally conjugated rational maps. Let $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a conformal map such that $R = h \circ Q \circ h^{-1}$. If $z_R \in \hat{\mathbb{C}}$ is a fixed point of R, and $z_Q := h(z_R)$, we have that $R'(z_R) = Q'(z_Q)$.*

Remark 2.1.6. *The map $M(z) = \frac{1}{z}$ is used to study the multiplier of the point $z = \infty$, for any rational map.*

Our goal is to understand the asymptotic behaviour of the orbits of all points $z \in \hat{\mathbb{C}}$. In other words, we want to understand the dynamical plane, realizing a partition based on different dynamical behaviours. The main tool for this was introduced by Fatou and Julia, following Montel's theory of normality of families of holomorphic maps. We start by defining the concept of normal family of holomorphic maps.

Definition 2.1.7. Let $U \subset \hat{\mathbb{C}}$ be a domain. Let $\mathcal{F}$ be a family of holomorphic maps such that any sequence $\{f_n\}_{n \in \mathbb{N}}$ has a subsequence which converges uniformly on compact sets of U to a limit map. Then, $\mathcal{F}$ is called a *normal family* and the set U is called a *domain of normality* for the family F.

The concept of normality is tied to equicontinuity by the Arzelá-Ascoli theorem. We first recall the definition of equicontinuity in complex spaces.

Definition 2.1.8. Let $U \subset \hat{\mathbb{C}}$ be a domain. Let $\mathcal{F}$ be a family of holomorphic maps defined on U. If for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_{\hat{\mathbb{C}}}(f(z_1),\, f(z_2)) < \varepsilon$, for any $f \in \mathcal{F}$, and $z_1,\, z_2 \in U$ with $d_{\hat{\mathbb{C}}}(z_1,\, z_2) < \delta$, then $\mathcal{F}$ is said to be *locally equicontinuous.*

The Arzelá-Ascoli theorem gives necessary and sufficient conditions to determine existence of a uniformly convergent subsequence of a family of continuous maps. We give a statement suited to our particular case.

Theorem 2.1.9 (Arzelá-Ascoli)**.** *Let $U \subset \hat{\mathbb{C}}$ be a domain. Let $\mathcal{F}$ be a family of holomorphic maps defined on U. Then, $\mathcal{F}$ is normal if and only if $\mathcal{F}$ is locally equicontinuous.*

The following theorem of Montel, also known as the fundamental normality test, is a sufficient criterion for a family of holomorphic maps to be normal.

Theorem 2.1.10 (Montel)**.** *Let $U \subset \hat{\mathbb{C}}$ be a domain, and let $\mathcal{F}$ be a family of holomorphic functions defined on U. If there exist three omitted values, that is, three different points w_1, w_2, and w_3, such that $f(U) \subset \hat{\mathbb{C}} \setminus \{w_1,\, w_2,\, w_3\}$ for all $f \in \mathcal{F}$, then the family $\mathcal{F}$ is normal.*

We are interested in the iteration of holomorphic functions. Julia and Fatou applied the theory of Montel to the family of iterations of a given holomorphic map, i.e.,

$$\mathcal{F} = \{f,\, f^2,\, f^3,\, \dots\}.$$

This leads to a partition of the dynamical plane in 2 sets, the Julia and Fatou sets. Before introducing these sets, let us proceed with a trivial example of using normality to have a better understanding of the dynamical plane.

Example 2.1.11. *Consider the map $Q_0 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $Q_0(z) = z^2$. For any point $z \in \mathbb{D}$, there exists an open neighbourhood $N(z)$ such that for any $z \in N(z)$, $\lim_{n\to\infty} Q_0^n(z) = 0$. Moreover,*

$$\bigcup Q_0^n\Big(N(z)\Big) \cap \Big(\hat{\mathbb{C}} \setminus \mathbb{D}\Big) = \emptyset.$$

Similarly, for all points $z \in \Big(\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}\Big)$ there exists an open neighbourhood $N(z)$ such that for any $z \in N(z)$, $\lim_{n\to\infty} Q_0^n(z) = \infty$. Furthermore,

$$\bigcup Q_0^n\Big(N(z)\Big) \cap \mathbb{D} = \emptyset.$$

It follows from Montel's Theorem that the interior and the exterior of the circle are domains of normality of $\{Q_0^n\}_n$.

We now prove that points on the unit circle do not belong to a domain of normality for the family $\{Q_0^n\}_n$. Let $\zeta \in \mathbb{S}^1$ and assume that ζ belongs to a domain of normality of $\{Q_0^n\}_n$, say $N(\zeta)$. Then, by definition of normality, there exists a subsequence $\{n_k\}_k$ and g a continuous map such that $Q_0^{n_k} \to g$ on compact subsets of $N(\zeta)$. Let $V \subset N(\zeta)$ be an open neighbourhood of ζ. Then, for points $z \in V \cap \{|z| < 1\}$, we get that $g(z) = 0$. Analogously, for points

$z \in V \cap \{|z| > 1\}$, we get that $g(z) = \infty$. It follows that the map g is not continuous at $z = \zeta$, which is a contradiction.

We proved that $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ are domains of normality for $Q_0(z) = z^2$. We also proved that points on the unit circle do not belong to a domain of normality. We have therefore realized a partition in the corresponding dynamical plane of the family of iterations of Q_0, using normality as a criterion.

We will now define the Fatou and Julia sets. This constitutes a natural dynamical partition of the dynamical plane. The study of the geometry and topology of the Julia set is the main topic of holomorphic dynamics.

Definition 2.1.12. Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. The maximal open subset of $\hat{\mathbb{C}}$ on which the family $\{R^n\}_n$ is normal is called the *Fatou set*, and it is denoted by $\mathcal{F}(R)$. The complement of the Fatou set is called the *Julia set*, $\mathcal{J}(R) = \hat{\mathbb{C}} \setminus \mathcal{F}(R)$.

The idea behind this partition is that the points in the Fatou set present stable behaviour, while the dynamics in any neighbourhood of a point $z \in \mathcal{J}(f)$ present chaotic behaviour. In the previous example, for $Q_0(z) = z^2$, the Julia set was precisely the unit circle. We will proceed with the basic non trivial example, the family $Q_c(z) = z^2 + c$, where $c \in \mathbb{C}$.

Example 2.1.13. *The family $Q_c(z) = z^2 + c$, where $c \in \mathbb{C}$, is as elementary as a rational map of degree $d \geq 2$ can be. Still, the geometry of Julia sets of maps in this family is far from simple (see Figure 2.1). It has precisely 2 critical points, $z = 0$, and $z = \infty$, which do not depend on the value of the parameter c. While $z = \infty$ is always a superattracting fixed point, it is the point $z = 0$ which governs the dynamics of the map. The boundedness of the orbit of the critical point $z = 0$ is a sufficient criterion to study whether the Julia set for a given parameter is connected or not. More precisely, the Julia set is connected if and only if $\mathcal{O}_{Q_c}(0)$ is bounded. This is related to the well known Dichotomy Theorem, the Julia set of a quadratic polynomial is always either connected, either totally disconnected (in fact, a set homeomorphic to a Cantor set). The set of points $c \in \mathbb{C}$ for which $\mathcal{J}(Q_c)$ is connected is called the Mandelbrot set (see Figure 2.2).*

The Julia set has many important properties, which will be useful later. We add here a brief enumeration of such properties.

Theorem 2.1.14. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. Then, the following statements hold:*

(i) The Julia set $\mathcal{J}(R)$ is non empty. Moreover, $\#\mathcal{J}(R)$ is infinite.

(ii) The Julia set $\mathcal{J}(R)$ and Fatou set $\mathcal{F}(R)$ are completely invariant.

(iii) The Julia set $\mathcal{J}(R)$ is the smallest closed and completely invariant set with at least 3 points (minimality of the Julia set).

(iv) The Julia set $\mathcal{J}(R)$ is either $\hat{\mathbb{C}}$, or it has empty interior.

(v) The Julia set $\mathcal{J}(R)$ is the closure of the repelling periodic points.

(vi) $\mathcal{J}(R) = \mathcal{J}(R^n)$, for $n \geq 1$ (and so, $\mathcal{F}(R) = \mathcal{F}(R^n)$).

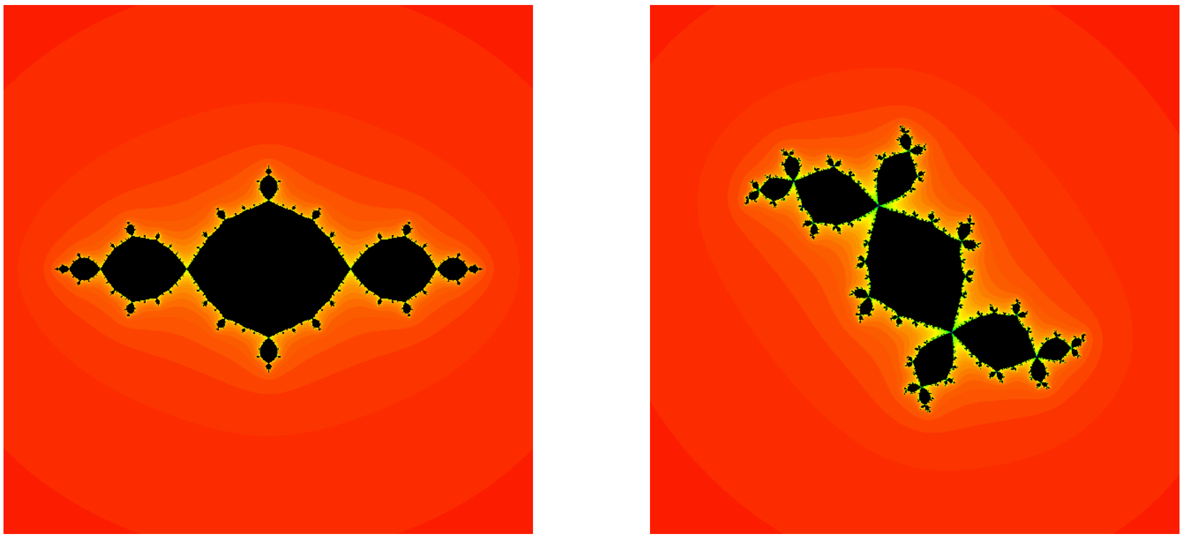


Figure 2.1: The left figure illustrates the dynamical plane of $Q_c = z^2 + c$ for $c = -1$, which is called the *basilica*. In the right figure we can see the *Douady rabbit*, for $c \approx -0.123 + 0.745i$. In black, we see the *filled Julia set*, the set of points with bounded orbit. The Julia set is precisely the boundary of this set.

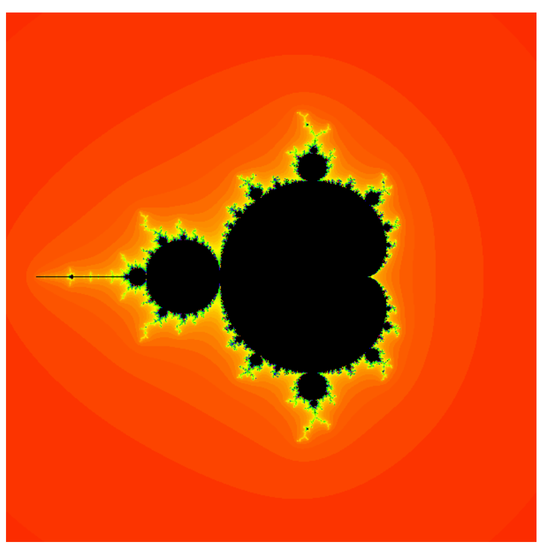


Figure 2.2: The Mandelbrot set.

Recall that in Example 2.1.11 we have studied the domains of normality for the map $Q_0(z) = z^2$. We are now able to say that the unit circle is the Julia set of the map Q_0. We highlight further the behaviour of dynamics of $Q_0(z)$ around points in $\mathcal{J}(Q_0)$ to give a trivial example of chaotic dynamics.

Lemma 2.1.15. *Let $Q_0 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $Q_0(z) = z^2$. Let $\zeta \in \mathcal{J}(Q_0)$ and let N be a neighbourhood of ζ. Then,*

$$\bigcup_{i=0}^{\infty} Q_0^n(N) = \hat{\mathbb{C}} \setminus \{0, \infty\}.$$

Proof. Let $\zeta = e^{i\theta_0}$ such that $0 \leq \theta_0 < 2\pi$, and let N be a small enough open neighbourhood of ζ such that $N \cap \{0, \infty\} = \emptyset$.

We first prove that for any $z \in \hat{\mathbb{C}} \setminus \{0, \infty\}$, there exists k large enough such that $z \in Q_0^i(S)$, for any $i \geq k$. Since $|\zeta| = 1$, there exists ε, $\delta > 0$ and $S \subset N$ an annulus sector such that

$$S = \left\{ z = re^{i\theta} \in \mathbb{C} \;\middle|\; 1 - \varepsilon < r < 1 + \varepsilon,\, \theta_0 - \delta < \theta < \theta_0 + \delta \right\}.$$

First, we point out that on the unit circle $\mathbb{S}^1$, the map Q_0 is conjugated to the *doubling map* T, $T : \mathbb{S}^1 \to \mathbb{S}^1$. It is easy to see then that the image of S through Q_0 is

$$Q_0(S) = \left\{ z = re^{i\theta} \in \mathbb{C} \;\middle|\; (1 - \varepsilon)^2 < r < (1 + \varepsilon)^2,\, 2(\theta_0 - \delta) < \theta < 2(\theta_0 + \delta) \right\}.$$

After $p := \lfloor |\log_2 \delta| \rfloor \geq 0$ more iterations, we have that $Q_0^{p+1}(S)$ is precisely the annulus

$$A = \left\{ z = re^{i\theta} \in \mathbb{C} \;\middle|\; (1 - \varepsilon)^{2^{p+1}} < r < (1 + \varepsilon)^{2^{p+1}} \right\}.$$

We have that for any $z \in \hat{\mathbb{C}} \setminus \{0, \infty\}$, there exists $n_0 \in \mathbb{N}$ such that $(1-\varepsilon)^{n_0} < |z| < (1+\varepsilon)^{n_0}$. Hence, for any $z \in \hat{\mathbb{C}} \setminus \{0, \infty\}$, there exists k large enough such that $z \in Q_0^n(S) \subset Q_0^n(N)$, for any $n \geq k$.

Now observe that $z = 0$ and $z = \infty$ are fixed points of Q_0, with no preimages other than themselves. Finally, since they do not lie in N, they do not lie in any of its eventual images. $\square$

We proved that the union of iterates of $Q_0(z)$ in any neighbourhood of a point in $\mathcal{J}(Q_0)$ contains the entire Riemann sphere $\hat{\mathbb{C}}$ except 2 points, $z = 0$ and $z = \infty$. This is a particular example of what is called *the blow-up property* of Julia sets.

Theorem 2.1.16. *(Blow-up property of Julia sets) Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. Let $U \subset \hat{\mathbb{C}}$ be an open set such that $U \cap \mathcal{J}(R) \neq \emptyset$. Then, there exist at most 2 points z_1, and z_2, such that*

$$\hat{\mathbb{C}} \setminus \{z_1, z_2\} \subset \bigcup R^n(U).$$

2.2 Local fixed point theory

To better understand the asymptotic behaviour of the sequence of iterates of a function in a neighbourhood of a fixed point, it is necessary to introduce some results in local fixed point theory (see [Mil06, Chapters 8 and 9]). In what follows, we assume $f : U \subset \mathbb{C} \to \mathbb{C}$ is a holomorphic map and $z_0 \in U$ is a fixed point, with multiplier $\lambda := f'(z_0)$. The dynamics in a neighborhood of z_0 depend on the multiplier λ (see Definition 2.1.3).

The attracting case: $0 < |\lambda| < 1$

The following result not only characterizes topologically the asymptotic behaviour of the points in a neighbourhood of the fixed point z_0, but it shows the existence of a change of coordinates between f in a neighbourhood of z_0 and the linear map $w \to \lambda w$ in a neighbourhood of $\lambda = 0$.

Theorem 2.2.1 (Koenigs' Linearization)**.** *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic map and let $z_0 \in$ be a fixed point. Assume that the multiplier $\lambda := f'(z_0)$ satisfies $|\lambda| \neq 0, 1$. Then, there exists $\varepsilon > 0$, a neighbourhood of z_0 $V \subset U$, and a local holomorphic change of coordinate $w = \phi(z)$, with $\phi(z_0) = 0$, such that $\phi \circ f \circ \phi^{-1}$ is the linear map $w \to \lambda w$ for all $w \in \mathbb{D}_\varepsilon$ with $\varepsilon > 0$, where $\mathbb{D}_\varepsilon := \{z \in \mathbb{C} \,|\, |z| < \varepsilon\}$. Furthermore, ϕ is unique up to multiplication by a non-zero constant.*

Any attracting fixed point has a corresponding *basin of attraction* (also known as *attracting basin*), the set of points which iterate towards the fixed point. Since the map is holomorphic, it is easy to show that $\mathcal{A}(z_0)$ is an open set which always includes a neighbourhood of z_0. The connected component of a basin of attraction containing the fixed point is known as *immediate basin of attraction.*

Definition 2.2.2 (Basin of attraction. Immediate basin of attraction.)**.** Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and let $z_0 \in \mathbb{C}$ be an attracting fixed point. We define the *basin of attraction* of z_0 as

$$\mathcal{A}(z_0) := \{z \in \hat{\mathbb{C}} \,|\, R^n(z) \xrightarrow[n\to\infty]{} z_0\}.$$

We also define the *immediate basin of attraction* as the connected component of $\mathcal{A}(z_0)$ which contains the fixed point z_0 and we denote it by $\mathcal{A}^*(z_0)$.

Local theory has important implications in iteration of rational maps. Suppose $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and $R(z_0) = z_0$, with $R'(z_0) = \lambda$, $0 < |\lambda| < 1$ (using, if needed, a Möbius transformation, we may assume that $z_0 \in \mathbb{C}$). We know from Koenigs' Theorem the local behaviour of R in a neighbourhood of z_0. We want to extend this to all points which eventually converge to z_0. More precisely, we extend the map ϕ (see Theorem 2.2.1) defined on a neighbourhood V of z_0 to a map $\hat{\phi}$ defined on the entire basin of attraction $\mathcal{A}(z_0)$. This is known as *global linearization* (see [Mil06, Corollary 8.4]).

Lemma 2.2.3. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. Let z_0 be an attracting fixed point, $\mathcal{A}(z_0)$ the corresponding basin of attraction, and $\mathcal{A}^*(z_0)$ the connected component of $\mathcal{A}(z_0)$ containing the fixed point. Then, there exists a holomorphic map $\hat{\phi} : \mathcal{A}(z_0) \to \mathbb{C}$, with $\hat{\phi}(z_0) = 0$, such that $\hat{\phi}\Big(R(z)\Big) = \lambda\hat{\phi}(z)$, for all $z \in \mathcal{A}(z_0)$. Furthermore, $\hat{\phi}$ maps a neighbourhood of z_0 biholomorphically onto a neighbourhood of $z = 0$ and it is unique up to multiplication by a constant.*

For a sufficiently small neighbourhood $\mathbb{D}_\varepsilon$ of 0, where the previously defined function $\hat{\phi}$ is biholomorphic, we can define the holomorphic function $\psi_\varepsilon : \mathbb{D}_\varepsilon \to \mathcal{A}(z_0)$, $\psi_\varepsilon(w) = \hat{\phi}^{-1}(w)$, for any $w \in \mathbb{D}_\varepsilon$. This map ψ_ε extends, by analytic continuation, to a maximal open disk $\mathbb{D}_r$. The set $\psi_\varepsilon(\mathbb{D}_r)$ is known as a *maximal domain of linearization.* The following lemma states the existence of this maximal domain and that its boundary contains a critical point, that is, a point c such that $R'(c) = 0$ (see [Mil06, Lemma 8.5]).

Lemma 2.2.4. *There exists a uniquely defined holomorphic map $\psi : \mathbb{D}_r \to \mathcal{A}(z_0)^*$, with $\psi(0) = z_0$, and $\psi\Big(\phi(w)\Big) \equiv w$. Furthermore, ψ extends holomorphically to a map $\hat{\psi}$ over the boundary circle $\partial\mathbb{D}_r$, and the image $\hat{\psi}(\partial\mathbb{D}_r) \subset \mathcal{A}^*(z_0)$ necessarily contains a critical point of R.*

The superattracting case: $\lambda = 0$

Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic map and let $z_0 \in U$. Assume that $f(z_0) = z_0$ and $f'(z_0) = 0$. Then, in a sufficiently small neighbourhood V of z_0, f can be written as

$$f(z) = z_0 + a_q(z - z_0)^q + O(|z - z_0|^{q+1}),\ z \in V,\ a_q \neq 0,\ q \geq 2.$$

The natural number q is called the *local degree* of the superattracting fixed point z_0. We can now define *Böttcher coordinates* around a superattracting fixed point.

Theorem 2.2.5. *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic map and let z_0 be a superattracting fixed point of local degree $q \geq 2$. Then, there exists a holomorphic change of coordinate $w = \phi(z)$, with $\phi(z_0) = 0$, which conjugates f to the map $w \to w^q$ throughout some neighbourhood of $w = 0$. Furthermore, ϕ is unique up to multiplication by an $(q-1)$st root of unity.*

We may also define a maximal domain of Böttcher coordinates. For any small enough neighbourhood $\mathbb{D}_\varepsilon$ of 0 where the previously defined function ϕ is biholomorphic, we can define the holomorphic function $\psi_\varepsilon : \mathbb{D}_\varepsilon \to \mathcal{A}^*(z_0)$ as $\psi_\varepsilon(w) = \phi^{-1}(w)$ for any $w \in \mathbb{D}_\varepsilon$. This map ψ_ε extends, by analytic continuation, to a maximal open disk $\mathbb{D}_r$. The set $\psi_\varepsilon(\mathbb{D}_r)$ is known as a *maximal domain of Böttcher coordinates* (see [Mil06, Theorem 9.3]).

Theorem 2.2.6. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map, and let z_0 be a superttracting fixed point. Then, there exists a unique open disk $\mathbb{D}_r$ of maximal radius $0 < r < 1$ such that ψ_ε extends holomorphically to a map ψ from the disk $\mathbb{D}_r$ into the immediate basin $\mathcal{A}^*(z_0)$. If $r = 1$, then ψ maps the unit disk $\mathbb{D}_1$ biholomorphically onto $\mathcal{A}^*(z_0)$ and z_0 is the only critical point in this immediate basin. On the other hand, if $r < 1$, then there is at least one other critical point in $\mathcal{A}^*(z_0)$, lying on the boundary of $\psi(\mathbb{D}_r)$.*

The parabolic case: $\lambda = e^{2\pi i\alpha}$, $\alpha = \frac{p}{q} \in \mathbb{Q}$

If $f : U \subset \mathbb{C} \to \mathbb{C}$ has a parabolic fixed point z_0 with multiplier $\lambda := e^{2\pi i\alpha}$, by shrinking to a sufficiently small neighbourhood $V \subset U$, f can be written as

$$f(z) = z_0 + e^{2\pi i\frac{p}{q}}(z - z_0) + a_k(z - z_0)^k + O(|z - z_0|^{k+1}),\ k \geq 2,\ a_k \neq 0.$$

Hence, considering $g = f^q$ (and renaming g as f), we may assume that

$$f(z) = z + a(z - z_0)^k + \ldots,\ a \neq 0,$$

that is, z_0 is an isolated parabolic fixed point with multiplier $\lambda = 1$.

This case is very different to the previous cases since, as we will see, the linear part of the map (which is the identity in this case) does not explain the local behaviour of z_0. Instead, the dynamics around z_0 describe what are called "petals" (and all together a "flower"). This is why the main result describing the local dynamics around a parabolic fixed point is called the Fatou-Leau Flower Theorem. To state it, we must first define the notion of *petal*, an invariant set having z_0 in its boundary.

Definition 2.2.7 (Attracting and repelling petals)**.** Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic map such that $z_0 \in U$ is a fixed point with multiplier $\lambda = 1$. Let $N \subset U$ be a neighbourhood of z_0 containing no critical points (so, $f : N \to f(N) = N'$ is a diffeomorphism). We say that $V \subset N$ is an *attracting petal* if:

- V is open and connected;
- $\overline{V} \subset (N \cup N')$;
- $f(\overline{V}) \subset V \cup \{z_0\}$;
- $\bigcap\limits_{k \geq 0} f^k(\overline{V}) = \{z_0\}$.

We say that V is a *repelling petal* if it is an attracting petal for the map f^{-1}.

A preliminary form of the following theorem was first proved by Leau. Later on, Julia and Fatou have improved the result to the current form, that is generally the one used in complex dynamics.

Theorem 2.2.8. *(Leau-Fatou Flower Theorem) Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic map. Let $z_0 \in U$ be a parabolic fixed point with multiplier $\lambda = 1$, that is, by shrinking to a sufficiently small neighbourhood $V \subset U$, f can be written as*

$$f(z) = z + a(z - z_0)^{k+1} + O(|z - z_0|^{k+2}),\ k \geq 1,\ a \neq 0.$$

Then, there exist k attracting $\{U_1, U_2, \ldots, U_k\}$ and k repelling $\{U_1', \ldots, U_k'\}$ petals such that:

- *$U_i \cap U_j = \emptyset$ and $U_i' \cap U_j' = \emptyset$, $\forall i \neq j$.*
- *$U_j \cap U_\ell' \neq \emptyset$ for $\ell = j,\ j + 1$.*
- *$U_j \cap U_\ell' = \emptyset$ for $\ell \neq j,\ j + 1$.*
- *The union of all attracting and repelling petals gives a pinched neighbourhood of z_0 without critical points.*

The irrational case: $\lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$

According to the previously studied cases, we conclude that, except for the parabolic case, the behaviour of the iterates of points in a sufficiently small neighbourhood of the fixed point $z_0 \in U$ is governed by the behaviour of its linear part. Keeping this in mind, it makes sense to classify the irrationally indifferent fixed points with respect to the existence (or not) of a conformal change of coordinates of R (near z_0) to the map $w \to e^{2\pi i\alpha} w$ (that is, the irrational rotation of angle R_α).

Definition 2.2.9 (Siegel and Cremer points)**.** Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic map and let $z_0 \in U$ such that $f(z_0) = z_0$ and $f'(z_0) = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. We say that z_0 is a *Siegel point* if there exists a local (conformal) change of coordinates $\phi : V \subset U \to \mathbb{C}$ such that

$$\phi \circ f \circ \phi^{-1}(w) = e^{2\pi i\alpha} w,\ \forall w \in \phi(V).$$

Otherwise, we say that z_0 is a *Cremer point*.

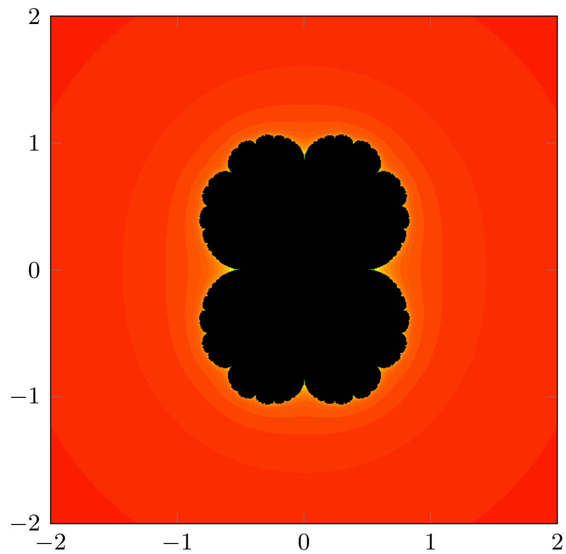


Figure 2.3: The figure illustrates the dynamical plane of $Q_c = z^2 + c$, for $c = 0.25$, which is coloquially known as the *cauliflower*. In black, we see the parabolic basin corresponding to the parabolic fixed point $z = \frac{1}{2}$.

The conditions (on α) that guarantee the existence of such change of coordinates rely on the arithmetic structure of α as an irrational number. For instance, it is known that if α is diophantine then z_0 is a Siegel point (see [Sie42]). More details can be found in [Mil06, Chapter 11]. Nevertheless, if z_0 is an irrationally indifferent fixed point of a rational map R, one can show the following dynamical characterization of R being linearizable in a neighbourhood of z_0. For the sake of completeness, we give the proof of the result (see also [Bea91, Theorem 6.6.2]).

Theorem 2.2.10. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. Let $z_0 \in \mathbb{C}$ (a Möebius transformation allows us to assume this without loss of generality) and assume that for $z \in U$ we have*

$$R(z) = z_0 + e^{2\pi i\alpha}(z - z_0) + O(|z - z_0|^2),\ \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Then, R is (locally) linearizable at z_0 if and only if $z_0 \in \mathcal{F}(R)$.

Proof. Assume that R is linearizable around z_0. this implies that there exist N a neighbourhood of z_0 and $\phi : N \to \phi(N)$ conformal such that

$$\phi \circ R \circ \phi^{-1}(w) = e^{2\pi i\alpha} w.$$

Let ε small enough such that $\phi^{-1}(\mathbb{D}_{z_0,\varepsilon}) \subset N$, where $\mathbb{D}_{z_0,\varepsilon} = \{z \in \mathbb{C} \,|\, |z - z_0| < \varepsilon\}$. Observe that $z_0 \in \phi^{-1}(\mathbb{D}_{z_0,\varepsilon})$. We have that $R\Big(\phi^{-1}(\mathbb{D}_{z_0,\varepsilon})\Big) = \phi^{-1}(\mathbb{D}_{z_0,\varepsilon})$. By Montel's theorem (see Theorem 2.1.10), the family $\{R^n\}_{n\geq 1}$ is normal in $\phi^{-1}(\mathbb{D}_{z_0,\varepsilon})$. Therefore, $z_0 \in \mathcal{F}(R)$.

Now we assume that $z_0 \in \mathcal{F}(R)$. Let $\lambda := R'(0)$. We have that $|\lambda| = 1$. Using eventually a Möbius transformation, we can assume without loss of generality that $z_0 = 0$. It follows that in a neighbourhood of $z = 0$, by Arzelá-Ascoli Theorem (see Theorem 2.1.9), the family of iterations $\{R^n\}_{n\geq 1}$ is locally equicontinous. Therefore, there exists a neighbourhood N of $z = 0$ such that $|R^n(z) - R^n(0)| = |R^n(z)| < 1$, for any $z \in N$.

