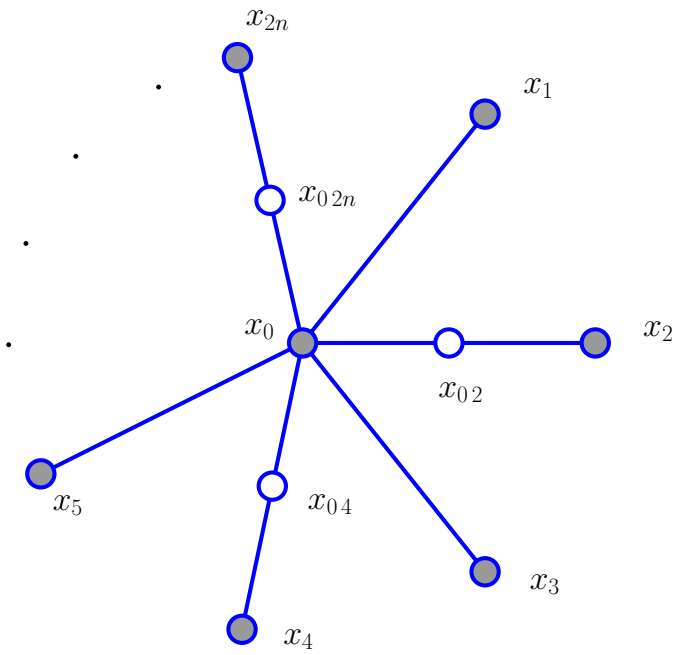

Discrete operators and distances on subdivision networks

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Universitat Politècnica de Catalunya, 2022

Thesis Advisors
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Agraïments

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i s'agraeix. I finalment, per ordre cronològic, l'Àlvar, a qui potser sabré retornar una mica d'allò que els altres tan sàviament m'han fet arribar.

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1
Abstract

The aim of the work we have developed is to contribute to the understanding of discrete elliptic boundary value problems on finite networks from an electrical point of view as we are concerned with a particular operation on networks that has a physical meaning in circuit theory.

Elliptic problems in the continuous field are a very well known item in physical mathematics. It is a very important tool in the development of so many situations of real interest and many efforts have been devoted to it from a long time ago. Even though our work is centered in the discrete field, we use notations that are inspired in the continuous setting for elliptic operators because there is a strong symbiosis between both fields. Indeed, sometimes solving a problem in the discrete setting can lead to the solution of its continuum version by a limit process, but sometimes the relation between these two worlds does not work so easily.

It is very clear that an electrical network may be viewed as a graph, and conversely, that every graph can be considered as a model of an electrical circuit, after considering that electrical components are identified with vertices and the interconnections between electrical components are described as edges. In other words, electrical circuits are naturally treated as graphs. When some item is connected to a circuit, an straightforward interpretation is to add one or various vertices to the former graph and/or consider new information defined on the edges of the graph. We are interested not only in graphs, discrete structures where vertices and nodes are considered all equal (only the number of incident edges to an specific vertex make a difference between different vertices), but in networks that are discrete structures where edges are equipped with a conductance (a positive value) that discriminates edges from other edges and is also possible to consider a value for every vertex that differentiate vertices among them (even further than for the adjacencies).

An elementary subdivision of a nonempty network is not electrically compatible as, when inserting a vertex in an edge, does not satisfy the total resistance series connection electrical law. We are interested in the subdivision

procedure on networks so as to model the physical situation of connecting items (to all edges or only to some of the edges) to an electrical network. These connections have to be electrically compatible and the concept of conductance associated with every edge has to satisfy the requirement of

$$\frac{1}{c(x,y)} = \frac{1}{c(x,v_{xy})} + \frac{1}{c(v_{xy},y)}$$

where x, y are vertices on the network, v_{xy} is the new vertex inserted in the edge $\{x, y\}$ after the elementary subdivision procedure, $\{x, v_{xy}\}$ and $\{v_{xy}, y\}$ are the resulting two new edges and provided that a conductance is the inverse value of a resistance. In this way the very well known physical law for electrical resistance in a series connection is satisfied.

All along our work we have considered global subdivision procedures first and a partial subdivision procedure of only some edges later, as this case is a more general one. We will continue our future work with an even more general subdivision operation where just some edges should be replaced by different length open paths.

Thus our task has consisted in stating the precise connection between solutions of elliptic problems on these related discrete structures, a given initial network and another one resulting of a subdivision procedure of the former.

Moreover, and again from the existing relationship between discrete structures and electrical circuits, a novel distance concept was introduced by exploiting the idea that, given an edge and its two adjacent vertices, the more the resistance value of and edge is, the further both adjacent vertices are. This so called resistance distance is proved to be thinner than the canonical shortest–path distance that is usually considered in graph theory but, what is more important to us, is suitable to treat diffusion problems on discrete structures as in most cases spreads try every single possibility of propagation at their hand and this distance is defined between two different vertices taking into account all possible paths that join them. Furthermore, very important topological information of the structure we are interested in is easily obtainable upon this resistance distance concept.

More specifically, we are focussed in determining the existing relationship between resistance distances and also between Kirchhoff indexes of these discrete structures, relating those parameters of the subdivided structures with their respective of the given initial structure.

In order to compute resistance distances, we take advantage of the so called Green's kernel of an elliptic operator. First for the Laplacian operator, then for Schödinger type operators, we have studied how to relate Green's kernel function for the subdivided network in terms of Green's kernel of the former discrete structure. As there exists a one-to-one identification between electrical circuits (graphs, networks) and M–matrices, and given that Green's

kernel of an elliptic operator is also identified with the Moore–Penrose inverse matrix, our problem can also be seen as a contribution to the computation of higher dimensional Moore–Penrose inverse matrices in terms of given lower order Moore–Penrose inverse matrices in a sort of Woodbury–Shermann–Morrison formula, a well known result but for singular matrices.

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Introduction

Despite the close relationship between graph theory and electrical circuit theory, with the well known analogy of vertices and edges standing for connectors and connections and despite the obvious equivalence between the elementary subdivision of an edge of a graph and the representation of a series connection of two elements in a given electrical circuit, when searching at Google for instance for any combination of the words "electric", "circuit", "graph" and "subdivision" it turns a very difficult task to find any reference to what has been the object of study of this thesis. Not even our published works are easy to find, and they are in the world wide web for sure!

Considering mathematical literature, we are aware of many papers where graph operations are considered, but these operations are generally other graph operations as union or intersection of two or more graphs, the join of two graphs, the cartesian product of graphs, the line graph of a graph, series or parallel graphs compositions, corona product of graphs, rooted product of graphs, ... The subdivision procedure is seldomly treated. We certainly have the impression that the well known and elementary operation of subdividing edges on graphs is devoted only to define homomorphism between graphs. It is definitely a no small matter and especially taking into account its relation to planarity of structures after Kuratowski's theorem.

However, papers devoted to the study of different parameters of the subdivision of a graph do exist. For instance, Chen in [31], obtained a formula for the effective resistances of the subdivision graph in terms of the effective resistances of the original graph by using some nice sum rules. The Kirchhoff index of the subdivision graph is considered in different works under several hypothesis such as regular graphs in [49], bipartite graphs in [71], or operations between graphs that involve the subdivision concept as well, see [22] for instance. In [76], the author extends the previous results to general graphs and computes the Kirchhoff index of subdivision graph in terms of the Kirchhoff index, the multiplicative degree–Kirchhoff index, the additive degree–Kirchhoff index, the number of vertices, and the number of edges of

the graph. Also in [71] an expression for the Kirchhoff index in terms of a {1}–inverse of the combinatorial Laplacian is given.

Also, subdivision graphs and their spectra seem to be particularly important in the study of thermodynamic properties of crystalline solids. This practical problem led B.E. Eichinger and J.E. Martin in [41], to devise an algorithm for computing the Laplacian eigenvalues of a subdivided graph by applying numerical linear algebraic methods to the matrix of the unsubdivided graph.

But none of these works seem to realize the existence of the obvious coincidence we have mention in the initial paragraph of this introduction. As far as we know nobody has used graph subdivision to model series connection in an electrical circuit.

On the other hand complex systems are pervasive in our society. Everybody knows so many examples that support the previous sentence (the Internet System, the World Wide Web System, the electrical power system, any biological system, ...).

Possibly there are three main aspects to have in mind to study complex systems: the nature of the individual components of the systems, the nature of connections or interactions and the pattern of connections between components. And it is at these points where graphs should be equipped, for well modelling really interesting situations, with alternatives that allow the discrimination between vertices (per se and not just because every vertex has its own quantity of adjacent edges), and between edges (as different links may have different behaviours). Then a third actor has a role in the play, as networks should be a too simple tool to face the study of the dynamical behavior of an aggregation of nodes and links. In our vocabulary it is called Schrödinger type operators as a potential function is defined over the vertices of the structure, so as the particular value it attains at every single node may model a specific behaviour.

The nodes, for vertices, might be molecules or genes for biological systems, humans for social systems, routers or switches for communication systems, transformers for electrical systems. The links, for edges, might be contagions or synapses for biological system, friendships or other relationships for social systems, physical wires or wireless for communication systems, connections for electrical systems, etc. And the behaviour of whatever is defined on the structure (not the structure itself) is the third component to take into account in a serious analysis of a real problem.

Is in this direction, over the years, that scientists have developed a huge set of mathematical, computational, and statistical tools for analyzing, modeling and understanding networks. These tools work with networks in their abstract form and help in finding some crucial and useful information about them, for example, the critical node or edge in a network, the length of a

path from one node to another in a network, the flow of traffic over the network, clusters or communities in a network, etc. These tools can be applied to any systems that can be represented as networks. And we honestly think that the techniques of discrete potential theory applied in this setting, mimicking the theory of partial differential equations in the continuous case, can be considered as a modest contribution to this amount of knowledge. And our work in this thesis follows this line, as we have been concerned with the study of Poisson problems on discrete structures having in mind diffusion contexts as the spread of an epidemic, heat transfer or whatever that flows throughout the set of vertices and edges is distributed.

Some authors, for instance [44], support the idea that it is more important to understand the organizational principles of such systems on the basis of their connectivity than to understand the role of their individual components. And it is in this sense that discrete Laplacian–like operators appear in the mathematical description of the majority of dynamical processes occurring on these systems, becoming more and more popular from time to time but unfortunately at still low levels.

A third very profitable idea that we would like to expose at this particular moment is that an efficient way to get valuable information from a graph or a network is to associate the discrete structure with matrices. Then some characteristics of these matrices have a direct translation in terms of relevant information concerned with the discrete structure. When finite graphs are considered, these matrices can be related to kernels of linear operators that are defined on the vertex set, so as acting on the discrete structure. For instance, a Laplacian matrix associated with the Laplacian operator can be seen as an object that acts distributing whatever is considered throughout the network, in a so called diffusion problem.

There are quite a few very well known matrices commonly tied with graphs and networks, whose consideration has returned in a profit on the knowledge of discrete structures. The probably most basic one is the so called adjacency matrix, from what important properties of a graph can be revealed such as the order of the structure, the count of paths of a precise length in the graph, the number of clicks that exist or whether the graph is bipartite or not. Also the Laplacian matrix is widely used for counting the number of spanning trees, studying connectivity, counting the number of connected components and more. The normalized Laplacian matrix, see [33] is also a very well known matrix that has been shown to be adequate enough to analyze from a probabilistic point of view a discrete structure. The normalized Laplacian discriminates vertices not only by the connections established with other vertices, but also considering the different behaviour that the corresponding edges can have. Nevertheless it is not able to capture an intrinsic discrimination of the vertices between them beyond for connections.

A fourth argument, to include and briefly discuss in this not at all short introduction, is a sort of geometrical and topological one. For decades to know what a graph looks like has been a main objective. So, new tools have been developed so as to understand networks from some other points of view. In this sense, Klein and Randić in [58] but also Stephenson and Zellen in [70] introduced both, independently, with no apparent connection and from two very separate frameworks, a new distance function that we call resistance distance. The electrical version of it, based on electric network theory, is defined to be the effective electrical resistance between two vertices when each edge is replaced by a unit resistor. But the social networks version of the same distance, see [18, 37] where the equivalence is stated, insists in the interesting and profitable idea that distances in a discrete structure that are based upon consideration of all possible paths between two given vertices do worth a value. From this point of view, vertices in the discrete structure are closer (this distance is thinner than the canonical shortest–path distance) but, what is more important to us, the resistance distance reacts to changes in the overall structure, whether they are due to erased/incorporated edges, or a variation of the modelled behaviour. So, this new exciting context fits extremely well with the initial idea of our work.

Moreover, once a new distance is considered, new topological invariants can be defined and studied. And paradoxically some valuable ideas initially developed in a Chemistry theory framework, as for instance the Wiener index, can be reproduced in some other very different frameworks as electrical circuits, thermodynamics, random walks, general science networks or others. Thus, based upon this resistance distance, the Kirchhoff index, that can be defined as the sum of resistance distances between all pairs of vertices of the network, is a classifier which is worth computing and studying.

Not surprisingly, the different considerations that have been presented in this introduction of course can be intimately related. For example the calculation of resistance distances, and therefore of Kirchhoff indexes, is related to the Moore–Penrose inverse matrix of the Laplacian matrix associated with a network where a diffusion problem is posed. And in this context is where our work has to be interpreted and comprehended.

Moreover, despite the great interest generated by resistance distances and Kirchhoff index, and the obvious importance of diffusion problems modelled by operators as those we consider and the subdivision procedure, we have taken no profit at all of the published literature in relation with these, our concerns. Certainly, there are some works that point in this direction, in [22] the authors investigated resistance distance in subdivision–vertex join and subdivision–edge join of graphs, also in [61] the resistance distance and Kirchhoff index of R–vertex join and R–edge join of two graphs are given, in [78] the Kirchhoff index of some composite operations between two graphs such as product, lexicographic product, join, corona and cluster is considered.

For sure that in [49] the Kirchhoff index for graphs derived from a single graph is studied. But all literature, as far as we know, is only concerned with graphs. So, the opportunity of considering really interesting discrete structures is missed. Or to put in other words, the knowledge more or less related to a subdivision procedure and resistance distances is devoted just to the simple graph setting, with no possibility of satisfying the electrical compatibility condition that we are interested in.

Therefore, we have faced a completely exciting new paradigm: we have studied different diffusion problems (for different difference operators), resistance distances and Kirchhoff index in a setting of more sophisticated discrete structures and by applying the subdivision operation but with an electrical compatibility condition to fulfill.

Once we have explained the motivations and introduced our work, we point out that we have four research papers that are already published. One of these in an online journal (ENDM) and the other three on paper support classical very reputated journals in the scope of our work (LAMA and AAMD), both usually in the highest Q1 range. Next we give their references in chronological order of appearance and a very short explanation of the realized work.

1. Á. Carmona, M. Mitjana and E. Monsó.

   The group inverse of subdivision networks

   Electronic notes in discrete mathematics, **54**: 295–300, 2016.

   url = http://hdl.handle.net/2117/101529

   doi = 10.1016/j.endm.2016.09.051

   This first work to appear, in an online publication, is related to what we expose in this memory as our second case, since the normalized Laplacian operator on a subdivision network was treated there. It is a work written in a matrix scenario rather than our prototypical functional framework.

2. Á. Carmona, M. Mitjana and E. Monsó.

   Effective resistances and Kirchhoff index in subdivision networks.

   Linear and Multilinear Algebra, **65**: 1823–1837, 2017.

   https://doi.org/10.1080/03081087.2016.1256945

   In second place we obtained the publication of our first work, where we studied total electrical compatible subdivision of networks and the Laplacian operator was considered. The definition of resistance distance is the classical one, as in [58].

3. Á. Carmona, M. Mitjana and E. Monsó.

   Green's function in partial subdivision networks.

   Linear and Multilinear Algebra, **68**: 94–112, 2020.

   https://doi.org/10.1080/03081087.2016.1256945

   Our third published work corresponds with the third problem we have addressed. In this work the structure is the toughtest, with a potential value considered on every vertex, thus the operator is a positive semi–definite Schrödinger type operator and a partial electrical subdivision (only in some edges) is applied. Of course, the generalized resistance distance is the one that fits in this environment.

4. Á. Carmona, M. Mitjana and E. Monsó.

   Group inverse matrix of the normalized Laplacian on subdivision networks.

   Applicable analysis and discrete mathematics, **14**: 272–286, 2020.

   doi = 10.2298/AADM180420023C

   Finally, our fourth published work is devoted to the functional expression of the results of our second problem, where a simple electrically compatible subdivision of a network and the normalized Laplacian operator are considered.

To end this chapter, this thesis memory is organized as follows. Chapter 3 gives an overview of Discrete Potential Theory, the mathematical framework we have worked in. Even though inspired in the works of Choquet and Deny about two centuries ago for the continuous field, our references here are from the colleagues of the MAPTHE research group, to which we proudly belong, [7, 8, 11, 12]. A brief introduction to the discrete counterpart of elliptic operators is presented: Laplacian and Schrödinger type. We justify the names. Moreover we prove the existence of their generalized inverse operators, called Green's operator in every case. Also their associated kernels are presented.

Chapter 4 is the core of our work. There the different cases that have produced our four now published works are treated. Case by case, the electrically compatible subdivision procedure is defined and the corresponding Poisson problem on the subdivided network is solved. This is done by taking advantage of a particular solution of a related Poisson problem posed on a given initial network, to which the subdivision operation is applied. The different cases of the three well known difference operators, combinatorial Laplacian, normalized Laplacian and Schrödinger type operators, are considered and the correspondent Green's functions are obtained. We give a matrix interpretation of our results, as Green kernels can be identified with Moore–Penrose inverse matrices (or group inverse matrices) of the respective symmetric matrices associated with the mentioned operators.

In Chapter 5 the definitions of the resistance distance and the Kirchhoff index are provided. The ideas behind the definitions are explained and their computation is adressed. In the case of networks, they are a quite straightforward generalization of the respective concepts well established on graphs in the seminal paper [58]. But in the case of the Schrödinger operator, the richness of the structure demands a brand new definition provided in [12]. Then, the computation of resistance distances and Kirchhoff indices for subdivision networks are obtained in the three reiteratively referred cases.

The three last chapters contained in this memory are devoted to the presentation of the application of our results to some simple scenarios, to establish the future work in which we are determined to devote our efforts, and to explain some conclusions that we have obtained from our work during this time, respectively. Thus, in Chapter 6 simple networks as Star networks and Wheel networks are taken into account, and examples of the obtained results are provided so as the reader can get a better idea of the work we have done. Then, in Chapter 7, some more works that we are to be submited soon, open problems and further works that are still in mind are described at the last chapter. To end with, in Chapter 8 we have listed some interesting ideas, remarks and conclusions that we have obtained from the work done.

The Bibliography used in our research is listed in the very last pages of this work, too.

