Chapter 5

Quadratic Extended Kalman Filtering

As it was explained in Section 2.5.2, a tracker is a closed-loop estimator that is able to follow the variations of the parameters of interest. To do so, the tracker is composed of a discriminator and a loop filter (see Fig. 2.4). In that scheme, the discriminator is designed to deliver unbiased estimates that are further integrated at the loop filter according to the known parameter dynamics.

In Chapter 4, the optimal second-order *discriminator* was formulated by minimizing the *steady-state* variance subject to the unbiasedness constraint. In this optimization, it was assumed that the small-error condition is satisfied in the steady-state. Implicitly, this assumption means that all the parameters have been initially acquired and the tracker is following accurately their temporal evolution. However, the tracker optimization was carried out without taking into account the acquisition and tracking performance. For this reason, the loop filter was not involved in the design.

Alternatively, the Kalman filter is designed considering globally both the acquisition and steady-state performance. In the Kalman filter theory, the parameter is modelled as a random variable of known statistics [And79][Kay93b, Ch. 13]. The Kalman filter, which is linear in the observed data, is known to be the optimal tracker if the parameters and observations are Gaussian random variables of known *a priori* mean and variance. In that case, the optimality of the Kalman filter means that it provides minimum variance unbiased estimates in the steady-state as well as minimum MSE estimates during the acquisition.

From the results in Chapter 4, the prior distribution about the parameters is useless once the small-error regime is attained. However, this information is very relevant during the acquisition, that is, in the large-error regime. The Kalman filter is considered in this thesis because it performs a gradual transition from the large-error regime in Chapter 3 to the small-error regime in Chapter 4 as the observation length increases. As stated before, this transition is optimal if and only if all the random variables are Gaussian distributed.

Unfortunately, the Gaussian condition is quite restrictive because it implies linear models for the observation as well as for the parameter dynamics. Otherwise, the observation and dynamics equations have to be linearized in order to derive the so-called Extended Kalman filter (EKF) [And79][Kay93b, Sec. 13.7]. It can be shown that the EKF is solely the best linear tracker in the steady-state independently of the parameter and observation distribution. This statement is verified because, whatever the parameterization and dynamical model at hand, the observation and dynamics equations are always linear in the vector of parameters if these equations are approximated around the true value of the parameters (small-error assumption). On the other hand, nothing can be stated about the EKF optimality during the acquisition stage (large-error regime), which is actually uncertain.

In the context of blind parameter estimation, second-order methods are mandatory because the observation is zero mean. Thus, the EKF is extended in this chapter to deal with quadratic observation models. The result is the so-called *Quadratic* Extended Kalman Filter (QEKF) that constitutes an alternative deduction for the optimal second-order tracker studied in Chapter 4. The main advantage is that the QEKF adjusts automatically its response during the acquisition phase in order to speed up the tracker convergence without altering the (optimal) steady-state solution. On the other hand, in Chapter 4, the tracker response was specifically designed for the steady-state (small-error regime) and it was not changed during all the operation time. Therefore, the QEKF can be seen as a *time-variant* quadratic tracker that automatically adjusts the loop bandwidth depending on the current uncertainty on the parameters (Section 2.5.2). Thus, the QEKF bandwidth is progressively decreased during the acquisition time and is finally "frozen" in the steady-state. Another important feature is that, assuming a successful acquisition, the QEKF provides a recursive low-cost implementation of the minimum variance unbiased estimator when the observation time increases indefinitely and the parameters remain stationary.

The main criticism about the EKF/QEKF tracker is that the acquisition cannot be guaranteed. Effectively, even in the noiseless case, the *linearized* model assumed in the EKF/QEKF formulation is not correct when the tracker operates out of the small-error regime, e.g., during the acquisition. To overcome this inconvenient, the Unscented Kalman Filter (UKF) is proposed in [Jul97][Wan00]. The UKF applies the actual nonlinear observation model to propagate correctly the mean *as well as the covariance* of the Gaussian parameter. The important point is that the convergence of the UKF is guaranteed under some mild conditions [Mer00, Sec. 5].

Implicityly, the UKF is still assuming Gaussian parameters. For other statistical distributions, sequential Monte Carlo estimators –also named particle filters– can be applied

[Mer00][Mer01]. The complexity of these methods is usually much greater than that of the well-known EKF. Anyway, the UKF and other particle filters were not considered in this thesis because they are actually higher-order techniques in which the observed samples are plugged into nonlinear posterior distributions to approximate the MMSE estimator, i.e., $\widehat{\boldsymbol{\theta}} = E_{\boldsymbol{\theta}} E\left\{\boldsymbol{\theta}/\mathbf{y}\right\}$. A tutorial article on the UKF and related sequential Monte Carlo methods is provided in [Mer03].

Finally, the QEKF is deduced and evaluated in the context of DOA estimation and tracking. The Gaussian assumption on the nuisance parameters is tested once more showing the significant improvement in terms of acquisition time as well as steady-state variance when the received signals are digitally modulated and this information is correctly exploited.

The results in this section were presented in the 3rd IEEE Sensor Array and Multichannel Signal Processing Workshop that was held in Barcelona in 2004 [Vil04a]:

- "On the Quadratic Extended Kalman Filter", J. Villares, G. Vázquez. *Proc. of the Third IEEE Sensor Array and Multichannel Signal Processing Workshop* (SAM 2004). July 2004. Sitges, Barcelona (Spain).

5.1 Signal Model

Let us consider a time-variant scenario in which the observed vector at time n is given by

$$\mathbf{y}_n = \mathbf{A}\left(\boldsymbol{\theta}_n\right)\mathbf{x}_n + \mathbf{w}_n \qquad n = 1, 2, 3, ... \tag{5.1}$$

where the transfer matrix $\mathbf{A}\left(\boldsymbol{\theta}_n\right)$ is known except for a vector of P real-valued parameters $\boldsymbol{\theta}_n$, $\mathbf{x}_n$ is the vector of K unknown *zero-mean* inputs and, $\mathbf{w}_n$ is the vector of Gaussian noise samples. The covariance matrix of $\mathbf{x}_n$ and $\mathbf{w}_n$ is given by

$$\begin{aligned} E\left\{\mathbf{w}_n\mathbf{w}_{n+k}^H\right\} &= \mathbf{R}_w\delta\left(k\right) \\ E\left\{\mathbf{x}_n\mathbf{x}_{n+k}^H\right\} &= \mathbf{I}_K\delta\left(k\right), \end{aligned}$$

respectively. Therefore, we are assuming that the noise and the nuisance parameters are uncorrelated in the time domain.

In order to track the parameter evolution in time, the estimator is provided with the following dynamical model or *state equation* [And79][Kay93b, Ch.13]:

$$\boldsymbol{\theta}_n = \mathbf{f}\left(\boldsymbol{\theta}_{n-1}\right) + \mathbf{u}_n \tag{5.2}$$

where $\mathbf{u}_n$ is a zero-mean random variable of known covariance matrix $\mathbf{R}_u \triangleq E\left\{\mathbf{u}_n\mathbf{u}_n^H\right\}$ modeling the uncertainty about the assumed model. The initial state $\boldsymbol{\theta}_0$ is also a random variable of known mean $\boldsymbol{\mu}_0 \triangleq E\{\boldsymbol{\theta}_0\}$ and covariance matrix $\Sigma_{0/0} \triangleq E\{\boldsymbol{\theta}_0\boldsymbol{\theta}_0^H\}$. These two quantities summarize all the available *prior* information about the parameter $\boldsymbol{\theta}_n$.