First, for $n \geq 1$, we define the functions

$$T_n(z) := \frac{1}{n}\Big(z + \frac{R(z)}{\lambda} + \frac{R^2(z)}{\lambda^2} + \cdots + \frac{R^{n-1}(z)}{\lambda^{n-1}}\Big).$$

We have that $(R^k)'(0) = \lambda^k$, for $k \geq 1$. It follows that $T_n'(0) = 1$ and $|T_n(z) \leq 1|$, since $N \subset \mathbb{D}_1$. Using that

$$\frac{n}{\lambda}T_n\Big(R(z)\Big) + z = nT_n(z) + \frac{R^n(z)}{\lambda^n},$$

we can write

$$T_n\Big(R(z)\Big) - \lambda T_n(z) = \frac{\lambda}{n}\Big(\frac{R^n(z)}{\lambda^n} - z\Big).$$

We get that $\Big(T_n\Big(R(z)\Big) - \lambda T_n(z)\Big) \to 0$ uniformly on N when $n \to \infty$. By Montel's theorem, the family $\{T_n\}_{n\geq 1}$ is normal. By definition of normal families (see Definition 2.1.7), there exists a subsequence $\{T_{n_k}\}$ which converges uniformly on compact sets to a limit map, say ϕ. We have that $\phi\Big(R(z)\Big) = \lambda\phi(z)$. Since $\phi'(0) = 1$, the map ϕ is not a constant map, therefore the map R can be liniarized around $z = 0$. □

If we assume that z_0 is a Siegel point of a rational map, it is natural to ask about the maximal domain of definition of the linearizing map ϕ.

Definition 2.2.11 (Siegel disk)**.** *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. Assume that z_0 is a Siegel point. The maximal domain of definition of the local change of coordinates described in Definition* 2.2.9 *is called a Siegel disk, say $\Delta(z_0)$. By definition, $\Delta(z_0)$ is simply connected and, also by definition, $f|_{\Delta(z_0)}$ is conformal.*

One major problem in holomorphic dynamics is to give geometrical, topological, and measure properties of $\partial\Delta(z_0)$. From the dynamical point of view, the interest is centered on understanding the relationship between $\Delta(z_0)$ and the orbits of the critical points. A cornerstone result is the following.

Theorem 2.2.12. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. Assume that z_0 is a Siegel point, and denote by $\Delta(z_0)$ the Siegel disk associated to z_0. Then, there exist $c \in \mathbb{C}$ such that*

$$\partial\Delta(z_0) \supset \overline{\bigcup_{n\geq 0} R^n(c)}.$$

2.3 Fatou components

We study the topology and geometry of the Fatou set using its connected components. They are known as *Fatou components* and are a central topic of interest in complex dynamics.

Definition 2.3.1. *(Fatou component)* A connected component of the Fatou set is called a *Fatou component.*

Since the Fatou set is completely invariant, in a given dynamical plane, the image of a Fatou component may only be a Fatou component. We have already introduced various types of Fatou components and dynamics of maps inside these Fatou components. Our goal now is to introduce the three major results about Fatou components (No Wandering Domains, Classification, and Fatou-Shishikura inequality).

We begin by extending the concept of periodicity to Fatou components.

Definition 2.3.2. Let f be a holomorphic map, and let $U \subset \mathcal{F}(f)$ be a Fatou component. Then, U is:

1. *fixed*, if $f(U) = U$;
2. *periodic*, if there exists $p \geq 1$ such that $f^p(U) = U$. If U is periodic, but not fixed, we say that U is *strictly periodic*;
3. *eventually periodic*, if there exist $m > n \geq 1$ such that $f^m(U) = f^n(U)$. If U is eventually periodic, but not periodic, we say that U is *strictly preperiodic*;
4. *wandering*, if it is not eventually periodic.

Sullivan (see [Sul85]) proved that rational maps do not have wandering Fatou components (this was a problem posed by Fatou himself). This is a compelling argument for rational maps having *simpler* dynamics. This result is also the first major result in holomorphic dynamics proved using quasiconformal surgery.

Theorem 2.3.3 (No Wandering Domains Theorem)**.** *Every Fatou component of a rational map is eventually periodic.*

It follows from Theorem 2.3.3 that any Fatou component of a rational map R is either a branched covering or a biholomorphic copy of a periodic Fatou component. We now give a classification of all periodic Fatou components of rational maps (see Figure 2.4). This theorem is mainly due to the works of Fatou and Sullivan. By considering eventually R^p, we can state the result for invariant Fatou components.

Theorem 2.3.4 (Classification of invariant Fatou components of rational maps)**.** Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and let $U \subset \mathcal{F}(R)$ be a invariant Fatou component. Then, U is one of the following:

1. (immediate) attractive basin: U contains an attracting fixed point z_0 and $R^n(z) \to z_0$, as $n \to \infty$, for all $z \in U$;
2. parabolic basin (or Leau domain): ∂U contains a unique fixed point z_0 such that $R^n(z) \to z_0$, as $n \to \infty$, for all $z \in U$. Moreover, $R'(z_0) = 1$;
3. Siegel disk: there exists a conformal map $\phi : U \to \mathbb{D}$ such that $(\phi \circ R \circ \phi^{-1})(z) = e^{2\pi i \alpha} z$, for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$;
4. Herman ring: there exist $r > 1$ and a conformal map $\phi : U \to \{1 < |z| < r\}$ such that $(\phi \circ R \circ \phi^{-1})(z) = e^{2\pi i \alpha} z$, for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Already proposed by Fatou and Julia, a major problem when describing the dynamical plane of a rational map R is to know the number of invariant (periodic) Fatou components (that is, different "tame behaviours") that R may have. It was already noticed before that immediate basins of (super)-attracting fixed (periodic) points always have an associated critical orbit. Using a similar argument to the one used to prove this result, one can also show that if z_0 is a parabolic fixed point with multiplier $\lambda = e^{2\pi i \frac{p}{q}}$ then there should be at least q different critical orbits associated to it. A more refined result is the celebrated Fatou-Shishikura inequality (see Theorem 2.3.6, see also [Mil06, Theorem 8.6]).

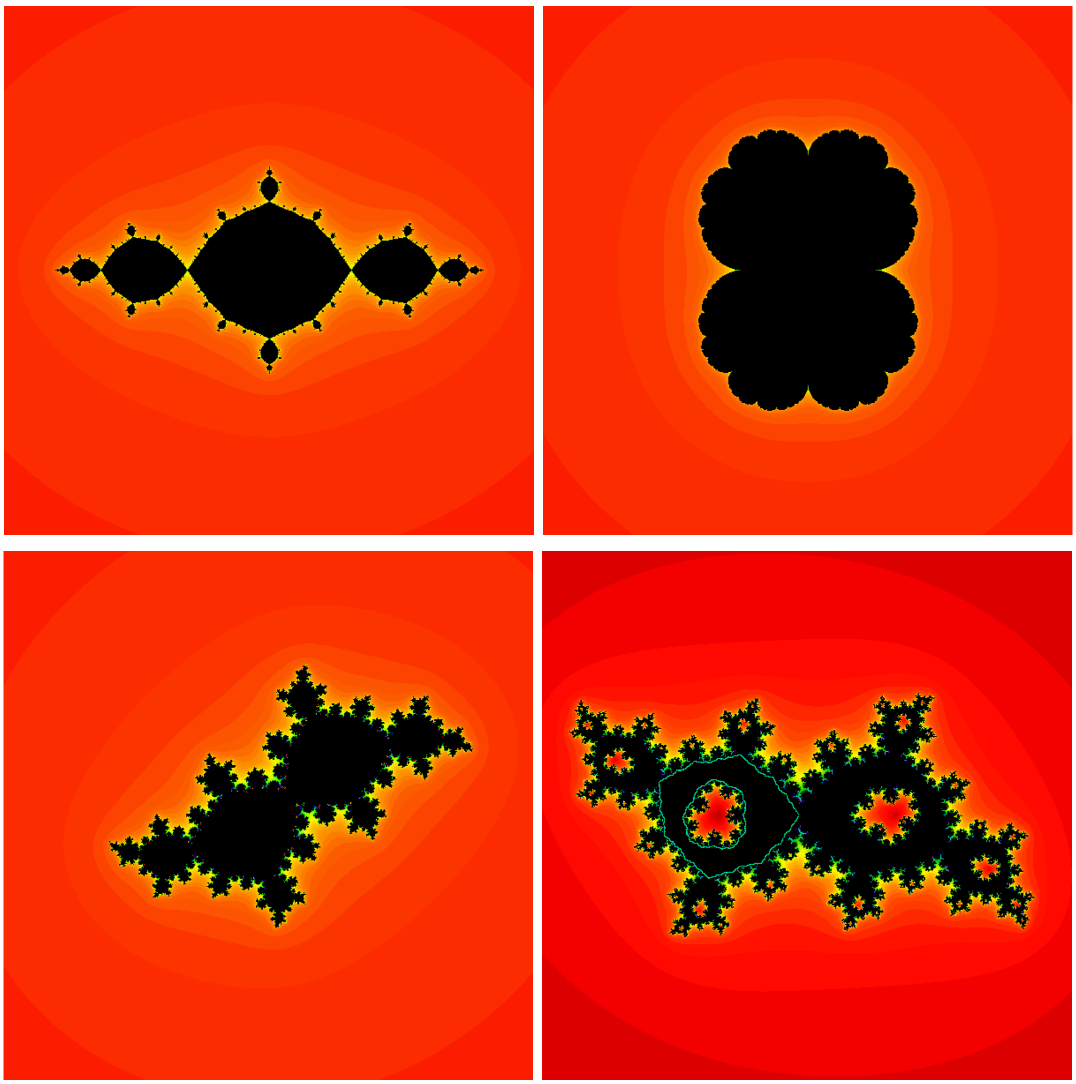


Figure 2.4: Examples of period Fatou components of rational maps. Top left corner, an attracting basin. Top right corner, a parabolic basin. Bottom left corner, a Siegel disk. Bottom right corner, a Herman ring.

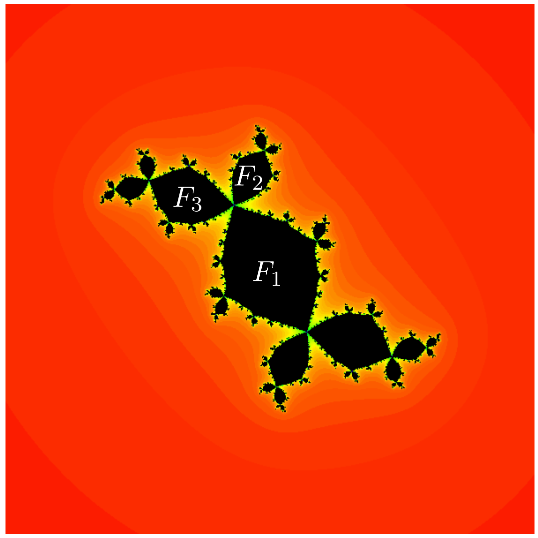


Figure 2.5: The figure illustrates the dynamical plane of $Q_c = z^2 + c$, for $c = -0.123 + 0.745i$, which is known as the *Douady rabbit*. In red, we see the basin of attraction of $z = \infty$. Since it is a simply connected set, it is also the immediate basin of attraction of $z = \infty$. In black, we see the basin of attraction corresponding to a periodic cycle of period 3. The 3 periodic points lie in the Fatou components F_1, F_2, and F_3. The corresponding immediate basin of attraction is the set $F_1 \cup F_2 \cup F_3$.

Theorem 2.3.5. *If R is a rational map of degree $d \geq 2$, then the immediate basin of every attracting periodic orbit contains at least one critical point. Hence, the number of attracting periodic orbits is less than or equal to the number of critical points.*

It was Fatou (see [Fat20]) who first attempted to give a bound, depending on the number of critical values, of the number of cycles of periodic Fatou components of a rational map. Later on, Douady and Hubbard expanded on this topic and proved a sharper bound, for polynomial maps. Their argument did not work, however, for rational maps having no attracting cycle. Sullivan also managed to find an upper bound in a general setting, and conjectured a sharp bound. It was finally Shishikura, who proved in his Master's thesis (see also [Shi87]) the precise upper bound of periodic cycles for rational maps. He accomplished this by using quasiconformal surgery to *transform* the Herman rings of a map to Siegel disks.

Theorem 2.3.6 (Fatou-Shishikura inequality)**.** *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree d. Let $n_{\mathbf{AB}}$ denote the number of attracting basins of R, $n_{\mathbf{PB}}$ denote the number of parabolic cycles of R, $n_{\mathbf{SD}}$ denote the number of Siegel disk cycles of R, $n_{\mathbf{HR}}$ denote the number of Herman ring cycles of R, and $n_{\mathbf{Cremer}}$ denote the number of Cremer fixed points. Then,*

$$n_{\mathbf{AB}} + n_{\mathbf{PB}} + n_{\mathbf{SD}} + 2n_{\mathbf{HR}} + n_{\mathbf{Cremer}} \leq 2(d-1).$$

In the dynamical planes that we will study, there will always be attracting fixed points. This implies the existence of immediate basins of attraction in the corresponding dynamical planes. The next result states that if an immediate basin of attraction is multiply connected, it is in fact infinitely connected (see [Mil06], Theorem 8.9).

Theorem 2.3.7. *Let $\mathcal{A}^*(z_0)$ be the immediate basin of an attracting fixed point z_0. Then, the complement $\hat{\mathbb{C}} \setminus A^*$ is either connected or else has uncountably many connected components.*

2.4 Riemann-Hurwitz formula

Recall that the *connectivity* of a set $U \subset \hat{\mathbb{C}}$ is the number of connected components of its complement $\hat{\mathbb{C}} \setminus U$. The Riemann-Hurwitz formula is the fundamental tool necessary to study connectivities of Fatou components. The Riemann-Hurwitz formula, as originally stated, has a much larger scope, regarding Riemann surfaces. We will use a particular form of it (see, for instance, [Ste93]), which will suffice with respect to our study of rational maps.

Theorem 2.4.1. *(Riemann-Hurwitz formula) Let $U, V \subset \hat{\mathbb{C}}$ be two connected domains of connectivity $m_U, m_V \in \mathbb{N}^*$ and let $f : U \to V$ be a degree d proper map branched over r critical points, counted with multiplicity. Then,*

$$m_U - 2 = d(m_V - 2) + r.$$

For our purposes, we add here two direct corollaries of the Riemann-Hurwitz formula, that we will use extensively in the following chapters.

Corollary 2.4.2. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree d. Then, R has at most $2d - 2$ critical points, counting multiplicity.*

Along the text we also use the following corollary of the Riemann-Hurwitz formula (compare [CFG15, Corollary 2.2]).

Corollary 2.4.3. *Let $U \subset \hat{\mathbb{C}}$ be an open set and let $f : U \to f(U)$ be a proper holomorphic map. Then, the following statements hold:*

(a) If $f(U)$ is doubly connected and f has no critical points in U, then U is doubly connected.

(b) If $f(U)$ is simply connected and f has at most one critical point in U (not counting multiplicities), then U is simply connected.

Proof. First we prove (a). We get from the Riemann-Hurwitz formula that $m_U - 2 = 0$. Therefore, the set U is doubly connected.

Second, we prove (b). By applying the Riemann-Hurwitz formula we obtain that

$$m_U - 2 = r - d.$$

Assume that there is no critical point in U. It follows that the map f is conformal on U, hence the degree of the map is $d = 1$. We get that U is simply connected.

Now consider the case that there exists a critical point c of multiplicity r. Therefore, there exists $W \subset U$ a neighbourhood of c such that the map $f|_W$ has degree $r + 1$. We get that $d \geq r + 1$ and, therefore, $m_U - 2 \leq 1$. It follows that $m_u = 1$ and U is simply connected. $\square$

2.5 Quasiconformal surgery

Quasiconformality can be described as a degree of regularity, used to study structural stability in holomorphic dynamics. Quasiconformal maps are continuous, but not necessary differentiable. Differentiable maps with a continuous derivative are quasiconformal on compact sets. The main reason that quasiconformal mappings are used in holomorphic dynamics are *quasiconformal conjugacies*. Conformal conjugacies for a holomorphic map are sparse, usually restricted to a few maps in the same family that we study. Quasiconformal conjugacies, however, are much more widespread and preserve many of the topological and dynamical properties that we are interested in. Another key advantage of quasiconformal maps is their flexibility. It follows from the Identity Theorem that it is impossible to paste different *holomorphic* maps together and obtain a holomorphic map. Meanwhile, pasting together *quasiregular* maps to obtain a new quasiregular map (ideally with specific properties, and conjugate to a given holomorphic map) is precisely the scope of *quasiconformal surgery*.

Quasiconformal mappings have been a much studied mathematical topic in the past century. The first to have studied them was Grötszch (see [Grö28]) in 1928, as a solution to the *Beltrami equation*

$$\partial_{\bar{z}}\phi(z) = \mu(z)\partial_z\phi(z). \tag{2.1}$$

Later on, Teichmüller (see [Tei40]) used the analytic definition of quasiconformal maps to study the function theory of Riemann surfaces. Pfluger (see [Pfl48]) and Ahlfors (see [Ahl54]) used the geometric definition of quasiconformal maps to study value distribution theory of holomorphic functions. The equivalence of the two definitions is not obvious, and was only proven in 1959, by Gehring and Lehto (see [GL59]).

The main result that connects quasiconformal maps and complex dynamics is the Measurable Riemann Mapping Theorem (see Theorem 2.5.29). It is the essential tool necessary for *quasiconformal surgery* and it is usually called the *Integrability Theorem.* The theorem is due to Morrey (see [Mor38]), Bojarski (see [Boj55]), and Ahlfors and Bers (see [AB60]). For more detailed explanations, see [Ahl06], [AIM09], and [BF14, Chapters 1 and 2].

Now a standard tool of complex dynamics, it was Sullivan who introduced *quasiconformal mappings* to complex dynamics, by doing a surgery construction to prove the No Wandering Domains Theorem (see Theorem 2.3.3). While Sullivan introduced what is known as *soft surgery*, Douady and Hubbard performed the first *cut-and-paste* surgery to prove the Straightening Theorem (see Theorem 2.6.4) using polynomial-like maps. Later on, Shishikura defined the principles of surgery and used *cut-and-paste* surgery to prove the Fatou-Shishikura inequality (see Theorem 2.3.6).

Quasiconformality

There exist several (equivalent) definitions of a quasiconformal map. We will start with an analytic definition, which, in turn, requires the concept of *absolute continuity on lines.*

Definition 2.5.1. *(Absolute continuity on an interval)* A continuous complex valued function f defined on an interval $I \subset \mathbb{R}$ is said to be *absolutely continuous* on I if it satisfies the following: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\sum_j |f(b_j) - f(a_j)| < \varepsilon$ for every finite sequence of nonintersecting intervals (a_j, b_j) whose closure are contained in I and have a total length $\sum |b_j - a_j| < \delta$.

Definition 2.5.2. *(Absolute continuity on lines, ACL)* A continuous function $f : U \to \mathbb{C}$ is said to be *absolutely continuous on lines* if for any family of parallel lines in any disc D compactly contained in U (that is, $D \subset U$), f is absolutely continuous on almost all of them.

We can now give an analytic definition of $K-$*quasiconformal mappings.*

Definition 2.5.3. *(Analytic definition of K-quasiconformal mapping)* Let U, $V \subset \mathbb{C}$ be two domains and let $1 \leq K < \infty$. A mapping $\phi : U \to V$ is K*-quasiconformal* if and only if:

1. ϕ is a homeomorphism;

2. ϕ is ACL;

3. $|\partial_{\bar{z}}\phi| \leq k|\partial_z\phi|$ almost everywhere, where $k \coloneqq \frac{K-1}{K+1} \in [0,\, 1]$.

To introduce a geometrical definition of quasiconformal maps, we first define the *modulus* of an annulus.

Definition 2.5.4. *(Modulus of an annulus) Let $A \subset \mathbb{C}$ be an open annulus, that is, a doubly connected domain in $\hat{\mathbb{C}}$. Then, A can be conformally mapped onto a standard annulus (the conformal map is unique up to multiplication by a positive real number)*

$$\mathbb{A}_{r,\,R} = \{z \in \mathbb{C} \,|\, 0 \leq r < |z| < R \leq \infty\}.$$

The conformal modulus of the annulus A is defined as the modulus of $\mathbb{A}_{r,\,R}$, i.e.

$$\mathrm{mod}A \coloneqq \mathrm{mod}\mathbb{A}_{r,\,R} \coloneqq \left\{ \begin{array}{ll} \frac{1}{2\pi}\log\frac{R}{r} & \text{if}\, r > 0 \,\text{and}\, R < \infty \\ \infty & \text{if}\, r = 0 \,\text{or}\, R = \infty. \end{array} \right\}$$

Definition 2.5.5. *(Geometric definition of K-quasiconformal mapping)* Let $U, V \subset \mathbb{C}$ be two domains and let $1 \leq K < \infty$. Then, $\phi : U \to V$ is *K-quasiconformal* if and only if ϕ is an orientation preserving homeomorphism satisfying

$$\frac{1}{K}\mathrm{mod}A \leq \mathrm{mod}\phi(A) \leq K\mathrm{mod}A,$$

for all annuli A compactly contained in U.

The analytic and geometric definitions of quasiconformal maps are equivalent, as proven by Gehring and Lehto (see [GL59]). We now give some basic properties of quasiconformal maps, which will be useful later.

Theorem 2.5.6. *Let U, V, and W be domains of $\mathbb{C}$ and let K, K_1, $K_2 \in [1, \infty)$. Let $\phi : U \to V$ be a $K-$quasiconformal map. Then:*

1. *The partial derivatives $\partial_z\phi$ and $\partial_{\bar{z}}\phi$ exist almost everywhere.*

2. *The inverse map ϕ^{-1} is K-quasiconformal.*

3. *If S is a set of measure zero, then $\phi(S)$ is also a set of measure zero.*

4. *Let $\phi_1 : U \to V$ and $\phi_2 : V \to W$ be K_1 and K_2-quasiconformal maps. Then the map $\phi_2 \circ \phi_1$ is K_1K_2-quasiconformal.*

The next result relates quasiconformality and conformality.

Theorem 2.5.7. *(Weyl's lemma) If ϕ is 1-quasiconformal, then ϕ is conformal. In other words, if ϕ is quasiconformal and $\partial_{\bar{z}}\phi = 0$ almost everywhere, then ϕ is conformal.*

We now give a well known theorem in complex analysis, proven by Koebe and Poincaré, which categorizes the topology of unidimensional complex spaces.

Theorem 2.5.8. *(Uniformization Theorem) Let S be a simply connected Riemann surface. Then, S is conformally equivalent to $\mathbb{D}$, $\mathbb{C}$ or $\hat{\mathbb{C}}$.*

Now we generalize the definition of K-quasiconformal mappings to Riemann surfaces.

Definition 2.5.9. *(K-quasiconformal mapping between Riemann surfaces) Let S and S' be two Riemann surfaces, and let $\phi : S \to S'$ be a homeomorphism. Then, ϕ is quasiconformal if and only if there exists a $K \geq 1$ so that ϕ is locally K-quasiconformal when expressed in charts.*

Quasiconformal maps are defined on domains. To extend them to the boundary, we require *local connectivity.*

Definition 2.5.10. *(Locally connected set)* A set $X \subset \mathbb{C}$ is *locally connected* if for every point $x \in X$ and any arbitrarily small $\varepsilon > 0$, the intersection $D_\varepsilon(x) \cap X$ is connected.

Now we state the extension of a conformal isomorphism to the boundary of a domain. The following theorem is due to Torhorst (see [Tor21]) and based on the work on prime ends of Caratheodory (compare with the discussion in [RG14]). It is used for extending Riemann maps to the boundary.

Theorem 2.5.11. *(Carathèodory-Torhorst Theorem) Let $G \subset \mathbb{C}$ be a simply connected, bounded domain. Let $f : \mathbb{D} \to G$ be a Riemann mapping for G (that is, $f : \mathbb{D} \to G$ is a conformal isomorphism). Then, f has a continuous extension $\bar{f} : \overline{\mathbb{D}} \to \overline{G}$ if and only if ∂G is locally connected. Moreover, f has a continuous and injective extension to $\overline{\mathbb{D}}$ if and only if ∂G is a Jordan curve.*

Before extending the previous result to quasiconformal maps, we need to define the concepts of *quasicircles*, *quasidisks*, and *quasiannuli.*

Definition 2.5.12. *(Quasicircle, quasidisk, quasiannulus)* Let $\gamma \subset \mathbb{C}$ be a Jordan curve. Then, γ is called a *quasicircle* if for some $C > 0$

$$\mathrm{diam}\gamma(z_1,\, z_2) \leq C|z_1 - z_2|, \quad \text{for } z_1,\, z_2 \in \gamma,$$

where $\gamma(z_1,\, z_2)$ is the arc of smaller diameter joining z_1 and z_2. The interior of γ, $\mathrm{Int}(\gamma)$, is known as a *quasidisk.* A bounded annulus such that its outer and inner boundary curves are quasicircles, is known as a *quasiannulus.*

Quasiconformality is a property of a map defined on a domain. A natural question arises, what degree of regularity must a map satisfy on the boundary of a domain (bounded by quasicircles), so that it is extended to a quasiconformal map on the domain. The answer is *quasisymmetry.*

Definition 2.5.13. A map $h : \mathbb{S}^1 \to \mathbb{C}$ is *quasisymmetric* if h is injective and if there exists a strictly increasing function $\lambda : [0,\, +\infty) \to [0,\, +\infty)$ such that

$$\frac{1}{\lambda\left(\dfrac{|z_2 - z_3|}{|z_1 - z_2|}\right)} < \frac{|h(z_1) - h(z_2)|}{|h(z_2) - h(z_3)|} < \lambda\left(\frac{|z_1 - z_2|}{|z_2 - z_3|}\right) \quad \text{for } z_1,\, z_2,\, z_3 \in \mathbb{S}^1.$$

We now connect the concepts of quasicircles and quasisymmetry. Quasicircles are precisely the non-self-intersecting closed curves with quasisymmetric parametrizations.

Proposition 2.5.14. *If $h : \mathbb{S}^1 \to \mathbb{C}$ is quasisymmetric, then $\gamma := h(\mathbb{S}^1)$ is a quasicircle. Conversely, if γ is a quasicircle, then it is image of the unit circle under some quasisymmetric map.*

The *Carathèodory-Torhorst Theorem* describes the extension of a Rieman mapping defined on a domain, up to the boundary. The following result (see, for instance, [BF14, Proposition 2.30]) states that given a quasisymmetric map defined on the boundary of a quasidisk (or a quasiannulus), there exists a quasiconformal interpolation of the map to the entire quasidisk (quasiannulus).

Proposition 2.5.15. *The following statements hold:*

1. *Suppose G_1 and G_2 are quasidisks bounded by γ_1 and γ_2, and let $f : \gamma_1 \to \gamma_2$ be quasisymmetric. Then, f extends to a quasiconformal map $\overline{f} : \overline{G_1} \to \overline{G_2}$.*
2. *For $j = 1, 2$ suppose that A_j are open quasiannuli bounded by the quasicircles γ_j^{in}, γ_j^{out}. Let $f^{\mathrm{in}} : \gamma_1^{\mathrm{in}} \to \gamma_2^{\mathrm{in}}$ and $f^{\mathrm{out}} : \gamma_1^{\mathrm{out}} \to \gamma_2^{\mathrm{out}}$ be quasisymmetric maps between the inner and outer boundaries respectively. Then, there exists a quasiconformal map $f : \overline{A}_1 \to \overline{A}_2$ such that $f|_{\gamma_1^{\mathrm{in}}} = f^{\mathrm{in}}$ and $f|_{\gamma_1^{\mathrm{out}}} = f^{\mathrm{out}}$.*

Almost complex structures

Let $\mathbb{C}_{\mathbb{R}}$ be the complex plane, viewed as the two-dimensional oriented Euclidean $\mathbb{R}$-vector space with the orthonormal positively oriented standard basis $\{1, i\}$. We define two attributes of an ellipse, *dilatation* and *Beltrami coefficient* (also known as *complex dilatation*). Dilatation of an ellipse describes the shape, but not the positioning of an ellipse in $\mathbb{C}_{\mathbb{R}}$. The Beltrami coefficient is sufficient to describe both the the shape, and the positioning of an ellipse in $\mathbb{C}_{\mathbb{R}}$.

Definition 2.5.16. *(Dilatation and Beltrami coefficient of an ellipse)* Let $\mathcal{E}$ be an ellipse and let M, respectively m, represent the major, respectively, the minor axis of $\mathcal{E}$. The dilatation $K_{\mathcal{E}}$ and the Beltrami coefficient $\mu(\mathcal{E})$ are given by:

$$K_{\mathcal{E}} = \frac{M}{m} > 1, \quad \mu(\mathcal{E}) = \frac{M-m}{M+m} e^{2i\theta} \in \mathbb{D},$$

where $\theta \in [0, \pi)$ is the argument of the direction of the minor axis.

At any point $u \in U \subset \mathbb{C}$ (U is usually assumed to be a domain), one may *attach* a corresponding copy of $\mathbb{C}_{\mathbb{R}}$, containing an ellipse $\mathcal{E}_u$. The properties that we will use to describe the ellipse $\mathcal{E}_u$ are precisely dilatation and Beltrami coefficient. Therefore, any scaled copy of the ellipse $\mathcal{E}_u$ would suffice, and we will call $\mathcal{E}_u$ an *infinitesimal ellipse.* Let $U \subset \mathbb{C}$ be a domain and σ be a *field of infinitesimal ellipses* $\mathcal{E}_u \in T_u U$ defined at almost every point $u \in U$, where $T_u U$ denotes the colelction of the tangent spaces over points $u \in U$, each one viewed as a copy of $\mathbb{C}_{\mathbb{R}}$. If the map $\mu : U \to \mathbb{D}$ defined by the Beltrami coefficient of the ellipse $\mathcal{E}_u$ is measurable with respect to Lebesgue measure, we say that σ is a *measurable field of ellipses.* Quasiconformal surgery constructions involve defining such measurable fields of infinitesimal ellipses, known as *almost complex structures.*

Definition 2.5.17. *(Almost complex structure)* Let $U \subset \mathbb{C}$, and let

$$TU = \bigcup_{u \in U} T_u U$$

be the tangent bundle over U, i.e. the collection of the tangent spaces over points $u \in U$, each one viewed as a copy of $\mathbb{C}_{\mathbb{R}}$. An *almost complex structure* σ on U is a measurable field of infinitesimal ellipses $\mathcal{E} \subset TU$.

To introduce a notion of boundedness for an almost complex structure, we must extend the definition of dilatation to the entire almost complex structure. Note that the notion of Beltrami coefficient can be extended to an almost complex structure by using the map $\mu : U \to \mathbb{D}$, $\mu(u) := \mu(\mathcal{E}(u))$, where $\mathcal{E}(u)$ denotes the ellipse at the point u.

Definition 2.5.18. *(Dilatation of an almost complex structure)* Let $U \subset \mathbb{C}$, and let TU be the tangent bundle over U. Let σ be an almost complex structure on U. The *dilatation* of σ is defined as

$$K(\sigma) := \operatorname*{ess\,sup}_{u \in U} K(u), \qquad \textit{where} \quad K(u) := \frac{1 + |\mu(u)|}{1 - |\mu(u)|}$$

denotes the dilatation of $\mathcal{E}(u)$.

There exists one particular almost complex structure that always appears in surgery constructions (as we will later see, due to the Integrability Theorem). Its name is the *standard complex structure* and it is trivially obtained by having each ellipse $\mathcal{E}(u)$ be a circle. Furthermore, a quasiconformal map preserves the standard complex structure if and only if it is a conformal map.