3

Discrete Potential Theory

Mathematical community has been for a long time interested in finding out explicit expressions for solutions of partial differential equations. Many efforts have been devoted to their resolution from a long time ago, many hurdles encountered have been fortunately overcame, but many others remain unfortunately unsolved. Computing solutions of such partial differential equations is a challenging task, even in numerical mathematics, mainly because of the physical domain where they are established.

As many situations of real interest are treated, modelled and solved, considering elliptic partial differential equations in the continuous field, elliptic problems play a central role in mathematical physics. Two very well known examples are the Poisson equation

$$\Delta u = f \quad \text{on } \mathcal{D}$$

where $\mathcal{D} \subset \mathbb{R}^s$ is a given domain and $f \in \mathcal{C}(\mathcal{D})$ is a given data function, and also the eigenvalue problem or Schrödinger's equation

$$-\Delta u = \lambda u \text{ on } \mathcal{D},$$

with $\lambda \in \mathbb{R}$. In both cases Δ denotes the classical Laplace operator given by

$$\Delta u = \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} + \cdots + \frac{\partial^2 u}{\partial {x_s}^2}.$$

The principal aim of the work we have developed is to contribute to the understanding of discrete elliptic problems on finite networks, where a particular operation that has a circuit theory inspiration has been applied. For this purpose, we have taken advantage of the so called discrete potential theory techniques.

Thus, even though our work is centered in the discrete field, we use notations that are inspired in the continuous setting for elliptic operators because

there is a strong symbiosis between both frameworks, see [47, 55] as an example. Indeed, sometimes solving a problem in the discrete setting can lead to the solution of its continuum version by a limit process. On the contrary, sometimes the relation between these two worlds does not work so easily.

In this chapter, our goal is to introduce the terminology and results on discrete vector calculus on finite networks that we have used in this work. We define what a network is, the functional notation we use, and the geometric concepts (up to tangent space and vector fields) to justify basic difference operators that mimic the usual differential operators in the continuous case, in particular gradient and divergence. Thus, we explain that these discrete operators satisfy analogue properties to those fulfilled by self–adjoint second order elliptic continuous operators. In particular, they have associated quadratic forms and resolvent or Green kernel, too.

The results we are about to show have been obtained by some members of our research group in [6, 7, 8, 9, 11, 23]. Particularly, the concept of tangent space is an identifying characteristic of our group. Other authors have also treated some of the topics presented in this chapter, see for instance [16, 19, 34, 36, 59].

3.1 Preliminaries

In this manuscript, we prefer to use a *functional* notation that emphasizes similarities between the situation of discrete structures (graphs and networks) and the continuous case (manifolds). We will consider real–valued functions defined on the vertex set of the graph, that can be identified so handled as real finite dimensional vectors, and real–valued functions defined on the edges set of the discrete structure as well. Moreover, we will also use the word Laplacian, as our principal discrete operator can be viewed as a proper discretization of the usual Laplace–Beltrami differential operator. Finally, operators between functional spaces will have their corresponding, kernels and inverses. Finally, a matrix version is also at disposal after giving a labelling on the network vertex set.

3.1.1 Graphs and networks

A *graph* $\Gamma = (V, E)$ consists of a finite nonempty set $V = \{x, y, z, \ldots\}$ of *vertices* (or nodes), and a second set E of *edges* (or links) that are conceived as relationships between vertices. Thus $E \subseteq V \times V$ so that $\{x, y\} \in E$ if and only if vertices x and y are to be considered as linked in some sense. Therefore, $(y, x) \in E$ too. The number of vertices, $|V|$, is known as the *order* of the graph while the number of edges, $|E|$, is referred as the *size* of Γ.

Given $x, y \in V$ two different vertices in a graph, they are told to be *adjacent* (or neighbours) if and only if there exists an edge relating them. In this case, we write $x \sim y$ and the edge $\{x, y\} \in E$ is said to be *incident* on both x and y. The set of adjacent vertices to a given vertex x is $N(x) = \{y \in V : y \sim x\}$ and the *degree* of a vertex is defined as the cardinality of $N(x)$, that is $\deg(x) = |N(x)|$. When a couple of vertices in V, say x and y, be connected by a sequence of $\ell + 1$ different vertices $\{x = x_1, x_2, \cdots, x_\ell, x_{\ell+1} = y\} \subseteq V$ such that $x_i \sim x_{i+1}$ for all $i = 1, \ldots, \ell$, then we say that there is a *path* between x and y and we write P_{xy} when referring to the set of all $\ell + 1$ vertices together with the ℓ edges $\{x_i, x_{i+1}\}$ for all $i = 1, \ldots, \ell$. We also say that $\ell(P_{xy}) = \ell$ is the *length* of this path P_{xy}.

Graphs can be sketched in two dimensional representations by drawing points for vertices, and segments for the edges that join corresponding neighbours. See [20, 29] for basic concept on graph theory.

In graph theory, vertices are all considered identical in nature, whatever they represent, as if they behave all exactly in the same way, with no established differences among them. Hence, no different roles can be assumed, except by the number of their connections to other vertices, as they may have different degree.

Similarly, edges are considered solely as connections. They are entities that just establish (or not) a relationship between vertices. There is no possibility of discrimination among them, as if all links were equal, and the sole question that can be treated is whether they exist or not.

When a positive value is assigned to every edge of a given graph Γ introducing the possibility of differentiating between connections, allowing the possibility of modelling links between vertices differently, the discrete structure is called network. Thus, a *network* $\Gamma = (V, E, c)$ is a graph (V, E) endowed with a nonnegative function $c : V \times V \to [0, +\infty)$, such that $c(x, y) > 0$ for every pair x, y when $\{x, y\} \in E$ and, therefore, $c(x, y) = 0$ when $\{x, y\} \notin E$. We call this function *conductance*, and consider the value $c(x, y)$ as a weight assigned to the corresponding edge $\{x, y\}$ whenever it exists. Moreover, this conductance function is symmetric as $c(x, y) = c(y, x)$ because $\{x, y\}$ and $\{y, x\}$ are considered as the same edge. Then, the *generalized degree* of a vertex $x \in V$ is $\kappa(x) = \sum_{y \sim x} c(x, y)$, and the reciprocal function r defined as $r(x, y) = 1/c(x, y)$ for $x \sim y$, is called *resistance* function.

As the particular case of $c(x, y)$ being one on every edge where is non–null puts us back in the case of a simple graph, where no differences between edges can be considered, we will also use Γ to denote a graph.

In this work we will only consider simple networks; that is, with no loops, *i.e.* no edges that link a vertex with itself will be considered, nor multiples edges, so there will exist at most one connection between every possible pair

of vertices. Therefore, the definition of conductance is consistent.

3.1.2 Connectness, geodesic distance

Related to paths (both in graphs but also in networks) there are two concepts that deserve a mention in our work: connectness and distance.

We say that Γ is *connected* if any two vertices x and y of V can be joined by a path, P_{xy}. By abuse of notation, it is said that V is connected. Furthermore, given a vertex subset $F \subseteq V$ we say that F is connected if each pair of vertices of F is joined by a path, entirely contained in F; that is, the vertices that compound such a path are all vertices in the subset F. We will consider only connected networks.

As the number of edges in a path P_{xy} connecting two vertices x and y is said to be the *length* of this path, $\ell(P_{xy})$, a *geodesic distance* function can be defined on a graph relating every pair x and y of two vertices of V to the minimum length among all paths that join x and y; that is, $d\colon V \times V \to \mathbb{N}$ such that $d(x,y) = \min\limits_{P_{xy}} \ell(P_{xy})$ when paths joining x and y do exist[1].

It is well known that this distance function satisfies the properties of a distance, so $d(x,y) = 0$ if and only if $x = y$, $d(x,y) = d(y,x)$, for all $x, y \in V$ and $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in V$.

The corresponding extension of former shortest–path distance to networks is easily acomplished by defining adequately the length of a path. Given $P_{xy} = \{x = x_1 \sim x_2 \sim \cdots \sim x_k = y\}$ a path joining vertices x and y, it is convenient to consider that the length of the path P_{xy} is $\ell_c(P_{xy}) = \displaystyle\sum_{i=1}^{k-1} \frac{1}{c(x_i, x_{i+1})}$. In such a way, we relate the length of an edge with the electrical resistance of it and consider then the sum of these distances as they are connected in series. Thus, on networks, the *geodesic distance* between two vertices x and y is defined as the least resistive path; that is,

$$d_c(x,y) = \min \{\ell_c(P_{xy}) : P_{xy} \text{ is a path between } x \text{ and } y\}.$$

The function d_c determines a distance on the network as it fulfills the three required properties for a function to be a distance. In particular, the triangular inequality is an equality when the central node separates the two others; that is, $d_c(x,y) = d_c(x,z) + d_c(z,y)$ if every path between x and y passes through z or, in other words, is a cut–vertex because its elimination disconnects the network.

From now on, we will make no difference between d and d_c, the two shortest–path distances defined for graphs and networks. These shortest–path distances are only one possibility to define whether two different vertices are

[1] It is accepted that $d(x,y) = +\infty$ when do not (only on non connected networks).

close or far away in a given discrete structure. We will also consider resistance distances (or effective resistances) in networks further on within this work.

3.1.3 Boundary of a vertex subset

For our purposes it is now timely and appropriate to introduce the concept of the *boundary of a subset* of V. Given $F \subset V$, a proper subset, for every $x \notin F$ the distance from x to F is defined canonically as $d(x, F) = \min_{y \in F} d(x, y)$. Then the *boundary* of a proper subset F, is denoted and defined by $\partial F = \{x \in V \setminus F \colon d(x, F) = 1\}$. Moreover, the *interior* and the *exterior* of $F \subset V$ are $\overset{\circ}{F} = \{x \in F \colon \text{ if } y \sim x \text{ then } y \in F\}$ and $\mathrm{Ext}(F) = \{x \in V \colon d(x, F) \geq 2\}$ consequently. Please, observe that $\overset{\circ}{F}$ is not necessarily connected even when F is. Finally, the closure of F is defined naturally as $\overline{F} = F \cup \partial F$ and coincides with the set $\{x \in V : d(x, F) \leq 1\}$.

3.2 Functions and linear operators on networks

3.2.1 Functions on networks

The set of real–valued *functions* $u \colon V \to \mathbb{R}$ where $u(x) \in \mathbb{R}$ for every $x \in V$, is denoted by $\mathcal{C}(V)$ and clearly is a real vector space that is isomorphic to $\mathbb{R}^{|V|}$ (and thus we will –by abuse of notation– simply denote this vector space by $\mathbb{R}^{|V|}$). Similarly, $\mathcal{C}(V \times V)$ is the real vector space of all real functions defined on edges, from $V \times V$ into $\mathbb{R}$. We will use $\mathcal{C}(V \times V)$ or $\mathcal{C}(E)$ indistinctly.

The *support* of $u \in \mathcal{C}(V)$ is $\mathsf{supp}(u) = \{x \in V \colon u(x) \neq 0\} \subset V$. For every $F \subsetneq V$ it is possible to identify the corresponding $\mathcal{C}(F)$ with the set $\{u \in \mathcal{C}(V) \colon \mathsf{supp}(u) \subseteq F\}$.

Dirac's delta function $\varepsilon_x \colon V \to \mathbb{R}$ is defined for every vertex $x \in V$ such that its support is exclusively the vertex x, with an assigned value of one. Similarly there also is the Dirac's delta function on edges $\varepsilon_e \colon E \to \mathbb{R}$, vanishing at all edges except the one from it is named and where attains a unitary value.

Both sets of Dirac's delta functions can be considered as standard or canonical bases for their respective vector spaces $\mathcal{C}(V)$ and $\mathcal{C}(E)$. Hence, every $u \in \mathcal{C}(V)$ can be expressed by $u = \sum_{x \in V} u(x)\varepsilon_x$ for some coordinates $u(x) \in \mathbb{R}$ and, analogously, every $u \in \mathcal{C}(E)$ can be expressed by $u = \sum_{e \in E} u(e)\varepsilon_e$ for some coordinates $u(e) \in \mathbb{R}$.

Moreover, we shall endow these two vector spaces with a correspondingly *inner product* $\langle u, v\rangle_{\mathcal{C}(V)} = \sum_{x\in V} u(x)v(x)$ and $\langle u, v\rangle_{\mathcal{C}(E)} = \sum_{e\in E} u(e)v(e)$, respectively, in which standard basis are orthonormal. As usual then, $||u||^2_{\mathcal{C}(V)} = \langle u, u\rangle_{\mathcal{C}(V)}$ and $||u||^2_{\mathcal{C}(E)} = \langle u, u\rangle_{\mathcal{C}(E)}$ define a *norm* on the vector space where they respectively have sense.

3.2.2 Linear operators on networks

Given two vertex sets $F_1, F_2 \subset V$, a *linear operator* $\mathcal{K}\colon \mathcal{C}(F_2) \to \mathcal{C}(F_1)$ is just a structure–preserving morphism such that $\mathcal{K}(u) \in \mathcal{C}(F_1)$ for every $u \in \mathcal{C}(F_2)$.

It is important for us to point out that, associated with a linear operator, there always exists a kernel $K \in \mathcal{C}(F_1 \times F_2)$, defined in the following way: if $y \in F_2$, then $\mathcal{K}(\varepsilon_y) \in \mathcal{C}(F_1)$ is a function that may be evaluated for every $x \in F_1$ hence

$$K(x, y) = \mathcal{K}(\varepsilon_y)(x)$$

for all $x \in F_1$ and all $y \in F_2$. It is called the discrete version of Schwarz's kernels theorem, as varying $y \in F_2$ we evaluate different functions over their whole domain, for every $x \in F_1$.

The function $K(x, y)$ is called a *kernel* (associated with $\mathcal{K}$) because the linear operator can be restituted from it

$$(\mathcal{K}(u))(x) = \sum_{y\in F_2} K(x, y)u(y) \text{ for all } x \in F_1 \text{ and } u \in \mathcal{C}(F_2)$$

as $\mathcal{K}(u) \in \mathcal{C}(F_1)$ is defined for every $x \in F_1$ by combining all the attained values $u(y)$ in the way that the kernel $K(x, y)$ specifies.

Moreover we can introduce two families of functions associated with the kernel K of the linear operator $\mathcal{K}$, the so called components of K with respect to y *i.e.* $K^y(x) = K(x, y) = \mathcal{K}(\varepsilon_y)(x)$ and with respect to x, *i.e.* $K_x(y) = K(x, y)$.

3.3 Difference operators on networks

Now we proceed to describe discrete analogues of the fundamental first order differential operators on Riemannian manifolds, specifically the derivative and the gradient, the divergence and the curl. Time for second order differential operators as Laplacian and Schrödinger type operators will come in the next section.

Before that we should start by defining what is the tangent space of a given network and so, how to consider vector fields.

3.3.1 Tangent space and vector fields

For every $x \in V$, the *tangent space* at vertex x of a graph is defined as the formal linear combinations of the edges that are incident with x. We denote it by $T_x(\Gamma)$, and consider that the set of edges $\{e_{xy} = \{x, y\} : y \in N(x)\}$ is a *coordinate basis* of the tangent space. Hence $\dim T_x(\Gamma) = \deg(x)$, varying with x unless the graph is regular.

Therefore, a *vector field* on Γ is an application $\mathsf{f} : V \to \bigcup_{x \in V} T_x(\Gamma)$ such that $\mathsf{f}(x) \in T_x(\Gamma)$, that is, $\mathsf{f}(x) = \sum_{y \sim x} \lambda_y e_{xy}$, where the sum is restricted only to the vertices of V that are adjacent to x. We will denote by $\chi(\Gamma)$ the space of vector fields. Moreover $\mathsf{supp}(\mathsf{f}) = \{x \in V : \mathsf{f}(x) \neq 0\}$ will stand for the *support* of a vector field f. A vector field on Γ is uniquely determined by its components in the (local) coordinate basis. Therefore, a *component function* may be associated with a vector field as $\mathsf{f}(x) = \sum_{y \sim x} f(x, y) e_{xy}$. As this relation is one–to–one, $\chi(\Gamma)$ can be identified with $\mathcal{C}(E)$.

A vector field is called a *flow* when its component functions satisfy $f(x, y) = -f(y, x)$ for any $x, y \in V$. Analogously, a vector field f is called *symmetric* when its corresponding component function is symmetric.

It is also possible to relate a vector field $\mathsf{f} \in \chi(\Gamma)$ with two other vector fields: f^s is the *symmetric* and f^a is the *antisymmetric* vector fields associated with f. We define both fields by expliciting their components functions as

$$f^s(x, y) = \frac{f(x, y) + f(y, x)}{2} \text{ and } f^a(x, y) = \frac{f(x, y) - f(y, x)}{2}.$$

Observe that $\mathsf{f} = \mathsf{f}^s + \mathsf{f}^a$ for any $\mathsf{f} \in \chi(\Gamma)$.

If $u \in \mathcal{C}(V)$ and $\mathsf{f} \in \chi(\Gamma)$ that has $f \in \mathcal{C}(E)$ as its component function, the field $u\mathsf{f} \in \chi(\Gamma)$ and has uf as its component function.

For any two vector fields $\mathsf{f}, \mathsf{g} \in \chi(\Gamma)$, we define $\langle \mathsf{f}, \mathsf{g} \rangle$ a function in $\mathcal{C}(V)$ such that

$$\langle \mathsf{f}, \mathsf{g} \rangle(x) = \sum_{y \sim x} r(x, y)\, f(x, y) g(x, y) \text{ for any } x \in V,$$

in terms of its component functions and of the resistance function. Clearly, $\langle \cdot, \cdot \rangle(x)$ determines an inner product on $T_x(\Gamma)$. Therefore, on a network, we can consider the following inner products on $\mathcal{C}(V)$ and on $\chi(\Gamma)$,

$$\langle u, v \rangle = \sum_{x \in V} u(x) v(x), \quad u, v \in \mathcal{C}(V) \quad \text{and} \quad \frac{1}{2} \sum_{x \in V} \langle \mathsf{f}, \mathsf{g} \rangle(x), \quad \mathsf{f}, \mathsf{g} \in \chi(\Gamma),$$

where the factor $\frac{1}{2}$ is due to the fact that information on each edge is con-

sidered twice. Moreover given $\mathsf{f}, \mathsf{g} \in \chi(\Gamma)$, then

$$\sum_{x \in V} \langle \mathsf{f}, \mathsf{g} \rangle(x) = \sum_{x \in V} \langle \mathsf{f}^s, \mathsf{g}^s \rangle(x) + \sum_{x \in V} \langle \mathsf{f}^a, \mathsf{g}^a \rangle(x).$$

Finally, and in particular, if f is symmetric and g is a flow, it holds that $\sum_{x \in V} \langle \mathsf{f}, \mathsf{g} \rangle(x) = 0$.