It is important to note that consistent estimates cannot be obtained using linear schemes because the observation $\mathbf{y}_n$ is zero-mean. Thus, blind estimation imposes the need of second-order tecnniques that are known to be optimal for low-SNR and/or Gaussian data (Section 2.4.1 and Section 2.4.3). Accordingly, the following *quadratic measurement equation* is considered:

$$\mathbf{r}_n \triangleq \text{vec}\left[\mathbf{y}_n\mathbf{y}_n^H\right] = \mathbf{h}(\boldsymbol{\theta}_n) + \mathbf{v}_n(\boldsymbol{\theta}_n) \tag{5.3}$$

where

$$\begin{aligned}
\mathbf{h}(\boldsymbol{\theta}_n) &\triangleq E\{\mathbf{r}_n\} = \text{vec}\left[\mathbf{A}(\boldsymbol{\theta}_n)\mathbf{A}^H(\boldsymbol{\theta}_n) + \mathbf{R}_w\right] \\
\mathbf{v}_n(\boldsymbol{\theta}_n) &\triangleq \mathbf{r}_n - E\{\mathbf{r}_n\} \\
&= \text{vec}\left[\mathbf{A}(\boldsymbol{\theta}_n)\left(\mathbf{x}_n\mathbf{x}_n^H - \mathbf{I}_K\right)\mathbf{A}^H(\boldsymbol{\theta}_n) + \mathbf{A}(\boldsymbol{\theta}_n)\mathbf{x}_n\mathbf{w}_n^H + \mathbf{w}_n\mathbf{x}_n^H\mathbf{A}^H(\boldsymbol{\theta}_n) + \mathbf{w}_n\mathbf{w}_n^H - \mathbf{R}_w\right]
\end{aligned}$$

are the signal and noise components of the measurement equation, respectively. Notice that the observation noise $\mathbf{v}_n(\boldsymbol{\theta}_n)$ is zero-mean and it depends on the wanted parameters in the considered quadratic model.

5.2 Background and Notation

Following the classical notation in [And79], $\widehat{\mathbf{a}}_{n/m}$ will denote the *linear MMSE* estimate of a given random vector $\mathbf{a}_n$ based on the quadratic observations $\mathbf{r}_1, \ldots, \mathbf{r}_m$. This means that $\widehat{\mathbf{a}}_{n/m}$ is an affine transformation of the sample covariance vectors $\mathbf{r}_1, \ldots, \mathbf{r}_m$ in (5.3) or, equivalently, a *quadratic* transformation of the input data $\mathbf{y}_1, \ldots, \mathbf{y}_m$ (5.1).

It is well-known that the MMSE estimator $E\{\mathbf{a}_n/\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ is linear in $\mathbf{r}_1, \ldots, \mathbf{r}_m$ if and only if $\mathbf{a}_n$ and $\mathbf{r}_1, \ldots, \mathbf{r}_m$ are jointly Gaussian distributed. However, the Gaussian assumption is not satisfied most times. In that case, it is convenient to introduce the following notation

$$\widehat{\mathbf{a}}_{n/m} = E_L\{\mathbf{a}_n/\mathbf{r}_1, \ldots, \mathbf{r}_m\}$$

to refer to the *linear MMSE* estimator $\widehat{\mathbf{a}}_{n/m}$, bearing in mind that $E_L\{\mathbf{a}_n/\mathbf{r}_1, \ldots, \mathbf{r}_m\} = E\{\mathbf{a}_n/\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ in the Gaussian case [And79, Sec. 5.2].

The Kalman filter can be seen as a *sequential* implementation of the linear MMSE estimator of $\boldsymbol{\theta}_n$ that, using the notation above, is given by

$$\widehat{\boldsymbol{\theta}}_{n/n} = E_L\{\boldsymbol{\theta}_n/\mathbf{r}_1, \ldots, \mathbf{r}_n\}.$$

From a complexity point of view, the sequential computation of $\widehat{\boldsymbol{\theta}}_{n/n}$ is unavoidable as the number of observations augments ($n \rightarrow \infty$).

The Kalman filter recursion is based on two facts:

- The orthogonalization (decorrelation) of the original observations $\mathbf{r}_1, \ldots, \mathbf{r}_n$ using the Gram-Schmidt method[1]. The transformed observations are the so-called *innovations* $\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_n$ [And79][Kay93b][Hay91], which are computed as

$$\widetilde{\mathbf{r}}_n = \mathbf{r}_n - \widehat{\mathbf{r}}_{n/n-1} \tag{5.4}$$

with

$$\widehat{\mathbf{r}}_{n/n-1} = E_L\left\{\mathbf{r}_n/\mathbf{r}_1, \ldots, \mathbf{r}_{n-1}\right\}$$

the *linear MMSE* prediction of $\mathbf{r}_n$ based on the past observations $\mathbf{r}_1, \ldots, \mathbf{r}_{n-1}$. Thus, the innovation $\widetilde{\mathbf{r}}_n$ supplies the *new information* contained in the observation $\mathbf{r}_n$ or, in other words, it yields the unpredictable component of $\mathbf{r}_n$. It can be shown that the innovation $\widetilde{\mathbf{r}}_n$ is *zero-mean* and it is *uncorrelated* with both $\widetilde{\mathbf{r}}_m$ and $\mathbf{r}_m$ for any $m \neq n$. Using this property, it is easy to show that

$$\begin{aligned}\widehat{\boldsymbol{\theta}}_{n/n} &= E_L\left\{\boldsymbol{\theta}_n/\mathbf{r}_1, \ldots, \mathbf{r}_n\right\} = E_L\left\{\boldsymbol{\theta}_n/\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_n\right\} = \sum_{k=1}^{n} E_L\left\{\boldsymbol{\theta}_n/\widetilde{\mathbf{r}}_k\right\} \\ &= E_L\left\{\boldsymbol{\theta}_n/\mathbf{r}_1, \ldots, \mathbf{r}_{n-1}, \widetilde{\mathbf{r}}_n\right\} = \widehat{\boldsymbol{\theta}}_{n/n-1} + E_L\left\{\boldsymbol{\theta}_n/\widetilde{\mathbf{r}}_n\right\} \\ &= \widehat{\boldsymbol{\theta}}_{n/n-1} + E_L\left\{\widetilde{\boldsymbol{\theta}}_n/\widetilde{\mathbf{r}}_n\right\},\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}\widehat{\boldsymbol{\theta}}_{n/n-1} &= E_L\left\{\boldsymbol{\theta}_n/\mathbf{r}_1, \ldots, \mathbf{r}_{n-1}\right\} \\ \widetilde{\boldsymbol{\theta}}_n &\triangleq \boldsymbol{\theta}_n - \widehat{\boldsymbol{\theta}}_{n/n-1}\end{aligned} \tag{5.6}$$

are the linear MMSE prediction of $\boldsymbol{\theta}_n$ –based on the past observations $\mathbf{r}_1, \ldots, \mathbf{r}_{n-1}$– and the resulting prediction error, respectively. It can be shown that both $\widehat{\boldsymbol{\theta}}_{n/n-1}$ and $\widetilde{\boldsymbol{\theta}}_n$ are *zero mean* and they are *uncorrelated* with both $\widetilde{\mathbf{r}}_n$ and $\mathbf{r}_n$. In fact, this property has been applied to obtain the final expression in (5.5) considering that $E_L\left\{\widehat{\boldsymbol{\theta}}_{n/n-1}/\widetilde{\mathbf{r}}_n\right\} = \mathbf{0}$.