Definition 2.5.19. *(Standard complex structure)* Let $U \subset \mathbb{C}$ be a domain and TU its tangent bundle. The almost complex structure σ_0 such that $\mu_0(u) = 0$, for any $u \in U$ is known as the *standard complex structure.*

The usual way to construct a non-trivial almost complex structure is by using *pullback* of quasiconformal maps. Before introducing pullback, we state formally the existence of the partial derivatives of quasiconformal maps a.e (see [BF14, Corollary 1.16]).

Proposition 2.5.20. *Let U be a domain. If $\phi : U \to \mathbb{C}$ is a quasiconformal map, then $\partial_z \phi(z) \neq 0$ almost everywhere. Moreover, $\operatorname{Jac} \phi(z) > 0$ almost everywhere.*

Next we introduce the *pullback* of an almost complex structure by a quasiconformal map $\phi : U \to V$. The preimage of a set of measure zero through a quasiconformal map is also a set of measure zero. Moreover, because of Proposition 2.5.20, $D_u\phi$ is well defined and invertible almost everywhere. It follows that if U is a domain and σ is an almost complex structure in V, the following definition of pullback of σ by ϕ is well defined almost everywhere in U.

Definition 2.5.21. *(Pullback of an almost complex structure by a quasiconformal map)* Let $U, V \subset \mathbb{C}$ be domains. Let $\phi : U \to V$ be a quasiconformal map and let σ be an almost complex structure on V. The *pullback* of σ by ϕ is the almost complex structure σ' defined on U which associates, to almost every $u \in U$, the infinitesimal ellipse

$$\sigma' := (D_u\phi)^{-1}(\mathcal{E}_{\phi(u)}).$$

Remark 2.5.22. *If μ is the Beltrami coefficient of the almost complex structure σ, the expression of pullback by a quasiconformal map ϕ is*

$$\phi^*\mu(u) = \frac{\partial_{\bar{z}}\phi(u) + \mu(\phi(u))\overline{\partial_z\phi(u)}}{\partial_z\phi(u) + \mu(\phi(u))\overline{\partial_{\bar{z}}\phi(u)}}.$$

Note that if σ_0 is the standard structure ($\mu_0 \equiv 0$), the expression of the pullback is

$$\phi^* \mu_0(u) = \frac{\partial_{\bar{z}} \phi(u)}{\partial_z \phi(u)},$$

which highlights the connection between quasiconformal maps and the Beltrami equation (see 2.1).

Remark 2.5.23. *If f is a conformal map, then the expression of the pullback further simplifies to*

$$f^* \mu(u) = \mu(f(u)) \frac{\overline{\partial_z f(u)}}{\partial_z f(u)}.$$

Therefore, pullback by conformal ($1-$quasiconformal) maps only rotates the corresponding ellipse.

The families of maps that we study are rational maps, that is, the maps holomorphic on the Riemann sphere. We now extend our definition of quasiconformal maps to Riemann surfaces.

Definition 2.5.24. *(Quasiconformal maps on Riemann surfaces)* Let S and S' be two Riemann surfaces, and let $\phi : S \to S'$ be a homeomorphism. Then ϕ is quasiconformal if and only if there exists a $K \geq 1$ so that ϕ is locally K-quasiconformal when expressed in charts.

We also need to extend Beltrami coefficients, defined on subsets of $\mathbb{C}$, to *Beltrami forms*, defined on Riemann surfaces. Keeping in mind that quasiconformal maps were introduced as solutions to the Beltrami equation (see 2.1), it follows naturally expressing the Beltrami form as $\mu(z) dz^{-1} d\bar{z}$ (a $(-1, 1)$-differential).

Definition 2.5.25. *(Beltrami form)* A *Beltrami form* μ, also called a Beltrami differential, on a Riemann surface S is a $(-1,\ 1)$-differential on S.

Quasiconformal surgery constructions involve constructing Beltrami forms with specific properties. We accomplish this by generalizing pullback of quasiconformal maps to Riemann surfaces.

Definition 2.5.26. *(Pullback of Beltrami forms) Let S and S' be two Riemann surfaces, and let $\phi : S \to S'$ be a quasiconformal mapping. If μ' is a Beltrami form on S' then $\varphi^* \mu'$ is defined as the Beltrami form on S, which when expressed in charts fits with the previous pullback definition.*

Sometimes an almost complex structure is preserved under iteration of a map (for instance, the standard complex structure under iteration of conformal maps). This property is known as *invariance*. This is possible in the setting of $D_0^+(U,\ U)$, the space of continuous orientation preserving functions f from U onto U which are $\mathbb{R}$-differentiable a.e, have a non-singular differential $D_u f$ a.e. depending measurably on u, and are absolute continuous with respect to Lebesgue measure (the preimages of sets of measure zero also have measure zero).

Definition 2.5.27. *(f-invariant almost complex structure)* Let U be an open subset of $\mathbb{C}$ and $f : U \to U$ a map in $D_0^+(U,\ U)$. Let σ be an almost open structure in U with Beltrami coefficient μ. We say that μ (or σ) is *f-invariant* if $f^* \mu(u) = \mu(u)$ for almost every $u \in U$. We also write $f^* \sigma = \sigma$.

The Integrability Theorem

Pullback of Beltrami forms allows the generation of new Beltrami forms using a given quasiconformal map. A natural question arises, whether one may pullback from the standard complex structure to a given complex structure σ defined on a set U. The Integrability Theorem (see [AB60]), also known as Measurable Riemann Mapping Theorem, states that all that is required is for σ to have uniformly bounded dilation. Moreover, it provides a quasiconformal map ϕ which is the solution of the *Beltrami equation*

$$\partial_{\bar{z}}\phi(z) = \mu(z)\partial_z\phi(z)$$

for almost every $z \in U$, where μ is the Beltrami coefficient corresponding to σ. The map ϕ *integrates* μ and it is called an *integrating map*.

Theorem 2.5.28. *(Integrability Theorem-local version)*

Let $U \subset \mathbb{C}$ be an open set such that $U \simeq \mathbb{D}$ (respectively $U = \mathbb{C}$). Let σ be an almost complex structure on U corresponding to the Beltrami coefficient μ. Suppose the dilatation of σ is uniformly bounded, that is, $K(\sigma) < \infty$ or, equivalently, the essential supremum of $|\mu|$ on U is

$$||\mu||_\infty = k < 1.$$

Then, μ is integrable, i.e. there exists a quasiconformal homeomorphism $\phi : U \to \mathbb{D}$ (respectively onto $\mathbb{C}$) which solves the Beltrami equation, i.e. such that

$$\mu(z) = \frac{\partial_{\bar{z}}\phi(z)}{\partial_z\phi(z)}$$

for a.e. $z \in U$. Moreover, ϕ is unique up to post-composition with automorphisms of $\mathbb{D}$ (respectively $\mathbb{C}$).

We have stated the Integrability Theorem for subsets of the complex plane. Since the maps we study are rational maps, defined on the Riemann sphere, we also need the so called *global* version of the Integrability Theorem (see [Ahl06]).

Theorem 2.5.29. *(Integrability Theorem-global version) Let S be a simply connected Riemann surface and σ be an almost complex structure on S with measurable Beltrami form μ. Suppose the dilatation of σ is uniformly bounded, i.e. $K(\sigma) < \infty$ or, equivalently, the essential supremum of $|\mu|$ on S is*

$$||\mu||_\infty = k < 1.$$

Then, μ is integrable, i.e. there exists a quasiconformal homeomorphism $\phi : S \to \mathbb{D}$ (respectively onto $\mathbb{C}$ or $\hat{\mathbb{C}}$) which satisfies

$$\phi^*\mu_0 = \mu.$$

If S is isomorphic to $\mathbb{D}$ (respectively to $\mathbb{C}$ or $\hat{\mathbb{C}}$) then $\phi : S \to \mathbb{D}$ (respectively to $\mathbb{C}$ or $\hat{\mathbb{C}}$) is unique up to post-composition with automorphisms of $\mathbb{D}$ (respectively to $\mathbb{C}$ or $\hat{\mathbb{C}}$).

Quasiregular maps

The quasiconformal surgery constructions that we will realize involve defining specific Beltrami forms, so that the Integrability Theorem can be used. The definition of these Beltrami forms is done using pullback of so called *quasiregular maps*. These maps are quasiconformal everywhere (thus injective), except a discrete set of points.

Definition 2.5.30. *(Quasiregular map)* Let $U \subseteq \mathbb{C}$ be an open set and $K < \infty$. A mapping $g : U \to \mathbb{C}$ is *K-quasiregular* if and only if g can be expressed as

$$g = f \circ \phi,$$

where $\phi : U \to \phi(U)$ is K-quasiconformal and $f : \phi(U) \to g(U)$ is holomorphic.

Since quasiregular maps are also in $D_0^+(S, S)$, it follows that we may define pullback under a quasiregular map as in Definition 2.5.21.

Lemma 2.5.31. *Quasiregular mappings and their inverse branches send sets of measure zero to sets of measure zero. Consequently, the pullback of a Beltrami form defined a.e. by a quasiregular map is well defined a.e.*

Since rational maps of degree $d \geq 2$ are not injective, they cannot be quasiconformally conjugated to a quasiconformal map. We list some properties of quasiregular maps which allow final step of our surgery constructions: conjugating the quasiregular map with a holomorphic map.

Lemma 2.5.32. *(Weyl's lemma for quasiregular maps) Let U be a domain of $\mathbb{C}$. If $g : U \to \mathbb{C}$ is a quasiregular map such that $g^*\mu_0 = \mu_0$ almost everywhere in U, then g is holomorphic.*

Finally, we introduce what is referred to as the *Key Lemma for surgery* (see [BF14, Lemma 1.39]). This lemma states the properties that a quasiregular map and a Beltrami form must satisfy, so that a quasiregular map can be quasiconformally conjugated to a holomorphic map.

Lemma 2.5.33. *(Key Lemma for surgery) Let S be a Riemann surface isomorphic to $\mathbb{C}$ or $\hat{\mathbb{C}}$ and let $g : S \to S$ be quasiregular. Let μ be a g-invariant Beltrami form on S such that $||\mu||_\infty := k < 1$. Then, there exists a holomorphic mapping $f : X \to X$, where $X \in \{\mathbb{C}, \hat{\mathbb{C}}\}$, such that g and f are quasiconformally conjugate.*

2.6 Polynomial-like mappings

We now recall the basis of the theory of polynomial-like mappings. This theory was introduced by Douady and Hubbard in [DH85] (see also [BF14, Chapter 7]).

Definition 2.6.1. *(Polynomial-like mapping)* Let U and V be simply connected domains in $\hat{\mathbb{C}}$, bounded by analytic curves, such that $\overline{U} \subset V$. If $f : U \to V$ is holomorphic and proper of degree d, then $(f; U, V)$ is called a *polynomial-like mapping* of degree d.

For polynomials, the point $z = \infty$ is always a superattracting fixed point. Therefore, partitioning the dynamical plane in the basin of attraction of $z = \infty$ and the *filled Julia set*, the set of points which never escape to $z = \infty$, is often useful. An analogous partition is used for polynomial-like mappings.

Definition 2.6.2. *(Filled Julia set of a polynomial-like mapping)* The *filled Julia set of a polynomial-like mapping* $(f; U, V)$ is defined as

$$\mathcal{K}_f := \bigcap_{n>0} f^n(V),$$

i.e. the set of points $z \in U$ whose orbits under f never escape the set U. The Julia set $\mathcal{J}_f$ is the boundary of $\mathcal{K}_f$.

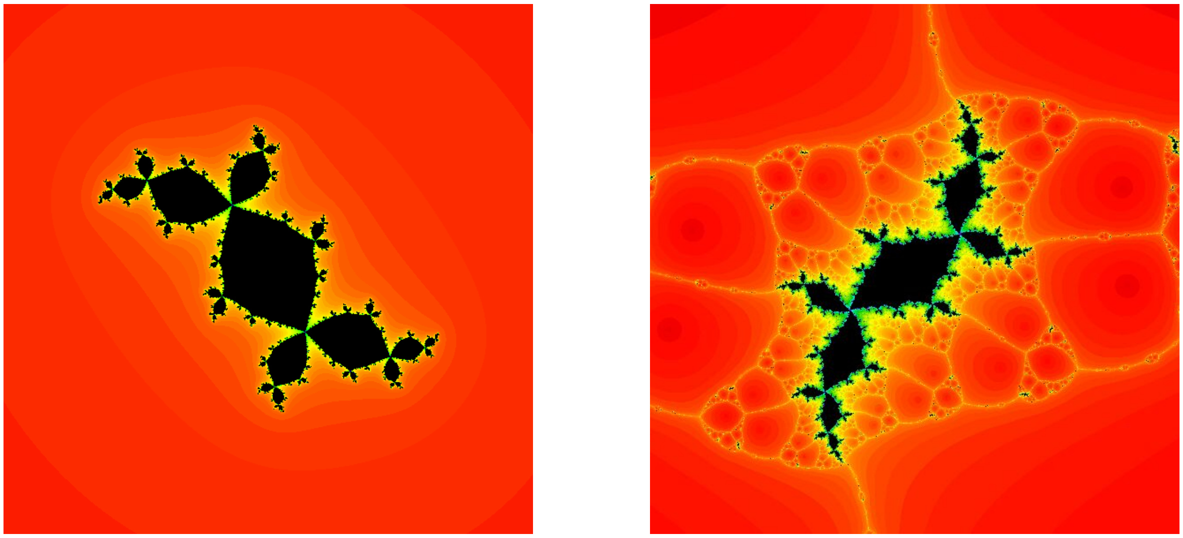


Figure 2.6: The left figure illustrates the dynamical plane of $Q_c = z^2 + c$ for $c \approx -0.123 + 0.745i$, which is called the *Douady rabbit*. In the right figure we can see the dynamical plane of $N(z) = \frac{z^2(1+a-2z)}{-a+2+2az-3z^2}$, where $a = 0.53725 + 2.26255i$ ($N(z)$ is the map obtained by applying Newton's method to $p(z) = z(z-1)(z-a)$).

By studying the dynamical planes for various families of holomorphic map, one may find *copies* of Julia sets of quadratic maps (see Figure 2.6). This is a consequence of the Straightening Theorem, but first we need to define *hybrid equivalence* for two polynomial maps.

Definition 2.6.3. *(Hybrid equivalence)* Two polynomials f and g in Pol_d are *hybrid equivalent* if there exist neighbourhoods U_f and U_g of $\mathcal{K}_f$ and $\mathcal{K}_g$ respectively, and a quasiconformal conjugacy $\phi : U_f \to U_g$ between f and g, satisfying $\bar{\partial}\phi = 0$ almost everywhere on $\mathcal{K}_f$.

The following theorem is the cornerstone of understanding the existence of quasiconformal copies of quadratic maps Julia sets in dynamical planes of other families. The result was proven by Douady and Hubbard in [DH85] using the theory of polynomial-like mappings. One of the now several avalaible proofs is also generally used as the standard example of *cut-and-paste* quasiconformal surgery construction.

Theorem 2.6.4. *(The Straightening Theorem) Let $(f; U, V)$ be a polynomial-like mapping of degree d. Then, the following statements hold:*

(a) The polynomial-like mapping $(f; U, V)$ is hybrid equivalent to a polynomial P of degree d.

(b) If $\mathcal{K}_f$ is connected, then P is unique up to affine conjugation.

The Mandelbrot set is one of the most famous fractals. One can easily see the small *Mandelbrot-like* copies inside the set. By studying many other families of maps, one can observe *Mandelbrot-like* copies in the corresponding parameter plane (see Figure 2.7). It has been proven by Douady and Hubbard that the *Mandelbrot-like* copies are, in fact, quasiconformal copies of the Mandelbrot set. They have accomplished this using the notion of *holomorphic family of polynomial-like mappings*, a tool that we will also use in later chapters.

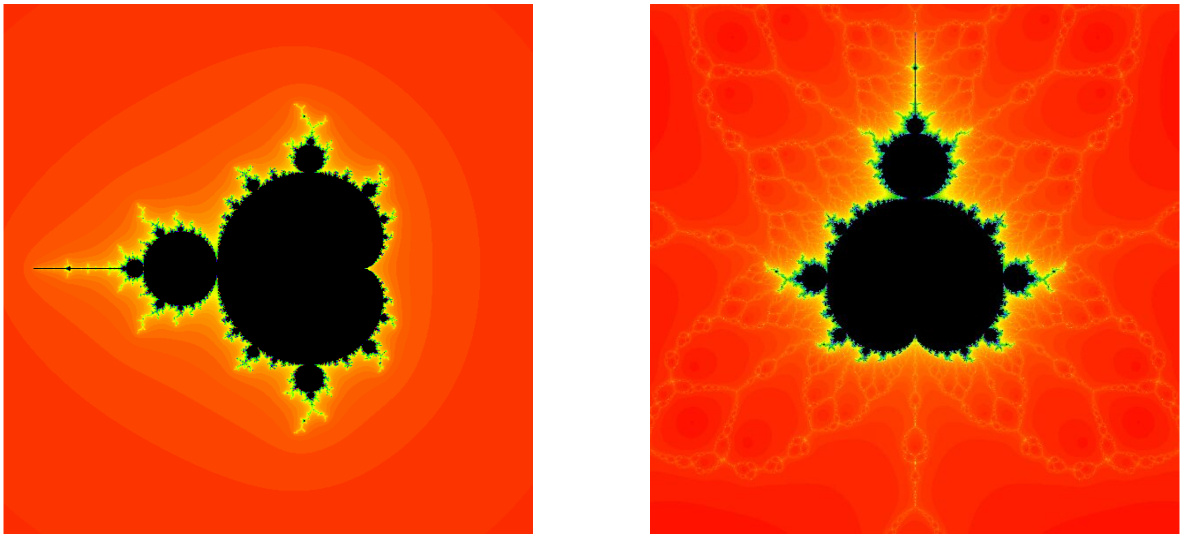


Figure 2.7: The left figure illustrates the Mandelbrot set. In the right figure we can observe a quasiconformal copy of the Mandelbrot set, in the parameter plane of $N(z) = \frac{z^2(1+a-2z)}{-a+2+2az-3z^2}$ ($N(z)$ is the map obtained by applying Newton's method to $p(z) = z(z-1)(z-a)$).

Definition 2.6.5. *(Holomorphic family of polynomial-like mappings) Let Λ be a complex analytic manifold and $\mathcal{F} = (f_\lambda; U_\lambda; V_\lambda)$ a family of polynomial-like mappings. Set*

$$\mathcal{V} = \{(\lambda, z) | z \in V_\lambda\}$$

$$\mathcal{U} = \{(\lambda, z) | z \in U_\lambda\}$$

$$f(\lambda, z) = (\lambda, f_\lambda(z)).$$

If $\mathcal{F}$ satisfies the following properties:

1. *$\mathcal{U}$ and $\mathcal{V}$ are homeomorphic over Λ to $\Lambda \times \mathbb{D}$;*

2. *The projection from the closure of $\mathcal{U}$ in $\mathcal{V}$ to Λ is proper;*

3. *The map $f : \mathcal{U} \to \mathcal{V}$ is holomorphic and proper;*

then $\mathcal{F}$ is called a holomorphic family of polynomial-like mappings, which has the property that all the polynomial-like mappings have the same degree, called the degree of $\mathcal{F}$.

It was Mañé, Sad, and Sullivan who first partitioned the parameter plane of families of rational maps based on structural stability on a neighbourhood of the Julia set (see [MSS83]). Douady and Hubbard observed that the $\mathcal{J}-$stability results by Mañé, Sad, and Sullivan also apply to polynomial-like mappings (see [DH85, Proposition 10]). This is the content of the next result. Before, we introduce the concept of *persistent indifferent periodic point*. A point z_0 is called *persistent indifferent periodic point* if for each neighbourhood V of z_0, there exists a neighbourhood W of λ_0 such that, for each $\lambda \in W$, the function R_λ has in V an indifferent periodic point of the same period.

Proposition 2.6.6. *Let $\mathcal{F} = \{P_\lambda : U'_\lambda \to U_\lambda\}_{\lambda\in\Lambda}$ be a holomorphic family of polynomial-like mappings. Let $\mathcal{J}$ be the set of parameters λ such that P_λ has a non-persistent periodic point. Let $\mathcal{N} = \overline{\mathcal{J}}$ and $\mathcal{R} = \Lambda - \mathcal{N}$. Then, we have that:*

(a) The open set $\mathcal{R}$ is dense in Λ;

(b) For any λ_0, there exists a neighbourhood W of λ_0 in $\mathcal{R}$, a neighbourhood V of $\mathcal{J}_{\lambda_0}$ in U_{λ_0} and an embedding $\tau : (\lambda, z) \to (\lambda, \tau_\lambda(z))$ of $W \times V$ into $\mathcal{U}$ such that:

(i) $\tau(\lambda, z)$ is holomorphic in λ and quasiconformal in z, with dilatation ration bounded by a constant independent of λ.

(ii) The image of τ is a neighbourhood of $\mathcal{J}_W = \{(\lambda, z) \mid \lambda \in W, z \in \mathcal{J}_\lambda\}$, which is closed in $\mathcal{U} \cap (W \times \mathbb{C})$.

(iii) the map τ_{λ_0} is the identity of V, and for all $\lambda \in W$ we have $f_\lambda \times \tau_\lambda = \tau_\lambda \circ f_{\lambda_0}$.

Observe that $\mathcal{R}$ is the set of parameters for which there is $\mathcal{J}-$stability. Indeed, if λ_1 and λ_2 belong in the same connected component of $\mathcal{R}$, then the previous results provides a quasiconformal conjugacy between P_{λ_1} and P_{λ_2}. This conjugacy is defined over a neighbourhood of the corresponding Julia sets. In particular, $\mathcal{J}(P_{\lambda_2})$ is a quasiconformal copy of $\mathcal{J}(P_{\lambda_1})$.

Chapter 3

Achievable connectivities of Fatou components for a family of singular perturbations

In the recent decades there has been an increasing interest in studying families of rational maps usually called *singular perturbations.* Roughly speaking, a family is called a singular perturbation if it is defined by a *base* family (called the *unperturbed family* and for which we have a deep understanding of the dynamical plane) plus a *local* perturbation, that is, a perturbation which has a significant effect on the orbits of points in some part(s) of the dynamical plane, but a very small dynamical relevancy on other regions.

Singular perturbations, no matter the concrete formulas, have some common properties which make their study interesting. On the one hand, the degree of the unperturbed family is smaller than the degree of the perturbed one. Consequently, one should expect richer dynamics for singular perturbations than for the unperturbed maps. On the other hand, most of this new freedom arising from the perturbation may be captive of the dynamical properties of the unperturbed family. The balance between these two scenarios has become very successful in finding new dynamical phenomena.

The relation between the topology of the dynamically invariant sets (Fatou and Julia set) and the behaviour of the critical orbit(s) is an important issue when studying the dynamical plane of a particular rational map. A paradigmatic example of this is the Dichotomy Theorem for the quadratic family. In this way, singular perturbations are somehow a perfect scenario to observe new phenomena for the invariant sets with respect to the unperturbed maps, for which we usually observe a tame topology. Indeed, the main goal of this chapter is to investigate in this direction and to prove that for a certain family of singular perturbations we can construct examples for which, in the same dynamical plane, there are Fatou components with given arbitrarily high connectivities.

The connectivity of a domain $D \subset \hat{\mathbb{C}}$ is defined as the number of connected components of its boundary. It is known that periodic Fatou components have connectivity 1, 2, or ∞. Indeed, Siegel disks have connectivity 1, Herman rings have connectivity 2, and immediate basins of attraction have connectivity 1 or ∞. Preperiodic Fatou components can have finite connectivity greater than 2. The first such example, with connectivities 3 and 5, was presented in [Bea91]. Moreover, for any given $n \in \mathbb{N}$, there are examples of rational maps with Fatou components of connectivity n. These examples can either be obtained by quasiconformal surgery (see [BKL91]) or by giving explicit families of rational maps (see [QG04] and [Ste93]). However, the degree of the rational maps obtained in all previous examples grows rapidly with n. To our knowledge, the first example of rational map whose dynamical plane contains Fatou components of arbitrarily large finite connectivities was presented in [Can17] (see also [Can18]) by using singular perturbations. However, in these papers it is not shown which precise connectivities can actually be attained. The goal of this chapter is to study the attainable connectivities for a wider family of singular perturbations which includes the ones studied in [Can17, Can18]. We also want to remark that while this work was being prepared we knew that, independently, professor Hiroyuki has obtained another family of rational maps with Fatou components of arbitrarily large connectivity [Hir].

Singular perturbations of rational maps were introduced by McMullen in [McM88]. He proposed the study of the family

$$Q_{n,d,\lambda}(z) = z^n + \frac{\lambda}{z^d}, \tag{3.1}$$

where $n, d \geq 2$ and $\lambda \in \mathbb{C}$, $|\lambda|$ small. Observe that in (3.1) the unperturbed map is the *simplest* possible: z^n. He considered the case $n = 2$ and $d = 3$ and he proved that if $|\lambda|$ is small enough then the Julia set is a Cantor sets of quasicircles (the result actually holds for n and d satisfying $1/n + 1/d < 1$ (compare [DLU05])). Later, Devaney, Look, and Uminsky [DLU05] considered (3.1) as a λ-family of rational maps and they extended McMullen's result by proving the Escape Trichotomy. More specifically, they showed that if all critical points belong to the basin of attraction of infinity then the Julia set is a Cantor set, a Sierpinski carpet, or a Cantor set of quasicircles (McMullen's case). The proof relies on the fact that there is a symmetry in the dynamical plane which implies that there is a unique free critical orbit (the symmetry forces all critical points to follow symmetric orbits). Other models similar to (3.1) have also been considered. For instance, in [BDGR08, GMR13] the authors consider singular perturbations of polynomials of the form $z^n + c,\ c \in \mathbb{C}$, choosing c appropriately. Those examples have shown Julia and Fatou sets with new and rich topology, but the connectivity of the Fatou components is kept as 1, 2 or ∞.

The examples mentioned in the previous paragraph are done perturbing maps with no free critical points: one or more poles are added to superattracting cycles which contain no critical points, other than the critical points of the cycle, in their basins of attraction. A next natural step is to consider singular perturbations of maps with free critical points. A good candidate for such a perturbation is the family of Blaschke products

$$B_{n,a}(z) = z^n \frac{z-a}{1-\bar{a}z}, \quad \text{where } a \in \mathbb{C} \text{ and } n \geq 2.$$

See [CFG15] for an introduction to the dynamics of these maps for $n = 3$. If a belongs to the punctured unit disk $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$, the maps $B_{n,a}$ restrict to automorphisms of the unit disk whose dynamical plane is trivial. Indeed, its Fatou set consists of two invariant components given by the immediate basin of attraction $\mathcal{A}^*(0)$ of $z = 0$ (the unit disk) and the immediate basin of attraction $\mathcal{A}^*(\infty)$ of $z = \infty$ (the complement of the closed unit disk). Their common boundary component, the unit circle, is the Julia set of these maps. Moreover, if $a \in \mathbb{D}^*$ the map $B_{n,a}$ has only two simple critical points $c_- \in \mathcal{A}^*(0)$ and $c_+ \in \mathcal{A}^*(\infty)$, other than the superatracting fixed points $z = 0$ and $z = \infty$. In [Can17, Can18], Canela studied the family of singular perturbations of the maps $B_{n,a}$ given by

$$B_{n,d,a,\lambda}(z) = z^n \frac{z-a}{1-\bar{a}z} + \frac{\lambda}{z^d},$$

where $a \in \mathbb{D}^*$ and $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, for $n = 3$ and $d = 2$. Compared to McMullen's singular perturbations, these maps can present a much richer dynamics since their free critical points (which come from the the singular perturbation and the continuous extension of $c_\pm$) are not tied by any kind of symmetry. Despite that, in [Can17] it was proven that if $|\lambda|$ is small enough the family $B_{3,2,a,\lambda}(z)$ is essentially unicritical: all critical points but the continuous extension $c_-(\lambda)$ of c_- belong to the basin of attraction of infinity $\mathcal{A}(\infty)$. In that case, if $c_-(\lambda)$ belongs to a Fatou component in $\mathcal{A}(\infty)$ which surrounds $z = 0$, the dynamical plane has Fatou components of arbitrarily large finite connectivity. The actual existence of parameters for which this actually happens was proven in [Can18]. We want to point out that the same results can be proven for $n, d \geq 2$ such that $1/n + 1/d < 1$. In Figure 3.1 we illustrate the dynamical plane of $B_{n,d,a,\lambda}(z)$ for $a = 0.5$, $d = 3$, and different values of n and λ.

The goal of this chapter is to extend the results in [Can17, Can18] to a wider family of singular perturbations and to study which connectivities are attainable for such family. With this aim we consider the family of degree $n + 1$ rational maps given by

$$S_{n,a,Q}(z) = \frac{z^n(z-a)}{Q(z)}, \tag{3.2}$$

where $n \geq 2$, $a \in \mathbb{C}^*$, and Q is a polynomial of degree at most n, with $Q(0)Q(a) \neq 0$. On the one hand it is clear that the family $S_{n,a,Q}$ contains the family $B_{n,a}$. On the other hand it is worth to be noticed that $S_{n,a,Q}$ also includes the family

$$M_{n,a}(z) = z^n(z-a),$$

where $n \geq 2$ and $a \in \mathbb{C}$. This family was first introduced by Milnor in 1991 (see [Mil09]) when studying cubic polynomials ($n = 2$) and was later studied by Roesch [Roe07] for $n \geq 2$. If $a \neq 0$, these maps have $z = 0$ and $z = \infty$ as superattracting fixed points of local degree n and $n+1$, respectively. Moreover, they have a unique free critical point $c_a \neq 0$ and the global phase portrait settles down on its dynamical behaviour. It is easy to see that, if $|a|$ is small enough, c_a belongs to the immediate basin of attraction of $z = 0$ and the Julia set consists of a quasicircle which separates the immediate basins of attraction of $z = 0$ and $z = \infty$ (see Corollary 3.1.2). In this sense, for $|a|$ small the family $M_{n,a}$ can be understood as a simplified version of $B_{n,a}$, $|a| < 1$, where there is no free critical point in $\mathcal{A}^*(\infty)$ but the Julia set is a quasicircle instead of a circle.

We now turn to the unperturbed family $S_{n,a,Q}$. Inspired by the work in [Can17, Can18] we will impose the following conditions to be satisfied for the maps in $S_{n,a,Q}$.

(a) The point $z = 0$ is a superattracting fixed point of degree n of $S_{n,a,Q}$.

(b) The fixed point $z = \infty$ is (super)attracting. In particular the coefficient of z^n of Q, say b_n, satisfies $0 \leq |b_n| < 1$.

(c) There are exactly two Fatou components: the immediate basins of attraction $\mathcal{A}^*(0)$ and $\mathcal{A}^*(\infty)$ of $z = 0$ and $z = \infty$, respectively.