3.3.2 Derivative, Gradient, Divergence and Curl

The *derivative operator* is the linear map $\mathsf{d} : \mathcal{C}(V) \to \chi(\Gamma)$ that assigns to every $u \in \mathcal{C}(V)$ the flow $\mathsf{d}u$, called derivative of u, and given by

$$\mathsf{d}u(x) = \sum_{y \sim x} (u(y) - u(x)) \; e_{xy}, \quad \text{for every } x \in V.$$

Then, the *gradient operator* is the linear map $\nabla : \mathcal{C}(V) \to \chi(\Gamma)$ that assigns to every $u \in \mathcal{C}(V)$ the flow ∇u, called gradient of u, and defined as

$$\nabla u(x) = \sum_{y \sim x} c(x, y) \, (u(y) - u(x)) \; e_{xy}, \quad \text{for every } x \in V.$$

Clearly, it is verified that $\mathsf{d}u = 0$, or equivalently $\nabla u = 0$, if and only if u is a constant function.

The *divergence operator* is defined, mimicking the continuous operator, as the linear map $\mathsf{div} : \chi(\Gamma) \to \mathcal{C}(V)$, given by $\mathsf{div} = -\nabla^*$. So, it is the one that assigns to every $\mathsf{f} \in \chi(\Gamma)$ the function $\mathsf{div}\,\mathsf{f}$ called *divergence of* f, given by the relation

$$\sum_{x \in V} u(x) \mathsf{div}\,\mathsf{f}(x) = -\frac{1}{2} \sum_{x \in V} \langle \nabla u, \mathsf{f} \rangle(x) \quad \text{for every } u \in \mathcal{C}(V).$$

Taking u constant above, we obtain the Divergence Theorem

$$\sum_{x \in V} \mathsf{div}\,\mathsf{f}(x) = 0 \quad \text{for every } \mathsf{f} \in \chi(\Gamma).$$

Moreover, we have $\mathsf{div}\,\mathsf{f}(x) = \sum_{y \sim x} f^a(x, y)$, for every $x \in V$.

The *curl* of a vector field is, finally, the linear map $\mathsf{curl} : \chi(\Gamma) \to \chi(\Gamma)$ that assigns to every $\mathsf{f} \in \chi(\Gamma)$ the symmetric vector field $\mathsf{curl}\,\mathsf{f}$, called curl of f, and defined by

$$\mathsf{curl}\,\mathsf{f}(x) = \sum_{y \sim x} r(x, y) f^s(x, y) \; e_{xy}, \quad \text{for every } x \in V.$$

As desired, the above defined difference operators satisfy analogous properties to that ones satisfied by their differential counterparts. For instance, $\mathsf{curl}^* = \mathsf{curl}$, $\mathsf{div} \circ \mathsf{curl} = 0$ and $\mathsf{curl} \circ \nabla = 0$, see [13].

3.4 Second order difference operators

It is time now to introduce the fundamental elliptic second order difference operator on $\mathcal{C}(V)$, which is obtained by composition of the last two first order operators previously presented.

3.4.1 The Laplace operator

Specifically, we consider the endomorphism of $\mathcal{C}(V)$ given by

$$\mathcal{L} = -\mathsf{div} \circ \nabla,$$

that we call the *Laplace operator* of Γ and which is also known as the *combinatorial Laplacian* of Γ. We point out the analogy of this definition with the definition of its continuous counterpart, the well known Laplace–Beltrami elliptic operator for Riemannian manifolds, that can be also defined as the divergence of the gradient $\Delta = \mathsf{div} \circ \nabla$, but for operators in the continuous setting of course. Hence the name given to the discrete operator. Please note the opposite sign, due to the former definition of the discrete divergence operator.

Its expression and some critical properties of this operator, see [7], are shown just right now. For every $u \in \mathcal{C}(V)$ and for every $x \in V$ we get that

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x,y)\big(u(x) - u(y)\big)$$

with the sum extended to all $y \in V$, provided that $c(x,y)$ vanishes when $\{x,y\} \notin E$. Moreover, given $u, v \in \mathcal{C}(V)$ the Laplacian of Γ satisfies analogous properties to those fulfilled by its continuous counterpart. The First Green Identity can be expressed as

$$\begin{aligned}\sum_{x \in V} v(x)\mathcal{L}(u)(x) &= \frac{1}{2}\sum_{x \in V}\langle \nabla u, \nabla v\rangle(x) \\ &= \frac{1}{2}\sum_{x,y \in V} c(x,y)\big(u(x) - u(y)\big)\big(v(x) - v(y)\big).\end{aligned}$$

Also a Second Green Identity can be established in the following terms

$$\sum_{x \in V} v(x)\mathcal{L}(u)(x) = \sum_{x \in V} u(x)\mathcal{L}(v)(x).$$

And finally the corresponding version of Gauss theorem is

$$\sum_{x\in V} \mathcal{L}(u)(x) = 0.$$

Easily from the previous results, we also obtain that the Laplacian of Γ is self–adjoint and positive semi–definite.

Moreover $\mathcal{L}(u) = 0$ if and only if u is constant. Hence, when restricted to its kernel, the Laplacian operator is an automorphism.

3.4.2 Schrödinger type operators

Given a function $q \in \mathcal{C}(V)$, we define a *Schrödinger operator* on Γ as the linear operator $\mathcal{L}_q : \mathcal{C}(V) \to \mathcal{C}(V)$ that assigns to every $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}_q(u)(x) = \mathcal{L}(u)(x) + q(x)u(x).$$

While we refer to q as the *potential*, some authors use the term ground–state as it might be interpreted as a connection of each vertex in a network with a conductor medium with null potential. Instead of considering a Schrödinger operator as a perturbation of the combinatorial Laplacian, which it is, we prefer to look at it as an assignation of a real number (positive, negative or zero) to every single vertex in the structure, allowing the possibility of differentiating between vertices somehow (and not only because of having different degree).

Hence, by introducing Schrödinger operators we get to satisfy our wish of working with discrete structures that consider the possibility of distinguishing both constitutive elements of a network; vertices by q and also edges by c.

Clearly $\mathcal{L}_q$ is also a self–adjoint operator and defines a bilinear form $\mathcal{E}_q(u, v) = \langle u, \mathcal{L}_q(v)\rangle$ that is called the *energy* of $\mathcal{L}_q$. Applying the first Green identity for the Laplacian it turns out that

$$\mathcal{E}_q(u, v) = \frac{1}{2} \sum_{x,y\in V} c(x, y)\big(u(x) - u(y)\big)\big(v(x) - v(y)\big) + \sum_{x\in V} q(x)u(x)v(x)$$

so if $q \geq 0$ then the energy of $\mathcal{L}_q$ is positive semi–definite.

What we will call a *weight* is a function $\omega \in \mathcal{C}(V)$, such that $\omega > 0$ on V and also $\langle \omega, \omega\rangle = 1$. We will denote as $\Omega(V)$ the set of positive and unitary, weight functions. For every weight ω, we define q_ω the *potential associated with* ω, as

$$q_\omega(x) = -\frac{1}{\omega(x)}\mathcal{L}(\omega)(x),\ x \in V.$$

Therefore, it is $q_\omega(x) = -\kappa(x) + \dfrac{1}{\omega(x)} \displaystyle\sum_{y \in V} c(x,y)\omega(y)$ for every $x \in V$. So a weight and its associated potential are orthogonal functions and therefore q_ω must take positive and negative values except when the weight is constant, in which case its associated potential vanishes at every vertex and its corresponding Schrödinger operator $\mathcal{L}_{q_\omega} = \mathcal{L}_0 = \mathcal{L}$ is the Laplacian.

Potentials associated with a weight do determine, up to a multiplicative positive constant, the weight function. That is, if ω_1 and ω_2 are weight functions, it is $q_{\omega_1} = q_{\omega_2}$ if and only if $\omega_1 = a\omega_2$ for some $a > 0$. On the other hand when $q_{\omega_1} \neq q_{\omega_2}$, q_{ω_1} determines a family of functions q for which $\mathcal{L}_q$ is positive semi–definite and is essentially different from the family determined by q_{ω_2}. More properties of these potential associated with weight functions can be found in [7].

By using the well known Perron–Frobenius theorem every potential function q is related to a potential function associated with a weight q_ω in the following terms: given $q \in \mathcal{C}(V)$, there exists a unique $\omega \in \Omega(V)$ and $\lambda \in \mathbb{R}$ such that

$$q = q_\omega + \lambda$$

(see [7] again for a detailed justification of this very important result).

So there is another characterization of Schrödinger operators,

$$\mathcal{L}_q(u)(x) = \mathcal{L}(u)(x) + q(x)u(x) = \mathcal{L}_{q_\omega} u(x) + \lambda u(x).$$

An expression for $\mathcal{L}_{q_\omega}$ is obtained by applying the so called Doob's Transform once a weight is given. Hence, for every $u \in \mathcal{C}(V)$ it is

$$\mathcal{L}(u)(x) + q_\omega(x)u(x) = \frac{1}{\omega(x)} \sum_{y \in V} c(x,y)\omega(x)\omega(y) \left(\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)} \right), \quad x \in V.$$

In addition, for every $u, v \in \mathcal{C}(V)$ we obtain that

$$\mathcal{E}_{q_\omega}(u,v) + \sum_{x \in V} q_\omega(x)u(x)v(x) = \frac{1}{2} \sum_{x,y \in V} c(x,y)\omega(x)\omega(y) \left(\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)} \right) \left(\frac{v(x)}{\omega(x)} - \frac{v(y)}{\omega(y)} \right).$$

So now we are ready to set a necessary and sufficient condition for the positive semi–definiteness of Schrödinger operators, which is a result that evidently has a continuous counterpart known as the Energy principle. Once $q = q_\omega + \lambda$, then $\mathcal{L}_q$ is positive semi–definite if and only if $\lambda \geq 0$, and positive definite if and only if $\lambda > 0$. Moreover, when $\lambda = 0$, $\mathcal{L}_q(u) = 0$ if and only if $u = a\omega$ for some $a \in \mathbb{R}$.

Also, as $\min_{\langle u,u\rangle=1}\{\mathcal{E}_q(u)\} \geq \lambda$, it is $\mathcal{E}_q(u) = \lambda$ if and only if $u = \pm\omega$. Hence $\mathcal{L}_q(\omega) = \lambda\omega$ and λ turns to be the lowest eigenvalue of $\mathcal{L}_q$, and it is simple.

In the positive semi–definite case, when $\lambda = 0$ and $q = q_\omega$, it is $\mathcal{L}_{q_\omega} \perp \omega$ and therefore if $u \in \mathcal{C}(V)$ is such that $\mathcal{L}_{q_\omega}(u) \geq 0$, then $\mathcal{L}_{q_\omega}(u) = 0$ and $u = a\omega$ with $a \in \mathbb{R}$.

3.4.3 The normalized Laplacian

The concept of Schrödinger operator encompasses other widely used discrete operators as, for example, the so called *normalized Laplacian* introduced in [35] and defined as

$$\mathscr{L}u(x) = \frac{1}{\kappa(x)} \sum_{y\in V} c(x,y) \left(\frac{u(x)}{\sqrt{\kappa(x)}} - \frac{u(y)}{\sqrt{\kappa(y)}} \right)$$

for a connected network with conductance function c. If the size of the subjacent graph is m, then the normalized Laplacian on (V, E, c) coincides with the non–singular positive semi–definite Schrödinger operator on $(V, E, \hat{c})$ where the conductance function is $\hat{c}(x,y) = \dfrac{c(x,y)}{\sqrt{\kappa(x)}\sqrt{\kappa(y)}}$, considering $\omega = \dfrac{1}{2m}\sqrt{\kappa}$ and obviously $\lambda = 0$, thus $\mathscr{L} = \mathcal{L}_{q_\omega}$, applied on the same graph but considered as two different networks.

3.5 Green's function for Poisson and Dirichlet problems

In this work we have considered solving Poisson and Dirichlet boundary value problems on networks associated with linear operators as the combinatorial Laplacian, the normalized Laplacian and Schrödinger type operators as well.

In this section we introduce the required concepts to solve these problems. The so called Green's function turn to be critical in the sense it can be considered as the universal solver.

3.5.1 Poisson and Dirichlet boundary value problems on networks

Roughly speaking, a Poisson problem on Γ is when the domain where the condition has to be validated is the hole vertex set V, whereas a Dirichlet problem establishes a condition only in a proper subset $F \subset V$ while adding the values of the solution on the boundary ∂F.

More explicitly, let $\mathcal{L}$ denote now whatever of our three second order difference linear operators, the combinatorial Laplacian or the normalized Laplacian or more generalized Schrödinger type operator. Then, given $f \in \mathcal{C}(V)$ a *Poisson problem* with data f is set so as we want to find out which is $u \in \mathcal{C}(V)$ such that

$$\mathcal{L}(u) = f$$

on the whole V.

On the other hand, a *Dirichlet boundary value problem* is posed once a proper subset $F \subset V$ and two data functions $f \in \mathcal{C}(V)$ and $g \in \mathcal{C}(\overline{F})$ are given. Therefore, we are interested in finding $u \in \mathcal{C}(V)$ such that

$$\begin{aligned} \mathcal{L}(u) &= f \text{ on } F \\ u &= g \text{ on } \partial F. \end{aligned}$$

It is worth to mention that both Poisson and Dirichlet boundary value problems are related not only to a domain (in our case the vertex set associated with the network Γ, or a subset of it), but also to a particular linear operator to be considered and solved.

3.5.2 Green operators and Green's kernel functions

Now we assume that both, combinatorial Laplacian and Schrödinger type operators[2], are positive semi–definite. We have seen that it is always the case for the first one, and that we have to consider $q = q_\omega + \lambda$, for some $\omega \in \Omega(V)$ and $\lambda \geq 0$, for the second one. Provided this situation, we construct kernels associated with their respective inverse operators that correspond to Poisson problems. Analogously to the well known continuous case, such inverse operators will be called *Green operators.*

We start recalling fundamental notions about operators and their associated kernels that we have exposed in section 3.2.2 and assuming $F_1 = F_2 = V$. We will remark the conditions that assure existence and uniqueness of the inverse operators, list a few properties that are satisfied by them and finally build the associated kernel that is known as *Green's kernel function* or Green's function for short.

Generally speaking, if $K : V \times V \to \mathbb{R}$ is a kernel in V then its associated endomorphism $\mathcal{K} : \mathcal{C}(V) \to \mathcal{C}(V)$ is defined as

$$(\mathcal{K}(u))(x) = \sum_{y \in V} K(x,y)u(y), \text{ for every } x \in V.$$

Conversely, any endomorphism in $\mathcal{C}(V)$ determines a kernel in V by $K(x,y) = \mathcal{K}(\varepsilon_y)(x)$ for every pair $x, y \in V$.

[2]We consider the normalized Laplacian as a "particular" case

In this context, an operator $\mathcal{K}$ is self–adjoint (that is $\langle \mathcal{K}(u), v\rangle = \langle u, \mathcal{K}(v)\rangle$ for every $u, v \in \mathcal{C}(V)$) if and only if its related kernel K is a symmetric function ($K(x,y) = K(y,x)$ for all $x, y \in V$).

3.5.3 Inverse operator for the Laplacian operator

Let us now denote $\mathsf{1} \in \mathcal{C}(V)$ the constant function $\mathsf{1}(x) = 1$, for every $x \in V$. It turns out that $||\mathsf{1}||^2 = |\mathsf{V}|$. Given $f \in \mathcal{C}(V)$, the Poisson problem on Γ for the Laplace operator consists in finding $u \in \mathcal{C}(V)$ such that

$$\mathcal{L}(u)(x) = f(x), \text{ for every } x \in V.$$

As we have previously seen, the kernel of the Laplacian operator is the set of constant functions, so we can consider the *orthogonal projection* onto $\mathsf{ker}\mathcal{L} = \mathsf{span}\{\mathsf{1}\}$, $\mathcal{P} : \mathcal{C}(V) \to \mathsf{ker}(\mathcal{L})$ defined as $\mathcal{P}(f) = \dfrac{\langle f, \mathsf{1}\rangle}{\langle \mathsf{1}, \mathsf{1}\rangle} \cdot \mathsf{1}$, for every $f \in \mathcal{C}(V)$ so that $(\mathcal{P}(f))(x) = \dfrac{\sum_{y\in V} f(y)\mathsf{1}(y)}{|V|} \cdot \mathsf{1}(x) = \dfrac{1}{|V|}\sum_{y\in V} f(y)$ is the constant function with value equal to the sum of all values that f attains.

It is clear, see below in Figure 3.1 for a representation of the Laplacian operator, that $\mathcal{L}\circ\mathcal{P} = 0$. As Gauss theorem holds for the Laplacian operator, $\mathcal{L}(u) \in (\mathsf{ker}\mathcal{L})^{\perp}$ or equivalently $\mathcal{P} \circ \mathcal{L} = 0$, as well.

Consequently we have a characterization for data function on Poisson problems for the corresponding problem being compatible and therefore solutions exist.

Therefore, the set of all solutions to a Poisson problem $\mathcal{L}(u) = f - \mathcal{P}(f)$ is a one–parameter family $\{u + \mu\mathsf{1}, \mu \in \mathbb{R}\}$, with u the unique function that exists by the previous results. Again Figure 3.1 illustrates this fact, as there is exactly one point (for a function) in every line parallel to $\mathsf{1}$ (for a one–parameter family) that is in $\mathsf{1}^{\perp}$.

Thus it is possible to fix a particular solution in some sense because an orthogonal to $\mathsf{1}$ solution of a compatible Poisson problem can be always found.

So there is a Fredholm's alternative discrete version that also applies in this discrete setting, see [8]. Given $f \in \mathcal{C}(V)$, the Poisson problem $\mathcal{L}(u) = f$ is consistent if and only if $\mathcal{P}(f) = 0$. Then, there exists a unique solution such that $\mathcal{P}(u) = 0$.

Therefore and so to speak, in some sense, there is existence and uniqueness of solutions for a Poisson problem given the Laplacian operator.

Now we consider the operator that assigns to every function $f \in \mathcal{C}(V)$ the unique $u \in \mathcal{C}(V)$ such that

$$\mathcal{L}(u) = (\mathcal{I} - \mathcal{P})(f) \text{ and } \langle u, \mathsf{1}\rangle = 0.$$

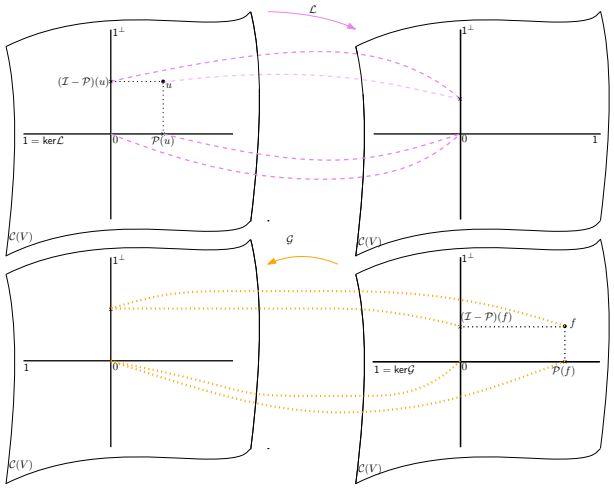


Figure 3.1 Two pictures about the Laplacian and Green operators in relation with a Poisson problem $\mathcal{L}(u) = f$

We will call it the *Green operator* and will denote it by $\mathcal{G}$ so as $\mathcal{G} : \mathcal{C}(V) \to \mathcal{C}(V)$, such that $\mathcal{G}(f) = u$. See again Figure 3.1, now above, for a geometric interpretation.