- The existence of a *linear* state equation (5.2) as well as a *linear* measurement equation (5.3). When this is possible, $\widehat{\boldsymbol{\theta}}_{n/n}$ (5.5) can be obtained from $\widehat{\boldsymbol{\theta}}_{n-1/n-1}$ (i.e., the previous estimate) and $\mathbf{r}_n$ (i.e., the new datum) bearing in mind that

$$E_L\left\{\mathbf{M}\mathbf{a}_n/\mathbf{r}_1, \ldots, \mathbf{r}_n\right\} = \mathbf{M}E_L\left\{\mathbf{a}_n/\mathbf{r}_1, \ldots, \mathbf{r}_n\right\}. \tag{5.7}$$

Unfortunately, the state and measurement equations in (5.2)-(5.3) are generally nonlinear in the parameters of interest. Consequently, these two equations have to be linearized in order to apply the Kalman filter formulation. This matter is addressed in the next section.

[1] Although $\mathbf{x}_n$ and $\mathbf{w}_n$ are uncorrelated in Sec. 5.1, the observations $\mathbf{r}_1, \ldots, \mathbf{r}_n$ are correlated because they depend on the random parameters $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_n$, which are correlated in the assumed dynamical model (5.2).

5.3 Linearized Signal Model

In order to have linear state and measurement equations, the original *nonlinear equations* (5.2)-(5.3) are expanded in a first-order Taylor series at the points $\boldsymbol{\theta}_{n-1} = \widehat{\boldsymbol{\theta}}_{n-1/n-1}$ and $\boldsymbol{\theta}_n = \widehat{\boldsymbol{\theta}}_{n/n-1}$, respectively. These points are selected because $\widehat{\boldsymbol{\theta}}_{n-1/n-1}$ and $\widehat{\boldsymbol{\theta}}_{n/n-1}$ are the linear MMSE estimates of $\boldsymbol{\theta}_{n-1}$ and $\boldsymbol{\theta}_n$ before the new datum $\mathbf{r}_n$ is processed. By definition, $\widehat{\boldsymbol{\theta}}_{n-1/n-1}$ and $\widehat{\boldsymbol{\theta}}_{n/n-1}$ are given by

$$\begin{aligned}\widehat{\boldsymbol{\theta}}_{n-1/n-1} &= E_L\left\{\boldsymbol{\theta}_{n-1}/\mathbf{r}_1,\ldots,\mathbf{r}_{n-1}\right\}\\ \widehat{\boldsymbol{\theta}}_{n/n-1} &= E_L\left\{\boldsymbol{\theta}_{n}/\mathbf{r}_1,\ldots,\mathbf{r}_{n-1}\right\}.\end{aligned}$$

Thus, assuming that $\widehat{\boldsymbol{\theta}}_{n-1/n-1}$ and $\widehat{\boldsymbol{\theta}}_{n/n-1}$ are previously computed at time $n-1$, the QEKF will be derived from the linearized state and quadratic measurement equations given next:

$$\boldsymbol{\theta}_n \approx \mathbf{f}\left(\widehat{\boldsymbol{\theta}}_{n-1/n-1}\right) + \mathbf{F}\left(\widehat{\boldsymbol{\theta}}_{n-1/n-1}\right)\left[\boldsymbol{\theta}_{n-1} - \widehat{\boldsymbol{\theta}}_{n-1/n-1}\right] + \mathbf{u}_n \tag{5.8}$$

$$\mathbf{r}_n \approx \mathbf{h}\left(\widehat{\boldsymbol{\theta}}_{n/n-1}\right) + \mathbf{H}_n\left(\widehat{\boldsymbol{\theta}}_{n/n-1}\right)\left[\boldsymbol{\theta}_{n} - \widehat{\boldsymbol{\theta}}_{n/n-1}\right] + \mathbf{v}_n\left(\widehat{\boldsymbol{\theta}}_{n/n-1}\right) \tag{5.9}$$

where $\mathbf{F}(\boldsymbol{\theta}_{n-1})$ and $\mathbf{H}_n(\boldsymbol{\theta}_n)$ are the Jacobian of $\boldsymbol{\theta}_n$ and $\mathbf{r}_n$, respectively, that is given by

$$\begin{aligned}\mathbf{F}(\boldsymbol{\theta}_{n-1}) &\triangleq \frac{\partial \boldsymbol{\theta}_n}{\partial \boldsymbol{\theta}_{n-1}^T} = \frac{\partial \mathbf{f}(\boldsymbol{\theta}_{n-1})}{\partial \boldsymbol{\theta}_{n-1}^T}\\ \mathbf{H}_n(\boldsymbol{\theta}_n) &\triangleq \frac{\partial \mathbf{r}_n}{\partial \boldsymbol{\theta}_n^T} = \frac{\partial \mathbf{h}(\boldsymbol{\theta}_n)}{\partial \boldsymbol{\theta}_n^T} + \frac{\partial \mathbf{v}_n(\boldsymbol{\theta}_n)}{\partial \boldsymbol{\theta}_n^T}.\end{aligned}$$

From the linearized state equation (5.8), the prediction $\widehat{\boldsymbol{\theta}}_{n/n-1}$ in (5.9) can be computed as

$$\widehat{\boldsymbol{\theta}}_{n/n-1} = \mathbf{f}\left(\widehat{\boldsymbol{\theta}}_{n-1/n-1}\right), \tag{5.10}$$

using (5.7) and taking into account that the noise $\mathbf{u}_n$ is zero mean. On the other hand, the Jacobian $\mathbf{H}_n(\boldsymbol{\theta}_n)$ is calculated from (5.3), obtaining

$$\begin{aligned}\left[\mathbf{H}_n(\boldsymbol{\theta})\right]_p = \operatorname{vec}\Bigg[&\frac{\partial \mathbf{A}(\boldsymbol{\theta})}{\partial \theta_p}\mathbf{x}_n\mathbf{x}_n^H\mathbf{A}^H(\boldsymbol{\theta}) + \mathbf{A}(\boldsymbol{\theta})\mathbf{x}_n\mathbf{x}_n^H\frac{\partial \mathbf{A}^H(\boldsymbol{\theta})}{\partial \theta_p}\\ &+ \frac{\partial \mathbf{A}(\boldsymbol{\theta})}{\partial \theta_p}\mathbf{x}_n\mathbf{w}_n^H + \mathbf{w}_n\mathbf{x}_n^H\frac{\partial \mathbf{A}^H(\boldsymbol{\theta})}{\partial \theta_p}\Bigg]\end{aligned}$$

where θ_p stands for the p-th component of $\boldsymbol{\theta}$. Note that the transfer matrix $\mathbf{H}_n(\boldsymbol{\theta})$ appearing in (5.9) is noisy because it depends on the random terms $\mathbf{x}_n$ and $\mathbf{w}_n$. This particularity is a consequence of the original quadratic observation model (5.3).