Remark 3.0.1. *We can deduce the following observations from the above conditions. Since the maps $S_{n,a,Q}$ have degree $n+1$, the immediate basins of attraction are mapped onto themselves with degree $n+1$ and, hence, each of them contains exactly n critical points counting multiplicity. In particular, the basin of attraction of $z = 0$ (which is a critical point of multiplicity $n-1$) contains a simple critical point $\nu_0 \neq 0$.*

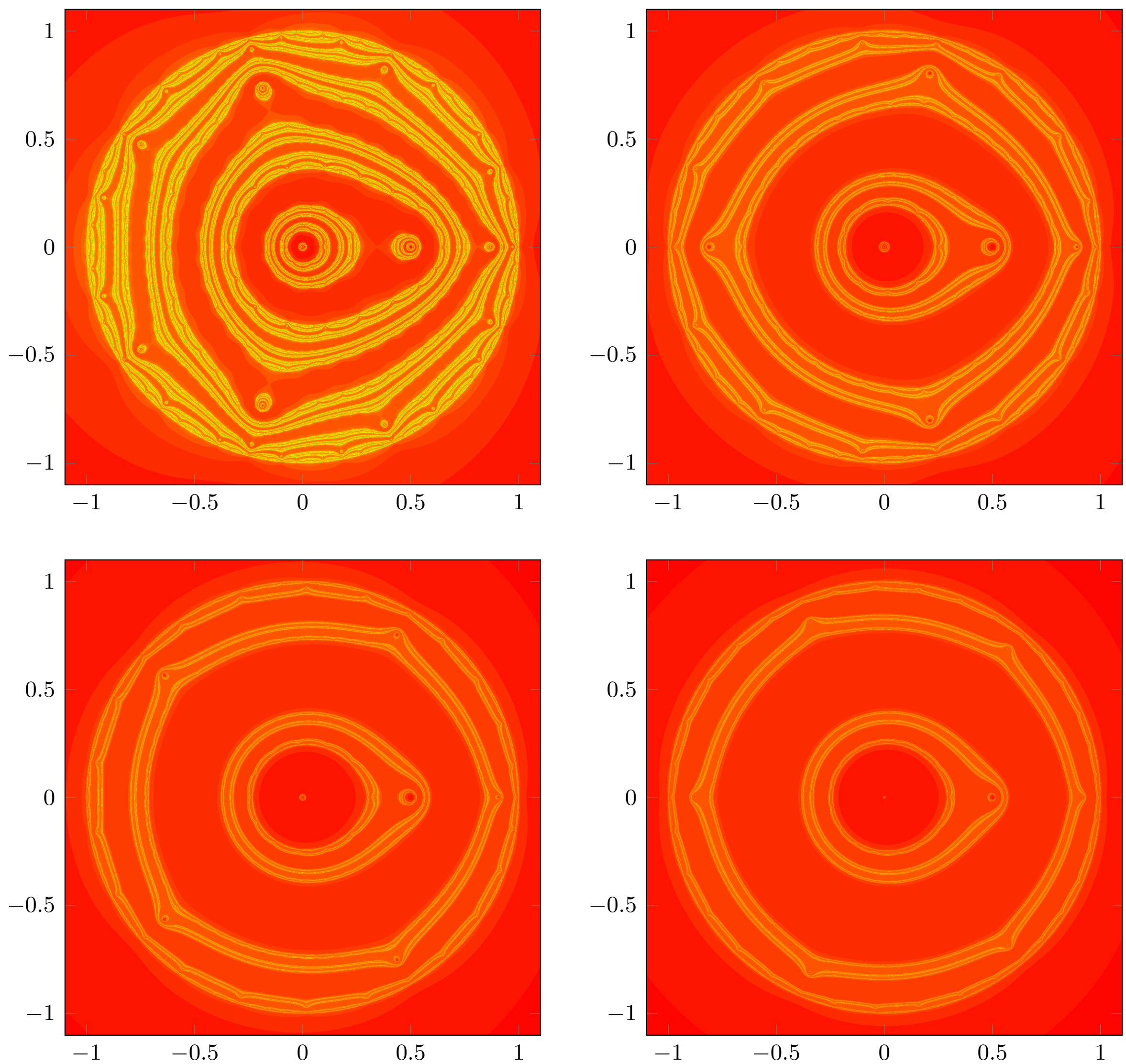


Figure 3.1: Dynamical planes of the family $B_{n,d,a,\lambda}(z)$ for $d = 3$. The top-left figure corresponds to $n = 2$ and $\lambda = 2 \cdot 10^{-8}$; the top-right corresponds to $n = 3$ and $\lambda = -5 \cdot 10^{-8}$; the bottom-left figure corresponds to $n = 4$ and $\lambda = -6.3 \cdot 10^{-9}$; and bottom-right corresponds to $n = 5$ and $\lambda = -1.2 \cdot 10^{-10}$. In all cases we can see the triply connected regions (where the critical point ν_λ lies) and their eventual preimages, which are Fatou components with increasing connectivity.

Once the unperturbed family has been described, we now consider the singular perturbation

$$\mathcal{S}_{n,d,\lambda}(z) = S_{n,a,Q}(z) + \frac{\lambda}{z^d}, \quad \lambda \in \mathbb{C}^*,\ d \geq 2. \tag{3.3}$$

Notice that to simplify notation we do not specify the dependence on a and Q of the family $\mathcal{S}_{n,d,\lambda}$. Notice also that the family $\mathcal{S}_{n,d,\lambda}$ includes the family $B_{n,d,a,\lambda}$. It follows immediately that all maps $\mathcal{S}_{n,d,\lambda}$ have degree $n+d+1$ and that for $\lambda \neq 0$ the point $z=0$ is a pole of degree d. We will say that $\mathcal{S}_{n,d,\lambda}$ *satisfies (a), (b), and (c)* if $S_{n,a,Q}$ satisfies the conditions (a), (b), and (c) explained above. Analogously to the condition needed to obtain a Cantor set of quasicircles for McMullen's family (see [DLU05]), we have to add a fourth condition to the family:

(d) The numbers $n, d \geq 2$ are such that $\frac{1}{n} + \frac{1}{d} < 1$. In other words, we exclude $n = d = 2$.

Since the critical points are not tied by any relation, for $|\lambda|$ big the dynamics can be very rich. Despite that, if $|\lambda|$ is small the family is essentially unicritical. Indeed, there exists a constant $\mathcal{C} > 0$ such that if $|\lambda| < \mathcal{C}$, $\lambda \neq 0$, the following hold (see Proposition 3.1.8):

- The continuous extensions of the n critical points which belong to the immediate basin of attraction $\mathcal{A}_0^*(\infty)$ of $z=\infty$ before perturbation belong to the immediate basin of attraction $\mathcal{A}_\lambda^*(\infty)$ of $z=\infty$ after perturbation. Moreover, $\mathcal{A}_\lambda^*(\infty)$ is a quasidisk.

- The pole $z=0$ belongs to a quasidisk T_λ (usually called *trap door*) which is mapped onto $\mathcal{A}_\lambda^*(\infty)$ under $\mathcal{S}_{n,d,\lambda}$.

- The $n+d$ critical points which appear around $z=0$ after perturbation belong to a doubly connected Fatou component A_λ which is mapped onto T_λ under $\mathcal{S}_{n,d,\lambda}$.

The previous points actually coincide with the skeleton of the dynamics in the Cantor set of quasicircles case of McMullen's family (see [McM88]). This is why the dynamical planes for this perturbed family resemble the dynamical planes for the Cantor set of quasicircles with extra decorations (see Figure 3.1 and Figure 3.2). These decorations come from the presence of the extra critical point ν_λ, which comes from the continuous extension of the critical point ν_0 that belongs to the basin of attraction of $z=0$ before perturbation. This is the only critical point which may not belong to the basin of attraction $\mathcal{A}_\lambda(\infty)$ of $z=\infty$ after perturbation if $|\lambda|$ is small. We want to remark that the main difference between $\mathcal{S}_{n,d,\lambda}$ and the particular family $B_{n,d,a,\lambda}$ is that we allow certain degrees of freedom in the n critical points that lie in $\mathcal{A}_\lambda^*(\infty)$. For instance, if the degree of Q is 0, then $z=\infty$ is a superatracting fixed point of local degree n. On the other hand, if the degree of Q is n, then $z=\infty$ is attracting (but not superattracting) and there are n critical points which move in $\mathcal{A}_\lambda^*(\infty)$. Also, the shape of the Julia set before perturbation affects the shape of the Julia set of the perturbed map (see Figure 3.1 and Figure 3.2). Recall that in the Blaschke case the unperturbed Julia set is the unit circle.

The goal of this chapter is to analyse the connectivities which can be achieved with these singular perturbations. The critical point ν_λ is crucial in order to increase the connectivities beyond 2. Indeed, if ν_λ belongs to a preimage U_ν of A_λ then the Fatou component U_ν is triply connected. Moreover, if U_ν surrounds $z=0$ then we can find sequences of iterated preimages of U_ν which increase the connectivity with every iteration. The next theorems describe the connectivities which can be achieved with this process. We denote by $\mathrm{Fill}(A_\lambda)$ the union of

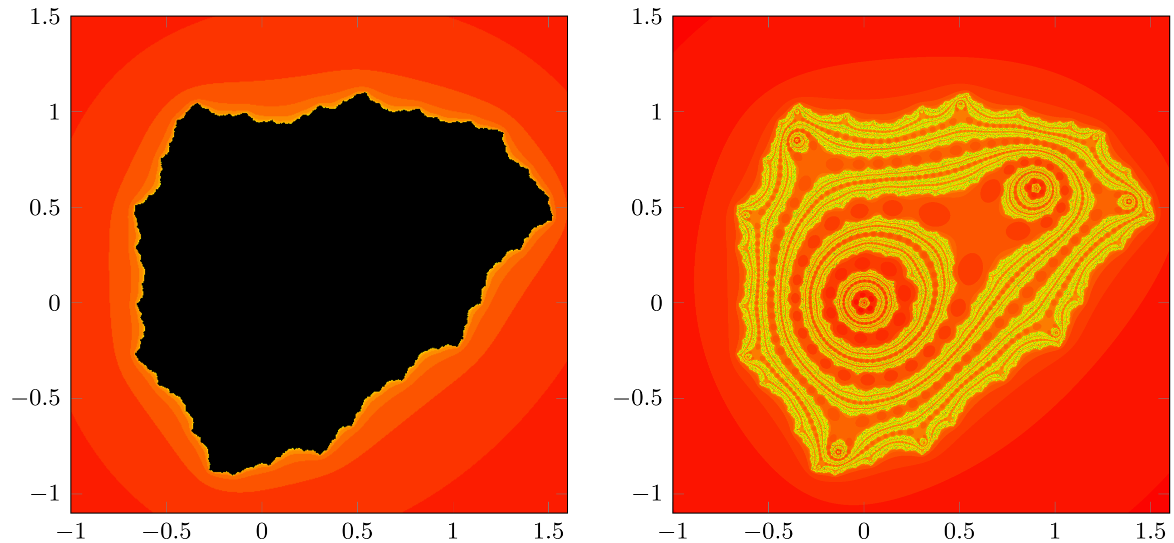


Figure 3.2: Left figure illustrates the dynamical planes of $M_{n,a}$ for $n = 2$ and $a = (0.9 + 0.6i)$. Right picture illustrates the dynamical plane of the (perturbed) family $\mathcal{S}_{n,d,\lambda}$ when the unperturbed map is precisely $M_{2,a}$, and the perturbation corresponds to $d = 3$ and $\lambda = -10^{-7}$. We can see in the right figure the triply connected Fatou component which contains ν_λ and its eventual preimages with higher connectivity.

the connected component of the complement of A_λ not containing $z = \infty$ and the annulus itself.

Theorem A. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d) and let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. Assume also that $\nu_\lambda \in \mathcal{U}_\nu$, where $\mathcal{U}_\nu$ is an iterated preimage of A_λ which surrounds $z = 0$. Let k be the minimal number of iterations needed by the free critical point ν_λ to be mapped into Fill(A_λ). Let U be a Fatou component of connectivity $\kappa > 2$. Then, there exist $i, j, \ell \in \mathbb{N}$ such that $\kappa = (n+1)^i d^j n^\ell + 2$ and $\ell \leq jk$.*

In other words, Theorem A is telling us all *potential* connectivities $\kappa > 2$ for a Fatou component of a map in $\mathcal{S}_{n,d,\lambda}$ for $|\lambda|$ sufficiently small; but it is not claiming the existence of a Fatou component of each (i, j, ℓ)-connectivity. The next result complements Theorem A and it gives the connectivities that are certainly achieved for any parameter λ as long as $|\lambda|$ is sufficiently small and ν_λ satisfies certain dynamical conditions.

Theorem B. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d) and let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. Assume also that $\nu_\lambda \in \mathcal{U}_\nu$, where $\mathcal{U}_\nu$ is an iterated preimage of A_λ which surrounds $z = 0$. Let $k \geq 1$ be the minimal number of iterations needed by the free critical point ν_λ to be mapped into Fill(A_λ). For any given $i, j, \ell \in \mathbb{N}$ such that $\ell \leq j(k-1)$, there exists a Fatou component U of connectivity $\kappa = (n+1)^i d^j n^\ell + 2$.*

Remark 3.0.2. *Observe that Theorem A is stated for $\ell \leq jk$, while Theorem B is stated for $\ell \leq j(k-1)$. It can be proven that, depending on the particular orbit of the free critical point, the case $\ell = jk$ may not be achieved.*

In Theorem A and Theorem B the achievable connectivities depend on the minimal number of iterations $k > 0$ needed by the free critical point ν_λ to be mapped into $\mathrm{Fill}(A_\lambda)$. However, choosing λ appropriately we can make this k as big as desired. Therefore, for any ℓ and j we can find λ so that the inequality $\ell \leq j(k-1)$ is satisfied. From this, we obtain Theorem C.

Theorem C. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d) and let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. For any given $i, l \geq 0$ and $j > 0$, there exists a parameter λ such that $\mathcal{S}_{n,d,\lambda}(z)$ has a Fatou component of connectivity $\kappa = (n+1)^i d^j n^\ell + 2$, and a Fatou component of connectivity $\kappa = (n+1)^i + 2$.*

The chapter is organised as follows. In Section 3.1 we describe in detail the skeleton of the dynamical plane of $\mathcal{S}_{n,d,\lambda}$ satisfying the conditions (a), (b), (c) and (d) for $|\lambda|$ small enough. In Section 3.2 we prove Theorems A and B. Finally, in Section 3.3 we prove Theorem C.

3.1 The dynamical plane of the perturbed family $\mathcal{S}_{n,d,\lambda}$

Let $\mathcal{S}_{n,d,\lambda}$ satisfying conditions (a), (b), and (c) described above. This section contains the main properties of the dynamical plane for parameters belonging to a neighbourhood of $\lambda = 0$. To do so, we first describe the dynamical plane of the unperturbed family $S_{n,a,Q}$, that is, when $\lambda = 0$. The next result due to Sullivan provides a sufficient criterion for the Julia set of a map to be a quasicircle. Its proof involves several quasiconformal surgery constructions as in the Straightening Theorem (see Theorem 2.6.4, see also [BF14, Theorem 7.4] for the proof).

Theorem 3.1.1. *[CG93, Theorem 2.1, page 102] If the Fatou set of a rational map R contains exactly two Fatou components and the map R is hyperbolic on its corresponding Julia set $J(R)$, then $J(R)$ is a quasicircle.*

We can immediately conclude that the Julia sets of the maps $S_{n,a,Q}$ are quasicircles.

Corollary 3.1.2. *Let $S_{n,a,Q}$ satisfying (a), (b), and (c). Then, its Julia set is a quasicircle.*

We first describe the immediate basin of attraction of $z = \infty$, which we further denote by $\mathcal{A}^*_\lambda(\infty)$, and its boundary. The proof of Proposition 3.1.4 uses the theory of holomorphic families of polynomial-like mappings introduced by Douady and Hubbard (see Definition 2.6.5 and Proposition 2.6.6).

Remark 3.1.3. *Along the chapter, when we say that a compact set moves continuously with respect to parameters, we use the topology induced by the Hausdorff metric for compact sets.*

Proposition 3.1.4. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying conditions (a), (b), and (c). Then, for $|\lambda|$ small enough, the following hold:*

(i) The Fatou component $\mathcal{A}^*_\lambda(\infty)$ is mapped onto itself with degree $n+1$.

(ii) The boundary of $\mathcal{A}^*_\lambda(\infty)$ is a quasicircle that moves continuously with respect to λ.

(iii) The set $\mathcal{A}^*_\lambda(\infty)$ contains exactly n critical points counting multiplicity. Each critical point of $\mathcal{S}_{n,d,\lambda}$ in $\mathcal{A}^*_\lambda(\infty)$ is a continuous extension of a critical point of $S_{n,a,Q}$ in $\mathcal{A}^*(\infty)$.

Proof. Observe that the three statements are trivially satisfied (by definition and Corollary 3.1.2) for the unperturbed family. So this proposition says that this conditions are still true if the perturbation is small enough. To prove the proposition we show the existence of a holomorphic family of polynomial-like maps which ensures the continuous deformation of the key dynamical objects.

Fix $S_{n,a,Q}$ and let U be the maximal domain of Bottcher coordinates around $z = 0$. The critical point ν_0 lies on $\partial U \subset \mathcal{A}^*(0)$. Let γ be an analytic Jordan curve surrounding the origin such that $\gamma \in U \setminus \overline{S_{n,a,Q}(U)}$. We now show that the preimage of γ has a unique connected component. Let $A := A(\gamma, \partial\mathcal{A}^*(0))$. Notice that $A \subset \mathcal{A}^*(0)$. Observe that the annulus A does not include any critical value. By Corollary 2.4.3(i), its preimage A^{-1} (under $S_{n,a,Q}$) is also an annulus. Since $\partial\mathcal{A}^*(0)$ is mapped by $S_{n,a,Q}$ onto itself with degree $n+1$, A^{-1} is also mapped onto A with degree $n+1$. Let γ_0^{-1} be the connected component of ∂A^{-1} other than $\partial\mathcal{A}^*(0)$. Since A^{-1} is mapped onto A with degree $n+1$ under $S_{n,a,Q}$, then γ_0^{-1} is mapped onto γ with degree $n+1$ under $S_{n,a,Q}$. Since $S_{n,a,Q}$ has (global) degree $n+1$, we conclude that there is no other preimage of γ than γ_0^{-1} under the map $S_{n,a,Q}$.

Let $\mathcal{V} = \operatorname{Ext}(\gamma)$ and $\mathcal{U}_0 = \operatorname{Ext}(\gamma_0^{-1})$. It follows from the construction that $\mathcal{U}_0$ is compactly contained in $\mathcal{V}$ and $S_{n,a,Q}|_{\mathcal{U}_0} : \mathcal{U}_0 \mapsto \mathcal{V}$ is a proper map of degree $n+1$. Therefore, the triple $(S_{n,a,Q}|_{\mathcal{U}_0}, \mathcal{U}_0, \mathcal{V})$ is a degree $n+1$ polynomial-like map. We want to extend (for $|\lambda|$ small enough) this map to a $\mathcal{J}$-stable holomorphic family of polynomial like mappings. Observe that the map $\mathcal{S}_{n,d,\lambda}$ depends analytically on λ for all $z \in \widehat{\mathbb{C}} \setminus \mathbb{D}_\varepsilon$, where $\mathbb{D}_\varepsilon$ denotes the disk of radius ε centered at $z = 0$, for all $\lambda \in \mathbb{C}$ and all $\varepsilon > 0$. Recall that $S_{n,a,Q} = \mathcal{S}_{n,d,0}$. Therefore, if $|\lambda|$ is small enough, the continuous extensions of the n critical points (counting multiplicity) which lie in $\mathcal{A}^*(\infty) = \mathcal{A}_0^*(\infty)$ for $S_{n,a,Q}$ lie in $\mathcal{A}_\lambda^*(\infty)$ for $\mathcal{S}_{n,d,\lambda}$. Moreover, if $|\lambda|$ is small enough then there exists a unique connected component γ_λ^{-1} of γ under the map $\mathcal{S}_{n,d,\lambda}^{-1}$ which is an analytic Jordan curve. In fact, γ_λ^{-1} is a continuous deformation of γ_0^{-1} and it is mapped with degree $n+1$ onto γ under $\mathcal{S}_{n,d,\lambda}$. Let $\mathcal{U}_\lambda = \operatorname{Ext}(\gamma_\lambda^{-1})$. Decreasing $|\lambda|$ if necessary, we can ensure the following. The set $\mathcal{U}_\lambda$ is compactly contained in $\mathcal{V}$ and the only critical points of $\mathcal{S}_{n,d,\lambda}$ in $\mathcal{U}_\lambda$ are the ones which come from the continuous extension of the critical points in $\mathcal{A}^*(\infty)$. Moreover, $\mathcal{S}_{n,d,\lambda}|_{\mathcal{U}_\lambda} : \mathcal{U}_\lambda \mapsto \mathcal{V}$ is a proper map of degree $n+1$ and the triple $(\mathcal{S}_{n,d,\lambda}|_{\mathcal{U}_\lambda}, \mathcal{U}_\lambda, \mathcal{V})$ is a degree $n+1$ polynomial-like mapping.

Let Λ be an open disk centered at $\lambda = 0$ compactly contained in the open set of parameters for which the previous conditions hold. Then, $\{(\mathcal{S}_{n,d,\lambda}|_{\mathcal{U}_\lambda}, \mathcal{U}_\lambda, \mathcal{V})\}_{\lambda\in\Lambda}$ defines a holomorphic family of polynomial like mappings (see Definition 2.6.5). Let $\mathcal{K}_\lambda := \{z \in \mathcal{U}_\lambda \mid \mathcal{S}_{n,d,\lambda}^n(z) \in \mathcal{U}_\lambda \text{ for all } n \geq 0\}$ and $\mathcal{J}_\lambda = \partial\mathcal{K}_\lambda$ denote the filled Julia set and the Julia set of the polynomial like map $(\mathcal{S}_{n,d,\lambda}, \mathcal{U}_\lambda, \mathcal{V})$, respectively. Notice that $\mathcal{K}_0 = \overline{\mathcal{A}^*(\infty)}$. Notice also that all connected components of the interior of $\mathcal{K}_\lambda$ are Fatou components of $\mathcal{S}_{n,d,\lambda}$. Therefore, since the point $z = \infty$ belongs to $\mathcal{K}_\lambda$ for all $\lambda \in \Lambda$, we conclude that $\mathcal{A}_\lambda^*(\infty) \subset \mathcal{K}_\lambda$.

To finish the proof, we observe that since all critical points of $\mathcal{S}_{n,d,\lambda}|_{\mathcal{U}_\lambda}$ belong to $\mathcal{A}_\lambda^*(\infty)$ it follows that the holomorphic family of $\{(\mathcal{S}_{n,d,\lambda}, \mathcal{U}_\lambda, \mathcal{V})\}_{\lambda\in\Lambda}$ is $\mathcal{J}$-stable. In particular, the Julia sets $\mathcal{J}_\lambda$ are quasicircles which are continuous deformations of $\mathcal{J}_0 = \partial\mathcal{A}^*(\infty)$ (see Proposition 2.6.6). Notice that, by Corollary 3.1.2, $\partial\mathcal{A}^*(\infty)$ is a quasicircle. Since $\mathcal{A}_\lambda^*(\infty) \subset \mathcal{K}_\lambda$, we can conclude that $\partial\mathcal{A}_\lambda^*(\infty) = \mathcal{J}_\lambda$ for all $\lambda \in \Lambda$. This proves (ii). Statements (i) and (iii) follow from the choice of the set of parameters Λ.

□

The first part of the following lemma describes a neighbourhood of $z = \infty$ which, for $|\lambda|$ small enough, always lies in the interior of $\mathcal{A}_\lambda^*(\infty)$. The second part shows that $z = 0$ lies in a preimage of $\mathcal{A}_\lambda^*(\infty)$, different from it.

Lemma 3.1.5. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying conditions (a), (b), and (c). Then, for $|\lambda|$ small enough, the following happen:*

(i) There exists a constant K, which only depends on n, a, and Q, such that $z \in \mathcal{A}^*_\lambda(\infty)$ if $|z| > K$.

(ii) Assume that $\mathcal{S}_{n,d,\lambda}$ also satisfies condition (d). For any constant $K_1 > 0$, if $|\lambda|$ is small enough, the disk $\left\{z \in \mathbb{C};\ |z| < K_1 |\lambda|^{\frac{n}{n+d}}\right\}$ belongs to a Fatou component T_λ. The Fatou component T_λ is mapped onto $\mathcal{A}^*_\lambda(\infty)$ and it is different from $\mathcal{A}^*_\lambda(\infty)$.

Proof. We begin with statement *(i)*. From condition (b) we know that, for fixed $S_{n,a,Q}$, there exists a constant K such that the set $\{z \in \mathbb{C};\ |z| > K\}$ is compactly contained in the immediate basin of attraction of ∞. By continuity with respect to λ, for $|\lambda|$ small enough, this set is also contained in $\mathcal{A}^*_\lambda(\infty)$.

For statement *(ii)*, let $K_1 > 0$. Assume that λ is such that *(i)* is satisfied for the constant K above. Let $z \in \mathbb{C}$ such that $|z| < K_1 |\lambda|^{\frac{n}{n+d}}$. It follows that

$$|\mathcal{S}_{n,d,\lambda}(z)| > \left|\frac{\lambda}{z^d}\right| - \left|\frac{z^n(z-a)}{Q(z)}\right| > \frac{|\lambda|^{1-\frac{nd}{n+d}}}{K_1^d} - \frac{|\lambda|^{\frac{n^2}{n+d}} K_1^n (|a|+1)}{M} =: C_1(\lambda) + C_2(\lambda).$$

Notice that $C_2(\lambda)$ tends to 0 as λ tends to 0. Because of assumption (d), $C_1(\lambda)$ tends to ∞ as λ tends to 0. Shrinking $|\lambda|$ if necessary, if $|z| < |\lambda|^{\frac{n}{n+d}} K_1$, then $|\mathcal{S}_{n,d,\lambda}(z)| > K$. We conclude that the set $\left\{z \in \mathbb{C};\ |z| < K_1 |\lambda|^{\frac{n}{n+d}}\right\}$ belongs to a Fatou component. This Fatou component contains $z = 0$, which is mapped to ∞ with degree d. By continuity with respect to λ and Proposition 3.1.4, $\partial \mathcal{A}^*_\lambda(\infty)$ is a quasicircle which surrounds $z = 0$. It follows that $\mathcal{A}^*_\lambda(\infty)$ does not contain the origin and $z = 0$ belongs to a preimage of $\mathcal{A}^*_\lambda(\infty)$, different from $\mathcal{A}^*_\lambda(\infty)$, which we denote by T_λ.

□

Recall that each map of the perturbed family has global degree $n + d + 1$. Hence, it has $2(n + d)$ critical points (counting multiplicity). By Proposition 3.1.4, $n + d - 1$ of them lie in $\mathcal{A}^*_\lambda(\infty) \cup \{0\}$. By continuity with respect to λ, there is a (simple) critical point ν_λ which is the continuous extension of the critical point ν_0 of $S_{n,a,Q}$ in $\mathcal{A}^*(0)$. Each map has $n + d + 1$ zeros, one of which, say w_λ, corresponds to the continuous extension of $w_0 = a$. We now give a description of the position of the remaining $n + d$ critical points and the $n + d$ preimages of $z = 0$ for $\mathcal{S}_{n,d,\lambda}$.

Lemma 3.1.6. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying conditions (a), (b), and (c). Let $\xi = e^{\frac{2\pi i}{n+d}}$. Then, for $|\lambda|$ small enough, there exist $n + d$ free critical points, c_{λ,ξ^j}, and $n + d$ zeros, z_{λ,ξ^j}, given by*

$$c_{\lambda,\xi^j} = \xi^j \left(\frac{dQ(0)}{-na}\right)^{\frac{1}{n+d}} \lambda^{\frac{1}{n+d}} + o\left(\lambda^{\frac{1}{n+d}}\right),$$

$$z_{\lambda,\xi^j} = \xi^j \left(\frac{Q(0)}{a}\right)^{\frac{1}{n+d}} \lambda^{\frac{1}{n+d}} + o\left(\lambda^{\frac{1}{n+d}}\right),$$

where $j \in \{0, 1, \ldots, n + d - 1\}$.

Proof. In order to avoid problems with the determinations of the $n+d$ roots, within the proof we assume that λ is of the form $\lambda = re^{2\pi i\theta}$ where $r > 0$ and $\theta \in [0, 1)$ is fixed. In particular, when we write $\lambda \to 0$ we are taking a radial limit by making $r \to 0$.