As the network is connected and as happens with the Laplacian operator, the Green operator, when restricted to the space $1^\perp$, is also an automorphism. Then it is clear that $\mathcal{G} \circ \mathcal{P} = 0$. By definition $\mathcal{G}(f) \in 1^\perp$ so $\mathcal{P} \circ \mathcal{G} = 0$ as well.

Moreover, this Green operator satisfies nice properties such that for instance being self–adjoint and positive semi–definite. Also the Green operator of $f \in \mathcal{C}(V)$ is orthogonal to f only when f is constant, or $\langle \mathcal{G}(f), f \rangle = 0$ if and only if $f = a1$ for $a \in \mathbb{R}$.

Therefore, the associated kernel $G : V \times V \to \mathbb{R}$ defined for every pair $x, y \in V$ as $G(x, y) = \mathcal{G}(\varepsilon_y)(x)$ is called *Green's kernel function.* We note that it is symmetric, as $\mathcal{G}$ is self–adjoint, and that $u(x) = \sum_{y \in V} G(x, y) f(y)$ is the unique solution orthogonal to 1 of the Poisson problem $\mathcal{L}(u) = (\mathcal{I} - \mathcal{P})(f)$ for every $f \in \mathcal{C}(V)$.

Hence, the well known relation between an operator and its associated kernel enables us, again, to characterize the Green kernel of Γ by considering a family of solutions of suitable consistent Poisson problems.

Let us define for every $y \in V$, the function $G_y \in \mathcal{C}(V)$ defined by $G_y(x) = G(x, y), x \in V$. It can be characterized by

$$\mathcal{L}(G_y) = \varepsilon_y - \frac{1}{|V|} 1, \quad \langle G_y, 1 \rangle = 0.$$

So, from the relationship between $\mathcal{L}$ and $\mathcal{G}$, it turns out that $\mathcal{L}\circ\mathcal{G} = \mathcal{G}\circ\mathcal{L} = \mathcal{I} - \mathcal{P}$. Thus, when restricted to $1^{\perp}$, it is $\mathcal{L}\circ\mathcal{G} = \mathcal{G}\circ\mathcal{L} = \mathcal{I}$ obviously. Therefore, it is

$$\mathcal{L}\circ\mathcal{G}\circ\mathcal{L} = \mathcal{L}\circ(\mathcal{I}-\mathcal{P}) = \mathcal{L}, \text{ and } \mathcal{G}\circ\mathcal{L}\circ\mathcal{G} = \mathcal{G}\circ(\mathcal{I}-\mathcal{P}) = \mathcal{G}$$

by applying that $\mathcal{L}\circ\mathcal{P} = \mathcal{P}\circ\mathcal{L} = \mathcal{G}\circ\mathcal{P} = \mathcal{P}\circ\mathcal{G} = 0$. So both operators are generalized inverse operators one of each other and our goal is accomplished.

3.5.4 Inverse operator for a Schrödinger type operator

Once we have presented the so called Green operator and its respective Green's kernel for $\mathcal{L}$, the Laplacian operator on a network Γ, we face now the study of their counterparts in the case of a Schrödinger operator $\mathcal{L}_q$ by reproducing previous arguments and definitions.

Now a Poisson problem consists in, given $f \in \mathcal{C}(V)$, figuring out $u \in \mathcal{C}(V)$, satisfying

$$\mathcal{L}_q(u) = f, \text{ for all } x \in V.$$

We are concerned with Schrödinger type operators that have good properties, so we consider that the potential $q \in \mathcal{C}(V)$ is $q = q_\omega + \lambda$ for some $\omega \in \Omega(V)$ and $\lambda \geq 0$. Two cases have to be treated separately: when $\lambda > 0$ the self–adjoint operator is positive definite, so invertible; while when $\lambda = 0$ it is self–adjoint and positive semi–definite, with a null simple eigenvalue as the network is connected.

As we shall see, in both cases the Green operator is well defined, self–adjoint, and denoted by $\mathcal{G}_q$ with no ambiguity because of the scenario. Moreover, the associated kernel $G_q : V \times V \to \mathbb{R}$, that is constructed by analogous procedures so as $G_q(x,y) = \mathcal{G}_q(\varepsilon_y)(x)$, for any $x, y \in V$, is called the Green function and it turns out to be symmetric.

We notably remark, in a unifying notation, that $\mathcal{G}_q(\omega) = \lambda^{\dagger}\omega$, where $\lambda^{\dagger} = \lambda^{-1}$ in the invertible case $\lambda > 0$, while $\lambda^{\dagger} = 0$ in the singular case, that is $\lambda = 0$.

Case (i): $\lambda > 0$, $\mathcal{L}_q$ positive definite

When $\lambda > 0$ the Schrödinger operator $\mathcal{L}_q$ is an isomorphism on $\mathcal{C}(V)$, as there is no kernel to consider, (*i.e.* $\mathsf{ker}(\mathcal{L}_q) = \mathsf{0}$). Then, a Poisson problem $\mathcal{L}_q(u) = f$, for all $x \in V$, is always, for every data function $f \in \mathcal{C}(V)$, consistent. Therefore, we can define its inverse operator $\mathcal{G}_q : \mathcal{C}(V) \to \mathcal{C}(V)$, such that $\mathcal{G}_q(f) = u$, for every $f \in \mathcal{C}(V)$. This inverse operator will be called *Green operator of* Γ.

Obviously it is $\mathcal{L}_q \circ \mathcal{G}_q = \mathcal{G}_q \circ \mathcal{L}_q = \mathcal{I}$. The Green operator is positive definite in this case by applying the positive semi–definiteness of $\mathcal{E}_q$. Moreover it is also a self–adjoint operator in the sense that

$$\sum_{x \in V} g(x)\mathcal{G}_q(f)(x) = \sum_{x \in V} f(x)\mathcal{G}_q(g)(x), \quad \text{for all } f, g \in \mathcal{C}(V).$$

The *Green kernel* of Γ is the kernel associated with the previous Green operator, is denoted by $G_q = G_q(x, y)$ and is symmetric again. Then, the sole solution to the Poisson problem $\mathcal{L}_q(u) = f$ on V, can be recovered from the data function $f \in \mathcal{C}(V)$ by

$$u(x) = \sum_{y \in V} G_q(x, y)\, f(y), \quad \text{for all } x \in V.$$

Finally, the relation between operators and their associated kernels enables us to characterize the Green kernel as the set of solutions of a suitable battery of Poisson problems. As $\mathcal{L}_q$ is an isomorphism, we consider for every $y \in V$, the function $(G_q)_y(x) = \mathcal{G}_q(\varepsilon_y)(x)$ for every x on V, and finally we define $G_q(x, y) = (G_q)_y(x)$ so this component function is symmetric and characterized by $\mathcal{L}_q(G_y) = \varepsilon_y$ on V.

Case (ii): $\lambda = 0$, $\mathcal{L}_q$ positive semi–definite

When $\lambda = 0$, the Schrödinger operator is a singular, self–adjoint and positive semi–definite operator, as the Laplacian operator is. We will denote again as $\mathcal{L}_q$ even though, as λ vanishes, it should be a $\mathcal{L}_{q_\omega}$ in fact, for some $\omega \in \Omega(V)$.

Let us consider the vector space $\mathsf{ker}(\mathcal{L}_q)$ which is now the linear space generated by ω. Therefore, the *orthogonal projection* onto $\mathsf{ker}(\mathcal{L}_q) = \mathsf{span}\{\omega\}$, is $\mathcal{P} : \mathcal{C}(V) \to \langle\omega\rangle$ defined as $\mathcal{P}(f) = \langle f, \omega\rangle \cdot \omega$, for every $f \in \mathcal{C}(V)$. It is clear that $\mathcal{L}_q \circ \mathcal{P} = 0$, and also that $\mathcal{P} \circ \mathcal{L}_q = 0$.

Consequently, we have a characterization for data function on Poisson problems so there exist a solution. Moreover, as in the Laplacian case, a Fredholm's alternative holds: consistency is assured if and only if data functions are orthogonal to the weight function, and there is a unique orthogonal to the weight function solution of the problem (when there is) or mathematical spoken given $\omega \in \Omega(V)$ and $f \in \mathcal{C}(V)$, then the corresponding Poisson problem for the Schrödinger operator $\mathcal{L}_q(u) = f$ is consistent if and only if $\mathcal{P}(f) = 0$. Moreover, there exists a unique solution such that $\mathcal{P}(u) = 0$.

Hence, for each $f \in \mathcal{C}(V)$, as $f - \mathcal{P}(f)$ is in $(\mathsf{ker}(\mathcal{L}_q))^\perp$, there exists $u \in \mathcal{C}(V)$ such that $\mathcal{L}_q(u) = f - \mathcal{P}(f)$. Then, the set of all solutions of the Poisson problem is $\big\{v \in \mathcal{C}(V) : v = u + a\omega \text{ for some } a \in \mathbb{R}\big\}$. And there is just one function in this set that is orthogonal to ω.

Now is clear how to define the desired inverse operator. Let us consider the operator that assigns to every function $f \in \mathcal{C}(V)$ the unique $u \in \mathcal{C}(V)$ such that

$$\mathcal{L}_q(u) = (\mathcal{I} - \mathcal{P})(f) \text{ and } \langle u, \omega \rangle = 0.$$

Of course, we will call it the *Green operator* and we will denote it by $\mathcal{G}_q$ so as $\mathcal{G}_q : \mathcal{C}(V) \to \mathcal{C}(V)$, such that $\mathcal{G}_q(f) = u$. As the network is connected, the Green operator, when restricted to the space $\omega^\perp$, is an automorphism. So $\mathcal{G}_q \circ \mathcal{P} = 0$ and $\mathcal{P} \circ \mathcal{G}_q = 0$ as well.

Moreover, this Green operator also satisfies the nice properties of being self–adjoint and positive semi–definite. And it sends $\langle \omega \rangle$ to $\langle \omega \rangle^\perp$, or more precisely $\langle \mathcal{G}_q(f), f \rangle = 0$ if and only if $f = a\omega$ for $a \in \mathbb{R}$.

With all that, its corresponding kernel $G_q : V \times V \to \mathbb{R}$ defined as $G_q(x, y) = \mathcal{G}_q(\varepsilon_y)(x)$ for every pair $x, y \in V$ is called *Green's kernel function.* We note that it is symmetric, as $\mathcal{G}_q$ is self–adjoint, and hence

$$u(x) = \sum_{y \in V} G_q(x, y) f(y)$$

is the unique solution orthogonal to ω of the Poisson problem $\mathcal{L}_q(u) = (\mathcal{I} - \mathcal{P})(f)$ for every $f \in \mathcal{C}(V)$, now in the case that $q = q_\omega$ for some weight function $\omega \in \Omega(V)$.

As in the previous cases, the relation between an operator and its associated kernel enables us, again, to characterize the Green kernel of Γ as solutions of a battery of appropriate consistent Poisson problems. Let us define for every $y \in V$, then function $(G_q)_y \in \mathcal{C}(V)$ defined by $(G_q)_y(x) = G_q(x, y), x \in V$ is characterized by

$$\mathcal{L}((G_q)_y) = (\mathcal{I} - \mathcal{P})\varepsilon_y = \varepsilon_y - \omega(y)\omega, \quad \langle (G_q)_y, \omega \rangle = 0.$$

Finally, to end this section and this chapter too, from the relationship between the singular Schrödinger operator $\mathcal{L}_q$ and its inverse operator $\mathcal{G}_q$, it turns out that $\mathcal{L}_q \circ \mathcal{G}_q = \mathcal{G}_q \circ \mathcal{L}_q = \mathcal{I} - \mathcal{P}$. Analogously to the Laplacian case, it is $\mathcal{L}_q \circ \mathcal{G}_q = \mathcal{G}_q \circ \mathcal{L}_q = \mathcal{I}$ when restricted to $\omega^\perp$. Therefore, it is true that

$$\mathcal{L}_q \circ \mathcal{G}_q \circ \mathcal{L}_q = \mathcal{L}_q \circ (\mathcal{I} - \mathcal{P}) = \mathcal{L}_q, \text{ and } \mathcal{G}_q \circ \mathcal{L}_q \circ \mathcal{G}_q = \mathcal{G}_q \circ (\mathcal{I} - \mathcal{P}) = \mathcal{G}_q$$

and we can state that both operators are generalized inverse operators one of each other.

3.6 Matrix interpretation

As we are concerned with finite discrete structures, both size and order of the subjacent graph are finite. Thus it is quite straightforward to obtain a

vectorial version of what functions are and also a matrix interpretation for linear operators and their correspondent kernels.

Hence, given a labelling of a connected network Γ, that is supposing that $V = \{x_1, ..., x_n\}$, the conductance $c(x_i, x_j) \geq 0$, for every $i, j = 1, \ldots, n$ then each $u \in \mathcal{C}(V)$ can be identified with a vector of n components (now $|V| = n$). Hence

$$u = [u(x_1), \ldots, u(x_n)]^T \in \mathbb{R}^n.$$

Therefore, the combinatorial Laplacian of Γ is identified with the singular irreducible M–matrix

$$\mathsf{L} = \begin{bmatrix} \kappa(x_1) & -c(x_1, x_2) & \cdots & -c(x_1, x_n) \\ -c(x_2, x_1) & \kappa(x_2) & \cdots & -c(x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ -c(x_n, x_1) & -c(x_n, x_2) & \cdots & \kappa(x_n) \end{bmatrix}.$$

Clearly, this matrix is symmetric and diagonally dominant and hence it is positive semi–definite. Moreover, it is singular and 0 is a simple eigenvalue whose associated eigenvectors are constant.

In the particular case of a graph, that is when $c(x, y) = 1$ when non–null, the Laplacian matrix is a very powerful tool that is used to analyze structures from the connectivity point of view (counting connected parts, algebraic connectivity, expanding properties, isoperimetric number and many more).

Similarly, we can also have in mind that its inverse operator, what is called the Green operator, has its own matrix counterpart, say G corresponding to the Green's kernel function of course, which is symmetric. Therefore, so it is G.

Moreover, the very well known Moore–Penrose conditions for a generalized inverse hold, as in this case it is

$$\begin{aligned} \mathsf{LGL} &= \mathsf{L} \\ \mathsf{GLG} &= \mathsf{G} \\ (\mathsf{LG})^{\mathsf{T}} &= \mathsf{LG} \\ (\mathsf{GL})^{\mathsf{T}} &= \mathsf{GL}. \end{aligned}$$

Hence G is the Moore–Penrose generalized inverse of the Laplacian matrix L, denoted as $\mathsf{G} = \mathsf{L}^{\dagger}$. See [17] for a general introduction to generalized inverses.

It is possibly to translate the former discussion to the case of a Schrödinger type operator.

So again, after a labelling on the vertex set, $\mathcal{L}_q$ can be identified with the matrix

$$\mathsf{L}_q = \begin{bmatrix} \kappa(x_1)+q(x_1) & -c(x_1,x_2) & \cdots & -c(x_1,x_n) \\ -c(x_2,x_1) & \kappa(x_2)+q(x_2) & \cdots & -c(x_2,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ -c(x_n,x_1) & -c(x_n,x_2) & \cdots & \kappa(x_n)+q(x_n) \end{bmatrix}$$

which is an irreducible, symmetric, Z–matrix, but not necessarily diagonally dominant. Depending on the values $q(x_i), i = 1, \ldots, n$, the matrix is singular or non singular. As we are interested only in positive definite or positive semi–definite Schrödinger type operators, and after the identity $q = q_\omega + \lambda$, we can think on L_q as its correspondent L_{q_ω} assuming $\lambda = 0$, thus semi–definiteness or as $\mathsf{L}_{q_\omega} + \lambda\mathsf{I}$ for the non–singular case, thus invertible.

As we did before, in the Laplacian matrix case, the corresponding inverse operator, what is called the Green operator, has its own matrix counterpart, say $\mathsf{G_q}$ in perfect matching with the Green's kernel function of course, which is always symmetric. Therefore, so it is $\mathsf{G_q}$ either when $\lambda = 0$ or positive.

Thus, when referred to the matrix environment, we conclude again that the two matrices associated with both operators $\mathcal{L}_q$ and $\mathcal{G}_q$, L_q and $\mathsf{G_q}$ respectively, are of course symmetric matrices (as the corresponding kernel are symmetric).

Therefore, $\mathsf{G_q} = \mathsf{L_q^{-1}}$ in the positive definite scenario because in terms of the respective matrices, it comes out that $\mathsf{L}_q\,\mathsf{G}_q = \mathsf{G}_q\,\mathsf{L}_q = \mathsf{I}$, from the properties operators satisfy.

And finally $\mathsf{G_q} = \mathsf{L_q^\dagger}$ in the positive semi–definite case as matrices satisfy the four Moore–Penrose conditions for a generalized inverse, by mimicking the relations established previously for the operators. So it is, again

$$\begin{aligned} \mathsf{L_q G_q L_q} &= \mathsf{L_q} \\ \mathsf{G_Q L_q G_q} &= \mathsf{G_q} \\ (\mathsf{L_q G_q})^\mathsf{T} &= \mathsf{L_q G_q} \\ (\mathsf{G_q L_q})^\mathsf{T} &= \mathsf{G_q L_q}. \end{aligned}$$

With all this in mind, it is clear that figuring out the Green's kernel related to a difference operator in this boundary value problems posed on networks has a computing generalized inverses counterpart. This is what allows our work to be interesting in the Matrix theory framework as a possible interpretation of it is as a sort of Sherman–Morrison–Woodbury formulae for structured matrices, as it will be clear from the sequel.

4

Poisson problems and Green's functions on Subdivision networks

Now that we have presented what is, given a network and a differential operator defined on it, a Poisson problem and it has also been stated that the solution of such a problem is possible to be achieved by using the so called Green's function, it is time now to expose the most important part of the work we have done.

Given the relationship between graph theory and electrical circuit theory, which is probably dating back to Kirchhoff's time, in this work we have inspected the possibilities of generating subdivisions of discrete structures that, as electrical circuits, can be considered equivalent. So while it does not seem like an overly original approach, we have already discussed on the difficulty of finding references to the issue in a scientific–mathematical context.