5.4 Quadratic Extended Kalman Filter (QEKF)

In this section, the Kalman filter is derived from the *quadratic and linearized* model introduced in the last two sections. The resulting tracker is named the Quadratic Extended Kalman Filter (QEKF) because it corresponds to the so-called *Extended* Kalman Filter (EKF) [And79]

[Kay93b, Sec. 13.7] in case of having quadratic observations (5.3). The QEKF is thus obtained from (5.5) after solving the second term as indicated now:

$$E_L\left\{\widetilde{\boldsymbol{\theta}}_n/\widetilde{\mathbf{r}}_n\right\} = \mathbf{M}_n^H\widetilde{\mathbf{r}}_n \tag{5.11}$$

where $\mathbf{M}_n$ is the so-called *Kalman gain matrix*,

$$\mathbf{M}_n \triangleq \mathcal{Q}_n^{-1}\mathbf{S}_n \tag{5.12}$$

$$\mathbf{S}_n \triangleq E\left\{\widetilde{\mathbf{r}}_n^H\widetilde{\boldsymbol{\theta}}_n\right\} \tag{5.13}$$

$$\mathcal{Q}_n \triangleq E\left\{\widetilde{\mathbf{r}}_n\widetilde{\mathbf{r}}_n^H\right\}, \tag{5.14}$$

and $E\{\cdot\}$ stands for the expectation with respect to all the random terms inside the brackets, namely $\boldsymbol{\theta}_0, \ldots, \boldsymbol{\theta}_n$ and $\mathbf{r}_1, \ldots, \mathbf{r}_n$.

The Kalman gain matrix (5.12) has been derived using the following well-known result [Kay93b, Eq. 12.6]:

$$E_L\{\mathbf{x}/\mathbf{y}\} = E\{\mathbf{x}\} + E\left\{\mathbf{x}\mathbf{y}^H\right\}E^{-1}\left\{\mathbf{y}\mathbf{y}^H\right\}(\mathbf{y}-E\{\mathbf{y}\})$$

particularized for the *zero-mean* random vectors $\widetilde{\boldsymbol{\theta}}_n$ and $\widetilde{\mathbf{r}}_n$ introduced in (5.6) and (5.4), respectively. This abbreviated deduction of the extended Kalman filter [Kay93b, App. 13.B] is based on the following two important equations:

$$E\{\widetilde{\mathbf{r}}_n\} = E_{\mathbf{r}_1,\ldots,\mathbf{r}_{n-1}}E\{\widetilde{\mathbf{r}}_n/\mathbf{r}_1,\ldots,\mathbf{r}_{n-1}\} = \mathbf{0}$$

$$E\left\{\widetilde{\boldsymbol{\theta}}_n\right\} = E_{\mathbf{r}_1,\ldots,\mathbf{r}_{n-1}}E\left\{\widetilde{\boldsymbol{\theta}}_n/\mathbf{r}_1,\ldots,\mathbf{r}_{n-1}\right\} = \mathbf{0}$$

since $E\{\widetilde{\mathbf{r}}_n/\mathbf{r}_1,\ldots,\mathbf{r}_{n-1}\}$ and $E\left\{\widetilde{\boldsymbol{\theta}}_n/\mathbf{r}_1,\ldots,\mathbf{r}_{n-1}\right\}$ are strictly zero in view of their definitions in (5.4) and (5.6), respectively.

Therefore, plugging (5.10) and (5.11) into (5.5), we obtain the QEKF recursion:

$$\widehat{\boldsymbol{\theta}}_{n/n} = \mathbf{f}\left(\widehat{\boldsymbol{\theta}}_{n-1/n-1}\right) + \mathbf{S}_n^H\mathcal{Q}_n^{-1}\left(\mathbf{r}_n - \widehat{\mathbf{r}}_{n/n-1}\right) \tag{5.15}$$

where

$$\widehat{\mathbf{r}}_{n/n-1} = \mathbf{h}\left(\widehat{\boldsymbol{\theta}}_{n/n-1}\right) = \mathbf{h}\left(\mathbf{f}\left(\widehat{\boldsymbol{\theta}}_{n-1/n-1}\right)\right)$$

is obtained from (5.15) and (5.10).

5.4.1 Another QEKF derivation

Thus far, the classical formulation of the QEKF is sketched introducing some simplifications. In this section, a simpler derivation of the QEKF is proposed based on the general formulation in Section 3.2. In fact, the solution in (5.15) is obtained by considering a generic second-order estimator,

$$\widehat{\boldsymbol{\theta}}_{n/n} = \mathbf{b}_n + \mathbf{M}_n^H \mathbf{r}_n,$$

and solving the following optimization problem:

$$\mathbf{b}_n, \mathbf{M}_n = \arg\max_{\mathbf{b},\mathbf{M}} E\left\{ \left\| \mathbf{b} + \mathbf{M}^H \mathbf{r}_n - \boldsymbol{\theta}_n \right\|^2 / \mathbf{r}_1, ..., \mathbf{r}_{n-1} \right\}$$

where $\mathbf{b}_n$ and $\mathbf{M}_n$ are the independent and quadratic components of the *second-order MMSE* estimator of $\boldsymbol{\theta}_n$, respectively. In (3.16), it was obtained that the optimal independent term is $\mathbf{b}_n = \widehat{\boldsymbol{\theta}}_{n/n-1} - \mathbf{M}_n^H \widehat{\mathbf{r}}_{n/n-1}$. On the other hand, the optimal quadratic term $\mathbf{M}_n$ was derived in (3.25) obtaining precisely the Kalman gain matrix in (5.12).

The conditional expectation in the last equation suggests that the random parameters are averaged by means of the *prior distribution* $f_{\boldsymbol{\theta}_n/\mathbf{r}_1,...,\mathbf{r}_{n-1}}(\boldsymbol{\theta}_n/\mathbf{r}_1, ..., \mathbf{r}_{n-1})$, which has all the existing knowledge on $\boldsymbol{\theta}_n$ before processing $\mathbf{r}_n$. In that way, the QEKF provides a means of updating the prior distribution every time a new datum is incorporated. In case the QEKF converges to the true parameter, the sequence of priors $f_{\boldsymbol{\theta}_n/\mathbf{r}_1,...,\mathbf{r}_{n-1}}(\boldsymbol{\theta}_n/\mathbf{r}_1, ..., \mathbf{r}_{n-1})$ becomes progressively more informative until the small-error regime is attained (Chapter 4). Moreover, if the Gaussian assumption applies, the prior updating is optimal, minimizing so the acquisition time. Definitely, this was the motivation of considering in this thesis the Kalman filter formulation: the QEKF provides the transition from the MMSE *large-error* solution in Chapter 3 to the *small-error* BQUE solution in Chapter 4.

An evident connection is observed between (5.15) and the expression obtained for the optimal second-order *discriminator* in (4.12). However, there are some important differences. First of all, the so-called Kalman gain matrix $\mathbf{M}_n$ appearing in (5.15) includes both the discriminator and the loop filter of a classical closed-loop implementation. Moreover, $\mathbf{M}_n$ is *time-varying* and, therefore, the QEKF is able to adjust online the *overall* tracker response in view of the instantaneous uncertainty about the parameters.

It can be shown that the QEKF and the closed-loop implementation in Chapter 4 become equivalent in the steady-state if they are arranged to have the same (noise equivalent) loop bandwidth. Formally, it is verified that

$$\lim_{n\to\infty} \mathbf{M}_n = \operatorname{diag}(\boldsymbol{\mu})\,\mathbf{M}$$

where $\mathbf{M}$ is the optimal second-order discriminator obtained in (4.12) and, the vector of step-sizes $\boldsymbol{\mu}$ is determined by the state equation noise covariance matrix $\mathbf{R}_u \triangleq E\left\{\mathbf{u}_n \mathbf{u}_n^H\right\}$ (5.2). The proof of this important statement would require to solve properly the Ricciati steady-state equation [And79, Ch.4] and it suggests an in-depth study that is still incomplete.