Let us start with the zeros. Notice that all the zeros of $\mathcal{S}_{n,d,\lambda}$ (except for w_λ) must converge to $z = 0$ when λ tends to 0. The zeros of $\mathcal{S}_{n,d,\lambda}(z)$ are the solutions of

$$z^{n+d}(z-a) = -\lambda Q(z).$$

Since a is away from $z = 0$, there are $n + d$ zeros bifurcating from $z = 0$, for $|\lambda|$ small enough. They are the fixed points of $n + d$ operators

$$T_{\lambda,\,\xi^j}(z) = \xi^j \left(\frac{Q(z)}{a-z}\right)^{\frac{1}{n+d}} \lambda^{\frac{1}{n+d}} = R_j(z)\lambda^{\frac{1}{n+d}},$$

where $j \in \{0,\, 1,\, \ldots,\, n+d-1\}$. Notice that $T_{\lambda,\,\xi^j}$ are well-defined for $|z|$ small, since $Q(0) \neq 0$. Observe that in a sufficiently small neighbourhood of $z = 0$, $R_j(z)$ is holomorphic and bounded (notice that $R_j(0) \neq 0$), so $T_{\lambda,\,\xi^j}(z) \to 0$ as $\lambda \to 0$. We can approximate $z_{\lambda,\,\xi^j}$ by $T_{\lambda,\,\xi^j}(0) = \xi^j \left(\frac{Q(0)}{a}\right)^{\frac{1}{n+d}} \lambda^{\frac{1}{n+d}}$. Indeed,

$$\begin{aligned}\left|z_{\lambda,\,\xi^j} - T_{\lambda,\,\xi^j}(0)\right| = \left|T_{\lambda,\,\xi^j}(z_{\lambda,\,\xi^j}) - T_{\lambda,\,\xi^j}(0)\right| &\leq \sup_{\omega\in[0,z_{\lambda,\,\xi^j}]} |T'_{\lambda,\,\xi^j}(\omega)||z_{\lambda,\,\xi^j} - 0| \\ &= |\lambda|^{\frac{1}{n+d}} \sup_{\omega\in[0,z_{\lambda,\xi^j}]} |R'_j(\omega)||z_{\lambda,\,\xi^j}|.\end{aligned}$$

For $|\lambda|$ small enough, there is no pole of R'_j in a neighbourhood of $z = 0$ containing the line segment $[0, z_{\lambda,\,\xi^j}]$, so it is bounded by a constant, say K_2. It follows that

$$\frac{1}{|\lambda|^{\frac{1}{n+d}}} \left| z_{\lambda,\,\xi^j} - \xi^j \left(\frac{Q(0)}{a}\right)^{\frac{1}{n+d}} \lambda^{\frac{1}{n+d}} \right| \leq K_2 |z_{\lambda,\,\xi^j}|.$$

Finally, since $\lim\limits_{\lambda\to 0} K_2 \left|z_{\lambda,\,\xi^j}\right| = 0$, it follows that

$$z_{\lambda,\,\xi^j} = \xi^j \left(\frac{Q(0)}{a}\right)^{\frac{1}{n+d}} \lambda^{\frac{1}{n+d}} + o\left(\lambda^{\frac{1}{n+d}}\right).$$

It can be shown analogously that the $n + d$ free critical points are solutions of the equation

$$\frac{1}{Q^2(z)} \left[(n+1)z^n Q(z) - z^{n+1}Q'(z) - anz^{n-1}Q(z) + az^n Q'(z)\right] - \frac{\lambda d}{z^{d+1}} = 0.$$

As before, we write the operators $S_{\lambda,\,\xi^j}$ as

$$S_{\lambda,\,\xi^j}(z) = \xi^j \left(\frac{dQ^2(z)}{(n+1)zQ(z) - anQ(z) - z^2Q'(z) + azQ'(z)}\right)^{\frac{1}{n+d}} \lambda^{\frac{1}{n+d}},$$

which have the critical points of $\mathcal{S}_{n,d,\lambda}$ as fixed points. The argument made is identical since $Q(0) \neq 0$, so $S_{\lambda,\,\xi^j}$ are holomorphic in the neighbourhood of $z = 0$. Finally, the critical points of the perturbation map are of the form

$$c_{\lambda,\,\xi^j} = \xi^j \left(\frac{dQ(0)}{-na}\right)^{\frac{1}{n+d}} \lambda^{\frac{1}{n+d}} + o\left(\lambda^{\frac{1}{n+d}}\right).$$

$\square$

Next we show that there exists a straight annulus (we will show later that it belongs to a doubly connected Fatou component) which is mapped into T_λ under $\mathcal{S}_{n,d,\lambda}$. Let

$$c_1 = \frac{1}{2}\min\left\{\left|\frac{dQ(0)}{na}\right|^{\frac{1}{n+d}}, \left|\frac{Q(0)}{a}\right|^{\frac{1}{n+d}}\right\} \quad \text{and} \quad c_2 = 2\max\left\{\left|\frac{dQ(0)}{na}\right|^{\frac{1}{n+d}}, \left|\frac{Q(0)}{a}\right|^{\frac{1}{n+d}}\right\}.$$

Lemma 3.1.7. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying conditions (a), (b), (c), and (d). Then, for $|\lambda|$ small enough, the straight annulus*

$$\Omega_\lambda := A\left(\mathbb{S}_{c_1|\lambda|^{\frac{1}{n+d}}}, \mathbb{S}_{c_2|\lambda|^{\frac{1}{n+d}}}\right) \tag{3.4}$$

contains the points c_{λ,ξ^j} and z_{λ,ξ^j} introduced in Lemma 3.1.6 *and it is mapped into T_λ under $\mathcal{S}_{n,d,\lambda}$.*

Proof. The first part of the statement follows directly from the algebraic expression of the points c_{λ,ξ^j} and z_{λ,ξ^j} in Lemma 3.1.6. The rest of the proof is devoted to show that $\mathcal{S}_{n,d,\lambda}(\Omega_\lambda) \subset T_\lambda$.

Let $m = \min\{|z|, Q(z) = 0\}$ and let $M = \min\{|Q(z)|, |z| < m/2\}$ (notice that $M > 0$ since $z = 0$ is not a root of Q). Let $z \in \Omega_\lambda$. For $|\lambda|$ small enough we have

$$|\mathcal{S}_{n,d,\lambda}(z)| < \frac{c_2^n|\lambda|^{\frac{n}{n+d}}(|a|+1)}{M} + \frac{|\lambda|^{\frac{n}{n+d}}}{c_1^d}.$$

We can rewrite this as $|\mathcal{S}_{n,d,\lambda}(z)| < K_1|\lambda|^{\frac{n}{n+d}}$, where K_1 depends on Q, c_1 and c_2, but it does not depend on z and λ. By Lemma 3.1.5, for $|\lambda|$ small enough, the disk centered at $z = 0$ and of radius $K_1|\lambda|^{\frac{n}{n+d}}$ lies in T_λ, as desired.

□

In the next proposition we describe the skeleton of the dynamical plane for $|\lambda|$ *small* (compare Figure 3.3). Recall that w_λ is the zero of $\mathcal{S}_{n,d,\lambda}$ which is the continuous extension of $w_0 = a$ and ν_λ is the continuous extension of the critical point ν_0 in $\mathcal{A}^*(0)$ of $S_{n,a,Q}$.

Proposition 3.1.8. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying conditions (a), (b), (c), and (d). Then, there exists a constant $\mathcal{C} = \mathcal{C}(a, Q, n, d)$ such that if $\lambda \neq 0$ and $|\lambda| < \mathcal{C}$ the following statements are satisfied:*

(i) The Fatou component T_λ is simply connected and it is mapped with degree d onto $\mathcal{A}^*_\lambda(\infty)$ under $\mathcal{S}_{n,d,\lambda}$. There are no other preimages of $\mathcal{A}^*_\lambda(\infty)$.

(ii) There exists a Fatou component A_λ which is doubly connected and contains exactly $n+d$ simple critical points, given by c_{λ,ξ^j}, and $n+d$ zeros, given by z_{λ,ξ^j}. Moreover, A_λ is mapped with degree $n+d$ onto T_λ and surrounds the origin.

(iii) Let A^{out} be the annulus bounded by $\overline{A_\lambda}$ and $\partial\mathcal{A}^*_\lambda(\infty)$. There exists a Fatou component $D_\lambda \subset A^{out}$ which is simply connected, is mapped with degree 1 onto T_λ, and contains w_λ.

(iv) The critical point ν_λ lies in $A^{out} \setminus D_\lambda$.

(v) There are no preimages of T_λ other than D_λ and A_λ.

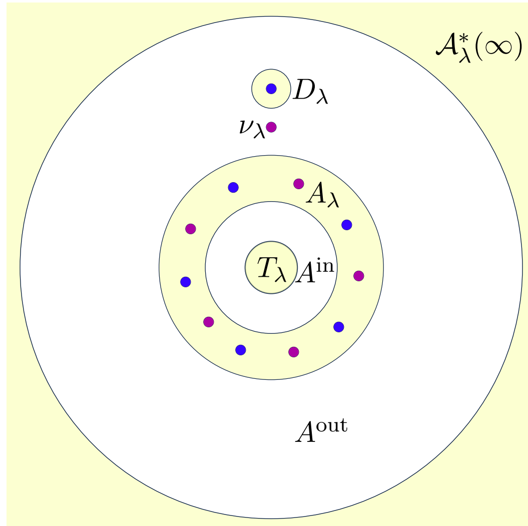


Figure 3.3: Partition of the dynamical plane with respect to $\mathcal{A}^*_\lambda(\infty)$, A_λ, T_λ, and D_λ, described in Proposition 3.1.8. Blue and purple points denote zeros and critical points, respectively.

(vi) Let A^{in} be the annulus bounded by ∂T_λ and $\overline{A_\lambda}$. Then, A^{in} is mapped onto the annulus $A(\partial T_\lambda, \partial\mathcal{A}^*_\lambda(\infty))$ with degree d. The set $A^{out} \setminus D_\lambda$ is also mapped onto the annulus $A(\partial T_\lambda, \partial\mathcal{A}^*_\lambda(\infty))$, with degree $n+1$.

Proof. Before proving the statements of the proposition we study the location and distribution of the critical points of $\mathcal{S}_{n,d,\lambda}$.

By Proposition 3.1.4, $\mathcal{A}^*_\lambda(\infty)$ is simply connected (in the Riemann sphere) and it is mapped onto itself with degree $n+1$. Since the global degree of the map $\mathcal{S}_{n,d,\lambda}$ is $n+d+1$, there exist exactly d preimages of ∞ outside $\mathcal{A}^*_\lambda(\infty)$, counting multiplicity. Since $z=0$ is mapped to ∞ with degree d, there exist no other preimages of ∞ (different from the ones in $\mathcal{A}^*_\lambda(\infty)$ and $z=0$). Moreover, T_λ is mapped with degree d onto $\mathcal{A}^*_\lambda(\infty)$ (observe that up to this point we still do not know if T_λ is simply connected).

Let Ω_λ be the annulus defined in (3.4). By Lemma 3.1.7, we know that $\mathcal{S}_{n,d,\lambda}(\Omega_\lambda) \subset T_\lambda$ and $\mathcal{S}_{n,d,\lambda}(T_\lambda) \subset \mathcal{A}^*_\lambda(\infty)$. Thus, $\Omega_\lambda \cap T_\lambda = \emptyset$ and Ω_λ is part of a multiply connected Fatou component which is a preimage of T_λ. We denote this Fatou component by A_λ (observe that up to this point we still do not know if A_λ is doubly connected).

We claim that w_λ and ν_λ do not belong to $T_\lambda \cup A_\lambda$. To see the claim we will prove that, for sufficiently small values of $|\lambda|$, w_λ and ν_λ belong to the annulus bounded by $\partial\mathcal{A}^*_\lambda(\infty)$ and $\overline{A_\lambda}$, denoted in what follows by A^{out}. Let γ be a smooth Jordan curve which separates $z=0$ from ν_0 and w_0, and such that $S^2_{n,a,Q}(\gamma)$ is a Jordan curve that surrounds $z=0$ and lies in $\text{Int}(\gamma)$. Its existence follows from the Bötcher coordinates of the fixed point $z=0$ for the unperturbed map. Notice that, by construction, γ does not depend on λ and it has a definite distance to $z=0$. By continuity with respect to λ, for $|\lambda|$ small enough, $\mathcal{S}^2_{n,d,\lambda}(\gamma) \subset \text{Int}(\gamma)$. Since $\mathcal{S}^2_{n,d,\lambda}(A_\lambda) \subset \mathcal{A}^*_\lambda(\infty)$ we conclude that $\gamma \cap A_\lambda = \emptyset$. Shrinking $|\lambda|$, if necessary, we claim that γ lies outside $\text{Fill}\,(\Omega_\lambda)$. Indeed, according to (3.4) the annulus Ω_λ collapses to the origin as $\lambda \to 0$ while γ keeps in a definite distance to $z=0$. Finally, notice that for $|\lambda|$ small ν_λ and w_λ remain as close as we want to ν_0 and w_0, respectively. Therefore, γ separates ν_λ and w_λ from T_λ and A_λ. Let $\mathcal{C}$ be a constant such that if $|\lambda| < \mathcal{C}$ all the above is true. Now we are ready to prove the statements.

Since T_λ contains only one critical point at $z = 0$ with multiplicity $d-1$ and it is mapped with degree d onto the topological disk $\mathcal{A}^*_\lambda(\infty)$, it follows from the Riemann-Hurwitz formula that T_λ is simply connected. This proves *(i)*. Similarly, A_λ contains exactly $n + d$ simple critical points and it is mapped with degree $n + d$ onto the topological disk T_λ. Thus, it is doubly connected by Riemann-Hurwitz formula. This proves *(ii)*.

The point w_λ is a preimage of $z = 0$ which lies in A^{out}, so it must belong to a Fatou component, denoted by D_λ, different from T_λ and A_λ. Moreover, $\mathcal{S}_{n,d,\lambda}(D_\lambda) = T_\lambda$. Since w_λ is the only (simple) preimage of $z = 0$ in D_λ we conclude that D_λ is mapped with degree 1 onto T_λ and is a conformal copy of T_λ. In particular, $\nu_\lambda \notin D_\lambda$ and all preimages of $z = 0$ belong to either A_λ and D_λ. This proves *(iii)*, *(iv)*, and *(v)*.

Finally, to prove statement *(vi)* we just notice that $\mathcal{S}_{n,d,\lambda}|_{A^{\text{in}}} : A^{\text{in}} \to A(\partial T_\lambda, \partial \mathcal{A}^*_\lambda(\infty))$ is a proper map. Since its degree is accomplished on the boundaries and ∂T_λ is mapped onto $\partial \mathcal{A}^*_\lambda(\infty)$ with degree d, it follows that A^{in} is mapped onto its image with degree d.

□

We will now show that the map $\mathcal{S}_{n,d,\lambda}$ is quasiconformally conjugated with a *finite generalized Blaschke product* in a specific annulus. We start by introducing generalized Blaschke products and stating a few properties. Let $\beta_\infty(z) := \frac{1}{z}$, and for $a \in \mathbb{C}$, let $\beta_a(z) := \frac{1-\bar{a}}{1-a}\frac{z-a}{1-\bar{a}z}$ be the Möbius maps which map the unit circle onto itself, fix $z = 1$, and send $z = a$ to $z = 0$.

Definition 3.1.9. *(generalized Blaschke product)* Let $\alpha \in [0, 1)$ and $|a_i| \in \mathbb{D}$, for all $i \in \{1, \ldots, d\}$. Then, the rational map

$$B(z) = e^{2\pi i\alpha} \prod_{i=1}^{d} \beta_{a_i}(z),$$

is called a degree d *Blaschke product*. If $\{a_i\}_i \subset (\hat{\mathbb{C}} \setminus \partial\mathbb{D})$, the map B is called a degree d *generalized Blaschke product*.

Generalized Blaschke products are a well studied topic of complex dynamics. This holds true because they are precisely the maps which map the unit circle onto itself. Therefore, they are a natural candidate for conjugacies of maps with Fatou components with invariant boundary.

Lemma 3.1.10. *[Mil06, Lemma 15.5] A rational map of degree d maps the unit circle onto itself if and only if it is a generalized Blaschke product.*

Generalized Blaschke products are symmetric with respect to the unit circle. This result follows directly from the Schwarz Reflection Principle since generalized Blaschke products map the unit circle onto itself.

Lemma 3.1.11. *Let B be a generalized Blaschke product, and let $\mathcal{I}(z) = \frac{1}{\bar{z}}$ denote the inversion with respect to the unit circle. Then, B is symmetric with respect to $\partial\mathbb{D}$, that is, $B(z) = \mathcal{I} \circ B \circ \mathcal{I}(z)$.*

We now prove that $\mathcal{S}_{n,d,\lambda}$ is conjugate to a finite Blaschke product on the annulus A^{out} introduced in the previous proposition.

Proposition 3.1.12. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d). Let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. Then, there exist an analytic Jordan curve $\Gamma \subset A_\lambda$ which surrounds $z = 0$, $b \in \mathbb{D}^*$, $\theta \in [0, 1)$, and a quasiconformal map $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $R_{b,\theta} \circ \varphi = \varphi \circ \mathcal{S}_{n,d,\lambda}$ on $A(\Gamma, \partial\mathcal{A}^*_\lambda(\infty))$, where*

$$R_{b,\theta}(z) = e^{2\pi i\theta} z^n \frac{z - b}{1 - \bar{b}z}$$

is a Blaschke product.

Proof. We claim that there exists an analytic Jordan curve $\Gamma \subset A_\lambda$ which surrounds $z = 0$ and the $n + d$ critical points c_{λ,ξ^j} and which is mapped with degree n to its image under $\mathcal{S}_{n,d,\lambda}$.

To see the claim let γ be an analytic Jordan curve in the interior of T_λ surrounding $z = 0$ and the $n+d$ critical values, images of the critical points c_{λ,ξ^j}. Clearly, the annulus $A(\gamma, \partial T_\lambda)$ contains no critical values and, from the Riemann-Hurwitz formula (compare Corollary 2.4.3), any connected component of its preimage is an annulus bounded by preimages of γ and ∂T_λ. It follows from Proposition 3.1.8 that two (among a total of three) of those preimages are disjoint annuli in A_λ, one associated to the internal boundary of A_λ and another associated to the external one. Denote them by $\mathcal{G}^{\text{in}}$ and $\mathcal{G}^{\text{out}}$. By construction, $\mathcal{S}_{n,d,\lambda}$ restricted to those two preimages is a proper map. We know that $\mathcal{S}_{n,d,\lambda}$ restricted to A_λ is proper of degree $n+d$ (see Proposition 3.1.8 *(ii)*) while $\mathcal{S}_{n,d,\lambda}$ restricted to $\mathcal{G}^{\text{in}}$ is proper of degree d (notice that the degree is achieved in the boundary, compare with Proposition 3.1.8*(vi)*). All together implies that $\mathcal{S}_{n,d,\lambda}$ restricted to $\mathcal{G}^{\text{out}}$ is proper of degree n. Let Γ be the inner boundary of $\mathcal{G}^{\text{out}}$. Then, Γ is an analytic Jordan curve, it maps to γ with degree n, and it surrounds the origin as well as all critical points c_{λ,ξ^j} (and zeros z_{λ,ξ^j}), as desired.

The remaining part of the proof is analogous to the one of [Can18, Proposition 3.1]. The strategy is to use a similar construction to the one of the Straightening Theorem for polynomial-like mappings (compare [BF14, Theorem 7.4]) to glue a dynamics conjugated to the one of the map $z \to z^n$ inside the curve γ, keep $\mathcal{S}_{n,d,\lambda}$ outside Γ, and interpolate using a quasi-conformal map in the annulus $A(\gamma, \Gamma)$. In this way we obtain a quasiregular map F of the Riemann sphere which has $z = 0$ as superattracting fixed point of local degree n (F is actually holomorphic around $z = 0$). The map F coincides with $\mathcal{S}_{n,d,\lambda}$ on $\text{Ext}(\Gamma)$, all points in $\text{Int}(\partial\mathcal{A}^*_\lambda(\infty))$ converge to $z = 0$ under iteration of F, and it maps $\text{Int}(\partial\mathcal{A}^*_\lambda(\infty))$ onto itself with degree $n + 1$ (since we have that $z = 0$ maps to itself with degree n and w_λ is the only further preimage of $z = 0$).

The map F is conjugate to a holomorphic function f via a quasiconformal φ map fixing $z = 0$. The basin of attraction of $z = 0$ under f is given by $\varphi(\text{Int}(\partial\mathcal{A}^*_\lambda(\infty)))$. Since the basin of attraction is simply connected and bounded by a quasicircle, f is conjugate to a Blaschke product in $\varphi(\text{Int}(\partial\mathcal{A}^*_\lambda(\infty)))$ (use a Rieman mapping to send $\varphi(\text{Int}(\partial\mathcal{A}^*_\lambda(\infty)))$ onto $\mathbb{D}$ and apply Lemma 3.1.10). Observe that $\varphi(\partial\mathcal{A}^*_\lambda(\infty)) = \partial\mathbb{D}$. Since $z = 0$ is superattracting of local degree n and f maps $\varphi(\text{Int}(\partial\mathcal{A}^*_\lambda(\infty)))$ onto itself with degree $n+1$, the Blaschke product has the form $R_{b,\theta} = e^{2\pi i\theta} z^n \frac{z-b}{1-\bar{b}z}$, where $b \in \mathbb{D}^*$ and θ satisfies $|e^{2\pi i\theta}| = 1$. Since F coincides with $\mathcal{S}_{n,d,\lambda}$ in $A(\Gamma, \partial\mathcal{A}^*_\lambda(\infty))$, it follows that $\mathcal{S}_{n,d,\lambda}$ is conjugate to $R_{b,\theta}$ in $A(\Gamma, \partial\mathcal{A}^*_\lambda(\infty))$.

□

It will be crucial in what follows to have a deep understanding of the preimages of curves which surround the origin $z = 0$ (as well as Fatou components). The following proposition describes this in a precise way.

Proposition 3.1.13. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d). Let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. Let $\gamma \subset A(\partial T_\lambda, \partial \mathcal{A}^*_\lambda(\infty))$ be a Jordan curve which surrounds $z = 0$. Then, $\mathcal{S}^{-1}_{n,d,\lambda}(\gamma)$ contains a single connected component in A^{in}, which surrounds $z = 0$ and is mapped with degree d onto γ. The other components of $\mathcal{S}^{-1}_{n,d,\lambda}(\gamma)$ lie in A^{out} and, depending on the location of the free critical value, i.e. $\mathcal{S}_{n,d,\lambda}(\nu_\lambda)$, one of the following holds:*

(i) If $\mathcal{S}_{n,d,\lambda}(\nu_\lambda) \in Int(\gamma)$, then $\mathcal{S}^{-1}_{n,d,\lambda}(\gamma)$ has a single connected component in A^{out}. Indeed, it is a Jordan curve which surrounds $z = 0$ and it is mapped with degree $n + 1$ onto γ under $\mathcal{S}_{n,d,\lambda}$.

(ii) If $\mathcal{S}_{n,d,\lambda}(\nu_\lambda) \in \gamma$, then $\mathcal{S}^{-1}_{n,d,\lambda}(\gamma)$ has a single connected component in A^{out} consisting of 2 Jordan curves intersecting precisely at ν_λ. One is a Jordan curve γ_0^{-1} which surrounds $z = 0$, but not w_λ, and it is mapped with degree n onto γ. The other is a Jordan curve γ_w^{-1} which surrounds w_λ, but not $z = 0$, and it is mapped with degree 1 onto γ. The curve γ_0^{-1} does not surround γ_w^{-1}.

(iii) If $\mathcal{S}_{n,d,\lambda}(\nu_\lambda) \in Ext(\gamma)$, then $\mathcal{S}^{-1}_{n,d,\lambda}(\gamma)$ has 2 disjoint components in A^{out}. One is a Jordan curve γ_0^{-1} which surrounds $z = 0$, but not w_λ, and it is mapped with degree n onto γ. The other is a Jordan curve γ_w^{-1} which surrounds w_λ, but not $z = 0$, and it is mapped with degree 1 onto γ. The curve γ_0^{-1} does not surround γ_w^{-1}.

Proof. We first notice that given any Jordan curve in $A(\partial T_\lambda, \partial \mathcal{A}^*_\lambda(\infty))$ all preimages should be located either in A^{in} or A^{out} since T_λ, A_λ and $\mathcal{A}^*_\lambda(\infty)$ are mapped outside $A(\partial T_\lambda, \partial \mathcal{A}^*_\lambda(\infty))$. Moreover, by Proposition 3.1.8*(vi)* any Jordan curve in $A(\partial T_\lambda, \partial \mathcal{A}^*_\lambda(\infty))$ should have preimage(s) in A^{in} as well as in A^{out}.

Let $\gamma \in A(\partial T_\lambda, \partial \mathcal{A}^*_\lambda(\infty))$ be a Jordan curve surrounding the origin. First we study the topology of the preimage(s) of γ in A^{in}. By Proposition 3.1.8, γ has exactly d preimages in A^{in}. Let γ_0 be one of the preimages of γ in A^{in}. The goal is to show that in fact γ_0 is mapped by T_λ with degree d, so there are no other preimages whatsoever. Observe that I$nt\,(\gamma_0)$ should contain either a pole, a zero, or $z = 0$, otherwise it cannot be mapped to γ which surrounds $z = 0$. Therefore, γ_0 surrounds $z = 0$. Take the annulus $A(\gamma, \partial \mathcal{A}^*_\lambda(\infty))$ and consider its preimage in A^{in}. Since the only preimage of $\partial \mathcal{A}^*_\lambda(\infty)$ in $\overline{A^{\text{in}}}$ is ∂T_λ and A^{in} contains no critical point, the preimage should be the annulus $A(\partial T_\lambda, \gamma_0)$. The map $\mathcal{S}_{n,d,\lambda}|_{A(\partial T_\lambda, \gamma_0)}$ is proper of degree d since $\mathcal{S}_{n,d,\lambda}$ maps ∂T_λ onto $\partial \mathcal{A}^*_\lambda(\infty)$ with degree d. We conclude that γ_0 is mapped with degree d onto γ, as desired.

The proof of statements *(i)*-*(iii)* about the topology of the preimage(s) of γ in A^{out} is analogous to the one of [Can18, Proposition 3.3] by using that $\mathcal{S}_{n,d,\lambda}$ is conjugate to the Blaschke product $R_{\theta,b}$ in A^{out} (see Proposition 3.1.12).

□

Remark 3.1.14. *It follows from Proposition* 3.1.13 *that each Fatou component different from T_λ and $\mathcal{A}^*_\lambda(\infty)$ which surrounds $z = 0$ (and so it contains a Jordan curve which surrounds $z = 0$) has exactly two preimages which also surround $z = 0$. One of them lies in A^{in} and the other lies in A^{out}.*

The following lemma shows that Fatou components which do not surround the origin do not have preimages which surround it.

Lemma 3.1.15. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d). Let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. Let U be a multiply connected Fatou component. If U does not surround $z = 0$, then no component of its preimage $\mathcal{S}_{n,d,\lambda}^{-1}(U)$ surrounds $z = 0$.*

Proof. Suppose that U does not surround $z = 0$ and let V be a preimage of U which surrounds $z = 0$. Let $U' = \text{Fill}(U)$ and V' the preimage of U' which contains V. Observe that $U' \subset A(\partial T_\lambda, \partial \mathcal{A}_\lambda^*(\infty))$. Since $V' \subset A^{\text{in}} \cup A^{\text{out}}$, it can contain at most one critical point. Since $\mathcal{S}_{n,d,\lambda}|_{V'} : V' \to U'$ is proper and V' contains at most one critical point, it follows from the Riemann Hurwitz formula that V' is simply connected (compare Corollary 2.4.3). Since V surrounds the origin, then $z = 0$ lies in V'. However, this is impossible since $z = 0$ is mapped to ∞ and U' is bounded.

□

Proposition 3.1.8 tells us that for $|\lambda|$ small enough the map $\mathcal{S}_{n,d,\lambda}$ is essentially uni-critical since all critical points except ν_λ belong to $A_\lambda(\infty)$. Up to now, however, we have not imposed any particular dynamical behaviour for ν_λ. With the aim of proving the main results of this chapter from now on we restrict ourselves to parameters for which the free critical point ν_λ belongs to $A_\lambda(\infty)$ (sometimes the term *captured parameters* is used).

Under this assumption, Proposition 3.1.8 implies $\nu_\lambda \in \mathcal{A}_\lambda(\infty) \setminus (\mathcal{A}_\lambda^*(\infty) \cup T_\lambda \cup D_\lambda \cup A_\lambda)$. We further assume that ν_λ belongs to a Fatou component which is an eventual preimage of A_λ that surrounds $z = 0$. The following result gives relevant notation and determines a partition of the dynamical plane (compare Figure 3.4) that will be extremely useful to study achievable connectivities of Fatou components.

Theorem 3.1.16. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d). Let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. Assume that $\nu_\lambda \in \mathcal{U}_\nu$, where $\mathcal{U}_\nu$ is a Fatou component which is eventually mapped onto A_λ and surrounds $z = 0$. Then, $\mathcal{U}_\nu$ is triply connected and $\mathcal{U}_\nu \subset A^{out}$. Moreover, the following statements hold.*

(i) The set $\overline{\mathcal{U}_\nu}$ bounds an open disk $\mathcal{U}_1$ which is mapped with degree 1 onto the open disk $\mathcal{V}_n \cup \overline{T_\lambda}$, where $\mathcal{V}_n$ is the annulus bounded by ∂T_λ and $\overline{\mathcal{S}_{n,d,\lambda}(\mathcal{U}_\nu)}$. In particular, $w_\lambda \in \mathcal{U}_1$.

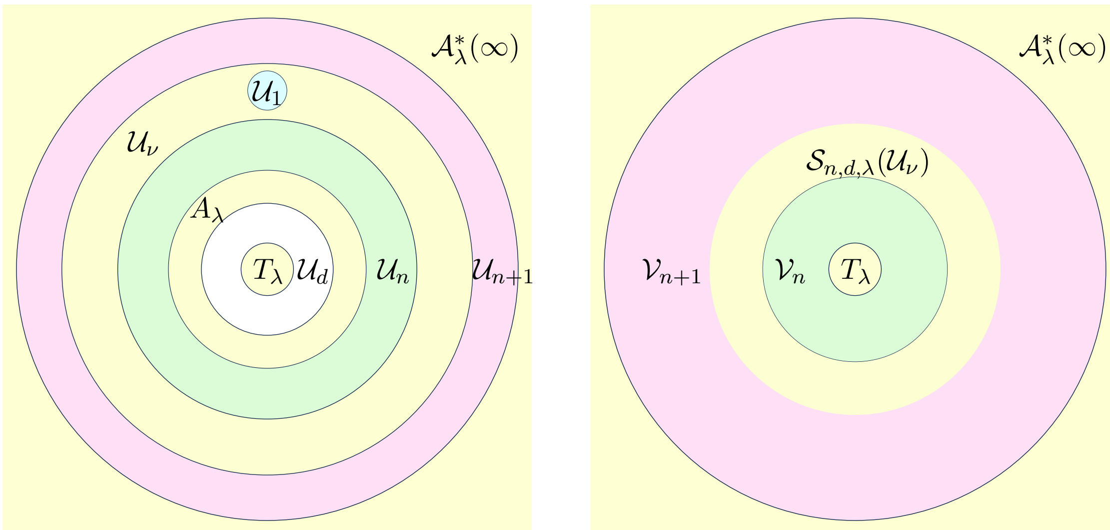
(ii) The annulus $\mathcal{U}_{n+1}$ bounded by $\overline{\mathcal{U}_\nu}$ and $\partial \mathcal{A}_\lambda^*(\infty)$ is mapped with degree $n+1$ onto the annulus $\mathcal{V}_{n+1}$ bounded by $\overline{\mathcal{S}_{n,d,\lambda}(\mathcal{U}_\nu)}$ and $\partial \mathcal{A}_\lambda^*(\infty)$.

(iii) The annulus $\mathcal{U}_n$ bounded by $\overline{A_\lambda}$ and $\overline{\mathcal{U}_\nu}$ is mapped with degree n onto the annulus $\mathcal{V}_n$ bounded by ∂T_λ and $\overline{\mathcal{S}_{n,d,\lambda}(\mathcal{U}_\nu)}$.

(iv) The annulus $\mathcal{U}_d$ bounded by ∂T_λ and $\overline{A_\lambda}$ (i.e., the annulus A^{in}) is mapped with degree d onto the annulus $A(\partial T_\lambda, \partial \mathcal{A}_\lambda^*(\infty))$.

(v) The Fatou component $\mathcal{U}_\nu$ lies in $\mathcal{V}_{n+1}$ and it is mapped under $\mathcal{S}_{n,d,\lambda}$ with degree $n+1$ onto its image. In particular, $\mathcal{U}_\nu$ surrounds $\mathcal{S}_{n,d,\lambda}(\mathcal{U}_\nu)$.