In this chapter we present the results we have obtained from the different possibilities that we have considered. These possibilities have been designed upon two factors that we had in mind: the network operation and the linear difference operator that models a physical situation object of study.

There is a first case, which became our first published work, when subdivision network and the combinatorial Laplacian operator were taken into account. So it can be understood as if the structural modification agrees with the idea of diffusion. Then, in our second work, we changed to the normalized Laplacian operator without modifying the network operation just to verify the compatibility of a random walk operator with an electrically equivalent modification of the domain. In third place, the problem was the hardest one

where we changed both, to a Schrödinger, more general, type operator and with a partial subdivision network operation, that is a slight generalization of the structural modification applied in the previous two cases offering the chance of expanding possibilities.

As it will be clear, the working way is quite similar in all the three situations. The starting point is a network Γ to be electrically subdivided and a difference linear operator. The given network is converted in Γ^S where a Poisson problem is posed for a consistent data function. The aim of the works (all three) is to obtain another consistent Poisson problem on the former, simpler, network. And then use a solution of it obtained by using the corresponding Green's function, to again obtain a solution of the initially posed Poisson problem on Γ^S. In this way we can set a relationship between both Green's functions that is interesting per se, of course, but that has a matrix lecture which is also worth it.

4.1 Subdivision networks for combinatorial Laplacian

The first and initiatory problem we faced in our study of subdivision networks, was the case of the combinatorial Laplacian operator $\mathcal{L}$ with a graph–theory inspired electrical subdivision of a given network Γ^S. That is, we considered a problem of diffusion on a simple network where vertices are indistinguishable except by their degree and all edges are substituted by a 2–length path setting, with its corresponding conductance function.

After the well known operation defined for graphs, we now say that a *subdivision network* $\Gamma^S = (V^S, E^S, c^S)$ of a given network $\Gamma = (V, E, c)$, is obtained by inserting a new vertex in every edge, so that all former edges $\{x, y\} \in E$ are replaced by only two new ones, say $\{x, v_{xy}\}$ and $\{y, v_{xy}\}$ being v_{xy} the new inserted vertex.

We denote by V' the new vertex set assuming that, $v_{xy} = v_{yx}$. Thus, $V^S = V \cup V'$, the order of the subdivision network is $n + m$, whereas the size is $2m$. Moreover, according to the well known rule that express the equivalent resistance of two resistors connected in series, we define the conductance function $c^S \colon V^S \times V^S \longrightarrow [0, +\infty)$ by choosing, for every pair of adjacent vertices, non–null values $c^S(x, v_{xy})$ and $c^S(y, v_{xy})$ such that

$$\frac{1}{c(x,y)} = \frac{1}{c^S(x, v_{xy})} + \frac{1}{c^S(y, v_{xy})}. \tag{4.1}$$

The definition of c^S (and those concepts depending upon it as the degree function for example) cannot be misunderstood because of notation as all the edges in E^S have both kind of vertices, one in V and the other in V'.

Hence, for the sake of simplicity, it will be denoted also as c (even though it has no the same meaning).

Moreover we point out that, for each edge, there exist infinitely many different choices of conductances fulfilling (4.1), so that different choices will lead to different subdivision networks.

Up to our knowledge, the only case that has been studied in the literature, ([31, 49, 71, 76]), is $c(x,y) = c(x,v_{xy}) = c(y,v_{xy}) = 1$, that not fulfills the electrical compatibility condition (4.1). In order to compare our study with the known results, we will consider as a particular case $c(x,y) = 1$ for all $x, y \in V$ and $c(x,v_{xy}) = c(y,v_{xy}) = 2$. We call it *standard subdivision graph.*

We remark that Γ^S is also a connected, finite, with no loops, nor multiple edges network.

4.1.1 Related Poisson problems

Let $\mathcal{L}^S$ be the combinatorial Laplacian of Γ^S. Then for any $u \in \mathcal{C}(V^S)$ we have that

$$\begin{aligned}\mathcal{L}^S(u)(x) &= \sum_{y\in V} c(x,v_{xy})\,(u(x) - u(v_{xy}))\,, \text{ for all } x \in V,\\ \mathcal{L}^S(u)(v_{xy}) &= c(x,v_{xy})\,(u(v_{xy}) - u(x))\\ &\quad + c(y,v_{xy})\,(u(v_{xy}) - u(y))\,, \text{ for all } v_{xy} \in V'.\end{aligned}$$

The aim of this section is to obtain a solution of a compatible Poisson problem in Γ^S in terms of the solution of an appropriate and also compatible Poisson problem on Γ.

It is helpful for the sequel to define, for each pair $x, y \in V$ with $x \sim y$, the coefficient

$$\alpha(x,y) = \frac{c(x,v_{xy})}{c(x,v_{xy}) + c(y,v_{xy})} = \frac{c(x,v_{xy})}{k(v_{xy})},$$

where $k(v_{xy}) = c(x,v_{xy}) + c(y,v_{xy})$, is the degree of v_{xy} in Γ^S. In additon, if $x \not\sim y$ we define $\alpha(x,y) = \alpha(y,x) = 0$.

Notice that trivially $\alpha(y,x) + \alpha(x,y) = 1$ and that $\alpha(x,y) = \alpha(y,x)$ iff $c(x,v_{xy}) = c(y,v_{xy}) = 2c(x,y)$ for any $x, y \in V$.

We will briefly take advantage of these two next definitions. In a first place and for every $h \in \mathcal{C}(V^S)$, we define the *contraction* of h to $V, \underline{h} \in \mathcal{C}(V)$, to be

$$\underline{h}(x) = h(x) + \sum_{y\sim x} \alpha(x,y) h(v_{xy}), \qquad \text{for all } x \in V. \tag{4.2}$$

Secondly, for every pair $u \in \mathcal{C}(V),\ h \in \mathcal{C}(V^S)$, we consider $u^h \in \mathcal{C}(V^S)$, the *extension of u to V^S with respect to h*, defined in the following sense

$$\begin{aligned} u^h(x) &= u(x), && \text{for all } x \in V; \\ u^h(v_{xy}) &= \frac{h(v_{xy})}{k(v_{xy})} + \alpha(x,y)u(x) + \alpha(y,x)u(y), && \text{for all } v_{xy} \in V'. \end{aligned} \tag{4.3}$$

Clearly the intention of the first definition is to adequately diffuse the value of a function h on the vertices v_{xy} in V' between its two neighbours. And we just use the coefficient $\alpha(x,y)$ to achieve this goal. On the contrary, the idea behind the concept of an extension with respect to, is just technical. We have created a function defined on V^S that clearly satisfies $\mathcal{L}^S(u^h)(v_{xy}) = h(v_{xy})$.

Thus, taking into account these definitions and the notation we have introduced we are now at the point we wanted to be and present our result.

Theorem 4.1.1. *Given $h \in \mathcal{C}(V^S)$ such that $\langle h, 1_{V^S}\rangle = 0$, then $\langle \underline{h}, 1_V\rangle = 0$. Moreover, $\overline{u} \in \mathcal{C}(V^S)$ is a solution of the Poisson equation $\mathcal{L}^S(\overline{u}) = h$ in V^S iff $u = \overline{u}_{|V}$ is a solution of the Poisson equation $\mathcal{L}(u) = \underline{h}$ in V. In this case, the identity $\overline{u} = u^h$ holds.*

Proof. Firstly we note that $\langle \underline{h}, 1_V\rangle = \langle h, 1_{V^S}\rangle$ as

$$\sum_{x\in V} \underline{h}(x) = \sum_{x\in V} h(x) + \sum_{x\in V}\sum_{y\sim x} \alpha(x,y)h(v_{xy}) = \sum_{x\in V} h(x) + \sum_{v_{xy}\in V'} h(v_{xy}).$$

So the first statement holds.

Given $h \in \mathcal{C}(V^S)$ such that $\langle h, 1_{V^S}\rangle = 0$ and $\overline{u}$ a solution of the Poisson equation $\mathcal{L}^S(\overline{u}) = h$ in V^S, then

$$\begin{aligned} h(v_{xy}) &= c(x,v_{xy})\left(\overline{u}(v_{xy}) - \overline{u}(x)\right) + c(y,v_{xy})\left(\overline{u}(v_{xy}) - \overline{u}(y)\right),\ v_{xy} \in V'; \\ h(x) &= \sum_{y\sim x} c(x,v_{xy})\left(\overline{u}(x) - \overline{u}(v_{xy})\right),\ x \in V. \end{aligned}$$

The first identity implies $\overline{u}(v_{xy}) = u^h(v_{xy})$, assuming $u = \overline{u}_{|V}$. Then, substituting the expression of $\overline{u}(v_{xy})$ in the second one, we obtain that

$$\begin{aligned} \mathcal{L}^S(\overline{u})(x) &= \sum_{y\sim x} c(x,v_{xy})\left(\overline{u}(x) - \frac{h(v_{xy})}{k(v_{xy})} - \alpha(x,y)\overline{u}(x) - \alpha(y,x)\overline{u}(y)\right) \\ &= \sum_{y\sim x} c(x,v_{xy})\alpha(y,x)\left(\overline{u}(x) - \overline{u}(y)\right) - \sum_{y\sim x}\frac{c(x,v_{xy})}{k(v_{xy})}h(v_{xy}) \\ &= \sum_{y\sim x} c(x,y)\left(u(x) - u(y)\right) - \sum_{y\sim x}\alpha(x,y)h(v_{xy}) \\ &= \mathcal{L}(u)(x) - \underline{h}(x) + h(x), \end{aligned}$$

for every $x \in V$. Therefore, $\mathcal{L}^S(\overline{u}) = h$ in V^S iff $\mathcal{L}(u) = \underline{h}$ in V. □

Once we have found out the intimate relationship between the solutions set of two Poisson problems posed on a subdivision network and on the network which has been electrically subdivided respectively, we are now ready to address the computations of the Green's function.

Thus, in order to obtain the Green's function that we are interested in, our next result shows how to obtain the unique solution of a Poisson problem on the subdivision network Γ^S that is orthogonal to 1_{V^S} in relation to the corresponding Green's function of a related Poisson problem on Γ.

Corollary 4.1.2. *Given $h \in \mathcal{C}(V^S)$, such that $\langle h, 1_{V^S}\rangle = 0$, let $u \in \mathcal{C}(V)$ be the unique solution of $\mathcal{L}(u) = \underline{h}$ that satisfies $\langle u, 1_V\rangle = 0$ and the constant*

$$\lambda = -\frac{1}{(n+m)} \sum_{x\sim y} \frac{h(v_{xy})}{k(v_{xy})} - \frac{1}{(n+m)} \sum_{x\sim y} \left(\alpha(x,y)u(x) + \alpha(y,x)u(y) \right).$$

Then, $u^\perp = u^h + \lambda$ is the unique solution of $\mathcal{L}^S(u^\perp) = h$ that satisfies $\langle u^\perp, 1_{V^S}\rangle = 0$.

Proof. As two solutions of a Poisson problem differ on a constant, we have that $u^\perp = u^h + \gamma 1_{V^S}$, $\gamma \in \mathbb{R}$. Then,

$$\begin{aligned} 0 = \langle u^\perp, 1_{V^S}\rangle &= \langle u^h, 1_{V^S}\rangle + (n+m)\gamma = \sum_{x\in V} u(x) + \sum_{x\sim y} u^h(v_{xy}) + (n+m)\gamma \\ &= \sum_{x\sim y} \frac{h(v_{xy})}{k(v_{xy})} + \sum_{x\sim y} (\alpha(x,y)u(x) + \alpha(y,x)u(y)) + (n+m)\gamma, \end{aligned}$$

because $\langle u, 1_V\rangle = 0$, and the result follows taking $\gamma = \lambda$. □

4.1.2 Related Green's functions

Taking into account the relation between Poisson problems on Γ^S and Γ, we obtain the expression of Green's kernel of a subdivision network, G^S, in terms of Green's kernel of the base network. From now on we consider the function on $\mathcal{C}(V)$, $\pi^S(x) = \sum_{y\sim x} \alpha(x,y)$ and the constant

$$\beta = \frac{1}{(n+m)^2} \sum_{x,y\in V} G(x,y)\pi^S(x)\pi^S(y) + \frac{1}{(n+m)^2} \sum_{x\sim y} \frac{1}{k(v_{xy})}.$$

Proposition 4.1.3. *Let Γ^S be the subdivision network of Γ, then for any $x, z \in V$ and $v_{xy}, v_{zt} \in V'$, the Green kernel of Γ^S is given by*

$$\begin{aligned}
G^S(x,z) &= G(x,z) - \frac{1}{n+m}\sum_{\ell\in V}\Big[G(x,\ell)+G(z,\ell)\Big]\pi^S(\ell) + \beta,\\
G^S(v_{xy},z) &= \alpha(x,y)G(x,z) + \alpha(y,x)G(y,z)\\
&\quad - \frac{1}{n+m}\sum_{\ell\in V}\Big[\alpha(x,y)G(x,\ell)+\alpha(y,x)G(y,\ell)+G(z,\ell)\Big]\pi^S(\ell)\\
&\quad - \frac{1}{(n+m)k(v_{xy})} + \beta,\\
G^S(v_{xy},v_{zt}) &= \alpha(z,t)\Big(\alpha(x,y)G(x,z)+\alpha(y,x)G(y,z)\Big)\\
&\quad + \alpha(t,z)\Big(\alpha(x,y)G(x,t)+\alpha(y,x)G(y,t)\Big)\\
&\quad - \frac{1}{n+m}\sum_{\ell\in V}\Big[\alpha(x,y)G(x,\ell)+\alpha(y,x)G(y,\ell)\Big]\pi^S(\ell)\\
&\quad - \frac{1}{n+m}\sum_{\ell\in V}\Big[\alpha(z,t)G(z,\ell)+\alpha(t,z)G(t,\ell)\Big]\pi^S(\ell)\\
&\quad + \frac{\varepsilon_{v_{zt}}(v_{xy})}{k(v_{xy})} - \frac{1}{(n+m)k(v_{xy})} - \frac{1}{(n+m)k(v_{zt})} + \beta.
\end{aligned}$$

Proof. Suppose $z \in V$, and let $h_z = \varepsilon_z - \dfrac{1}{n+m}$. Then, for every $x \in V$

$$\underline{h}_z(x) = \varepsilon_z(x) - \frac{1}{n+m} - \frac{1}{n+m}\sum_{y\sim x}\alpha(x,y) = \varepsilon_z(x) - \frac{1}{n+m}(1+\pi^S(x)).$$

Hence, we now need to solve the Poisson problem $\mathcal{L}(u_z) = \underline{h}_z$. Using the Green kernel for Γ, we obtain

$$u_z(x) = G(\varepsilon_z)(x) - \frac{1}{n+m}\sum_{\ell\in V}G(x,\ell)\pi^S(\ell) = G(x,z) - \frac{1}{n+m}\sum_{\ell\in V}G(x,\ell)\pi^S(\ell).$$

Then, from Corollary 4.3.2

$$\begin{aligned}
G^S_z(x) &= u^{h_z}_z(x) - \frac{1}{(n+m)}\sum_{r\sim s}\frac{h_z(v_{rs})}{k(v_{rs})}\\
&\quad - \frac{1}{(n+m)}\sum_{r\sim s}[\alpha(r,s)u_z(r)+\alpha(s,r)u_z(s)]
\end{aligned}$$

and

$$\begin{aligned}
G_z^S(x) &= G(x,z) - \frac{1}{n+m}\sum_{\ell\in V} G(x,\ell)\pi^S(\ell) + \frac{1}{(n+m)^2}\sum_{r\sim s}\frac{1}{k(v_{rs})} \\
&\quad - \frac{1}{(n+m)}\sum_{r\sim s}\alpha(r,s)\left[G(r,z) - \frac{1}{n+m}\sum_{\ell\in V} G(r,\ell)\pi^S(\ell)\right] \\
&\quad - \frac{1}{(n+m)}\sum_{r\sim s}\alpha(s,r)\left[G(s,z) - \frac{1}{n+m}\sum_{\ell\in V} G(s,\ell)\pi^S(\ell)\right] \\
&= G(x,z) - \frac{1}{n+m}\sum_{\ell\in V}\Big[G(x,\ell) + G(z,\ell)\Big]\pi^S(\ell) \\
&\quad + \frac{1}{(n+m)^2}\sum_{r,s} G(s,r)\pi^S(r)\pi^S(s) + \frac{1}{(n+m)^2}\sum_{r\sim s}\frac{1}{k(v_{rs})}.
\end{aligned}$$

Now if $z \in V$, then for every $v_{xy} \in V'$

$$\begin{aligned}
G_z^S(v_{xy}) &= \frac{h_z(v_{xy})}{k(v_{xy})} + \alpha(x,y)u_z(x) + \alpha(y,x)u_z(y) \\
&\quad - \frac{1}{(n+m)}\sum_{r\sim s}\frac{h_z(v_{rs})}{k(v_{rs})} - \frac{1}{(n+m)}\sum_{r\sim s}[\alpha(r,s)u_z(r) + \alpha(s,r)u_z(s)]
\end{aligned}$$

$$\begin{aligned}
G_z^S(v_{xy}) &= -\frac{1}{(n+m)k(v_{xy})} + \alpha(x,y)G(x,z) + \alpha(y,x)G(y,z) \\
&\quad - \frac{1}{n+m}\sum_{\ell\in V}\big[\alpha(x,y)G(x,\ell) + \alpha(y,x)G(y,\ell)\big]\pi^S(\ell) \\
&\quad + \frac{1}{(n+m)^2}\sum_{r\sim s}\frac{1}{k(v_{rs})} \\
&\quad - \frac{1}{(n+m)}\sum_{r\sim s}\alpha(r,s)\left[G(r,z) - \frac{1}{n+m}\sum_{\ell\in V} G(r,\ell)\pi^S(\ell)\right] \\
&\quad - \frac{1}{(n+m)}\sum_{r\sim s}\alpha(s,r)\left[G(s,z) - \frac{1}{n+m}\sum_{\ell\in V} G(s,\ell)\pi^S(\ell)\right]
\end{aligned}$$

$$\begin{aligned}
G_z^S(v_{xy}) &= -\frac{1}{(n+m)k(v_{xy})} + \alpha(x,y)G(x,z) + \alpha(y,x)G(y,z) \\
&\quad - \frac{1}{n+m}\sum_{\ell\in V}\big[\alpha(x,y)G(x,\ell) + \alpha(y,x)G(y,\ell)\big]\pi^S(\ell) \\
&\quad + \frac{1}{(n+m)^2}\sum_{r\sim s}\frac{1}{k(v_{rs})} - \frac{1}{n+m}\sum_{\ell\in V}\Big[G(z,\ell)\Big]\pi^S(\ell) \\
&\quad + \frac{1}{(n+m)^2}\sum_{r,s} G(s,r)\pi^S(r)\pi^S(s)
\end{aligned}$$