5.4.2 Kalman gains recursion

The linearized model in Section 5.3 allows obtaining $\mathbf{M}_n = \mathcal{Q}_n^{-1}\mathbf{S}_n$ recursively. In this section, the QEKF deduction is completed by making this recursion explicit.

Let us study first the cross-correlation matrix $\mathbf{S}_n$ (5.13). It is easy to prove that

$$\mathbf{S}_n = \mathbf{D}_r\left(\widehat{\boldsymbol{\theta}}_{n/n-1}\right)\Sigma_{n/n-1} = \mathbf{D}_r\left(\mathbf{f}\left(\widehat{\boldsymbol{\theta}}_{n-1/n-1}\right)\right)\Sigma_{n/n-1}$$

where

$$\Sigma_{n/n-1} \triangleq E\left\{\widetilde{\boldsymbol{\theta}}_n\widetilde{\boldsymbol{\theta}}_n^H/\mathbf{r}_1,...,\mathbf{r}_{n-1}\right\} = \mathbf{F}\left(\widehat{\boldsymbol{\theta}}_{n-1/n-1}\right)\Sigma_{n-1/n-1}\mathbf{F}^H\left(\widehat{\boldsymbol{\theta}}_{n-1/n-1}\right) + \mathbf{R}_u \tag{5.16}$$

is the covariance matrix of the prediction error $\widetilde{\boldsymbol{\theta}}_n$ expressed as a function of the estimation MSE matrix[2] at time $n-1$:

$$\Sigma_{n-1/n-1} = E\left\{\left(\boldsymbol{\theta}_{n-1} - \widehat{\boldsymbol{\theta}}_{n-1/n-1}\right)\left(\boldsymbol{\theta}_{n-1} - \widehat{\boldsymbol{\theta}}_{n-1/n-1}\right)^H/\mathbf{r}_1,...,\mathbf{r}_{n-1}\right\}.$$

The linear relationship between $\Sigma_{n/n-1}$ and $\Sigma_{n-1/n-1}$ is a consequence of the linearized state equation (5.8).

On the other hand, $\mathbf{D}_r(\boldsymbol{\theta}) = E\{\mathbf{H}_n(\boldsymbol{\theta})\}$ was introduced in Chapter 4 as the matrix collecting the covariance matrix derivatives, i.e.,

$$[\mathbf{D}_r(\boldsymbol{\theta})]_p = \mathrm{vec}\left[\frac{\partial\mathbf{A}(\boldsymbol{\theta})}{\partial\theta_p}\mathbf{A}^H(\boldsymbol{\theta}) + \mathbf{A}(\boldsymbol{\theta})\frac{\partial\mathbf{A}^H(\boldsymbol{\theta})}{\partial\theta_p}\right].$$

Likewise, the innovations covariance matrix $\mathbf{Q}_{n/n-1}$ can be also computed from the last estimate $\widehat{\boldsymbol{\theta}}_{n-1/n-1}$ and the associated MSE matrix $\Sigma_{n-1/n-1}$. In the studied quadratic observation model, the deduction of $\mathbf{Q}_{n/n-1}$ results a little bit more involved because $\mathbf{H}_n(\boldsymbol{\theta})$ is random (noisy) and the measurement noise $\mathbf{v}_n\left(\widehat{\boldsymbol{\theta}}_{n/n-1}\right)$ depends on the parameterization. Ommiting the dependence on $\widehat{\boldsymbol{\theta}}_{n/n-1} = \mathbf{f}\left(\widehat{\boldsymbol{\theta}}_{n-1/n-1}\right)$ for the sake of clarity, it follows that

$$\mathcal{Q}_{n/n-1} \triangleq E\left\{\widetilde{\mathbf{r}}_n\widetilde{\mathbf{r}}_n^H/\mathbf{r}_1,...,\mathbf{r}_{n-1}\right\} = E\left\{\mathbf{H}_n\Sigma_{n/n-1}\mathbf{H}_n^H\right\} + E\left\{\mathbf{v}_n\mathbf{v}_n^H\right\}$$

Regarding now the second term, $E\{\mathbf{v}_n\mathbf{v}_n^H\}$ is the measurement noise covariance (Section 5.1). It is easy to realize that $E\{\mathbf{v}_n\mathbf{v}_n^H\}$ is the fourth-order matrix $\mathbf{Q}(\boldsymbol{\theta})$ introduced in (3.9) for $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}_{n/n-1}$. In that chapter, a closed-form expression was deduced for $\mathbf{Q}(\boldsymbol{\theta})$ in equation (3.10), obtaining

$$\mathbf{Q}(\boldsymbol{\theta}) = \mathbf{R}^*(\boldsymbol{\theta}) \otimes \mathbf{R}(\boldsymbol{\theta}) + \mathcal{A}(\boldsymbol{\theta})\mathbf{K}\mathcal{A}^H(\boldsymbol{\theta}) \tag{5.17}$$

[2] Due to the original nonlinear signal model, $\Sigma_{n-1/n-1}$ is not the tracker covariance matrix. However, following the original nomenclature in the Kalman filter theory, $\Sigma_{n-1/n-1}$ will be referred to as the MSE matrix.

where $\mathbf{R}(\boldsymbol{\theta}) = \mathbf{A}(\boldsymbol{\theta})\mathbf{A}^H(\boldsymbol{\theta}) + \mathbf{R}_w$, $\mathcal{A}(\boldsymbol{\theta}) = \mathbf{A}^*(\boldsymbol{\theta}) \otimes \mathbf{A}(\boldsymbol{\theta})$ and $\mathbf{K}$ is the so-called kurtosis matrix, that supplies all the non-Gaussian information about the nuisance parameters $\mathbf{x}_n$. Once more, $\mathbf{K}$ plays a prominent role in this chapter in case of non-Gaussian nuisance parameters.

Regarding now the first term of $\mathcal{Q}_{n/n-1}$, it follows that

$$E\left\{\mathbf{H}_n \Sigma_{n/n-1} \mathbf{H}_n^H\right\} = \sum_{p,q=1}^{P} \left[\Sigma_{n/n-1}\right]_{p,q} \mathcal{H}_{p,q}(\boldsymbol{\theta})\Bigg|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}_{n/n-1}}$$

where, after some tedious manipulations,

$$\begin{aligned}\mathcal{H}_{p,q}(\boldsymbol{\theta}) &\triangleq E\left\{[\mathbf{H}_n(\boldsymbol{\theta})]_p [\mathbf{H}_n(\boldsymbol{\theta})]_q^H\right\} \\ &= \left(\mathbf{A}^*(\boldsymbol{\theta}) \otimes [\mathbf{D}_r(\boldsymbol{\theta})]_p + [\mathbf{D}_r(\boldsymbol{\theta})]_p^* \otimes \mathbf{A}(\boldsymbol{\theta})\right) \mathcal{K} \left(\mathbf{A}^*(\boldsymbol{\theta}) \otimes [\mathbf{D}_r(\boldsymbol{\theta})]_q + [\mathbf{D}_r(\boldsymbol{\theta})]_q^* \otimes \mathbf{A}(\boldsymbol{\theta})\right)^H \\ &\quad + \mathbf{R}_w^* \otimes [\mathbf{D}_r(\boldsymbol{\theta})]_p [\mathbf{D}_r(\boldsymbol{\theta})]_q^H + \left([\mathbf{D}_r(\boldsymbol{\theta})]_p [\mathbf{D}_r(\boldsymbol{\theta})]_q^H\right)^* \otimes \mathbf{R}_w \end{aligned} \quad (5.18)$$

with

$$\mathcal{K} \triangleq E\left\{\operatorname{vec}\left[\mathbf{x}_n\mathbf{x}_n^H\right] \operatorname{vec}^H\left[\mathbf{x}_n\mathbf{x}_n^H\right]\right\} = \mathbf{I}_{K^2} + \operatorname{vec}(\mathbf{I}_K)\operatorname{vec}^H(\mathbf{I}_K) + \mathbf{K}$$

being $\mathbf{K}$ the nuisance parameters kurtosis matrix.