Remark 3.1.17. *Statement (iv) in Theorem 3.1.16 coincides with statement (vi) of Proposition 3.1.8. We use this double naming ($\mathcal{U}_d$ and A^{in}) in order to uniformize notation in what follows. Notice that every set $\mathcal{U}_i$, $i = d, n, n+1$, is mapped onto its image with degree i. This notation is particularly useful in Section 3.2. Also, notice that in order to simplify notation we avoid indicating the dependence of $\mathcal{U}_\nu$ and $\mathcal{U}_i$, $i = 1, d, n, n+1$, with respect to the parameter λ.*



(a) The partition with respect to $\mathcal{U}_\nu$ and A_λ.

(b) The partition with respect to $\mathcal{S}_{n,d,\lambda}(\mathcal{U}_\nu)$.

Figure 3.4: Partitions of the dynamical plane introduced in Theorem 3.1.16.

Proof. By the Riemann-Hurwitz formula, the iterated preimages of A_λ are doubly connected unless they contain a critical point. Under our hypothesis, this occurs precisely at $\mathcal{U}_\nu$ (since the only critical point eventually mapped in A_λ is ν_λ). A direct application of the Riemann-Hurwitz formula implies that, since ν_λ is a simple critical point, $\mathcal{U}_\nu$ is triply connected. Moreover, $\mathcal{U}_\nu \subset A^{\text{out}}$ since $\nu_\lambda \in A^{\text{out}}$. This proves the first part of the statement.

From above $\partial\mathcal{U}_\nu$ has three components. Since $\mathcal{U}_\nu$ separates $z = 0$ from $z = \infty$, there should be exactly two components of $\partial\mathcal{U}_\nu$ surrounding $z = 0$. We denote them by γ_c^{in} and γ_c^{out}, where $\gamma_c^{\text{in}} \subset \text{Int}(\gamma_c^{\text{out}})$. The other component of $\partial\mathcal{U}_\nu$, denoted by γ_c^1, does not surround $z = 0$.

Set $\mathcal{U}_1 = \text{Int}(\gamma_c^1)$. Since γ_c^1 is mapped onto a component of $\partial\mathcal{S}_{n,d,\lambda}(\mathcal{U}_\nu)$, $\mathcal{U}_1$ is mapped either to the bounded or the unbounded component of the complement of $\mathcal{S}_{n,d,\lambda}(\mathcal{U}_\nu)$ (which is an annulus by hypothesis). However, since all poles are in $\mathcal{A}^*_\lambda(\infty) \cup T_\lambda$, then $\mathcal{U}_1$ should be mapped onto the bounded component of $\mathcal{S}_{n,d,\lambda}(\mathcal{U}_\nu)$. Therefore, $\mathcal{U}_1$ contains the zero w_λ (and no other preimages of $z = 0$). We conclude that $\mathcal{S}_{n,d,\lambda}|_{\mathcal{U}_1}$ has degree 1. In particular, γ_1 is mapped onto its image with degree 1. This proves *(i)*.

Let $\mathcal{U}_{n+1}$ be the annulus bounded by $\overline{\mathcal{U}_\nu}$ and $\overline{\partial\mathcal{A}^*_\lambda(\infty)}$, and let $\mathcal{V}_{n+1} = \mathcal{S}_{n,d,\lambda}(\mathcal{U}_{n+1})$. By construction, $\mathcal{V}_{n+1}$ is the annulus bounded by $\overline{\mathcal{S}_{n,d,\lambda}(\mathcal{U}_\nu)}$ and $\partial\mathcal{A}^*_\lambda(\infty)$. It is immediate that the map $\mathcal{S}_{n,d,\lambda}|_{\mathcal{U}_{n+1}} : \mathcal{U}_{n+1} \to \mathcal{V}_{n+1}$ is proper. Since the degree is accomplished on the boundaries and $\partial\mathcal{A}^*_\lambda(\infty)$ is mapped onto itself with degree $n+1$, $\mathcal{U}_{n+1}$ is mapped onto $\mathcal{V}_{n+1}$ with degree $n+1$. This proves *(ii)*. The proof of statement *(iii)* is similar and statement *(iv)* was already proven in Proposition 3.1.8.

Finally, we prove statement *(v)* by contradiction. Assume that $\mathcal{U}_\nu$ does not lie in $\mathcal{V}_{n+1}$. Then, either $\mathcal{U}_\nu$ maps onto itself (which is impossible) or $\mathcal{V}_{n+1} \subset \mathcal{U}_{n+1}$. This would imply that $\mathcal{U}_{n+1}$ is mapped under $\mathcal{S}_{n,d,\lambda}$ on itself and, hence, there exists a periodic Fatou component different from $\mathcal{A}_\lambda(\infty)$. This is impossible since, by assumption, the orbits of all critical points converge to $z = \infty$. $\square$

Remark 3.1.18. *Under the assumptions of Theorem* 3.1.16*, if* U *is a Fatou component which surrounds* $z = 0$ *and lies in* $\mathcal{U}_{n+1}$ *or* $\mathcal{U}_n$*, then it follows from Proposition* 3.1.12 *that its image lies in Fill(*U*). Indeed,* $\mathcal{U}_n$ *and* $\mathcal{U}_{n+1}$ *are contained in* A^{out} *(see Figure 3.4), and* A_{out} *belongs to the region where the dynamics are conjugate to the ones of a Blaschke product.*

As it will become clear in the next sections devoted to prove the main results of this chapter, the presence of Fatou components with high connectivity in the dynamical plane is based on taking special iterated preimages of $\mathcal{U}_\nu$. With this in mind we end the section by stating the following corollary of Proposition 3.1.13 and Theorem 3.1.16.

Corollary 3.1.19. *Let* $\mathcal{S}_{n,d,\lambda}$ *satisfying (a), (b), (c), and (d). Let* $\lambda \neq 0$*,* $|\lambda| < \mathcal{C}$*. Assume that* $\nu_\lambda \in \mathcal{U}_\nu$*, where* $\mathcal{U}_\nu$ *is an iterated preimage of* A_λ *which surrounds* $z = 0$*. Let* W *be a Fatou component which surrounds* $z = 0$*, different from* T_λ *and* A_λ*. Then, the following statements hold.*

(i) If $W \subset \mathcal{V}_{n+1}$*, then it has a unique preimage in* $\mathcal{U}_d$*, which surrounds* $z = 0$ *and is mapped under* $\mathcal{S}_{n,d,\lambda}$ *onto* W *with degree* d*, and a unique preimage in* $\mathcal{U}_{n+1}$*, which surrounds* $z = 0$ *and is mapped under* $\mathcal{S}_{n,d,\lambda}$ *onto* W *with degree* $n + 1$*.*

(ii) If $W \subset \mathcal{V}_n$*, then it has a unique preimage in* $\mathcal{U}_d$*, which surrounds* $z = 0$ *and is mapped under* $\mathcal{S}_{n,d,\lambda}$ *onto* W *with degree* d*, and two further preimages. One lies in* $\mathcal{U}_n$*, surrounds* $z = 0$ *and is mapped under* $\mathcal{S}_{n,d,\lambda}$ *onto* W *with degree* n*. The other one lies in* $\mathcal{U}_1$*, does not surround* $z = 0$*, and is mapped under* $\mathcal{S}_{n,d,\lambda}$ *onto* W *with degree* 1*.*

3.2 Proofs of theorems A and B

In this section we prove Theorem A and Theorem B. We first show that Fatou components which do not surround $z = 0$ cannot be used to achieve higher connectivities.

Lemma 3.2.1. *Let* $\mathcal{S}_{n,d,\lambda}$ *satisfying (a), (b), (c), and (d). Let* $\lambda \neq 0$*,* $|\lambda| < \mathcal{C}$*. Assume also that* $\nu_\lambda \in \mathcal{U}_\nu$*. Let* V *be a Fatou component which does not surround* $z = 0$*. Then* V *and all of its eventual preimages have the same connectivity.*

Proof. Let V be a Fatou component which does not surround $z = 0$ and let U be a preimage of it. By Lemma 3.1.15, U does neither surround $z = 0$. It follows that Fill(U) does not contain any critical point. Therefore, the map $\mathcal{S}_{n,d,\lambda}|_{\text{Fill}(U)} : \text{Fill}(U) \to \text{Fill}(V)$ is a proper map of degree 1. We can conclude that $\mathcal{S}_{n,d,\lambda}|_U : U \to V$ is conformal, so U and V have the same connectivity. □

Next we give the form of the connectivities of Fatou components of $\mathcal{S}_{n,d,\lambda}$. We want to remark that not all these connectivities are achievable (see Theorem A).

Proposition 3.2.2. *Let* $\mathcal{S}_{n,d,\lambda}$ *satisfying (a), (b), (c), and (d). Let* $\lambda \neq 0$*,* $|\lambda| < \mathcal{C}$*. Assume also that* $\nu_\lambda \in \mathcal{U}_\nu$*, where* $\mathcal{U}_\nu$ *is an iterated preimage of* A_λ *which surrounds* $z = 0$*. All Fatou components have connectivity 1, 2, or* $\kappa = (n+1)^i n^j d^\ell + 2$*, where* $i, j, \ell \in \mathbb{N}$*.*

Proof. By Proposition 3.1.8, D_λ and all its eventual preimages have connectivity 1 since none of them can contain critical points. Analogously, all eventual preimages of A_λ other than $\mathcal{U}_\nu$ and its preimages have connectivity 2 since none of them contain critical points (see Corollary 2.4.3). Finally, we study the connectivity of the preimages of $\mathcal{U}_\nu$. By Lemma 3.2.1,

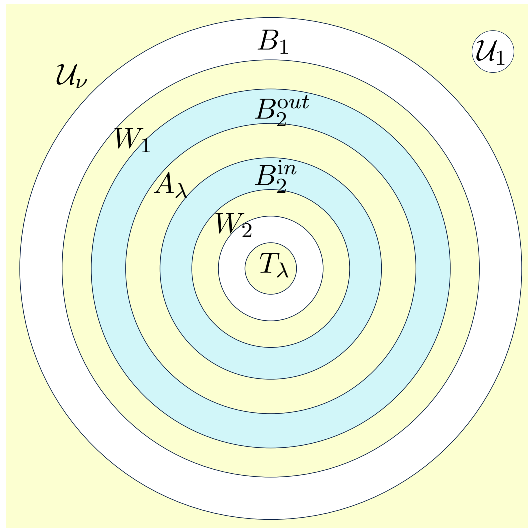


Figure 3.5: Description of the situation in the proof of Lemma 3.2.3, where $k = 2$ and $W_2 \subset \mathcal{U}_d$. In this case, $B_2 = B_2^{\text{out}} \cup \overline{A_\lambda} \cup B_2^{\text{in}}$.

it suffices to study preimages of $\mathcal{U}_\nu$ which surround $z = 0$. It follows from the Riemann-Hurwitz formula that if $f : U \to V$ is proper of degree q without critical points and V has connectivity $p + 2$, then U has connectivity $qp + 2$. By Corollary 3.1.19, all preimages of $\mathcal{U}_\nu$ which surround $z = 0$ map to their images with degree d, n, or $n + 1$. Since $\mathcal{U}_\nu$ has connectivity $3 = 1 + 2$, the possible connectivities of the preimages of $\mathcal{U}_\nu$ surrounding $z = 0$ are of the form $\kappa = (n + 1)^i n^j d^\ell + 2$, where $i, j, \ell \in \mathbb{N}$.

□

According to Corollary 3.1.19, if U is a iterative preimage of $\mathcal{U}_\nu$ which surrounds $z = 0$, its preimages which surround $z = 0$ may be located in $\mathcal{U}_d$, $\mathcal{U}_n$ or $\mathcal{U}_{n+1}$. Next lemma shows that there are achievable upper bounds for the itineraries of iterated preimages of $\mathcal{U}_\nu$. Recall from Remark 3.1.18 that if $U \subset \mathcal{U}_n$ is an iterated preimage of $\mathcal{U}_\nu$, then either $\mathcal{S}_{n,d,\lambda}(U) \subset \mathcal{U}_n$ or $\mathcal{S}_{n,d,\lambda}(U) \subset \mathcal{U}_d$. Let $k \geq 1$ be minimal such that $\mathcal{S}^k_{n,d,\lambda}(\mathcal{U}_\nu) \subset \text{Fill}(A_\lambda)$. The first half of Lemma 3.2.3 shows that if $U \subset \mathcal{U}_n$ is a preimage of $\mathcal{U}_\nu$ which surrounds $z = 0$, then the itinerary of U intersects $\mathcal{U}_d$ in $p \leq k$ iterations. The second half of Lemma 3.2.3 shows that if $U \subset \mathcal{U}_d$ is a preimage of $\mathcal{U}_\nu$ which surrounds $z = 0$, then there exist at least $k - 1$ consecutive backwards iterates of U which lie in $\mathcal{U}_n$.

Lemma 3.2.3. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d). Let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. Assume also that $\nu_\lambda \in \mathcal{U}_\nu$. Let $k \geq 1$ such that $\mathcal{S}^k_{n,d,\lambda}(\mathcal{U}_\nu) \subset$ Fill(A_λ) and $\mathcal{S}^j_{n,d,\lambda}(\mathcal{U}_c) \subset \mathcal{U}_n$ for $1 \leq j < k$.*

(i) Let $U \subset \mathcal{U}_n$ be an iterated preimage of $\mathcal{U}_\nu$ which surrounds $z = 0$. Then, there exists $p \geq 1$ such that $S^p(U) \subset \mathcal{U}_d$ and $S^{p'}(U) \in \mathcal{U}_n$ for $0 \leq p' < p$. Moreover, p satisfies $p \leq k$.

(ii) Let $U \subset \mathcal{U}_d$ be a preimage of $\mathcal{U}_\nu$ which surrounds $z = 0$. Then, there exists $U' \subset \mathcal{U}_n$ such that $\mathcal{S}^j_{n,d,\lambda}(U') \subset \mathcal{U}_n$ for $0 \leq j < k - 1$ and $\mathcal{S}^{k-1}_{n,d,\lambda}(U') = U$.

Proof. Set $W_i = \mathcal{S}^i_{n,d,\lambda}(\mathcal{U}_\nu)$, $i = 0, \ldots, k$. By Remark 3.1.18, W_i surrounds W_{i+1}, $i = 0, \ldots, k - 1$. Let B_{i+1} be the annulus bounded by $\overline{W_i}$ and $\overline{W_{i+1}}$, $i = 0, \ldots, k - 1$. It follows that $\mathcal{S}_{n,d,\lambda}(B_i) = B_{i+1}$, $i = 1, \ldots, k-1$. Observe that if $W_k \subset \mathcal{U}_d$, then $B_k = B_k^{out} \cup \overline{A_\lambda} \cup B_k^{in}$, where $B_k^{out} = B_k \cap \mathcal{U}_n$ and $B_k^{in} = B_k \cap \mathcal{U}_d$ (see Figure 3.5). Along the proof we distinguish the cases $W_k = A_\lambda$ and $W_k \subset \mathcal{U}_d$.

We now prove statement *(i)*. Assume first that $W_k = A_\lambda$. Let $U \subset B_i$, $i = 1, \ldots, k$, be an eventual preimage of $\mathcal{U}_\nu$ which surrounds $z = 0$. Then $\mathcal{S}_{n,d,\lambda}^{k+1-i}(U) \subset \mathcal{U}_d$. We can conclude that if $U \subset \mathcal{U}_n$ is a preimage of $\mathcal{U}_\nu$ which surrounds $z = 0$, then there exists $p \leq k$ such that $\mathcal{S}_{n,d,\lambda}^{p}(U) \subset \mathcal{U}_d$. In fact, $p = k - i$ if $U \subset B_i$. Now assume $W_k \subset \mathcal{U}_d$. For $k = 1$, $\mathcal{S}_{n,d,\lambda}(\mathcal{U}_n) \subset \mathcal{U}_d$ and the conclusion follows. For $k \geq 2$ we have $W_1 \subset \mathcal{U}_n$ (so B_{k-1} and B_k exist). Let $U \subset B_i$, $i = 1, \ldots, k-1$, be a preimage of $\mathcal{U}_\nu$ which surrounds $z = 0$ and let $V = \mathcal{S}_{n,d,\lambda}^{k-1-i}(U)$. Observe that $V \subset B_{k-1}$ is an eventual preimage of $\mathcal{U}_\nu$ which surrounds $z = 0$ and $\mathcal{U}_d$ is the disjoint union of B_k^{in}, $\overline{W_k}$, and $\mathcal{S}_{n,d,\lambda}(B_k^{out})$. Then, $\mathcal{S}_{n,d,\lambda}(V) \subset B_k^{out}$ (and $\mathcal{S}_{n,d,\lambda}^2(V) \subset \mathcal{U}_d$) or $\mathcal{S}_{n,d,\lambda}(V) \subset B_k^{in}$ (and $\mathcal{S}_{n,d,\lambda}(V) \subset \mathcal{U}_d$). Since $\mathcal{S}_{n,d,\lambda}^{k-1-i}(B_i) = B_{k-1}$, $i = 0, \ldots, k-1$, this concludes the proof of statement *(i)*.

Now we prove *(ii)*. Let $U \subset \mathcal{U}_d$ be a preimage of $\mathcal{U}_\nu$ which surrounds $z = 0$. Assume first that $W_k = A_\lambda$. Then $\mathcal{U}_d = \mathcal{S}_{n,d,\lambda}(B_k) = \mathcal{S}_{n,d,\lambda}^k(B_1)$. So there exists $U' \subset \mathcal{U}_1$ such that $\mathcal{S}_{n,d,\lambda}^j(U') \subset B_{j+1} \subset \mathcal{U}_n$ for $0 \leq j < k$ and $\mathcal{S}_{n,d,\lambda}^k(U') = U$. Now let $W_k \subset \mathcal{U}_d$. For $k = 1$ there is nothing to prove. For $k \geq 2$ we have $W_1 \subset \mathcal{U}_n$ (so B_{k-1} and B_k exist). Moreover, $\mathcal{U}_d = B_k^{in} \cup \overline{W_k} \cup \mathcal{S}_{n,d,\lambda}(B_k^{out})$. Since U is a preimage of $\mathcal{U}_\nu$ we have that $U \neq W_k$. We distinguish 2 cases. If $U \subset B_k^{in}$, then $U \subset \mathcal{S}_{n,d,\lambda}(B_{k-1})$ and we can take preimages through the sets B_i so that there exists $U' \subset B_1$ such that $\mathcal{S}_{n,d,\lambda}^j(U') \subset B_{j+1} \subset \mathcal{U}_n$ for $0 \leq j < k-1$ and $\mathcal{S}_{n,d,\lambda}^{k-1}(U') = U$. Finally, if $U \subset \mathcal{S}_{n,d,\lambda}(B_k^{out})$, then $U \subset \mathcal{S}_{n,d,\lambda}(B_k)$ and there exists $U' \subset B_1$ such that $\mathcal{S}_{n,d,\lambda}^j(U') \subset B_{j+1} \subset \mathcal{U}_n$ for $0 \leq j < k$ and $\mathcal{S}_{n,d,\lambda}^k(U') = U$. This concludes the proof of statement *(ii)*. □

We can now proceed with proof of Theorem A.

Proof of Theorem A. By Corollary 3.1.19, Lemma 3.2.1, and Proposition 3.2.2, if the connectivity of a Fatou component is different from 1 or 2, then it has to be of the form $\kappa = (n+1)^i d^j n^\ell + 2$ where $i, j, \ell \in \mathbb{N}$. It follows from the Riemann-Hurwitz formula that if $f : U \to V$ is proper of degree q without critical points and V has connectivity $p+2$, then U has connectivity $qp+2$. Moreover, these connectivities are achieved through preimages of $\mathcal{U}_\nu$. In order to increase the connectivity we have to take preimages of $\mathcal{U}_\nu$, which has connectivity $3 = 1 + 2$. It follows from Remark 3.1.18 that if $U \subset \mathcal{U}_s$, $s \in \{n+1, d, n\}$, is a Fatou component that surrounds $z = 0$, then it is mapped onto its image with degree s. Therefore, in order to increase the coeficient n in the expression of the connectivity, we have to take preimages in $\mathcal{U}_n$. By Remark 3.1.18, every Fatou component $U \subset \mathcal{U}_n$ is eventually mapped to $\mathcal{U}_d$, without passing through $\mathcal{U}_{n+1}$. By Lemma 3.2.3 *(i)*, for every backwards iteration through $\mathcal{U}_d$ there are at most k backwards iterations in $\mathcal{U}_n$. Since $\mathcal{U}_\nu \not\subset \mathcal{U}_d \cup \mathcal{U}_n$, it follows that $\ell \leq jk$. □

The final part of this section is devoted to the proof of Theorem B. The following lemma shows that there are no restrictions to the exponents of $n+1$ and d in respect to achievable connectivities.

Lemma 3.2.4. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d). Let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. Assume also that $\nu_\lambda \in \mathcal{U}_\nu$, where $\mathcal{U}_\nu$ is an iterated preimage of A_λ which surrounds $z = 0$. Then, the following hold:*

(i) There exists an eventual preimage of $\mathcal{U}_\nu$ which lies in $\mathcal{U}_{n+1}$, surrounds $z=0$, and has connectivity $\kappa=(n+1)^i+2, \quad \forall i\in\mathbb{N}$.

(ii) Let V be an eventual preimage of $\mathcal{U}_\nu$ which surrounds $z=0$ and let κ be the connectivity of V. Then, there exists a Fatou component, which surrounds $z=0$, of connectivity $d(\kappa-2)+2$.

In particular, there exists a Fatou component which surrounds $z=0$ and has connectivity $\kappa=(n+1)^i d^j+2, \quad \forall i,j\in\mathbb{N}$.

Proof. First we prove *(i)*. Recall that, by Theorem 3.1.16, $\mathcal{U}_\nu\subset\mathcal{V}_{n+1}$. By Corollary 3.1.19 and Remark 3.1.18, for any $i\geq 1$ there exists a Fatou component U which surrounds $z=0$ such that $\mathcal{S}^j_{n,d,\lambda}(U)\subset\mathcal{U}_{n+1}$, for $j=0,\dots i-1$, and $\mathcal{S}^i_{n,d,\lambda}(U)=\mathcal{U}_\nu$. Since $\mathcal{U}_\nu$ has connectivity 3 and no eventual preimage of $\mathcal{U}_\nu$ contains a critical point, it follows by succesively applying the Riemann-Hurwitz formula that the connectivity of U is $(n+1)^i+2$. This proves *(i)*.

To prove *(ii)*, let V be an eventual preimage of $\mathcal{U}_\nu$ which surrounds $z=0$, of connectivity κ. By Corollary 3.1.19, V has a preimage in $\mathcal{U}_d$ which surrounds $z=0$ and which is mapped onto it with degree d. Since V cannot contain any critical value, by the Riemann-Hurwitz formula the connectivity of this preimage of V is $d(\kappa-2)+2$. This concludes the proof of *(ii)*. □

We can now proceed with proof of Theorem B.

Proof of Theorem B. Fix $i\geq 0, j\geq 0, \ell\geq 0$ such that $\ell\leq j(k-1)$. We want to show that there exists a Fatou component U (which will be an iterated preimage of $\mathcal{U}_\nu$) of connectivity $\kappa=(n+1)^i d^j n^\ell+2$.

If $k=1$, then by Lemma 3.2.4 the conclusion holds. Otherwise, $\mathcal{S}_{n,d,\lambda}(\mathcal{U}_\nu)\subset\mathcal{U}_n$ and $\mathcal{U}_d\subset\mathcal{V}_n$. By Lemma 3.2.4 *(i)*, there exists a Fatou component V which is an iterated preimage of $\mathcal{U}_\nu$, surrounds $z=0$, lies in $\mathcal{U}_{n+1}$, and has connectivity $(n+1)^i+2$. This concludes the proof for $j=0$.

Assume that $j\neq 0$ (remember that $k>1$). By Corollary 3.1.19 *(i)*, there exists a preimage of V in $\mathcal{U}_d$ which surrounds $z=0$. By Lemma 3.2.3 *(ii)*, there exists $V^{(1)}\subset\mathcal{U}_n$ which surrounds $z=0$ such that

$$\mathcal{S}^r_{n,d,\lambda}(V^{(1)})\subset\mathcal{U}_n \text{ for } 0\leq r<k-1, \quad \mathcal{S}^{k-1}_{n,d,\lambda}(V^{(1)})\subset\mathcal{U}_d \quad \text{and} \quad \mathcal{S}^k_{n,d,\lambda}(V^{(1)})=V.$$

Recall that no iterated preimage of $\mathcal{U}_\nu$ contains a critical point and so, by Corollary 3.1.19, if they lie in $\mathcal{U}_i$, $i\in\{n+1,\, n,\, d\}$, they map forward with degree i. Applying this criteria to the iterated preimages of V up to $V^{(1)}$, we get from the Riemann-Hurwitz formula that

$$\kappa\left(\mathcal{S}^r_{n,d,\lambda}(V^{(1)})\right)=(n+1)^i d n^{k-1-r}+2,\ r=0,\dots k-1.$$

Starting the process with $V^{(1)}$ instead of V we can take a preimage of $V^{(1)}$ in $\mathcal{U}_d$ and then we can take up to $k-1$ iterated preimages in $\mathcal{U}_n$ to land on, say, $V^{(2)}$. As above, we get that

$$\kappa\left(\mathcal{S}^r_{n,d,\lambda}(V^{(2)})\right)=(n+1)^i d^2 n^{2(k-1)-r}+2,\ r=0,\dots k-1.$$

Repeating the same process j-times we conclude that there exist Fatou components with connectivity

$$\kappa\left(\mathcal{S}^r_{n,d,\lambda}(V^{(s)})\right)=(n+1)^i d^s n^{s(k-1)-r}+2,\ s=1,\dots j,\ r=0,\dots k-1.$$

If $(j-1)(k-1) < \ell \leq j(k-1)$ we are done (take $s = j$ in the previous formula). If $\ell \leq t(k-1)$ with $t \leq (j-1)$, then we stop the process at level t and take $j-t$ preimages in U_d to get the desired connectivity.

□

3.3 Proof of theorem C

In this section we prove Theorem C. We first show that there is a sequence of preimages of A_λ which surround $z = 0$ and accumulate on $\partial\mathcal{A}^*_\lambda(\infty)$. We want to remark that the set A^{out} depends on λ even if we do not indicate it in its notation.

Lemma 3.3.1. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d). Let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. Let $A_{0,\lambda} := A_\lambda$. Then there exist $\{A_{m,\lambda}\}_{m\geq 1}$ iterated preimages of $A_{0,\lambda}$, $\mathcal{S}_{n,d,\lambda}(A_{m+1,\lambda}) = A_{m,\lambda}$, such that the following properties are satisfied:*

(i) Each Fatou component $A_{m,\lambda}$ is surrounded by $A_{m+1,\lambda}$, that is, $A_{m,\lambda} \subset$ Fill$(A_{m+1,\lambda})$. In particular, all $\{A_{m,\lambda}\}_{m\geq 1}$ lie in A^{out}.

(ii) The sequence of Fatou components $\{A_{m,\lambda}\}_{m\geq 1}$ accumulate on $\partial\mathcal{A}^*_\lambda(\infty)$ as $m \to \infty$.

Proof. Every Fatou component which surrounds $z = 0$ has exactly 2 boundary components which surround $z = 0$. It follows from Proposition 3.1.13 that every Fatou component in A^{out} which surrounds $z = 0$ has exactly a preimage in A^{out} which surrounds $z = 0$. Let $\{A_{m,\lambda}\}_{m\geq 1}$ be the sequence of Fatou components obtained by taking consecutive preimages of $A_{0,\lambda}$ in A^{out} which surround $z = 0$. Since $\mathcal{S}_{n,d,\lambda}$ is conjugated to a Blaschke product on A^{out}, by Proposition 3.1.12, the Fatou components $A_{m,\lambda}$ accumulate on $\partial\mathcal{A}^*_\lambda(\infty)$ as m goes to ∞. It also follows from the conjugation with the Blaschke product that $A_{m,\lambda} \subset \text{Fill}(A_{m+1,\lambda})$ for all $m \geq 0$. □

The multiply connected sets $A_{m,\lambda}$ surround $z = 0$. Therefore, there are exactly 2 boundary components of $A_{m,\lambda}$ which surround $z = 0$. We denote them by $\partial^{\text{Int}}A_{m,\lambda}$ and $\partial^{\text{Ext}}A_{m,\lambda}$, where $\partial^{\text{Int}}A_{m,\lambda} \subset \text{Int}(\partial^{\text{Ext}}A_{m,\lambda})$. Next lemma tells as that if m is large enough then there are parameters λ such that $\nu_\lambda \in A_{m,\lambda}$.

Lemma 3.3.2. *Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d). Let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. Then, if $m \in \mathbb{N}^*$ is big enough, there exists a parameter λ such that $\nu_\lambda \in A_{m,\lambda}$.*

Proof. The idea of the proof is to show that, if m is big enough, we can find a parameter λ_0 such that $\nu_{\lambda_0} \in \text{Fill}(A_{m,\lambda_0})$ and a parameter λ_m such that ν_{λ_m} belongs to the unbounded component of $\mathbb{C}^* \setminus A_{m,\lambda_m}$. We will then conclude that there exists a parameter λ'_m such that $\nu_{\lambda'_m} \in A_{m,\lambda'_m}$.