So finally,

$$\begin{aligned} G_z^S(v_{xy}) = &-\frac{1}{(n+m)k(v_{xy})} + \alpha(x,y)G(x,z) + \alpha(y,x)G(y,z) \\ &- \frac{1}{n+m}\sum_{\ell\in V}\big[\alpha(x,y)G(x,\ell) + \alpha(y,x)G(y,\ell) + G(z,\ell)\big]\pi^S(\ell) \\ &+ \frac{1}{(n+m)^2}\sum_{r\sim s}\frac{1}{k(v_{rs})} + \frac{1}{(n+m)^2}\sum_{r,s}G(s,r)\pi^S(r)\pi^S(s). \end{aligned}$$

Suppose now we fix $v_{zt} \in V$. The compatible data function to take into account is $h_{v_{zt}} = \varepsilon_{v_{zt}} - \dfrac{1}{n+m}$. Then, for every $x \in V$ the contraction to be used in the basis network Γ is

$$\begin{aligned} \underline{h}_{v_{zt}}(x) &= \varepsilon_{v_{zt}}(x) - \frac{1}{n+m} + \sum_{y\in V}\alpha(x,y)\left(\varepsilon_{v_{zt}}(v_{xy}) - \frac{1}{n+m}\right) \\ &= -\frac{1}{n+m}(1+\pi^S(x)) + \alpha(z,t)\varepsilon_z(x) + \alpha(t,z)\varepsilon_t(x). \end{aligned}$$

Hence, the Poisson problem to solve is $\mathcal{L}(u_{v_{zt}}) = \underline{h}_{v_{zt}}$, and, using Green's kernel for Γ, we obtain that the solution to be extended is

$$u_{v_{zt}}(x) = -\frac{1}{n+m}\sum_{\ell\in V}G(x,\ell)\pi^S(\ell) + \alpha(z,t)G(x,z) + \alpha(t,z)G(x,t).$$

Then, by applying again Corollary 4.3.2 Green's function on Γ^S on the new generated vertices is

$$\begin{aligned} G_{v_{zt}}^S(v_{xy}) = &\frac{h_{v_{zt}}(v_{xy})}{k(v_{xy})} + \alpha(x,y)u_{v_{zt}}(x) + \alpha(y,x)u_{v_{zt}}(y) \\ &- \frac{1}{(n+m)}\sum_{r\sim s}\frac{h_{v_{zt}}(v_{rs})}{k(v_{rs})} \\ &- \frac{1}{(n+m)}\sum_{r\sim s}[\alpha(r,s)u_{v_{zt}}(r) + \alpha(s,r)u_{v_{zt}}(s)] \end{aligned}$$

Now substituting we get that

$$\begin{aligned}
G^S_{v_{zt}}(v_{xy}) = {} & \frac{\varepsilon_{v_{zt}}(v_{xy})}{k(v_{xy})} - \frac{1}{(n+m)k(v_{xy})} \\
& - \frac{1}{n+m}\sum_{\ell\in V}\Big(\alpha(x,y)G(x,\ell)+\alpha(y,x)G(y,\ell)\Big)\pi^S(\ell) \\
& + \alpha(z,t)\Big(\alpha(x,y)G(x,z)+\alpha(y,x)G(y,z)\Big) \\
& + \alpha(t,z)\Big(\alpha(x,y)G(x,t)+\alpha(y,x)G(y,t)\Big) \\
& - \frac{1}{(n+m)k(v_{zt})} + \frac{1}{(n+m)^2}\sum_{r\sim s}\frac{1}{k(v_{rs})} \\
& - \frac{1}{n+m}\sum_{\ell\in V}\Big(\alpha(z,t)G(z,\ell)+\alpha(t,z)G(t,\ell)\Big)\pi^S(\ell) \\
& + \quad \frac{1}{(n+m)^2}\sum_{r,s\in V} G(r,s)\pi^S(r)\pi^S(s).
\end{aligned}$$

And finally,

$$\begin{aligned}
G^S_{v_{zt}}(v_{xy}) = {} & \alpha(z,t)\Big(\alpha(x,y)G(x,z)+\alpha(y,x)G(y,z)\Big) \\
& + \alpha(t,z)\Big(\alpha(x,y)G(x,t)+\alpha(y,x)G(y,t)\Big) \\
& - \frac{1}{n+m}\sum_{\ell\in V}\Big(\alpha(x,y)G(x,\ell)+\alpha(y,x)G(y,\ell)\Big)\pi^S(\ell) \\
& - \frac{1}{n+m}\sum_{\ell\in V}\Big(\alpha(z,t)G(z,\ell)+\alpha(t,z)G(t,\ell)\Big)\pi^S(\ell) \\
& + \frac{\varepsilon_{v_{zt}}(v_{xy})}{k(v_{xy})} - \frac{1}{(n+m)k(v_{xy})} - \frac{1}{(n+m)k(v_{zt})} \\
& + \quad \frac{1}{(n+m)^2}\sum_{r\sim s}\frac{1}{k(v_{rs})} + \frac{1}{(n+m)^2}\sum_{r,s\in V} G(r,s)\pi^S(r)\pi^S(s). \quad \square
\end{aligned}$$

In particular, if Γ is a k–regular graph and we consider the standard subdivision graph; that is $c(x,v_{xy}) = c(y,v_{xy}) = 2$, we get the following result.

Corollary 4.1.4. *Let Γ^S be the standard subdivision graph of a k–regular graph, Γ; then for any $x,z\in V$ and $v_{xy},v_{zt}\in V'$, the Green kernel of Γ^S is*

given by

$$G^S(x,z) = G(x,z) + \frac{k}{2n(2+k)^2},$$

$$G^S(v_{xy},z) = \frac{1}{2}\Big(G(x,z) + G(y,z)\Big) - \frac{1}{n(2+k)^2},$$

$$G^S(v_{xy},v_{zt}) = \frac{1}{4}\Big(G(x,z) + G(y,z) + G(x,t) + G(y,t) + \varepsilon_{v_{zt}}(v_{xy})\Big) - \frac{(4+k)}{2n(2+k)^2}.$$

4.2 Subdivision networks for the normalized Laplacian

Our second step in exploring the possibilities for the electrical subdivision procedure was set in the context of random walks and Markov chains, as we studied also the compatibility of related Poisson problems on a given network and its subdivision counterpart for the well known normalized Laplacian operator. So, the idea of the modification of a given network was not at stake at that moment, but the idea of the Poisson problem was no more a typical diffusion setting.

The conclusion of our study was at first glance a little bit disappointing as the result we obtained forces a very restrictive idea of electrical subdivision for this operator. However, due to the particular use of the degree concept in the definition of the normalized Laplacian operator, with plenty of square roots of sums, subdivision of edges must follow a very particular pattern because of technicalities. Hence the exciting network operation of subdivision has a rather short run when used in such a random walks scenario. But this, at the end, is in perfect concordance with the mismatching of the concepts of random walks and electrical circuits or flows, and subdivision procedure as successful as it were in the previous case of a diffusion problem. Or, explained in other words, the relation of conductances (or resistances, we don't care) in electrical circuits with $\hat{c}(x,y) = \dfrac{c(x,y)}{\sqrt{\kappa(x)}\sqrt{\kappa(y)}}$, when considering $\omega = \dfrac{1}{2m}\sqrt{\kappa}$ and obviously $\lambda = 0$ is, let's say, at least rather intrincated.

Thus a *subdivision network* $\Gamma^S = (V^S, E^S, c^S)$ of Γ is now obtained in a very similar way than the previous scenario, by inserting a new vertex in every edge, so that each $\{x,y\} \in E$ is replaced by two new edges, say $\{x, v_{xy}\}$ and $\{v_{xy}, y\}$ where v_{xy} is the new inserted vertex, see [75] where a classical subdivision process is considered. The important point now is that we have had to define conductances on the new edges as $c^S(x,v_{xy}) = c^S(y,v_{xy}) =$

$2c(x,y)$, so that, electrical compatibility is still fulfilled, but in a somewhat graph but not network manner. So it is still $\dfrac{1}{c(x,y)} = \dfrac{1}{c^S(x,v_{xy})} + \dfrac{1}{c^S(y,v_{xy})}$ but more in detail it turns $\dfrac{1}{c(x,y)} = \dfrac{1}{2c(x,y)} + \dfrac{1}{2c(y,x)}$ and there is just one possibility to accomplish the network structural operation. Of course then the degree function on Γ^S, $k^S \in \mathcal{C}(V^S)$, satisfies $k^S(x) = 2k(x)$ for any $x \in V$, and $k^S(v_{xy}) = 4c(x,y)$ for those vertices in V'. Moreover, it holds that $\mathrm{vol}(\Gamma^S) = 4\mathrm{vol}(\Gamma)$.

So we now present the precise relationship between a solution of a compatible Poisson problem for the normalized Laplacian on a subdivision network Γ^S and a solution of a conveniently well posed Poisson problem for the normalized Laplacian on the base network Γ.

4.2.1 Related Poisson problems

Mimicking the preceding subsection 4.1.1, we also recall what is the normalized Laplacian operator for a subdivision network, denoted now as $\mathscr{L}^S$, and defined for any $u \in \mathcal{C}(V^S)$ as

$$\mathscr{L}^S(u)(x) = \frac{1}{\sqrt{\kappa^S(x)}} \sum_{y \in V} c(x, v_{xy}) \left(\frac{u(x)}{\sqrt{\kappa^S(x)}} - \frac{u(v_{xy})}{\sqrt{\kappa^S(v_{xy})}} \right), x \in V,$$

$$\begin{aligned} \mathscr{L}^S(u)(v_{xy}) = \frac{1}{\sqrt{\kappa^S(v_{xy})}} \Bigg\{ & c(x, v_{xy}) \left(\frac{u(v_{xy})}{\sqrt{\kappa^S(v_{xy})}} - \frac{u(x)}{\sqrt{\kappa^S(x)}} \right) \\ & + c(y, v_{xy}) \left(\frac{u(v_{xy})}{\sqrt{\kappa^S(v_{xy})}} - \frac{u(y)}{\sqrt{\kappa^S(y)}}) \right) \Bigg\}, v_{xy} \in V'. \end{aligned}$$

As we proceed similarly to the previous case, we also put in place two functional operators, let's say a contraction and an extension related to.

Let $h \in \mathcal{C}(V^S)$ we define its *contraction to* $\mathcal{C}(V)$ as

$$\underline{h}(x) = h(x) + \frac{1}{\sqrt{2k(x)}} \sum_{y \sim x} \sqrt{c(x,y)}\, h(v_{xy}).$$

Now unfortunately with no ideological context, just because it works. Honestly, we have never seen in the literature a significance for $\sqrt{c(x,y)}$ and we have arrived to non satisfactory conclusions when trying to understand what could be a possible meaning for it.

Also we briefly will consider, for a couple $u \in \mathcal{C}(V)$ and $h \in \mathcal{C}(V^S)$ the *extension* of u (the former) related to h (the latter) to $\mathcal{C}(V^S)$, such as

$$u^h(v_{xy}) = h(v_{xy}) + \frac{\sqrt{c(x,y)}}{\sqrt{2}} \left(\frac{u(x)}{\sqrt{k(x)}} + \frac{u(y)}{\sqrt{k(y)}} \right)$$

for $v_{xy} \in V'$, while $u^h(x) = u(x)$ for those vertices in V.

The following result links the solution of a given compatible Poisson problem in the subdivision network with an appropriate and also compatible Poisson problem on the base network.

Theorem 4.2.1. *Given $h \in \mathcal{C}(V^S)$ such that $\langle h, \sqrt{k^S}\rangle_{V^S} = 0$, then $\langle \underline{h}, \sqrt{k}\rangle_V = 0$. Moreover, $\overline{u} \in \mathcal{C}(V^S)$ is a solution of the Poisson equation $\mathscr{L}_S(\overline{u}) = h$ in V^S iff $u = \overline{u}_{|_V}$ is a solution of the Poisson equation $\mathscr{L}(u) = 2\underline{h}$ in V. In this case, the identity $\overline{u} = u^h$ holds.*

Proof. Firstly we note that $\left\langle \underline{h}, \sqrt{k} \right\rangle_V = \dfrac{1}{\sqrt{2}} \left\langle h, \sqrt{k^S} \right\rangle_{V^S}$ as

$$\begin{aligned}\sum_{x\in V} \underline{h}(x)\sqrt{k(x)} &= \sum_{x\in V} h(x)\sqrt{\frac{k^S(x)}{2}} \\ &\quad + \frac{1}{\sqrt{2}} \sum_{x\in V} \frac{1}{\sqrt{k(x)}} \sum_{y\sim x} \sqrt{c(x,y)} h(v_{xy}) \sqrt{k(x)} \\ &= \frac{1}{\sqrt{2}} \sum_{x\in V} h(x)\sqrt{k^S(x)} + \frac{1}{\sqrt{2}} \sum_{v_{xy}\in V'} h(v_{xy})\sqrt{k^S(v_{xy})}.\end{aligned}$$

So the first statement holds. Then,

$$\begin{aligned}\mathscr{L}^S\ \overline{u}(v_{xy}) &= \overline{u}(v_{xy}) - \frac{c^S(v_{xy},x)}{\sqrt{k^S(v_{xy})}\sqrt{k^S(x)}}\overline{u}(x) - \frac{c^S(v_{xy},y)}{\sqrt{k^S(v_{xy})}\sqrt{k^S(y)}}\overline{u}(y) \\ &= \overline{u}(v_{xy}) - \frac{\sqrt{c(x,y)}}{\sqrt{2}}\frac{\overline{u}(x)}{\sqrt{k(x)}} - \frac{\sqrt{c(x,y)}}{\sqrt{2}}\frac{\overline{u}(y)}{\sqrt{k(y)}}.\end{aligned}$$

So we obtain that,

$$\overline{u}(v_{xy}) = \mathscr{L}^S\ \overline{u}(v_{xy}) + \frac{\sqrt{c(x,y)}}{\sqrt{2}} \left(\frac{\overline{u}(x)}{\sqrt{k(x)}} + \frac{\overline{u}(y)}{\sqrt{k(y)}} \right).$$

Also, for the former vertex in the given network,

$$\mathscr{L}^S\ \overline{u}(x) = \frac{1}{\sqrt{k^S(x)}} \sum_{v_{xy}\sim x} c^S(x, v_{xy}) \left[\frac{\overline{u}(x)}{\sqrt{k^S(x)}} - \frac{\overline{u}(v_{xy})}{\sqrt{k^S(v_{xy})}} \right].$$

Substituting the precedent expression for $\overline{u}(v_{xy})$ we obtain

$$\begin{aligned}\mathscr{L}^S\ \overline{u}(x) &= \sum_{y\sim x}\frac{c(x,y)}{2\sqrt{k(x)}}\left[\frac{2\overline{u}(x)}{\sqrt{k(x)}} - \frac{\sqrt{2}\mathscr{L}^S\ \overline{u}(v_{xy})}{\sqrt{c(x,y)}} - \frac{\overline{u}(x)}{\sqrt{k(x)}} - \frac{\overline{u}(y)}{\sqrt{k(y)}}\right]\\ &= \frac{1}{2\sqrt{k(x)}}\sum_{y\sim x} c(x,y)\left(\frac{\overline{u}(x)}{\sqrt{k(x)}} - \frac{\overline{u}(y)}{\sqrt{k(y)}}\right)\\ &- \frac{1}{\sqrt{2k(x)}}\sum_{y\sim x}\sqrt{c(x,y)}\mathscr{L}^S\overline{u}(v_{xy}).\end{aligned}$$

Finally, we get, if $u = \overline{u}_{|_V}$

$$\mathscr{L}^S(u)(x) = \frac{1}{2}\mathscr{L}(u)(x) - \frac{1}{\sqrt{2k(x)}}\sum_{v_{xy}\sim x}\sqrt{c(x,y)}\mathscr{L}^S\overline{u}(v_{xy}). \quad \square$$

So, for short, we have proven that given a Poisson problem with a compatible data function on Γ^S, it can be contracted to a compatible data function on Γ so as a solution of this related Poisson problem on Γ expands to a solution of the initial given Poisson problem on the subdivided network.

Now, in order to choose a particular solution of the Poisson problem on the subdivided network, as in the previous section we can find the precise value for the constant because the following holds.

Corollary 4.2.2. *Given $h \in \mathcal{C}(V^S)$, such that $\langle h, \sqrt{k^S}\rangle_{V^S} = 0$, let $\underline{h} \in \mathcal{C}(V)$ be its contraction to V, let $u \in \mathcal{C}(V)$ be the unique solution of $\mathscr{L}(u) = 2\underline{h}$ that satisfies $\langle u, \sqrt{k}\rangle_V = 0$ and let be the constant*

$$\lambda = -\frac{1}{2\mathrm{vol}(\Gamma)}\sum_{r\sim s} h(v_{rs})\sqrt{c(r,s)}.$$

Then, $u^\perp = u^h + \lambda\sqrt{k^S} \in \mathcal{C}(V^S)$ is the unique solution of the Poisson problem $\mathscr{L}^S(u) = h$ that satisfies $\langle u^\perp, \sqrt{k^S}\rangle_{V^S} = 0$. Specifically,

$$\begin{aligned}u^\perp(x) &= u(x) - \frac{\sqrt{k(x)}}{\sqrt{2}\mathrm{vol}(\Gamma)}\sum_{r\sim s} h(v_{rs})\sqrt{c(r,s)},\\ u^\perp(v_{xy}) &= h(v_{xy}) + \frac{\sqrt{c(x,y)}}{\sqrt{2}}\left(\frac{u(x)}{\sqrt{k(x)}} + \frac{u(y)}{\sqrt{k(y)}}\right)\\ &- \frac{\sqrt{c(x,y)}}{\mathrm{vol}(\Gamma)}\sum_{r\sim s} h(v_{rs})\sqrt{c(r,s)},\end{aligned}$$

for any $x \in V$ and $v_{xy} \in V'$.