Therefore, it is found that the Kalman gains $\mathbf{M}_n$ can be computed from the previous estimate $\widehat{\boldsymbol{\theta}}_{n-1/n-1}$ and the associated covariance matrix $\Sigma_{n-1/n-1}$. In order to apply this recursion to $\mathbf{M}_{n+1}$ in the next time instant, it is necessary to evaluate the estimation MSE matrix at time n, which is given by

$$\Sigma_{n/n} \triangleq E\left\{\left(\boldsymbol{\theta}_n - \widehat{\boldsymbol{\theta}}_{n/n}\right)\left(\boldsymbol{\theta}_n - \widehat{\boldsymbol{\theta}}_{n/n}\right)^H / \mathbf{r}_1, ..., \mathbf{r}_n\right\} = \Sigma_{n/n-1} - \mathbf{M}_n^H \mathbf{S}_n$$

considering the QEKF solution in (5.15).

5.4.3 QEKF programming

In this section, the more important equations in the QEKF deduction are listed in order to facilitate its implementation in a hardware or software platform. Thus, assuming that $\widehat{\boldsymbol{\theta}}_{n-1/n-1}$ and $\Sigma_{n-1/n-1}$ were computed in the previous iterate, the following operations must be carried out when the new sample $\mathbf{y}_n$ is received:

1. **Prediction:**

$$\begin{aligned}\widehat{\boldsymbol{\theta}}_{n/n-1} &= \mathbf{f}\left(\widehat{\boldsymbol{\theta}}_{n-1/n-1}\right) \\ \widehat{\mathbf{r}}_{n/n-1} &= \mathbf{h}\left(\widehat{\boldsymbol{\theta}}_{n/n-1}\right) = \mathbf{h}\left(\mathbf{f}\left(\widehat{\boldsymbol{\theta}}_{n-1/n-1}\right)\right) \\ \Sigma_{n/n-1} &= \mathbf{F}\left(\widehat{\boldsymbol{\theta}}_{n-1/n-1}\right)\Sigma_{n-1/n-1}\mathbf{F}^H\left(\widehat{\boldsymbol{\theta}}_{n-1/n-1}\right) + \mathbf{R}_u\end{aligned}$$

2. **Kalman gain:**

$$\begin{aligned}\mathbf{M}_n &= \mathcal{Q}_n^{-1}\mathbf{S}_n \\ \mathbf{S}_n &= \mathbf{D}_r(\boldsymbol{\theta})\,\Sigma_{n/n-1}\big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}_{n/n-1}} \\ \mathcal{Q}_n &= \sum_{p,q=1}^{P}\left[\Sigma_{n/n-1}\right]_{p,q}\mathcal{H}_{p,q}(\boldsymbol{\theta}) + \mathbf{R}^*(\boldsymbol{\theta})\otimes\mathbf{R}(\boldsymbol{\theta}) + \mathcal{A}(\boldsymbol{\theta})\,\mathbf{K}\mathcal{A}^H(\boldsymbol{\theta})\Bigg|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}_{n/n-1}}\end{aligned}$$

where $\mathbf{M}_n$ is eventually a function of the signal model $\mathbf{A}(\boldsymbol{\theta})$ and its derivatives $\partial\mathbf{A}(\boldsymbol{\theta})/\theta_1,\ldots,\partial\mathbf{A}(\boldsymbol{\theta})/\theta_P$ –evaluated at $\widehat{\boldsymbol{\theta}}_{n/n-1}$– as well as the noise covariance $\mathbf{R}_w$ and the kurtosis matrix $\mathbf{K}$. The exact expressions of $\mathbf{S}_n$ and $\mathcal{Q}_n$ were deduced in Section 5.4.2.

3. **Estimation:**

$$\widehat{\boldsymbol{\theta}}_{n/n} = \widehat{\boldsymbol{\theta}}_{n/n-1} + \mathbf{M}_n^H\left(\mathbf{r}_n - \widehat{\mathbf{r}}_{n/n-1}\right)$$

with $\mathbf{r}_n = \operatorname{vec}\left[\mathbf{y}_n\mathbf{y}_n^H\right]$ the sample covariance matrix.

4. **MSE matrix update:**

$$\Sigma_{n/n} = \Sigma_{n/n-1} - \mathbf{M}_n^H\mathbf{S}_n = \Sigma_{n/n-1} - \mathbf{S}_n\mathcal{Q}_n^{-1}\mathbf{S}_n.$$

5.5 Simulations

Let us consider the problem of tracking the direction-of-arrival (DOA) of P mobile terminals transmitting toward a uniform linear array composed of $M > P$ antennas spaced $\lambda/2$ meters, with λ the wavelength of the received signals. The received signal is passed throught the matched-filter and then sampled at one sample per symbol. We will consider independent snapshots assuming that the actual modulation is ISI-free and the P signals are perfectly synchronized. Assuming for simplicity that all the users are received with the same power, the observed signal verifies the linear signal model in equation (5.1) with

$$\begin{aligned}\mathbf{A}(\boldsymbol{\theta}_n) &\triangleq \exp\left[j\pi\mathbf{d}_M\boldsymbol{\theta}_n^T\right] \\ \mathbf{d}_M &= [0,\ldots,M-1]^T\end{aligned}$$

being $\mathbf{x}_n$ the transmitted symbols for the P users and $\mathbf{w}_n$ the vector of AWGN samples with $E\left\{\mathbf{w}_n\mathbf{w}_n^H\right\} = \sigma_w^2\mathbf{I}_M$. Therefore, the SNR (per user) is given by σ_w^{-2} bearing in mind that $E\left\{\mathbf{x}_n\mathbf{x}_n^H\right\} \triangleq \mathbf{I}_K$ with $K = P$ in this case.

Several illustrative simulations have been carried out to evaluate the performance of the QEKF (5.15) when the transmitted signal is digitally modulated. The optimum QEKF is compared with the one based on the Gaussian assumption that is obtained imposing $\mathbf{K} = \mathbf{0}$ into

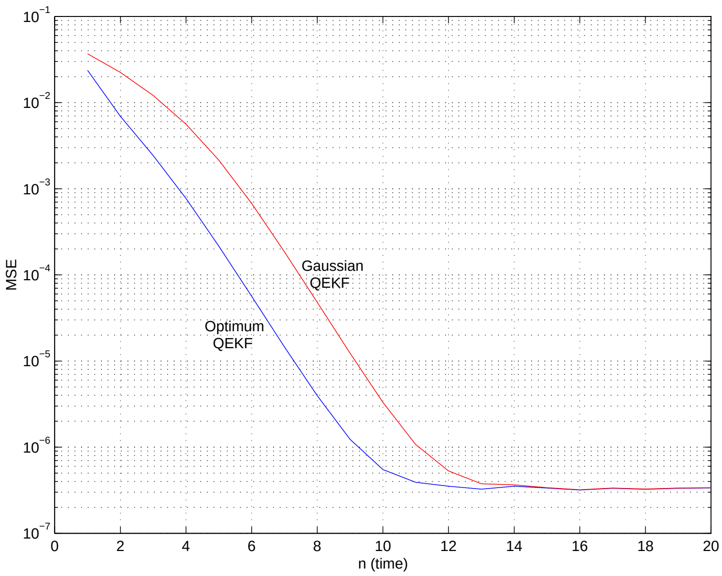

Figure 5.1: Estimation MSE as a function of time for the optimum and Gaussian QEKF in the case of a single static MPSK-modulated user with random DOA in the range ±0.4 and SNR=40dB.