Fix λ_0 such that all hypothesis hold. Then A_{0,λ_0} is well defined, and so are A_{m,λ_0}, $m > 0$. Let m_0 be such that $\nu_{\lambda_0} \in \text{Fill}(A_{m_0,\lambda_0})$. Then, for all $m \geq m_0$ we have $\nu_{\lambda_0} \in \text{Fill}(A_{m,\lambda_0})$. For fixed $m \geq m_0$, we want to find the parameter λ_m. If $\lambda = 0$, the critical point ν_0 belongs to the boundary of the maximum domain of definition of the Böttcher coordinate of $A^*(0)$ under $S_{n,a,Q}$. Therefore, the orbit of ν_0 under $S_{n,a,Q}$ accumulates on $z = 0$ but never maps onto it. Observe that $\mathcal{S}_{n,d,\lambda}$ converges uniformly to $S_{n,a,Q}$ on compact subsets of $\mathbb{C}^* \setminus \mathbb{D}_\epsilon$ as $\lambda \to 0$, where $\epsilon > 0$ is arbitrarily small and $\mathbb{D}_\epsilon$ denotes the disk of radius ϵ centered at $z = 0$. Consequently, for fixed $m \geq 0$ and $\epsilon > 0$, if $|\lambda|$ is small enough then $A_{m,\lambda} \subset \mathbb{D}_\epsilon$. Since $\nu_\lambda \to \nu_0$

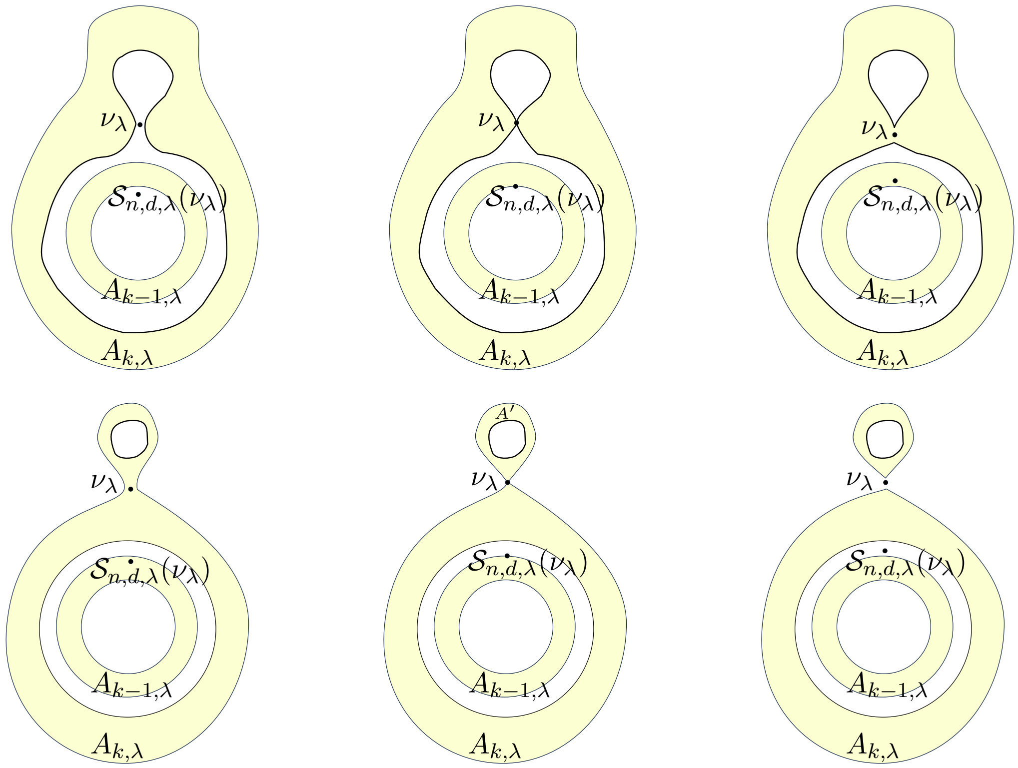


Figure 3.6: The top figures correspond to the possible cases of ν_λ lying in a neighbourhood of $\partial^{\text{Int}} A_{m,\lambda}$. The bottom figures correspond to the possible cases of ν_λ lying in a neighbourhood of $\partial^{\text{Ext}} A_{m,\lambda}$.

as $\lambda \to 0$, it follows that if $|\lambda|$ is small enough then ν_λ belongs to the unbounded component of $\mathbb{C}^* \setminus A_{m,\lambda}$. It is enough to take λ_m to be any such λ.

To finish the proof we need to show that when we move continuously the parameter from λ_0 to λ_m we need to find intermediate parameters λ'_m such that $\nu_{\lambda'_m} \in A_{m,\lambda'_m}$. By Proposition 3.1.4 we know that $\partial\mathcal{A}^*_\infty$ moves continuously with respect to λ. Since $\partial\mathcal{A}^*_\infty$ and ∂T_λ cannot contain critical values, it follows that both boundary components of $A_{0,\lambda}$ move continuously with respect to λ. For fixed λ', the set $\partial A_{m,\lambda}$, $m \geq 1$, moves continuously with respect to λ in a neighbourhood of λ' unless $\partial A_{m,\lambda'}$ (or an iterated image of $\partial A_{m,\lambda'}$) contains a critical point. Here by moving continuously we mean that every connected component of $\partial A_{m,\lambda}$ is a Jordan curve that moves continuously with respect to the Hausdorff metric (in particular, it does not pinch itself or split in several connected components). Notice that since there is only a free critical point, at most one of the 2 components of $\partial A_{m,\lambda}$ which surround $z = 0$ may not move continuously for λ in a neighbourhood of λ'. Using Proposition 3.1.12 it can be proven that $\mathcal{S}_{n,d,\lambda}(\partial^{\text{Ext}} A_{m,\lambda_0}) = \partial^{\text{Ext}} A_{m-1,\lambda_0}$ and $\mathcal{S}_{n,d,\lambda}(\partial^{\text{Int}} A_{m,\lambda_0}) = \partial^{\text{Int}} A_{m-1,\lambda_0}$. Assume that for λ' we have $\nu_{\lambda'} \in \partial A_{m,\lambda'}$. Then $A_{m-1,\lambda}$ is an annulus that moves continuously with respect to λ for all λ in a neighbourhood of λ'. If $\nu_{\lambda'} \in \partial^{\text{Int}} A_{m,\lambda'}$ then $\mathcal{S}_{n,d,\lambda}(\nu_{\lambda'}) \in \partial^{\text{Int}} A_{m-1,\lambda'}$. By Proposition 3.1.13, for λ in a neighbourhood of λ' exactly one of the following holds (see the three upper figures in Figure 3.6):

- If $\mathcal{S}_{n,d,\lambda}(\nu_\lambda) \in \text{Int}(\partial^{\text{Int}} A_{m-1,\lambda})$ then $\nu_\lambda \in \text{Int}(\partial^{\text{Int}} A_{m,\lambda})$ and $A_{m,\lambda}$ is doubly connected.

- If $\mathcal{S}_{n,d,\lambda}(\nu_\lambda) \in \partial^{\mathrm{Int}} A_{m-1,\lambda}$ then $\nu_\lambda \in \partial^{\mathrm{Int}} A_{m,\lambda}$. Then, $A_{m,\lambda}$ is doubly connected and $\partial^{\mathrm{Int}} A_{m,\lambda}$ consists of the union of 2 Jordan curves.

- If $\mathcal{S}_{n,d,\lambda}(\nu_\lambda) \in A_{m-1,\lambda}$ then $\nu_\lambda \in A_{m,\lambda}$ and $A_{m,\lambda}$ is triply connected.

On the other hand, if $\nu_{\lambda'} \in \partial^{\mathrm{Ext}} A_{m,\lambda'}$ then $\mathcal{S}_{n,d,\lambda}(\nu_{\lambda'}) \in \partial^{\mathrm{Ext}} A_{m-1,\lambda'}$. By Proposition 3.1.13, for λ in a neighbourhood of λ' exactly one of the following holds (see the three lower figures in Figure 3.6):

- If $\mathcal{S}_{n,d,\lambda}(\nu_\lambda) \in A_{m-1,\lambda}$ then $\nu_\lambda \in A_{m,\lambda}$ and $A_{m,\lambda}$ is triply connected.

- If $\mathcal{S}_{n,d,\lambda}(\nu_\lambda) \in \partial^{\mathrm{Ext}} A_{m-1,\lambda}$ then $\nu_\lambda \in \partial^{\mathrm{Ext}} A_{m,\lambda}$. Then, $A_{m,\lambda}$ is doubly connected, $\partial^{\mathrm{Ext}} A_{m,\lambda}$ is a Jordan curve, and there is an extra preimage A' of $A_{m-1,\lambda}$ such that $\partial A_{m,\lambda} \cap \partial A' = \nu_\lambda$.

- If $\mathcal{S}_{n,d,\lambda}(\nu_\lambda) \in \mathrm{Ext}(\partial^{\mathrm{Int}} A_{m-1,\lambda})$ then $\nu_\lambda \in \mathrm{Ext}(\partial^{\mathrm{Int}} A_{m,\lambda})$ and $A_{m,\lambda}$ is doubly connected.

It follows from the previous configurations that if we move continuously the parameter λ from λ_0 until λ_m we can find parameters λ'_m such that $\nu_{\lambda'_m} \in A_{m,\lambda'_m}$. This finishes the proof of the result.

□

We can now proceed with proof of Theorem C.

Proof of Theorem C. Fix i, j, ℓ. We have to prove that there exists λ for which there is a Fatou component of connectivity $\kappa = (n+1)^i d^j n^\ell + 2$ and a Fatou component of connectivity $\kappa = (n+1)^i + 2$. Recall that the results in Section 4 required the free critical point ν_λ to lie in a preimage of A_λ which surrounds $z = 0$. By Lemma 3.3.2, there exists $m > \ell$ such that $\nu_\lambda \in A_{m,\lambda}$. The existence of the Fatou component of connectivity $\kappa = (n+1)^i + 2$ is proven in Lemma 3.2.4*(i)*. Since $\nu_\lambda \in A_{m,\lambda}$ and $m > \ell$, the existence of a Fatou component of connectivity $\kappa = (n+1)^i d^j n^\ell + 2$ follows from Theorem B.

□

Chapter 4

Newton-like components in the Chebyshev-Halley family of degree n polynomials

Numerical methods have been extensively used to give accurate approximations of the solutions of systems of nonlinear equations. Those equations or systems of equations correspond to a wide source of scientific models from biology to engineering and from economics to social sciences, and so their solutions are a cornerstone of applied mathematics. One of the most studied families of numerical methods are the so called root-finding algorithms; that is, iterative methods which asymptotically converge to the zeros (or some of the zeros) of the non linear equation, say $g(z) = 0$. Although g could in general describe an arbitrary high dimensional problem, in this chapter we focus on the one dimensional case, i.e. $g : \mathbb{C} \to \mathbb{C}$.

The universal and most studied root-finding algorithm is known as *Newton's method*. If g is holomorphic, we generate a sequence $\{z_n\}_{n\geq 0}$ of approximations of a root of g, using Newton's method, defined as follows

$$z_{n+1} = z_n - \frac{g(z_n)}{g'(z_n)}, \quad z_0 \in \mathbb{C}.$$

It is well known that if $z_0 \in \mathbb{C}$ is chosen close enough to one of the solutions of the equation $g(z) = 0$, say α, then the sequence $\{z_n = g^n(z_0)\}_{n\geq 0}$ has the limit α when n tends to ∞. Moreover, the speed of (local) convergence is generically quadratic (see, for instance, [ABP04]). It was Cayley (see [Cay79]) the first to consider Newton's method as a (holomorphic) dynamical system, that is studying the convergence of these sequences for all possible seeds $z_0 \in \mathbb{C}$ at once, under the assumption that g was a degree 2 or 3 polynomial. This was known as *Cayley's problem*.

Many authors have studied alternative iterative methods having, for instance, a better local speed of convergence. Two of the best known root-finding algorithms of order of convergence 3 are *Chebyshev's method* and *Halley's method* (see [ABP04]). They are included in the *Chebyshev-Halley family* of root-finding algorithms, which was introduced in [CTV13] (see also [Ama16]), and is defined as follows. Let g be a holomorphic map. Then

$$z_{n+1} = z_n - \left(1 + \frac{1}{2}\frac{L_g(z_n)}{1 - \alpha L_g(z_n)}\right)\frac{g(z_n)}{g'(z_n)}, \tag{4.1}$$

where $\alpha \in [0, 1]$ and $L_g(z) = \frac{g(z)g''(z)}{(g'(z))^2}$. Notice that in a real setting, it suffices for g to be a doubly differentiable function such that $g''(x)$ is continuous.

For $\alpha = 0$, we have Chebyshev's method and for $\alpha = \frac{1}{2}$ Halley's method. As α tends to ∞, the Chebyshev-Halley algorithms tend to Newton's method. The main goal of the chapter is to show that the unbounded connected component of the Julia set of the Chebyshev-Halley maps applied to $z^n - c$ (for large enough α) is homeomorphic to the Julia set of the map obtained by applying Newton's method to $z^n - 1$.

Root-finding algorithms are a natural topic for complex dynamics. In particular, maps obtained by applying Newton's method to polynomials are a much studied topic (see [Shi09], [HSS01], [Tan97], [BFJK14]). It is proven in [Shi09] that the Julia set of such maps is connected, so all Fatou components are simply connected.

Previously, Campos, Canela, and Vindel have studied the Chebyshev-Halley family applied to $f_{n,c}(z) = z^n + c$, $c \in \mathbb{C}^*$ (see [CCV18], [CCV20]). The maps obtained by applying the Chebyshev-Halley family to $f_{n,c}$ are all conjugated to the map obtained by applying the Chebyshev-Halley family to $f_n(z) \coloneqq f_{n,-1}(z) = z^n - 1$ (see Lemma 4.1.1). By applying the Chebyshev-Halley method to $f_n(z) = z^n - 1$ we obtain the map:

$$O_{n,\alpha}(z) = \frac{(1-2\alpha)(n-1) + (2 - 4\alpha - 4n + 6\alpha n - 2\alpha n^2)z^n + (n-1)(1 - 2\alpha - 2n + 2\alpha n)z^{2n}}{2nz^{n-1}(\alpha(1-n) + (-\alpha - n + \alpha n)z^n)}. \tag{4.2}$$

The map $O_{n,\alpha}$ has degree $2n$ and it has $4n-2$ critical points, counting multiplicity. The point $z = 0$ is a critical point of multiplicity $n-2$, which is mapped to the fixed point $z = \infty$. The n-th roots of unity are superattracting fixed points of local degree 3, and therefore, they have multiplicity 2 as critical points. This leaves n free critical points. They are given by

$$c_{n,\alpha,\xi} = \xi \left(\frac{\alpha(n-1)^2(2\alpha - 1)}{n(2n-1) - \alpha(4n-1)(n-1) + 2\alpha^2(n-1)^2} \right)^{\frac{1}{n}}, \tag{4.3}$$

where $\xi^n = 1$. This family is symmetric with respect to rotation by the nth root of unity (see Lemma 4.1.2). This symmetry ties the orbits of the n free critical points, so the family $O_{n,\alpha}$ has only one degree of freedom (see Figure 4.2).

In [CCV20], the authors studied in detail the topology of the immediate basins of attraction of the fixed points of $O_{n,\alpha}(z)$ given by the nth root of unity, that is, the zeros of $f_n(z)$. In what follows we refer to these basins as

$$A_{n,\alpha}(\xi) \coloneqq A_{O_{n,\alpha}}(\xi) \left[A^*_{n,\alpha}(\xi) \coloneqq A^*_{O_{n,\alpha}}(\xi) \right],$$

where $\xi^n = 1$. For one particular case, the immediate basins of attraction are infinitely connected (see Figure 4.1). We study the Julia set of $O_{n,\alpha}$ for this particular case and relate it to the Julia set of the map obtained by applying Newton's method to f_n. We realise this using a quasiconformal surgery construction, which erases the holes in the immediate basins of attraction. The construction is a simpler case of one in [McM88]. However, realising the surgery is still needed, as we prove the uniqueness of the resulted quasiconformal map, to show that the quasirational map presents the necessary symmetries and is precisely N_{f_n}.

Theorem D. *Fix $n \geq 2$ and assume that $A^*_{n,\alpha}(1)$ is infinitely connected for some $\alpha \in \mathbb{C}$. Then there exists an invariant Julia component Π (which contains $z = \infty$) which is a quasiconformal copy of the Julia set of N_{f_n}, where N_{f_n} is the map obtained by applying Newton's method to the polynomial $f_n(z) = z^n - 1$.*

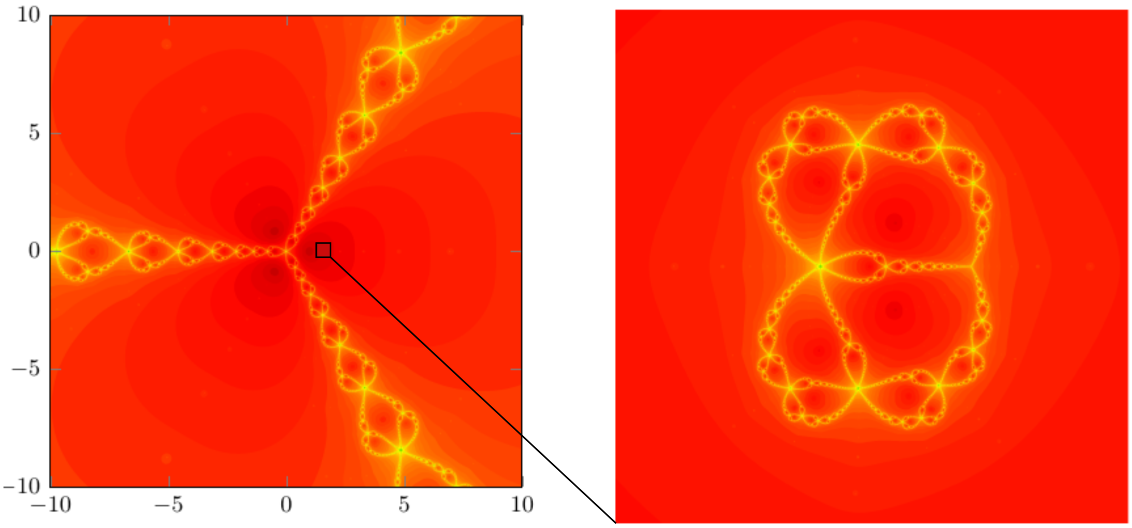


Figure 4.1: Left figure illustrates the dynamical plane of $O_{n,\alpha}$ for $n = 3$ and $\alpha = 10$. In the right figure (which shows $z \in \mathbb{C}$ such that $\mathrm{Re}(z) \in [1.620; 1.623]$ and $\mathrm{Im}(z) \in [-0.0015; 0.0015]$ in the same dynamical plane), we can see a component of the Julia set which lies in $A^*(1)$.

We finish the chapter by proving that there exist parameters such that the hypothesis of Theorem D holds. We split the proof of Theorem E in two cases, $n = 2$ and $n \geq 3$. For the case $n = 2$, much work was previously done in [CCV18], and the map is conjugate to a Blaschke product. For the case $n \geq 3$, the map is not conjugate to a Blaschke product. We provide a map conjugate to $O_{n,\alpha}$, for which we prove, using various properties and computational arguments, that the immediate basin of attraction of $z = \infty$ is infinitely connected. Numerical computations confirm to us the existence of such hyperbolic components (see Figure 4.2).

Theorem E. *Let $n \geq 2$. Then there exists $\alpha_0 > 0$ large enough such that for $\alpha > \alpha_0$, $\alpha \in \mathbb{R}$, $A^*_{n,\alpha}(1)$ is infinitely connected. Moreover, for $n = 2$, the statement is true for any $\alpha \in \mathbb{C}$ such that $|\alpha| > \alpha_0$.*

The chapter is organised as follows. In Section 4.1 we briefly introduce the tools later used in the chapter. In Section 4.2 we prove Theorem D. In Section 4.3 we prove Theorem E.

4.1 Preliminaries on Chebyshev-Halley family

In this section we present the main tools that we use along the chapter. Let $O_{n,\alpha,c}$ be the map obtained by applying the Chebyshev-Halley method with parameter α to the map $f_{n,c} = z^n + c$. The following lemma, indicated but not proven in [CCV18], states that for any $c \in \mathbb{C}^*$, the map $O_{n,\alpha,c}$ is conjugated to $O_{n,\alpha,-1} = O_{n,\alpha}$. We give the proof for the sake of completeness.

Lemma 4.1.1. *Let $c \in \mathbb{C}^*$, $c = re^{i2\pi k}$, where $r > 0$ and $k \in [0, 1]$. Let $u = \sqrt[n]{r}e^{i\frac{(2k+1)\pi}{n}}$ and $\eta_c(z) = uz$, $\eta_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Then $O_{n,\alpha,c}$ and $O_{n,\alpha,-1}$ are conjugated by the map η_c, i.e. $O_{n,\alpha,c} \circ \eta_c(z) = \eta_c \circ O_{n,\alpha,-1}(z)$.*

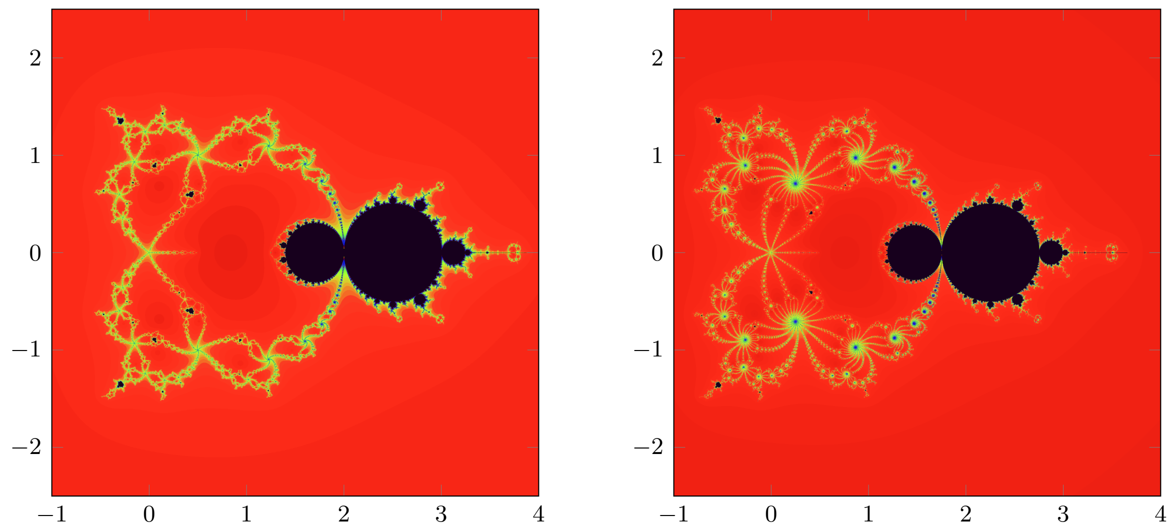


Figure 4.2: Left figure illustrates the parameter plane of $O_{n,\alpha}$ for $n=3$. In the right figure we can see the parameter plane of $O_{n,\alpha}$ for $n=5$.

Proof. First, we need to compute $O_{n,\alpha,c}$. We use the Chebyshev-Halley family of methods definition (see (4.1)), which can be rewritten as

$$O_{n,\alpha,c}(z) = \frac{2zf'_{n,c}(z) - 2f_{n,c}(z) - f_{n,c}(z)L_{f_{n,c}}(z) + 2\alpha L_{f_{n,c}}(z)(f_{n,c}(z) - zf'_{n,c}(z))}{2f'_{n,c}(z)(1-\alpha L_{f_{n,c}}(z))},$$

where $f_{n,c}(z) = z^n + c$, $f'_{n,c}(z) = nz^{n-1}$, $L_{f_{n,c}}(z) = \frac{(n-1)(z^n+c)}{nz^n}$. This gives us the expression of $O_{n,\alpha,c}$:

$$O_{n,\alpha,c}(z) = \frac{c^2(1-2\alpha)(n-1) - c(2-4\alpha-4n+6\alpha n-2\alpha n^2)z^n + (n-1)(1-2\alpha-2n+2\alpha n)z^{2n}}{2nz^{n-1}[-c\alpha(1-n) + (-\alpha-n+\alpha n)z^n]}.$$

Now we prove the conjugation. Observe that $u^n = -c$, and the map η_c maps roots of $f_{n,1}$ to roots of $f_{n,c}$ (therefore, it also maps the superattracting fixed points of $O_{n,\alpha}$ to the superattracting fixed points of $O_{n,\alpha,c}$). Then

$O_{n,\alpha,c}\Big(\eta_c(z)\Big) = \frac{c^2(1-2\alpha)(n-1)+c^2(2-4\alpha-4n+6\alpha n-2\alpha n^2)z^n+c^2(n-1)(1-2\alpha-2n+2\alpha n)z^{2n}}{u^{n-1}2nz^{n-1}[-c\alpha(1-n)-c(-\alpha-n+\alpha n)z^n]}$

$= u\frac{(1-2\alpha)(n-1)+(2-4\alpha-4n+6\alpha n-2\alpha n^2)z^n+(n-1)(1-2\alpha-2n+2\alpha n)z^{2n}}{2nz^{n-1}[\alpha(1-n)+(-\alpha-n+\alpha n)z^n]}$

$= \eta_c\Big(O_{n,\alpha}(z)\Big).$

□

The next lemma shows that the map $O_{n,\alpha}$ is symmetric with respect to rotation by an nth root of unity.

Lemma 4.1.2 ([CCV18], Lemma 6.2)**.** *Let $n \in \mathbb{N}$ and let ξ be an nth root of unity, i.e. $\xi^n = 1$. Then $I_\xi(z) \coloneqq \xi z$ conjugates $O_{n,\alpha}$ with itself, i.e. $O_{n,\alpha} \circ I_\xi(z) = I_\xi \circ O_{n,\alpha}(z)$.*

For $\alpha = \frac{1}{2}$ and $\alpha = \frac{2n-1}{2n-2}$, the family $O_{n,\alpha}$ degenerates to maps of a lower degree (see [CCV20], Lemma 4.1). For other values of α, the map $O_{n,\alpha}$ has degree $2n$, hence, it has $4n-2$ critical points. The point $z=0$ maps with degree $n-1$ to the fixed point $z=\infty$.

Since the n roots of f_n are superattracting fixed points of local degree 3, there remain precisely n free critical points. The next lemma follows directly from Lemma 4.1.2, since the orbits of the free critical points are symmetric.

Lemma 4.1.3 ([CCV20], Lemma 3.4)**.** *Let $n \geq 2$ and $\xi \in \mathbb{C}$, such that $\xi^n = 1$. For all $\alpha \in \mathbb{C}$, the basin of attraction $A_{n,\alpha}(\xi)$ contains at most one critical point other than $z = \xi$.*

The following proposition establishes a trichotomy for rational maps with the property described in Lemma 4.1.3. Based on the existence of the critical point and preimages of the superattracting fixed point in the immediate basin of attraction, we can establish if the immediate basin is simply connected.

Proposition 4.1.4 ([CCV20], Proposition 3.1)**.** *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and let $z = 0$ be a superattracting fixed point of f. Assume that $A_f(0)$ contains at most one critical point other than $z = 0$. Then, exactly one of the following statements holds.*

1. *The set $A^*_f(0)$ contains no critical point other than $z = 0$. Then $A^*_f(0)$ is simply connected.*
2. *The set $A^*_f(0)$ contains a critical point $c \neq 0$ and a preimage $z_0 \neq 0$ of $z = 0$. Then $A^*_f(0)$ is simply connected.*
3. *The set $A^*_f(0)$ contains a critical point $c \neq 0$ and no preimage of $z = 0$ other than $z = 0$ itself. Then $A^*_f(0)$ is multiply connected.*

Corollary 4.1.5 ([CCV20], Corollary 3.5)**.** *For fixed $n \geq 2$ and $\alpha \in \mathbb{C}$, the immediate basins of attraction of the roots of $z^n - 1$ under $O_{n,\alpha}$ are multiply connected if and only if $A^*_{n,\alpha}(1)$ contains a critical point $c \neq 1$ and no preimage of $z = 1$ other than $z = 1$ itself.*

Remark 4.1.6. *An immediate attracting basin may only have connectivity 1 or ∞ (see Theorem 2.3.7). Hence, if $A^*_{n,\alpha}(1)$ is multiply connected, then all the immediate basins of attraction corresponding to the roots of f_n are infinitely connected.*

The following lemma in [Tan97] is the critical criterion used to prove Theorem D (see also [Hea88]).

Lemma 4.1.7 ([Tan97], Lemma 2.2)**.** *Any rational map F of degree d having d distinct superattracting fixed points is conjugate by a Möbius transformation to N_P for a polynomial of degree d. Moreover, if $z = \infty$ is not superattracting for F and F fixes ∞, then $F = N_P$ for some polynomial P of degree d.*

4.2 Proof of Theorem D

We start with a proposition that describes two curves in the dynamical plane of $O_{n,\alpha}$. These curves are used in the proof of Theorem D, as part of a quasiconformal surgery construction. The proof follows closely an argument made in the proof of [CCV20, Proposition 3.1].

Proposition 4.2.1. *Let $O_{n,\alpha}$ such that $A^*_{n,\alpha}(1)$ is infinitely connected. Then there exist Γ and Γ^{-1}, analytic Jordan curves in $A^*_{n,\alpha}(1)$ which surround $z = 1$, such that $O_{n,\alpha}|_{\Gamma^{-1}} : \Gamma^{-1} \to \Gamma$ is a two-to-one map with $\Gamma \subset \mathrm{Int}(\Gamma^{-1})$.*

(a) $\partial U = \gamma_2$.

(b) $\partial U = \gamma_1 \cup \gamma_2$.

Figure 4.3: The two possible configurations, of preimages of γ, described in Proposition 4.2.1.

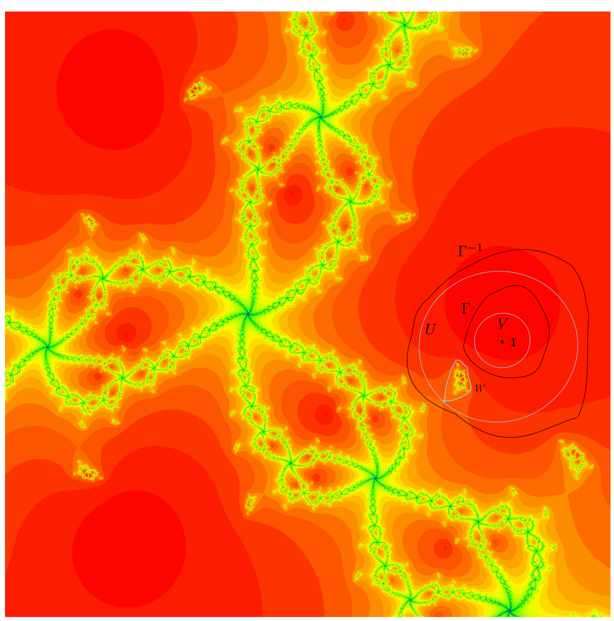


Figure 4.4: Description of the situation in Proposition 4.2.1, where $n = 3$ and $\alpha = 0.2{+}1.592i$.

Proof. Let U be the maximal domain of definition of the Böttcher coordinates of the super-attracting fixed point $z=1$. By hypothesis and Corollary 4.1.5, $A^*_{n,\alpha}(1)$ contains the critical point $c_1 \coloneqq c_{n,a,1}$ (see (4.3)), which lies on ∂U, and no other preimages of $z=1$. Since $z=1$ has local degree 3, the map $O_{n,\alpha}|_{A^*_{n,\alpha}(1)} : A^*_{n,\alpha}(1) \to A^*_{n,\alpha}(1)$ has degree 3. Let $V \coloneqq O_{n,\alpha}(U)$. Then $\gamma \coloneqq \partial V$ is a Jordan curve. Let γ^{-1} be the preimage of γ contained in $A^*_{n,\alpha}(1)$. Then $\gamma^{-1} = \gamma_1 \cup \gamma_2$ is the union of two simple closed curves which intersect at the critical point c_1. Exactly one of the two curves, say γ_2, contains $z=1$ in the Jordan domain bounded by it. There exist two possibilities: either $\partial U = \gamma_2$, or $\partial U = \gamma_1 \cup \gamma_2$ (see Figure 4.3). Assume that $\partial U = \gamma_2$. By hypothesis, the critical point c_1 lies in $A^*_{n,\alpha}(1)$ and γ_1 is contained in the Fatou set. Therefore, $\gamma_1 \subset A^*_{n,\alpha}(1)$ and there exists a preimage of V which lies in $\mathrm{Int}(\gamma_1)$. Hence, $A^*_{n,\alpha}(1)$ contains a preimage of $z=1$ other than itself, which is impossible according to Corollary 4.1.5. Consequently, we have that $\partial U = \gamma_1 \cup \gamma_2$. Let W be the connected component of $\hat{\mathbb{C}} \setminus \gamma_1$ which does not contain $z=1$. Then W is mapped to an open set which contains $\hat{\mathbb{C}} \setminus U$, so W contains a pole. Since $z=0$ is mapped to $z=\infty$ with degree $n-1$, $z=\infty$ is a fixed point, and the map has degree $2n$, there remain exactly n preimages for $z=\infty$. By symmetry, the pole in W is simple; therefore, ∂W is mapped onto ∂V with degree 1. Hence, γ_1 is mapped onto γ with degree 1 and γ_2 is mapped onto γ with degree 2.