Proof. As two solutions differ on a multiple of $\sqrt{k^S}$, we have that $u^\perp = u^h + \gamma\sqrt{k^S}$, $\gamma \in \mathbb{R}$. Then,

$$\begin{aligned} 0 &= \langle u^\perp, \sqrt{k^S}\rangle_{V^S} = \langle u^h, \sqrt{k^S}\rangle_{V^S} + \gamma \sum_{x\in V^S} k^S(x) \\ &= \sqrt{2}\sum_{x\in V} u(x)\sqrt{k(x)} + \sum_{v_{xy}\in V'} u^h(v_{xy})\sqrt{k^S(v_{xy})} + \gamma \mathrm{vol}(\Gamma^S) \\ &= 2\sum_{v_{xy}\in V'} u^h(v_{xy})\sqrt{c(x,y)} + 4\gamma \mathrm{vol}(\Gamma), \end{aligned}$$

because $\langle u, \sqrt{k}\rangle_V = 0$, and hence

$$\begin{aligned} \lambda &= -\frac{1}{2\mathrm{vol}(\Gamma)}\sum_{r\sim s} u^h(v_{rs})\sqrt{c(r,s)} \\ &= -\frac{1}{2\mathrm{vol}(\Gamma)}\sum_{r\sim s}\left[h(v_{rs})\sqrt{c(r,s)} + \frac{c(r,s)}{\sqrt{2}}\left(\frac{u(r)}{\sqrt{k(r)}} + \frac{u(s)}{\sqrt{k(s)}}\right)\right] \\ &= -\frac{1}{2\mathrm{vol}(\Gamma)}\sum_{r\sim s} h(v_{rs})\sqrt{c(r,s)} - \frac{1}{2\sqrt{2}\mathrm{vol}(\Gamma)}\sum_{r\in V}\frac{u(r)}{\sqrt{k(r)}}\sum_{s\sim r} c(r,s) \\ &= -\frac{1}{2\mathrm{vol}(\Gamma)}\sum_{r\sim s} h(v_{rs})\sqrt{c(r,s)} - \frac{1}{2\sqrt{2}\mathrm{vol}(\Gamma)}\sum_{r\in V} u(r)\sqrt{k(r)} \\ &= -\frac{1}{2\mathrm{vol}(\Gamma)}\sum_{r\sim s} h(v_{rs})\sqrt{c(r,s)}. \end{aligned}$$

□

4.2.2 Related Green's functions

Taking into account the relation between both Poisson problems for the normalized Laplacian on Γ^S and on Γ, we obtain the expression of the Green function for the normalized Laplacian of the subdivision network G^S, in terms of the Green function of the base network G.

Theorem 4.2.3. *Let Γ^S be the subdivision network of Γ, then for any $x, z \in V$ and $v_{xy}, v_{zt} \in V'$, the Green function of Γ^S is given by*

$$\begin{aligned} G^S(x,z) &= 2G(x,z) + \frac{\sqrt{k(x)}\sqrt{k(z)}}{4\mathrm{vol}(\Gamma)}, \\ G^S(v_{xy},z) &= \sqrt{2}\sqrt{c(x,y)}\left(\frac{G(x,z)}{\sqrt{k(x)}} + \frac{G(y,z)}{\sqrt{k(y)}} - \frac{\sqrt{k(z)}}{4\mathrm{vol}(\Gamma)}\right), \\ G^S(v_{xy},v_{zt}) &= \sqrt{c(x,y)c(z,t)}\left(\frac{G(x,z)}{\sqrt{k(x)k(z)}} + \frac{G(x,t)}{\sqrt{k(x)k(t)}} + \frac{G(y,z)}{\sqrt{k(y)k(z)}} + \frac{G(y,t)}{\sqrt{k(y)k(t)}}\right) \\ &\quad - \frac{3\sqrt{c(x,y)c(z,t)}}{2\mathrm{vol}(\Gamma)} + \varepsilon_{v_{zt}}(v_{xy}). \end{aligned}$$

Proof. For the first case, suppose $z \in V$, and let $h_z = \varepsilon_z - \dfrac{\sqrt{k^S(z)}}{4\mathrm{vol}(\Gamma)}\sqrt{k^S}$. After Theorem 4.2.1, for every $x \in V$ the data function to be used for the Poisson problem on Γ must be

$$\begin{aligned}\underline{h_z}(x) &= \varepsilon_z(x) - \frac{\sqrt{k^S(z)}\sqrt{k^S(x)}}{4\mathrm{vol}(\Gamma)} - \frac{1}{\sqrt{2}}\frac{\sqrt{k^S(z)}}{4\mathrm{vol}(\Gamma)}\sum_{y\sim x}\frac{\sqrt{c(x,y)}}{\sqrt{k(x)}}\sqrt{k^S(v_{xy})} \\ &= \varepsilon_z(x) - \frac{\sqrt{k^S(x)}\sqrt{k^S(z)}}{2\mathrm{vol}(\Gamma)} = \varepsilon_z(x) - \frac{\sqrt{k(x)}\sqrt{k(z)}}{\mathrm{vol}(\Gamma)}.\end{aligned}$$

The unique solution to the Poisson problem $\mathscr{L}(u_z) = 2\underline{h_z}$, orthogonal to $\sqrt{k}$, using the Green function for Γ, is $u_z(x) = 2G(x,z)$, and from Corollary 4.3.2

$$\begin{aligned}G^S(x,z) &= 2G(x,z) + \frac{\sqrt{k(x)}\sqrt{k(z)}}{4\mathrm{vol}(\Gamma)^2}\sum_{r\sim s}\sqrt{k^S(v_{rs})}\sqrt{c(r,s)} \\ &= 2G(x,z) + \frac{\sqrt{k(x)}\sqrt{k(z)}}{2\mathrm{vol}(\Gamma)^2}\sum_{r\sim s}c(r,s) = 2G(x,z) + \frac{\sqrt{k(x)}\sqrt{k(z)}}{4\mathrm{vol}(\Gamma)}.\end{aligned}$$

On the other hand, for every $v_{xy} \in V'$,

$$\begin{aligned}G^S(v_{xy},z) &= \sqrt{2}\sqrt{c(x,y)}\left(\frac{G(x,z)}{\sqrt{k(x)}} + \frac{G(y,z)}{\sqrt{k(y)}}\right) \\ &\quad - \frac{\sqrt{k(z)}\sqrt{c(x,y)}}{\sqrt{2}\mathrm{vol}(\Gamma)} + \frac{\sqrt{c(x,y)}\sqrt{k(z)}}{\sqrt{2}\mathrm{vol}(\Gamma)^2}\sum_{r\sim s}c(r,s) \\ &= \sqrt{2}\sqrt{c(x,y)}\left(\frac{G(x,z)}{\sqrt{k(x)}} + \frac{G(y,z)}{\sqrt{k(y)}} - \frac{\sqrt{k(z)}}{4\mathrm{vol}(\Gamma)}\right).\end{aligned}$$

Finally, we complete the proof by considering the case where the pole is a new generated vertex by the subdivision procedure. So suppose now $v_{zt} \in V'$, and let $h_{v_{zt}} = \varepsilon_{v_{zt}} - \dfrac{\sqrt{c(z,t)}}{2\mathrm{vol}(\Gamma)}\sqrt{k^S}$. Then, for every $x \in V$

$$\begin{aligned}
\underline{h_{v_{zt}}}(x) &= -\frac{\sqrt{2}\sqrt{c(z,t)}}{2\mathrm{vol}(\Gamma)}\sqrt{k(x)} + \frac{1}{\sqrt{2}}\sum_{y\sim x}\frac{\sqrt{c(x,y)}}{\sqrt{k(x)}} \\
&\quad - \left(\varepsilon_{v_{zt}}(v_{xy}) - \frac{\sqrt{c(z,t)}}{\mathrm{vol}(\Gamma)}\sqrt{c(x,y)}\right) \\
&= -\frac{2\sqrt{c(z,t)}}{\sqrt{2}\mathrm{vol}(\Gamma)}\sqrt{k(x)} + \frac{1}{\sqrt{2}}\left[\frac{\sqrt{c(z,t)}}{\sqrt{k(z)}}\varepsilon_z(x) + \frac{\sqrt{c(z,t)}}{\sqrt{k(t)}}\varepsilon_t(x)\right] \\
&= \frac{1}{\sqrt{2}}\frac{\sqrt{c(z,t)}}{\sqrt{k(z)}}\left(\varepsilon_z(x) - \frac{\sqrt{k(z)}}{\mathrm{vol}(\Gamma)}\sqrt{k(x)}\right) \\
&\quad + \frac{1}{\sqrt{2}}\frac{\sqrt{c(z,t)}}{\sqrt{k(t)}}\left(\varepsilon_t(x) - \frac{\sqrt{k(t)}}{\mathrm{vol}(\Gamma)}\sqrt{k(x)}\right).
\end{aligned}$$

Hence, the Poisson problem to solve is $\mathscr{L}(u_{v_{zt}}) = 2\underline{h_{v_{zt}}}$ and, using the Green function for Γ, we obtain

$$u_{v_{zt}}(x) = \sqrt{2}\sqrt{c(z,t)}\left(\frac{G(x,z)}{\sqrt{k(z)}} + \frac{G(x,t)}{\sqrt{k(t)}}\right).$$

Then, from Corollary 4.3.2, we get that

$$\begin{aligned}
G^S(v_{xy}, v_{zt}) &= \varepsilon_{v_{zt}}(v_{xy}) - \frac{\sqrt{c(x,y)}\sqrt{c(z,t)}}{\mathrm{vol}(\Gamma)} \\
&\quad + \sqrt{c(x,y)c(z,t)}\left(\frac{G(x,z)}{\sqrt{k(x)k(z)}} + \frac{G(x,t)}{\sqrt{k(x)k(t)}}\right) \\
&\quad + \sqrt{c(x,y)c(z,t)}\left(\frac{G(y,z)}{\sqrt{k(y)k(z)}} + \frac{G(y,t)}{\sqrt{k(y)k(t)}}\right) \\
&\quad - \frac{\sqrt{c(x,y)c(z,t)}}{2\mathrm{vol}(\Gamma)}. \quad \square
\end{aligned}$$

4.3 Partial subdivision for singular Schrödinger type operators

Our third goal in this thesis work was established again in a diffusion problems setting, with an extension of the combinatorial Laplacian operator to a more general Schrödinger type one and also by considering a partial subdivision structural operation that pretends a wider range of applicability by considering the standard subdivision for networks on only some prescribed edges.

It is at this moment when networks can be used as discrete structures where edges are distinguishable ones from the others because of the conductance

function, but also vertices can be treated differently, modelled as if their behaviour is not the same, not only because of the number of its connections (degree) but also because the real value a potential function q assigns to every one of them expresses this specificity.

Our objective in this section is, as in the previous cases, to relate the solution of a compatible Poisson problem on the partial subdivision network with the solution of an appropriate Poisson problem on the base network. In this way, we obtained a relationship between solutions of problems that differ in dimension in a more freely way. What it is meant to say with the previous sentence is that, till now, we were concerned in relation to Poisson problems of different dimensionality of course, but fixed in the sense that edges were always doubled and the number of added vertices was established by the base network Γ as it was the number of initially given edges. With this partial subdivision we will be in position to relate solutions of Poisson problems between discrete structures with a number of vertices and edges that can vary at our wish (with some upper bounds of course).

Let us begin with the definition of what a *partial subdivision network* $\Gamma^S = (V^S, E^S, c^S)$ of a given network $\Gamma = (V, E, c)$ is. The point now is that Γ^S is obtained by inserting a new vertex in some edges of Γ, we denote the set of subdivided edges by $E_1 \subset E$, so that each edge $\{x, y\} \in E_1$ is replaced by two new edges, say $\{x, v_{xy}\}$ and $\{y, v_{xy}\}$ where v_{xy} is the new inserted vertex. We denote by V' the new vertex set assuming that, $v_{xy} = v_{yx}$. Thus, $V^S = V \cup V'$, the order of the subdivision network is $n + |E_1|$, and the size is $m + |E_1|$.

When, $E_1 = E$, the partial subdivision network is nothing else but the so called *subdivision network* [26] and we return back to previous network operations. But when it is not, we have here a degree of freedom to create a new discrete structures of different size and order (always bounded of course).

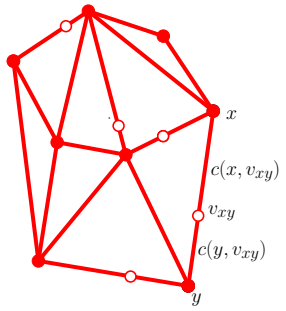


Figure 4.1 A partial subdivision network

As Schrödinger linear operators are intimately linked with weight functions after Doob's transform, and having in mind that we are now concerned with

singular operators (*i.e.* $\lambda = 0$) it is mandatory in this new context to define the extension of a given weight function on Γ^S so that it remains being a weight (of V^S now). So given a weight, that is $\omega \in \mathcal{C}(V)$ such that $\omega(x) > 0$ for every $x \in V$ and $\sum_{x \in V} \omega^2(x) = 1$, we now define an extension of this weight function, $\omega^S : V \cup V' \to [0, +\infty)$ by using a positive constant $\alpha > 0$ so that $\omega^S(x) = \alpha\omega(x)$, it is a reescaling of the given initial weight ω when $x \in V$, and also such that $\omega^S(v_{xy}) = \alpha\omega(v_{xy})$, where $\omega(v_{xy})$ is a positive absolutely arbitrary value, for $v_{xy} \in V'$. Hence, by choosing the factor α such that

$$\alpha^2 = \frac{1}{1 + \sum_{x \in V'} \omega(v_{xy})^2}$$

the extension of a weight function is also a weight function defined on the partial subdivided network Γ^S.

Moreover, according to the well known rule that express the equivalent resistance of two resistors connected in series and the expression for the Schrödinger operator, we define the conductance function $c^S \colon V^S \times V^S \longrightarrow [0, +\infty)$ by choosing, for every edge in E_1, $\{x, y\}$, non–null values $c^S(x, v_{xy})$ and $c^S(y, v_{xy})$ such that

$$\frac{1}{\omega(x)\omega(y)} \frac{1}{c(x,y)} = \frac{1}{\omega(x)\omega(v_{xy})} \frac{1}{c^S(x, v_{xy})} + \frac{1}{\omega(y)\omega(v_{xy})} \frac{1}{c^S(y, v_{xy})}, \quad (4.4)$$

whereas for every edge in $E^S \setminus E_1$ we define $c^S(x, y) = c(x, y)$.

Again, and as usual now, the definition of c^S cannot be misunderstood because of the chosen notation. Hence, by the sake of simplicity, it will be denoted as c. Moreover and for each edge, there exist infinitely many different choices of conductances fulfilling (4.4), so that different choices will lead to different partial subdivision networks.

Observe that Γ^S is still a connected, finite, with no loops, nor multiple edges network.

4.3.1 Related Poisson problems

We proceed now, similarly at what we did in the precedent subsections 4.1.1 and 4.2.1, to obtain the expression of a singular Schrödinger linear operator defined on the partial subdivided network. We recall that, given a potential $q \in \mathcal{C}(V)$ the corresponding Schrödinger operator is $\mathcal{L}_q = \mathcal{L} + q$ being $\mathcal{L}$ the combinatorial Laplacian operator defined on the network. Hence, when considering Γ^S, and if $\mathcal{L}^S$ denotes the combinatorial Laplacian on Γ^S, then for any $u \in \mathcal{C}(V^S)$ we have now that, for any $x \in V$

$$\mathcal{L}^S(u)(x) = \sum_{y \in V \setminus N(x)} c(x, y)\,(u(x) - u(y)) + \sum_{y \in N(x)} c(x, v_{xy})\,(u(x) - u(v_{xy}))$$

being $N(x)$ the set of adjacent vertices to x, see 3.1.1, and for any $v_{xy} \in V'$,

$$\mathcal{L}^S(u)(v_{xy}) = c(x, v_{xy})\left(u(v_{xy}) - u(x)\right) + c(y, v_{xy})\left(u(v_{xy}) - u(y)\right).$$

On the other hand, we consider the potential determined by ω^S,

$$q' = -(\omega^S)^{-1}\mathcal{L}^S(\omega^S) = -\omega^{-1}\mathcal{L}^S(\omega),$$

because of linearity. Hence, the expression of a singular Schrödinger operator on a partial subdivided network turns to be

$$\begin{aligned}\mathcal{L}^S_{q'}(u)(x) = & \frac{1}{\omega(x)} \sum_{y \in N(x)} c(x, v_{xy})\omega(x)\omega(v_{xy}) \left[\frac{u(x)}{\omega(x)} - \frac{u(v_{xy})}{\omega(v_{xy})}\right] \\ & + \frac{1}{\omega(x)} \sum_{y \in V \setminus N(x)} c(x, y)\omega(x)\omega(y) \left[\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right], \text{ for } x \in V.\end{aligned}$$

and

$$\mathcal{L}^S_{q'}(u)(v_{xy}) = \frac{c(v_{xy}, x)\omega(x) + c(v_{xy}, y)\omega(y)}{\omega(v_{xy})} u(v_{xy}) - c(v_{xy}, x)u(x) - c(v_{xy}, y)u(y),$$

for $v_{xy} \in V'$. Therefore, for any $v_{xy} \in V'$ and $u \in \mathcal{C}(V^S)$ we have that

$$\frac{u(v_{xy})}{\omega(v_{xy})} = \frac{\mathcal{L}^S_{q'}(u)(v_{xy}) + c(x, v_{xy})u(x) + c(y, v_{xy})u(y)}{c(x, v_{xy})\omega(x) + c(y, v_{xy})\omega(y)}.$$

Keeping in mind the compatibility equation (4.4) we can rewritte the expression for $\mathcal{L}^S_{q'}(u)(x)$ as

$$\mathcal{L}^S_{q'}(u)(x) = \mathcal{L}_{q_\omega}(u)(x) - \sum_{y \in N(x)} \frac{c(x, v_{xy})\omega(v_{xy})}{c(x, v_{xy})\omega(x) + c(y, v_{xy})\omega(y)} \mathcal{L}^S_{q'}(u)(v_{xy}). \quad (4.5)$$

This expression suggests to call *contraction of* $h \in \mathcal{C}(V^S)$ the function of $\mathcal{C}(V)$, $\underline{h}$, defined as

$$\underline{h}(x) = h(x) + \sum_{y \in N(x)} \alpha(x, y)h(v_{xy}),$$

with

$$\alpha(x, y) = \frac{c(x, v_{xy})\omega(v_{xy})}{c(x, v_{xy})\omega(x) + c(y, v_{xy})\omega(y)},$$

satisfying that

$$\alpha(x, y)\omega(x) + \alpha(y, x)\omega(y) = \omega(v_{xy}).$$

Moreover, we call *extension of* $u \in \mathcal{C}(V)$ *with respect to* $h \in \mathcal{C}(V^S)$ to the function of $\mathcal{C}(V^S)$, u^h, defined as

$$\begin{aligned} u^h(x) &= u(x), \ x \in V, \\ u^h(v_{xy}) &= \frac{h(v_{xy})c(x, y)}{c(x, v_{xy})c(y, v_{xy})} + \alpha(x, y)u(x) + \alpha(y, x)u(y), \ v_{xy} \in V' \end{aligned}$$

Using these definitions we obtain from (7.8) that for any $u \in \mathcal{C}(V)$ and $x \in V$,

$$\mathcal{L}_{q_\omega}(u)(x) = \underline{\mathcal{L}^S_{q'}(u)}(x).$$

This relation and the previous constructions, allow us to obtain the following result which stablishes the relation between solutions of related Poisson problems posed on both discrete structures, a given initial network Γ and a partial subdivision of it, Γ^S.