(5.17) and (5.18). This suboptimal QEKF will be referred to as the Gaussian QEKF in the sequel. The normalized mean square error (MSE) is adopted as the figure of merit, that at time n reads

$$MSE\left(n\right) \triangleq \frac{1}{P}E\left\{\left\|\boldsymbol{\theta}_n - \widehat{\boldsymbol{\theta}}_{n/n}\right\|^2\right\}.$$

- Simulation 1: in Fig. (5.1), a single user ($P = 1$) transmitting from a *static* DOA is simulated. The transmitted symbols are drawn from a phase shift keying (MPSK) constellation. The basestation array is composed of $M = 4$ antennas and the SNR per user is set to 40 dB. This very high SNR scenario is studied in order to analyze how the trackers cope with the random nuisance parameters, i.e., the so-called self-noise.

The estimator is initialized at $\widehat{\boldsymbol{\theta}}_{0/0} = 0$ with $\Sigma_{0/0} = 1000$. Then, 1000 realizations are run with $\boldsymbol{\theta}$ uniformly distributed within $(-0.4, 0.4)$. The parameter range is limited in this interval because the tracker acquisition margin is limited to $\pm 2/M = \pm 0.5$. In general, the QEKF solution is unique, whatever the initial start-up, if and only if $M = P + 1$. When $M > P + 1$ the array directivity is augmented but new sidelobes appear in the array beam pattern yielding spurious solutions.

Figure 5.1 depicts the estimated $MSE(n)$ for the optimum and Gaussian QEKF. The state equation noise $\mathbf{R}_u$ (5.2) is set up to attain the same steady-state variance in both cases. It

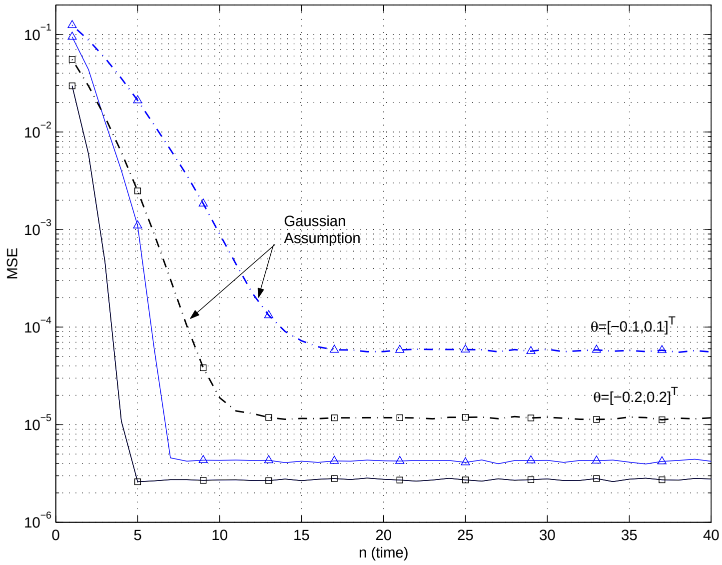

Figure 5.2: Estimation MSE as a function of time for the optimum and Gaussian QEKF in the case of two static MPSK-modulated users placed at ± 0.2, ± 0.1 and SNR=40dB.

becomes apparent that the acquisition time is reduced if the QEKF exploits the digital structure of the received signal by incorporating the kurtosis matrix $\mathbf{K}$. Alternatively, this improvement could be used to reduce the QEKF steady-state variance if the optimum and Gaussian QEKF trackers were adjusted to yield the same acquisition time.

- Simulation 2: in this simulation, we have $P = 2$ users transmitting from $\boldsymbol{\theta} = [-0.1, 0.1]^T$ or $\boldsymbol{\theta} = [-0.2, 0.2]^T$. The array size is $M = 4$ and the SNR per user is again 40 dB. The QEKF trackers are initialized at $\widehat{\boldsymbol{\theta}}_{0/0} = [-0.5, 0.5]^T$ with $\Sigma_{0/0} = 1000\mathbf{I}_P$ and $\mathbf{R}_u = 10^{-3}\mathbf{I}_P$. The resulting $MSE(n)$ is plotted in Fig. 5.2 for the optimum and Gaussian QEKF. Once more, the fourth-order information about the discrete symbols is shown to improve the QEKF performance in both the acquisition and steady-state regimes. As shown in figure 5.2, the closer are the two sources the higher is this improvement. Further simulations showed that the simulated Gaussian QEKF is unable to acquire the actual DOAs in some cases, e.g., $\boldsymbol{\theta} = [0.2, 0.4]^T$, whereas the optimum QEKF converges eventually to the true DOAs.

- Simulation 3: in this simulation, the state equation noise is removed ($\mathbf{R}_u \to 0$) in order to evaluate the estimator consistency when $n \to \infty$. First of all, the DOAs are acquired ($n < 0$) with all the QEKFs adjusted to yield the *same* steady-state variance ($0 < n < 20$). From this steady-state situation, $\mathbf{R}_u$ is set to zero at $n = 20$ so that the QEKF (noise equivalent) bandwidth is progressively reduced. At this moment, the optimum QEKF becomes an *order-recursive* implementation of the second-order tracker in Section 4.2 as the observation time goes

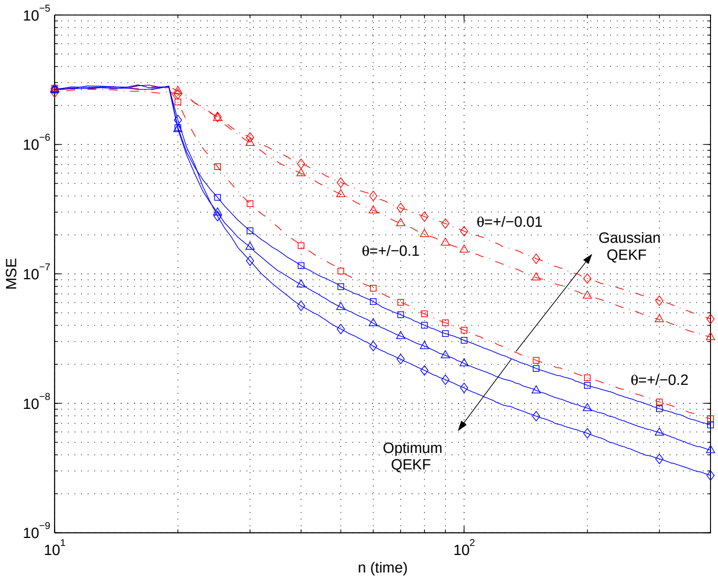

Figure 5.3: Estimation MSE as a function of time for the optimum and Gaussian QEKF (dashed) in the case of two static MPSK-modulated sources transmitting from ±0.2 (□), ±0.1 (△) and, ±0.01 (◇) when $\mathbf{R}_u$ is set to zero at n=20. SNR=40dB.

to infinity. Likewise, the Gaussian QEKF implements the well-known GML estimator explained in Section 2.4.3. These statements are based on the technical discussion in Section 5.4.1.