Let Γ be an analytic Jordan curve which surrounds $z=1$ such that $\Gamma \subset U \setminus V$, and let $\mathcal{A}$ be the open annulus bounded by Γ and γ. Then $\mathcal{A}$ has precisely 3 preimages in $A^*_{n,\alpha}(1)$. Since $\mathcal{A}$ does not contain any critical value, its preimages do not contain critical points. It follows from the Riemann-Hurwitz formula (see Theorem 2.4.1) that any preimage of $\mathcal{A}$ is also an annulus. One preimage of $\mathcal{A}$ lies in W and is mapped onto $\mathcal{A}$ with degree 1. There exists precisely one other preimage of $\mathcal{A}$ in $A^*_{n,\alpha}(1)$, which we denote by $\mathcal{A}^{-1}$. It lies in $A^*_{n,\alpha}(1) \setminus \mathrm{Fill}(U)$, surrounds $z=1$, and is mapped onto $\mathcal{A}$ with degree 2. Let Γ^{-1} be the boundary component of $\mathcal{A}^{-1}$ which is mapped onto Γ. Observe that Γ^{-1} is an analytic Jordan curve. Since Γ^{-1} surrounds $z=1$ and lies outside U, we have that $\Gamma \subset \mathrm{Int}(\Gamma^{-1})$ (see Figure 4.4). □

The main tool used in the proof of Theorem D is quasiconformal surgery. The strategy of the proof is as follows. We start by defining a quasiregular map $f : A^*_{n,\alpha}(1) \to A^*_{n,\alpha}(1)$ on the immediate basin of attraction of $z=1$, which we later extend to a quasiregular map $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Secondly, we construct a *symmetric* F-invariant Beltrami coefficient and prove, using the Integrability Theorem (see Theorem 2.5.29), the existence of a map N_P quasiconformally conjugate to F. Then, we use Lemma 4.1.7 to show that N_P is a map obtained by applying Newton's method to a polynomial of degree n, and it is quasiconformally conjugated to N_{f_n}. Finally, we compare the filled Julia sets of N_{f_n} and $O_{n,\alpha}$.

Proof of Theorem D. Let $0 < \rho < 1$. Let $R : \mathrm{Int}(\Gamma) \to \mathbb{D}_{\rho^2}$ be a Riemann map such that $R(1) = 0$. Since Γ is an analytic curve, the Riemann map R extends analitically to the boundary (see Theorem 2.5.11). Let $\psi_2 : \Gamma \to \mathbb{S}_{\rho^2}$ be the restriction of R to its boundary. Let $\psi_1 : \Gamma^{-1} \to \mathbb{S}_\rho$ be an analytic lift map such that $\psi_2\big(O_{n,\alpha}(z)\big) = \big(\psi_1(z)\big)^2$. Let $\mathcal{A} = \mathrm{Int}(\Gamma^{-1}) \setminus \overline{\mathrm{Int}(\Gamma)}$ and $\mathcal{A}_{\rho^2,\rho} = \mathbb{D}_\rho \setminus \overline{\mathbb{D}_{\rho^2}}$. Let $\psi : \partial\mathcal{A} \to \partial\mathcal{A}_{\rho^2,\rho}$, such that $\psi|_{\Gamma^{-1}} = \psi_1$ and $\psi|_\Gamma = \psi_2$. Since ψ_1 and ψ_2 are analytic maps, ψ extends quasiconformally to $\psi : \overline{\mathcal{A}} \to \overline{\mathcal{A}}_{\rho^2,\rho}$ (see Proposition 2.5.15).

We now define $f : A^*_{n,\alpha}(1) \to A^*_{n,\alpha}(1)$ quasiregular, as follows:

$$f(z) := \begin{cases} R^{-1}\big((R(z))^2\big) & \text{if } z \in \mathrm{Int}(\Gamma), \\ R^{-1}\Big(\big(\psi(z)\big)^2\Big) & \text{if } z \in \overline{\mathcal{A}}, \\ O_{n,\alpha}(z) & \text{if } z \in A^*_{n,\alpha}(1) \setminus \overline{\mathrm{Int}(\Gamma^{-1})}. \end{cases}$$

Now let $\xi := e^{\frac{2\pi i}{n}}$. We have that $I_{\xi^j}(z) = I^j_\xi(z)$, for $j \in \{0, 1, \dots\ n-1\}$, where I_ξ is defined as in Lemma 4.1.2. We extend to a quasiregular map $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined on the Riemann sphere, as follows:

$$F(z) := \begin{cases} I_{\xi^j} \circ f \circ I^{-1}_{\xi^j}(z) & z \in A^*_{n,\alpha}(\xi^j),\ j \in \{0, 1, ..., n-1\}, \\ O_{n,\alpha}(z) & \text{otherwise.} \end{cases}$$

Observe that F is a quasiregular map which coincides with $O_{n,\alpha}$ outside the immediate basins of the roots of unity. We intend to construct an $F-$invariant and $I_\xi-$ invariant Beltrami coefficient μ. We first define an F-invariant Beltrami coefficient, say μ_1, in $A_{n,\alpha}(1)$, as follows:

$$\mu_1(z) := \begin{cases} \psi^*\mu_0(z) & \text{if } z \in \mathcal{A}, \\ (F^m)^*\mu_0(z) & \text{if } F^{m-1}(z) \in \mathcal{A},\ \text{for } m \geq 2, \\ \mu_0(z) & \text{otherwise.} \end{cases}$$

Observe that for $z \in \mathcal{A}$, we have that $\psi^*\mu_0(z) = F^*\mu_0(z)$. Now, we extend the previous construction to the rest of the Fatou set, that is, the basins of attraction of the nth root of unity $\xi^j \neq 1$, for $1 \leq j \leq n-1$. In the following, instead of using I_{ξ^j}, we will only refer to invariance with respect to I_ξ. Since $I_{\xi^j} = \underbrace{I_\xi \circ I_\xi \circ \cdots \circ I_\xi}_{\text{j times}}$, it suffices to prove the symmetry for I_ξ. We define an $I_\xi-$invariant Beltrami coefficient in $A_{n,\alpha}(\xi^j)$:

$$\mu(z) := \begin{cases} \mu_1(z) & \text{if } z \in A_{n,\alpha}(1), \\ (I^{-1}_{\xi^j})^*\mu_1(z) & \text{if } z \in A_{n,\alpha}(\xi^j), \\ \mu_0(z) & \text{otherwise.} \end{cases}$$

For $z \in A_{n,\alpha}(\xi)$ we have that

$$(F)^*\mu = (F)^*(I^{-1}_{\xi^j})^*\mu_1 = (I^{-1}_{\xi^j}F)^*\mu_1 = (FI^{-1}_{\xi^j})^*\mu_1 = (I^{-1}_{\xi^j})^*F^*\mu_1 = (I^{-1}_{\xi^j})^*\mu_1 = \mu.$$

It follows that μ is also $F-$invariant. By hypothesis, the map $O_{n,\alpha}$ is hyperbolic, hence, the Julia set has measure 0. Since $I^n_\xi(z) = z$, by construction, μ is both $F-$invariant and $I^{-1}_\xi-$invariant, with bounded dilatation. By the Integrability Theorem (see Theorem 2.5.29), there exists $\phi_0 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ quasiconformal map such that $\phi_0^*\mu_0 = \mu$. We normalise ϕ_0 such that $\phi_0(0) = 0$, $\phi_0(\infty) = \infty$, and ϕ_0 is tangent to the identity at ∞. Let $\phi_\xi := I_\xi \phi_0 I^{-1}_\xi$. We prove that ϕ_ξ and ϕ_0 coincide by using the uniqueness part of the Integrability Theorem. First, we have that ϕ_ξ satisfies the same equation as ϕ_0:

$$\phi_\xi^*\mu_0 = (I^{-1}_\xi)^*\phi_0^*I_\xi^*\mu_0 = (I^{-1}_\xi)^*\phi_0^*\mu_0 = (I^{-1}_\xi)^*\mu = \mu.$$

We have that ϕ_ξ satisfies $\phi_\xi^*\mu_0 = \mu$, $\phi_\xi(\infty) = \infty$, $\phi_\xi(0) = 0$, and ϕ_ξ is tangent to the identity at ∞. It follows from the uniqueness up to post-composition with Möbius transformations of the Integrability Theorem that $\phi_\xi = \phi_0$; therefore, $I_\xi \circ \phi_0 = \phi_0 \circ I_\xi$.

Now let $N_P : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $N_P := \phi_0 \circ F \circ \phi_0^{-1}$. Observe that, by construction, $N_P \circ I_\xi = I_\xi \circ N_P$. The map N_P is quasiregular and satisfies $(N_P)^*\mu_0 = \mu_0$, therefore, by Weyl's lemma (see Lemma 2.5.32) it is holomorphic and quasiconformally conjugated to F by ϕ_0. Since $z = \infty$ is a fixed point of F which is topologically repelling, $z = \infty$ is a repelling (therefore, not superattracting) fixed point of N_P. It also follows from the conjugacy that N_P has precisely n distinct superattracting fixed points, given by the set $\{\xi^j \phi_0(1)\}$, where $j \in \{0, 1, \ldots n-1\}$.

By Lemma 4.1.7, the map N_P is the map obtained by applying Newton's method to

$$P(z) = \prod_{j=0}^{n-1} \Big(z - \xi^j \phi_0(1)\Big) = z^n - \phi_0(1)^n.$$

We prove that N_P and N_{f_n} are linearly conjugated by $\eta(z) \coloneqq \phi_0(1)z$. Analogously to the proof of Lemma 4.1.1, we first compute

$$N_P = \frac{n-1}{n} z + \frac{\phi_0(1)^n}{nz^{n-1}}.$$

Then,

$$N_P\Big(\eta(z)\Big) = \frac{n-1}{n}\phi_0(1)z + \frac{\phi_0(1)^n}{n\phi_0(1)^{n-1}z^{n-1}} = \phi_0(1)\Big(\frac{n-1}{n}z + \frac{1}{nz^{n-1}}\Big) = \eta\Big(N_{f_n}(z)\Big)$$

completes the proof of the linear conjugation.

The Julia set of N_{f_n}, $J(N_{f_n})$, is connected (see [Shi09], Theorem 3.1). Moreover, by construction, N_{f_n} and $O_{n,\alpha}$ are quasiconformally conjugate in a neighborhood of $J(N_{f_n})$, by a conjugacy, say ϕ. Since the conjugacy sends ∞ to ∞, we can conclude that there is an unbounded connected component Π of $J(O_{n,\alpha})$, which is a quasiconformal copy of $J(N_{f_n})$. The fact that $\phi(J(N_{f_n}))$ is a connected component of $J(O_{n,\alpha})$ follows from the surgery construction, since the surgery is done on the Fatou set of $O_{n,\alpha}$. $\square$

4.3 Proof of Theorem E

We begin by studying the case of $n = 2$. Let $\alpha > 2$ and let $M_2(z) \coloneqq \frac{z+1}{z-1}$ be the Möbius transformation which maps the superattracting fixed points $z = 1$ and $z = -1$, to $z = \infty$ and $z = 0$. Finally, set $a = 2(\alpha - 1) > 2$, and consider the map

$$B_a(z) = z^3 \frac{z-a}{1-az} \tag{4.4}$$

which is conjugated to $O_{2,\alpha}$ by M_2. Indeed, for $n = 2$, the map $O_{n,\alpha}$ is

$$O_{2,\alpha}(z) = \frac{(2\alpha-3)z^4 - 6z^2 + (1-2\alpha)}{4(\alpha-2)z^3 - 4\alpha z}.$$

We remark that if $z = \frac{a}{b}$, then $M_2(z) = \frac{a+b}{a-b}$. This gives us

$$M_2\Big(O_{n,\alpha}(z)\Big) = \frac{(2\alpha-3)z^4 + 4(\alpha-2)z^3 - 6z^2 - 4\alpha z + (1-2\alpha)}{(2\alpha-3)z^4 - 4(\alpha-2)z^3 - 6z^2 + 4\alpha z + (1-2\alpha)},$$

and

$$B_a\Big(M_2(z)\Big) = \frac{(z+1)^3\Big[z(3-2\alpha)+(2\alpha-1)\Big]}{(z-1)^3\Big[z(3-2\alpha)-(2\alpha-1)\Big]} = M_2\Big(O_{n,\alpha}(z)\Big).$$

The map $B_a(z) = z^3\frac{z-a}{1-az}$ is a rational map of degree 4 studied in [CFG15], [CFG16], and [CCV18]. In [CCV18, Section 4] it is proven that for $a \in \mathbb{C}$, $|a| > 15.133$, $c_+ \in A_a(\infty)$. More precisely, it is shown that there exists a critical point c_+, such that $B_a(c_+) \in A_a^*(\infty)$. We will prove that this is a sufficient condition for $A_a^*(\infty)$ to be infinitely connected. Therefore, to prove Theorem B when $n = 2$, it suffices to prove the statements for the family B_a, and by conjugacy, they hold for $O_{2,\alpha}$.

The map B_a is a rational map of degree 4, and it is symmetric with respect to $\mathbb{S}^1$. The points $z = 0$ and $z = \infty$ are superattracting fixed points of local degree 3. Moreover, $z_\infty = \frac{1}{a} \in (0, 1)$ is a pole, and $z_0 = a$ is a preimage of $z = 0$. Consequently, there are two free critical points given by

$$c_\pm = \frac{1}{3a}\Big(2 + a^2 \pm \sqrt{(a^2-4)(a^2-1)}\Big). \tag{4.5}$$

The following lemma is a particular case of [CCV18, Proposition 4.5].

Lemma 4.3.1. *Let $a > 1$. If $|z| > 2a$, then $z \in A_a^*(\infty)$. Equivalently, for $a > 1$, we have that $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}(0, 2a) \subset A_a^*(\infty)$.*

Proof. If $|z| > 2a$ then

$$|B_a(z)| = |z^3|\frac{|z-a|}{|1-az|} > |z-a||z|\frac{2a|z|}{|1-az|} > a|z|\frac{2a|z|}{1+a|z|} > a|z|.$$

Since $|B_a(z)| > |z|$, it follows that $z \in A_a^*(\infty)$.

□

In the proof of Proposition 4.6 in [CCV18], the authors show that for $a \in \mathbb{C}$ with $|a|$ large enough (indeed $|a| > 16$), we have $B_a(c_+) \in A_a^*(\infty)$. A similar proof was previously done in [CFG16, Lemma 2.6] for a family that includes B_a (but without providing an explicit bound). Here we present an easier proof, only for real values of the parameter a.

Lemma 4.3.2. *If $a \in \mathbb{R}_+$ is large enough, then $B_a(c_+) \in A_a^*(\infty)$.*

Proof. It follows from (4.5) that if $a > 2$, then $\frac{a}{2} < c_+ < a$. We write $B_a(z)$ as $B_a(z) = z^3h(z)$, where $h(z) = \frac{z-a}{1-az}$, and $h'(z) = -\frac{(a+1)(a-1)}{(az-1)^2}$. Then

$$B_a'(z) = 3z^2h(z) + z^3h'(z), \quad \text{so} \quad B_a(z) = \frac{zB'(z)}{3} - \frac{z^4}{3}h'(z).$$

We have that

$$B_a(c_+) = c_+^4\frac{(a+1)(a-1)}{3(ac_+-1)^2} > c_+^4\frac{a(a-1)}{3a^4}.$$

Since $c_+ > \frac{a}{2} > 1$, it follows that

$$B_a(c_+) > \frac{a-1}{48}a.$$

So, for $a > 97$, we have that $B_a(c_+) > 2a$. According to Lemma 4.3.1, we conclude that $B_a(c_+) \in A_a^*(\infty)$.

□

Proposition 4.3.3. *Assume $a \in \mathbb{R}_+$ is large enough such that Lemma 4.3.2 applies. Then $c_+ \in A_a^*(\infty)$ and $A_a^*(\infty)$ is infinitely connected.*

Proof. Observe that, for $a > 1$, we have that $0 < z_\infty < z_0 < 2a$. By Lemma 4.3.2, $B_a(c_+) \in A_a^*(\infty)$. Therefore, the critical point c_+ lies either in $A_a^*(\infty)$ or in a preimage of $A_a^*(\infty)$.

Assume that c_+ lies in a preimage of $A_a^*(\infty)$, distinct from $A_a^*(\infty)$, say U. Since U contains a critical point, it is mapped onto $A_a^*(\infty)$ with degree at least 2. Hence, U contains at least 2 preimages of $z = \infty$ (different from itself), a contradiction with $\deg(B_a) = 4$, and $z = \infty$ being a superattracting fixed point with local degree 3.

Since the map is real, by the Schwarz Reflexion Principle, the map is conjugated to itself by complex conjugation, i.e. $I(z) = \bar{z}$. Then, Fatou components intersecting the real line are symmetric with respect to the real line. Since $0 < c_+ < z_0$, it follows that 0 and z_0 belong to different connected components of the complement of $A_a^*(\infty)$. Thus, $A_a^*(\infty)$ is multiply connected, therefore, by Remark 4.1.6 it is infinitely connected. □

Remark 4.3.4. *It follows from [CCV18, Proposition 4.6] that all $a \in \mathbb{C}$, with $|a| > 15.133$, belong to the same hyperbolic component. Since the connectivity of $A_a^*(\infty)$ is an invariant topological property within hyperbolic components, we conclude from Proposition 4.3.3 that if $|a| > 15.133$, then $A_a^*(\infty)$ is infinitely connected. This completes the proof of Theorem E for $n = 2$.*

To finish the proof of Theorem E we now consider $n \geq 3$. As we did before, we consider a new map $R_{n,\alpha}$ which is conjugated to $O_{n,\alpha}$ via the Möbius map $M(z) = \frac{1}{z-1}$. More specifically, we consider $R_{n,\alpha} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, given by $R_{n,\alpha} = M \circ O_{n,\alpha} \circ M^{-1}$. Since M sends $z = 1$ to $z = \infty$ and $z = \infty$ to $z = 0$, it is clear from (4.2) that $z = \infty$ is a superattracting fixed point of $R_{n,\alpha}$ with local degree 3 and $z = 0$ is a fixed point of $R_{n,\alpha}$.

We have $M(z) = \frac{1}{z-1}$, $M^{-1}(z) = \frac{z+1}{z}$, and

$$O_{n,\alpha}(z) = \frac{(1-2\alpha)(n-1) + (2-4\alpha-4n+6\alpha n-2\alpha n^2)z^n + (n-1)(1-2\alpha-2n+2\alpha n)z^{2n}}{2nz^{n-1}(\alpha(1-n) + (-\alpha-n+\alpha n)z^n)}.$$

We write

$$O_{n,\alpha}(z) = \frac{E_3(n,\alpha) + E_4(n,\alpha)z^n + E_5(n,\alpha)z^{2n}}{2nz^{n-1}[E_1(n,\alpha) + E_2(n,\alpha)z^n]},$$

where $E_i(n,\alpha)$ are polynomials of degree 1 in α. It follows that
$R_\alpha(z) = M \circ S_{n,a,Q} \circ M^{-1}(z)$
$= M\left(\frac{E_3(n,\alpha)z^{2n} + E_4(n,\alpha)z^n(z+1)^n + E_5(n,\alpha)(z+1)^{2n}}{2nz(z+1)^{n-1}[E_1(n,\alpha)z^n + E_2(n,\alpha)(z+1)^n]}\right)$
$= \frac{2nz(z+1)^{n-1}[E_1(n,\alpha)z^n + E_2(n,\alpha)(z+1)^n]}{2nz(z+1)^{n-1}[E_1(n,\alpha)z^n + E_2(n,\alpha)(z+1)^n] - E_3(n,\alpha)z^{2n} + E_4(n,\alpha)z^n(z+1)^n + E_5(n,\alpha)(z+1)^{2n}}$
which we finally write as

$$R_{n,\alpha}(z) = \frac{2nz(z+1)^{n-1}[E_1(n,\alpha)z^n + E_2(n,\alpha)(z+1)^n]}{Q^1_{2n-3}(z) + \alpha Q^2_{2n-3}(z)} =: \frac{zP(z)}{Q(z)}, \tag{4.6}$$

where $Q^j_{2n-3}(z)$, $j = 1, 2$ are degree $2n-3$ polynomials, with coefficients independent of α. We can use this expression of $R_{n,\alpha}$, without further computations, since R_α is a rational map of degree $2n$, being conjugated to $O_{n,\alpha}$. Furthermore, $z = \infty$ corresponds to the superattracting fixed point of local degree 3 of $O_{n,\alpha}(z)$, $z = 1$. So the denominator of $R_{n,\alpha}$ has degree $2n - 3$ in z. Since $E_i(n,\alpha)$ are polynomials of degree 1 in α, this concludes the argument of writing $R_{n,\alpha}$ in this form.

We split the proof of the case $n \geq 3$ in several lemmas. We start by giving an estimate for a zero of R_α, which lies on the positive real line.

Lemma 4.3.5. *Let $\alpha > 2$ and let $S(z) = E_1(\alpha)z^n + E_2(\alpha)(z+1)^n$. Then $S\Big(\alpha(n-1)\Big) < 0 < S(\alpha n - \alpha - n)$. In particular, R_α has a zero on the interval $\Big(\alpha(n-1) - n,\, \alpha(n-1)\Big)$, for all $\alpha > 2$.*

Proof. Direct computations show that S writes as

$$S(z) = -nz^n + [\alpha(n-1) - n)]\sum_{k=0}^{n-1}\binom{n}{k}z^k. \tag{4.7}$$

On one hand,

$$S\Big(\alpha(n-1) - n\Big) = \sum_{k=0}^{n-2}\binom{n}{k}(\alpha n - \alpha - n)^k > 0.$$

On the other hand,

$$\begin{aligned}
&S\Big(\alpha(n-1)\Big) = -n\Big[\alpha(n-1)\Big]^n + [\alpha(n-1) - n]\textstyle\sum_{k=0}^{n-1}\binom{n}{k}[\alpha(n-1)]^k\\
&= -n\Big[\alpha(n-1)\Big]^n + [\alpha(n-1)]\textstyle\sum_{k=0}^{n-2}\binom{n}{k}[\alpha(n-1)]^k + [\alpha(n-1)]n\Big[\alpha(n-1)\Big]^{n-1} - n\textstyle\sum_{k=0}^{n-1}\binom{n}{k}[\alpha(n-1)]^k\\
&= \textstyle\sum_{k=0}^{n-2}\Big[\binom{n}{k} - n\binom{n}{k+1}\Big][\alpha(n-1)]^{k+1} - n\binom{n}{0} < 0.
\end{aligned}$$

□

The following technical lemma will be useful later.

Lemma 4.3.6. *Let $m, k \in \mathbb{N}^*$, $m > k$. Let $u, v_j \in \mathbb{C}$, $j = 1, \ldots, m$. If $|u| - \sum_{j=1}^{m}|v_j| > 0$, then*

$$\left|u - \sum_{j=1}^{k} v_j\right| > \left|\sum_{j=k+1}^{m} v_j\right|.$$

Proof. Since $|u| - \sum_{j=1}^{m}|v_j| > 0$, we have that

$$\left|u - \sum_{j=1}^{k} v_j\right| \geq |u| - \sum_{j=1}^{k}|v_j| > \sum_{j=k+1}^{m}|v_j| \geq \left|\sum_{j=k+1}^{m} v_j\right|.$$

□

We give a sufficient condition for points to lie in $A_\alpha^*(\infty)$.

Lemma 4.3.7. *Let $\alpha > 0$ large enough. If $|z| > n\alpha$, then $z \in A_\alpha^*(\infty)$.*

Proof. We show that if $|z| > n\alpha$, then $|R_\alpha(z)| > |z|$, which is a sufficient condition for $z \in A_\alpha^*(\infty)$. According to (4.6), we have to prove that, for α large enough, $\left|\frac{P(z)}{Q(z)}\right| > 1$. We write P as

$$P(z) = 2n\Big[-nz^{2n-1} + n\alpha(n-1)z^{2n-2} + P_{2n-2}(z) + \alpha P_{2n-3}(z)\Big].$$

Observe that $P_{2n-2}(z)$ and $P_{2n-3}(z)$ are polynomials of degree $2n-2$ and $2n-3$, respectively, with coefficients independent of α. For α large enough (recall that we are assuming $|z| > n\alpha$), the following statements hold:

1. $(n-1)|z|^{2n-1} > n\alpha(n-1)|z|^{2n-2}$.
2. $\frac{1}{3}|z|^{2n-1} > |P_{2n-2}(z)|$, since
$$\lim_{\alpha\to\infty} \frac{n(n-1)|z|^{2n-2}}{|z|^{2n-1}} = 0.$$
3. $\frac{1}{3}|z|^{2n-1} > |\alpha P_{2n-3}(z)|$, since
$$\lim_{\alpha\to\infty} \frac{P_{2n-3}(z)}{|z|^{2n-2}} = 0.$$
4. $\frac{1}{3}|z|^{2n-1} > |Q(z)|$, since Q^1_{2n-3} and Q^2_{2n-3} are polynomials of degree $2n-3$ with coefficients independent of α.

All together imply that
$$n|z|^{2n-1} > n\alpha(n-1)|z|^{2n-2} + |P_{2n-2}(z)| + \alpha|P_{2n-3}(z)| + |Q(z)|.$$
By using Lemma 4.3.6 (recall that for α large enough, there is no root of Q for $|z| > n\alpha$), we get that
$$\left|\frac{P(z)}{Q(z)}\right| = \left|2n\frac{-nz^{2n-1} + \alpha(n-1)z^{2n-2} + P_{2n-2}(z) + \alpha P_{2n-3}(z)}{Q(z)}\right| > 2n > 1.$$
Thus, for $|z| > n\alpha$, we have that $|R_\alpha(z)| > |z|$ and $z \in A^*_\alpha(\infty)$. □

The following proposition concludes the proof of Theorem E.

Proposition 4.3.8. *Let $\alpha > 0$ large enough. Then $A^*_\alpha(\infty)$ is infinitely connected.*

Proof. If $z \in (0, \alpha n - \alpha - n)$, then $nz^n < (\alpha n - \alpha - n)nz^{n-1}$. It follows that $S(z)$ (see (4.7)) has no zeros in $(0, \alpha n - \alpha - n)$. In particular, R_α has no zeros in $(0, \alpha n - \alpha - n)$. Let
$$I = \left\{ z \in \mathbb{C} \middle| z = n\alpha\left(\frac{1}{2} + it\right), \, t \in [-1, 1] \right\}.$$
We claim that $R_\alpha(I) \subset A^*_\alpha(\infty)$.

Let $T_\alpha(z) := \frac{1}{(1+z)^2} R_\alpha(z)$. Firstly, we prove that there exists a constant $\kappa > 0$ such that for $z \in I$, we have that $\left|T_\alpha(z)\right| > \kappa$. A direct computation shows that
$$\left|T_\alpha\left(n\alpha\left(\frac{1}{2} + it\right)\right)\right| := \frac{N(\alpha)}{M(\alpha)},$$
where N and M are polynomials of degree $2n-2$ in α with coefficients depending on t. Moreover, if we denote by $c(t)$ the degree $2n-2$ coefficient of N, we have:
$$c(t) = 2n^{2n-1}\left(\frac{1}{2} + it\right)^{2n-2}\left[-n\left(\frac{1}{2} + it\right) + n - 1\right].$$
Observe that
$$\min_{t\in[-1,\,1]} \left|c(t)\right| = \left|c(0)\right| := C > 0.$$

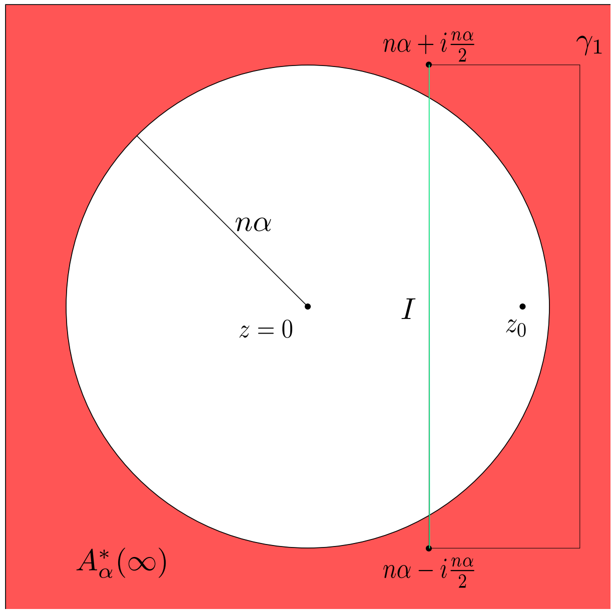


Figure 4.5: Description of the situation in proof of Proposition 4.3.8. The zero z_0 is separated by $(I \cup \gamma_1) \subset A^*_\alpha(\infty)$ from $z = 0$. Therefore, $A^*_\alpha(\infty)$ is multiply connected.

We denote by $d(t)$ the degree $2n-2$ coefficient of M. Let $D := \max\limits_{t\in[-1,1]} |d(t)|$, and let $\kappa := \frac{C}{2D}$. For large enough α, we have that $\left|T_\alpha\Big(n\alpha\left(\frac{1}{2}+it\right)\Big)\right| > \kappa$ and that

$$\left|R_\alpha\left(n\alpha\left(\frac{1}{2}+it\right)\right)\right| > \kappa\left|\frac{n\alpha}{2}+1+n\alpha ti\right|^2 > \frac{n^2}{4}\kappa\alpha^2 > n\alpha.$$

It follows from Lemma 4.3.7 that $R_\alpha(I) \subset A^*_\alpha(\infty)$. Hence, I is a subset of $A^*_\alpha(\infty)$ or a preimage of this Fatou component. Moreover, for $z_\pm = \frac{n}{2}\alpha \pm in\alpha$ we have $|z_\pm| > n\alpha$. We conclude from Lemma 4.3.7 that $z_\pm \in A^*_\alpha(\infty)$. Therefore, $I \subset A^*_\alpha(\infty)$. By Lemma 4.3.5, there exists a zero z_0 of R_α such that $\frac{n\alpha}{2} < z_0 < n\alpha$. Therefore, there exists a piece-wise smooth Jordan curve $\Gamma = I \cup \gamma_1 \subset A^*_\alpha(\infty)$ such that $z_0 \in \mathrm{Int}(\Gamma)$ and $0 \in \mathrm{Ext}(\Gamma)$ (see Figure 4.5). It follows that $A^*_\alpha(\infty)$ is multiply connected. By Remark 4.1.6, it is infinitely connected. $\square$

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