In Fig. 5.3, numerical results are provided for two quiet users at $\boldsymbol{\theta} = \pm 0.01$, ± 0.1 or ± 0.2 with $M = 4$ and SNR=40dB. We observe that the Gaussian assumption suffers a constant penalty as n is augmented. Consequently, the GML estimator is proved to be suboptimal at high SNR when the modulation has constant envelope (e.g., MPSK or CPM [Pro95]), even if the observation is arbitrarily large ($n \to \infty$). This result is further validated by means of the asymptotic study in Section 7.4.5.

Finally, notice that the incurred loss is a function of the users angular separation. Surprisingly, the variance of the optimal QEKF is improved as the user are closer. This abnormal result is a consequence of the secondary lobes of the array response when the number of antennas is small. The same effect will be observed in Section 7.5 when studying the asymptotic performance of the optimal small-error DOA tracker.

- Simulation 4: in order to validate that the simulated QEKFs are tracking the actual DOAs, the users are moved with constant angular speed from -0.8 to 0.8 with fixed angular separation (0.02). 50 trials are plot in figure 5.4 showing that the Gaussian QEKF fails in tracking the two users.

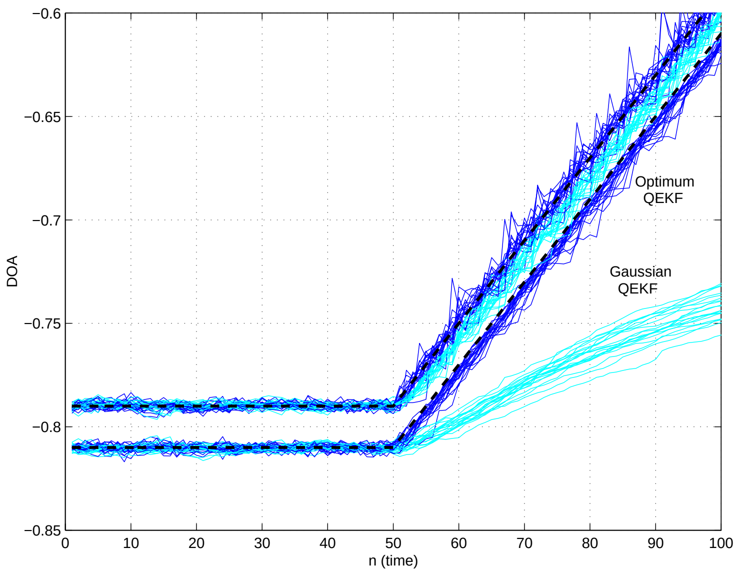

Figure 5.4: DOA Tracking of two close MPSK-modulated signals separated 0.02 using the optimum and Gaussian QEKF. SNR=40dB.

- Simulation 5: the same simulation in Fig. 5.3 has been carried out for a low signal-to-noise ratio (SNR=10 dB) and a multilevel modulation such as 16-QAM [Pro95]. Figures 5.5 and 5.6 manifest the optimality of the Gaussian assumption when multilevel constellations or low SNRs are considered, respectively.

5.6 Conclusions

The EKF formulation has been extended to deal with quadratic signal models that appear naturally in blind estimation problems. The resulting Quadratic EKF (QEKF) is found to exploit the fourth-order cumulants (kurtosis) of the unknown inputs whereas this information is implicitly omitted when the classical Gaussian assumption is adopted in the design. The QEKF is further applied to estimate and track the DOA of multiple *digitally-modulated* sources concluding that constant amplitude modulations (e.g., MPSK or CPM) yield a significant improvement in terms of acquisition and/or steady-state variance for moderate-to-high SNRs. In these scenarios, the Gaussian assumption is found to provide *suboptimal* DOA estimators or trackers even if the tracker bandwidth is indefinitely reduced or, in other words, the (effective) observation time is increased without limit.

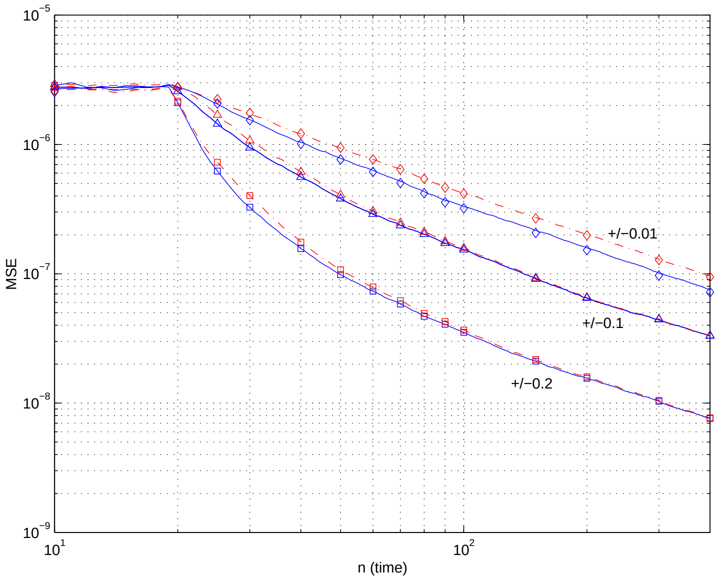

Figure 5.5: Estimation MSE as a function of time for the optimum and Gaussian QEKF (dashed) in the case of two 16-QAM modulated signals received from ±0.2 (□), ±0.1 (△) and ±0.01 (◇) when R_u is set to zero at time n=20. SNR=40dB.

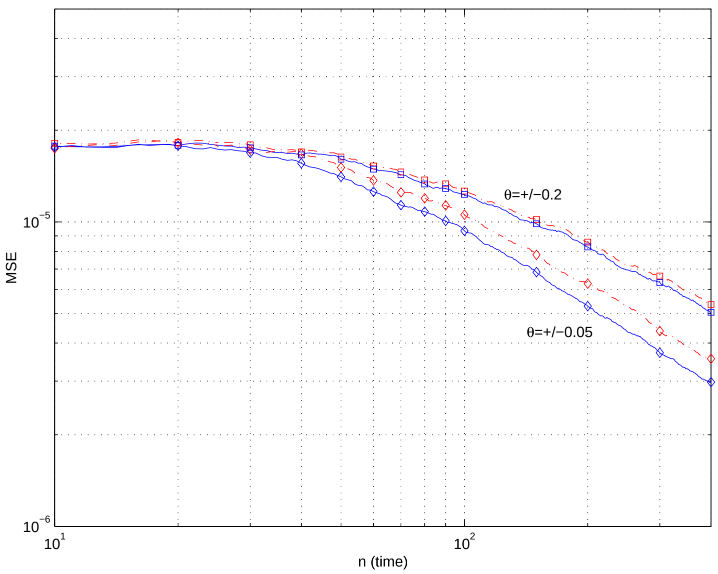

Figure 5.6: Estimation MSE as a function of time for the optimum and Gaussian QEKF (dashed) in the case of two MPSK-modulated signals received from ±0.2 (□) and ±0.05 (◇) when R_u is set to zero at time n=20 and the SNR is equal to 10dB.