Theorem 4.3.1. *Given $h \in \mathcal{C}(V^S)$ such that $\langle h, \omega^S \rangle = 0$, then $\langle \underline{h}, \omega \rangle = 0$. Moreover, $\overline{u} \in \mathcal{C}(V^S)$ is a solution of the Poisson equation $\mathcal{L}^S_{q'}(\overline{u}) = h$ in V^S iff $u = \overline{u}_{|V}$ is a solution of the Poisson equation $\mathcal{L}_{q_\omega}(u) = \underline{h}$ in V. In this case, the identity $\overline{u} = u^h$ holds.*

Proof. We only have to prove the first statement. For that we show that $\alpha \langle \underline{h}, \omega \rangle = \langle h, \omega^S \rangle$ as

$$\begin{aligned}\sum_{x\in V} \underline{h}(x)\omega(x) &= \sum_{x\in V} h(x)\omega(x) + \sum_{x\in V}\sum_{y\in N(x)} \alpha(x,y)h(v_{xy})\omega(x) \\ &= \frac{1}{\alpha}\left(\sum_{x\in V} h(x)\omega^S(x) + \sum_{v_{xy}\in V'} h(v_{xy})\omega^S(v_{xy})\right). \quad \square\end{aligned}$$

To end this subsection, and following the structure of the previous two subsections, we give our next result that shows how to obtain the unique solution of a Poisson problem on the partial subdivision network Γ^S orthogonal to ω^S.

Corollary 4.3.2. *Given $h \in \mathcal{C}(V^S)$, such that $\langle h, \omega^S \rangle = 0$, let $\underline{h} \in \mathcal{C}(V)$ be its contraction to V, $u \in \mathcal{C}(V)$ be the unique solution of $\mathcal{L}_{q_\omega}(u) = \underline{h}$ that satisfies $\langle u, \omega \rangle = 0$ and the constant*

$$\lambda = -\sum_{\{x,y\}\in E_1} \frac{c(x,y)\omega^S(v_{xy})}{c(x,v_{xy})c(y,v_{xy})}\big(h(v_{xy}) + c(x,v_{xy})u(x) + c(y,v_{xy})u(y)\big)$$

Then, $u^\perp = u^h + \lambda\omega^S$ is the unique solution of $\mathcal{L}^S_{q'}(u^\perp) = h$ that satisfies $\langle u^\perp, \omega^S \rangle = 0$.

Proof. As two solutions differ on a constant times the weight, we have that

$u^\perp = u^h + \gamma\omega^S$, $\gamma \in \mathbb{R}$. Then,

$$\begin{aligned}
0 = \langle u^\perp, \omega^S\rangle &= \langle u^h, \omega^S\rangle + \gamma \\
&= \alpha \sum_{x\in V} u(x)\omega(x) + \sum_{v_{xy}\in V'} u^h(v_{xy})\omega^S(v_{xy}) + \gamma \\
&= \sum_{\{x,y\}\in E_1} \frac{h(v_{xy})c(x,y)}{c(x,v_{xy})c(y,v_{xy})}\omega^S(v_{xy}) \\
&+ \sum_{\{x,y\}\in E_1} (\alpha(x,y)u(x) + \alpha(y,x)u(y))\omega^S(v_{xy}) + \gamma \\
&= \sum_{\{x,y\}\in E_1} \frac{h(v_{xy})c(x,y)}{c(x,v_{xy})c(y,v_{xy})}\omega^S(v_{xy}) \\
&+ \sum_{\{x,y\}\in E_1} c(x,y)\omega^S(v_{xy})\left(\frac{u(x)}{c(y,v_{xy})} + \frac{u(y)}{c(x,v_{xy})}\right) + \gamma,
\end{aligned}$$

because $\langle u, \omega\rangle = 0$, and the result follows taking $\gamma = \lambda$. □

4.3.2 Related Green's functions

The preceding results allows us to obtain the expression for the Green kernel of a partial subdivision network in terms of the Green kernel of the base network and some other parameters.

If we let

$$\pi^S(x) = \sum_{y\sim S(x)} \frac{c(x,y)\omega^S(v_{xy})}{c(y,v_{xy})} = \sum_{y\sim S(x)} \alpha(x,y)\omega^S(v_{xy})$$

and

$$\beta = \sum_{r,s\in V} G_{q_\omega}(s,r)\pi^S(r)\pi^S(s) + \sum_{\{r,s\}\in E_1} \frac{c(r,s)\omega^S(v_{rs})^2}{c(r,v_{rs})c(s,v_{rs})},$$

we get, in the next result, the desired expression.

Proposition 4.3.3. *Let Γ^S be the partial subdivision network of Γ, then for any $x, z \in V$ and $v_{xy}, v_{zt} \in V'$, the Green kernel of Γ^S is given by*

$$\begin{aligned}
G^S_{q'}(x,z) = G_{q_\omega}(x,z) &- \sum_{\ell\in V}\Big[\omega^S(z)G_{q_\omega}(x,\ell) + \omega^S(x)G_{q_\omega}(z,\ell)\Big]\pi^S(\ell) \\
&+ \beta\omega^S(x)\omega^S(z),
\end{aligned}$$

$$
\begin{aligned}
G^S_{q'}(v_{xy},z) &= \alpha(x,y)G_{q_\omega}(x,z)+\alpha(y,x)G_{q_\omega}(y,z)\\
&\quad-\sum_{\ell\in V}\omega^S(z)\Big(\alpha(x,y)G_{q_\omega}(x,\ell)+\alpha(y,x)G_{q_\omega}(y,\ell)\Big)\pi^S(\ell)\\
&\quad-\sum_{\ell\in V}\omega^S(v_{xy})G_{q_\omega}(z,\ell)\pi^S(\ell)\\
&\quad+\left(\beta-\frac{c(x,y)}{c(x,v_{xy})c(y,v_{xy})}\right)\omega^S(v_{xy})\omega^S(z),\\
G^S_{q'}(v_{xy},v_{zt}) &= \frac{\varepsilon_{zt}(v_{xy})c(x,y)}{c(x,v_{xy})c(y,v_{xy})}\\
&\quad+\omega^S(v_{zt})\omega^S(v_{xy})\left(\beta-\frac{c(x,y)}{c(x,v_{xy})c(y,v_{xy})}-\frac{c(z,t)}{c(z,v_{zt})c(t,v_{zt})}\right)\\
&\quad-\omega^S(v_{zt})\sum_{\ell\in V}\Big(\alpha(x,y)G_{q_\omega}(x,\ell)+\alpha(y,x)G_{q_\omega}(y,\ell)\Big)\pi^S(\ell)\\
&\quad-\omega^S(v_{xy})\sum_{\ell\in V}\Big(\alpha(z,t)G_{q_\omega}(z,\ell)+\alpha(t,z)G_{q_\omega}(t,\ell)\Big)\pi^S(\ell)\\
&\quad+\alpha(x,y)\big(\alpha(z,t)G_{q_\omega}(x,z)+\alpha(t,z)G_{q_\omega}(x,t)\big)\\
&\quad+\alpha(y,x)\big(\alpha(z,t)G_{q_\omega}(y,z)+\alpha(t,z)G_{q_\omega}(y,t)\big).
\end{aligned}
$$

Proof. Suppose $z\in V$, and let $h_z=\varepsilon_z-\omega^S(z)\omega^S$. Then, for every $x\in V$

$$
\begin{aligned}
\underline{h}_z(x) &= \varepsilon_z(x)-\omega^S(x)\omega^S(z)-\sum_{y\sim N(x)}\alpha(x,y)\omega^S(v_{xy})\omega^S(z)\\
&= \varepsilon_z(x)-\omega^S(x)\omega^S(z)-\sum_{y\sim N(x)}\frac{c(x,v_{xy})\omega^S(z)\omega^S(v_{xy})^2}{c(x,v_{xy})\omega(x)+c(y,v_{xy})\omega(y)}\\
&= \varepsilon_z(x)-\omega^S(x)\omega^S(z)-\omega^S(z)\sum_{y\sim N(x)}\frac{c(x,y)\omega^S(v_{xy})}{c(y,v_{xy})}\\
&= \varepsilon_z(x)-(\omega^S(x)+\pi^S(x))\omega^S(z),
\end{aligned}
$$

with $\pi^S(x)=\displaystyle\sum_{y\sim N(x)}\frac{c(x,y)\omega^S(v_{xy})}{c(y,v_{xy})}$.

Hence, from Theorem 4.3.1, the related Poisson problem to solve on Γ is $\mathcal{L}_{q_\omega}(u_z)=\underline{h}_z$ and, using the Green kernel G for Γ, we obtain

$$
\begin{aligned}
u_z(x) &= \mathcal{G}_{q_\omega}(\varepsilon_z)(x)-\sum_{\ell\in V}G_{q_\omega}(x,\ell)\pi^S(\ell)\omega^S(z)\\
&= G_{q_\omega}(x,z)-\sum_{\ell\in V}G_{q_\omega}(x,\ell)\pi^S(\ell)\omega^S(z).
\end{aligned}
$$

Hence, after corollary 4.3.2 the expression of the Green kernel on Γ^S is

$$\begin{aligned}
G^S_{q'}(v_{xy},z) &= \frac{h(v_{xy})c(x,y)}{c(x,v_{xy})c(y,v_{xy})} + \alpha(x,y)u(x) + \alpha(y,x)u(y) \\
&\quad - \sum_{\{r,s\}\in E_1} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})c(s,v_{rs})} h(v_{rs})\omega^S(v_{xy}) \\
&\quad - \sum_{\{r,s\}\in E_1} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})c(s,v_{rs})} \left(c(r,v_{rs})u(r) + c(s,v_{rs})u(s)\right)\omega^S(v_{xy}).
\end{aligned}$$

After substituting, our expression turns to be

$$\begin{aligned}
G^S_{q'}(v_{xy},z) &= -\frac{\omega^S(v_{xy})\omega^S(z)c(x,y)}{c(x,v_{xy})c(y,v_{xy})} + \alpha(x,y)G_{q_\omega}(x,z) + \alpha(y,x)G_{q_\omega}(y,z) \\
&\quad - \omega^S(z)\sum_{\ell\in V}\Big(\alpha(x,y)G_{q_\omega}(x,\ell) + \alpha(y,x)G_{q_\omega}(y,\ell)\Big)\pi^S(\ell) \\
&\quad + \omega^S(z)\sum_{\{r,s\}\in F} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})c(s,v_{rs})}\omega^S(v_{rs})\omega^S(v_{xy}) \\
&\quad - \sum_{\{r,s\}\in E_1} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})c(s,v_{rs})} \left(c(r,v_{rs})G_{q_\omega}(r,z)\right)\omega^S(v_{xy}) \\
&\quad - \sum_{\{r,s\}\in E_1} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})c(s,v_{rs})} \left(c(s,v_{rs})G_{q_\omega}(s,z)\right)\omega^S(v_{xy}) \\
&\quad + \sum_{\{r,s\}\in E_1} \frac{c(r,s)\omega^S(v_{rs})}{c(s,v_{rs})} \left(\sum_{\ell\in V} G_{q_\omega}(r,\ell)\pi^S(\ell)\omega^S(z)\right)\omega^S(v_{xy}) \\
&\quad + \sum_{\{r,s\}\in E_1} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})} \left(\sum_{\ell\in V} G_{q_\omega}(s,\ell)\pi^S(\ell)\omega^S(z)\right)\omega^S(v_{xy})
\end{aligned}$$

and can finally can be established as

$$\begin{aligned}
G^S_{q'}(v_{xy},z) &= -\frac{\omega^S(v_{xy})\omega^S(z)c(x,y)}{c(x,v_{xy})c(y,v_{xy})} + \alpha(x,y)G_{q_\omega}(x,z) + \alpha(y,x)G_{q_\omega}(y,z) \\
&\quad - \sum_{\ell\in V}\omega^S(z)\Big(\alpha(x,y)G_{q_\omega}(x,\ell) + \alpha(y,x)G_{q_\omega}(y,\ell)\Big)\pi^S(\ell) \\
&\quad - \sum_{\ell\in V}\Big(\omega^S(v_{xy})G_{q_\omega}(z,\ell)\Big)\pi^S(\ell) \\
&\quad + \omega^S(z)\omega^S(v_{xy})\sum_{\{r,s\}\in E_1} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})c(s,v_{rs})}\omega^S(v_{rs}) \\
&\quad + \omega^S(z)\omega^S(v_{xy})\sum_{r,\ell\in V} G_{q_\omega}(r,\ell)\pi^S(\ell)\pi^S(r).
\end{aligned}$$

On the other hand, for vertices from former structure

$$G^S_{q'}(x,z) = u_z^{h_z}(x) - \sum_{\{r,s\}\in E_1} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})c(s,v_{rs})} h(v_{rs})\omega^S(x) \\ - \sum_{\{r,s\}\in E_1} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})c(s,v_{rs})} \big(c(r,v_{rs})u(r) + c(s,v_{rs})u(s)\big)\omega^S(x).$$

This expression can be seen as

$$\begin{aligned} G^S_{q'}(x,z) &= G_{q_\omega}(x,z) - \sum_{\ell\in V} G_{q_\omega}(x,\ell)\pi^S(\ell)\omega^S(z) \\ &+ \omega^S(x)\omega^S(z) \sum_{\{r,s\}\in E_1} \frac{c(r,s)\omega^S(v_{rs})^2}{c(r,v_{rs})c(s,v_{rs})} \\ &- \sum_{\{r,s\}\in E_1} \alpha(r,s) G_{q_\omega}(r,z)\omega^S(v_{rs})\omega^S(x) \\ &+ \sum_{\{r,s\}\in E_1} \alpha(r,s) \left(\sum_{\ell\in V} G_{q_\omega}(r,\ell)\pi^S(\ell)\omega^S(z) \right) \omega^S(v_{rs})\omega^S(x) \\ &- \sum_{\{r,s\}\in E_1} \alpha(s,r) G_{q_\omega}(s,z)\omega^S(v_{rs})\omega^S(x) \\ &+ \sum_{\{r,s\}\in E_1} \alpha(s,r) \left(\sum_{\ell\in V} G_{q_\omega}(s,\ell)\pi^S(\ell)\omega^S(z) \right) \omega^S(v_{rs})\omega^S(x). \end{aligned}$$

Finally, by summing up related terms, we obtain the desired expression

$$\begin{aligned} G^S_{q'}(x,z) &= G_{q_\omega}(x,z) - \omega^S(x)\omega^S(z) \sum_{\ell\in V} \Big[\frac{G_{q_\omega}(x,\ell)}{\omega^S(x)} + \frac{G_{q_\omega}(z,\ell)}{\omega^S(z)} \Big] \pi^S(\ell) \\ &+ \omega^S(x)\omega^S(z) \sum_{r,s\in V} G_{q_\omega}(s,r)\pi^S(r)\pi^S(s) \\ &+ \omega^S(x)\omega^S(z) \sum_{\{r,s\}\in E_1} \frac{c(r,s)\omega^S(v_{rs})^2}{c(r,v_{rs})c(s,v_{rs})}. \end{aligned}$$

Once the first case is completed, suppose now $v_{zt} \in V'$, and consider as data function $h_{v_{zt}} = \varepsilon_{v_{zt}} - \omega^S(v_{zt})\omega^S$. Then, for every $x \in V$ the corresponding contraction results in

$$\begin{aligned} \underline{h}_{v_{zt}}(x) &= \varepsilon_{v_{zt}}(x) - \omega^S(v_{zt})\omega^S(x) \\ &+ \sum_{y\in S(x)} \alpha(x,y) \left(\varepsilon_{v_{zt}}(v_{xy}) - \omega^S(v_{zt})\omega^S(v_{xy}) \right). \end{aligned}$$

This previous expression can be developed and finally rewritten as follows

$$\begin{aligned}
\underline{h}_{v_{zt}}(x) &= -\omega^S(v_{zt})\omega^S(x) + \sum_{y\in S(x)} \alpha(x,y)\left(\varepsilon_{v_{zt}}(v_{xy}) - \omega^S(v_{zt})\omega^S(v_{xy})\right)\\
&= -\omega^S(v_{zt})\omega^S(x) + \sum_{y\in S(x)} \alpha(x,y)\varepsilon_{v_{zt}}(v_{xy})\\
&\quad - \omega^S(v_{zt}) \sum_{y\in S(x)} \alpha(x,y)\omega^S(v_{xy})\\
&= -\omega^S(v_{zt})\omega^S(x) + \alpha(z,t)\varepsilon_z(x) + \alpha(t,z)\varepsilon_t(x) - \omega(v_{zt})\pi^S(x)\\
&= -\omega^S(v_{zt})\big(\omega^S(x) + \pi^S(x)\big) + \alpha(z,t)\varepsilon_z(x) + \alpha(t,z)\varepsilon_t(x).
\end{aligned}$$

Hence, the Poisson problem to solve on Γ is stated as $\mathcal{L}_{q_\omega}(u_{v_{zt}}) = \underline{h}_{v_{zt}}$. Its solution, by using Green's kernel for Γ, is

$$\begin{aligned}
u_{v_{zt}}(x) &= -\omega^S(v_{zt}) \sum_{\ell\in V} G_{q_\omega}(x,\ell)\big(\omega(\ell) + \pi^S(\ell)\big)\\
&\quad + \alpha(z,t)G_{q_\omega}(x,z) + \alpha(t,z)G_{q_\omega}(x,t)\\
&= -\omega^S(v_{zt}) \sum_{\ell\in V} G_{q_\omega}(x,\ell)\pi^S(\ell) + \alpha(z,t)G_{q_\omega}(x,z) + \alpha(t,z)G_{q_\omega}(x,t).
\end{aligned}$$

Then, the result follows by applying once again Corollary 4.3.2.

$$\begin{aligned}
G^S_{v_{zt}}(v_{xy}) &= \frac{h_{zt}(v_{xy})c(x,y)}{c(x,v_{xy})c(y,v_{xy})} + \alpha(x,y)u_{zt}(x) + \alpha(y,x)u_{zt}(y)\\
&\quad - \sum_{\{r,s\}\in F} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})c(s,v_{rs})} h_{zt}(v_{rs})\omega(v_{xy})\\
&\quad - \sum_{\{r,s\}\in F} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})c(s,v_{rs})} \left(c(r,v_{rs})u_{zt}(r) + c(s,v_{rs})u_{zt}(s)\right)\omega(v_{xy})\\
G^S_{v_{zt}}(v_{xy}) &= \frac{\varepsilon_{zt}(v_{xy})c(x,y)}{c(x,v_{xy})c(y,v_{xy})} - \frac{\omega(v_{zt})\omega(v_{xy})c(x,y)}{c(x,v_{xy})c(y,v_{xy})}\\
&\quad - \omega(v_{zt}) \sum_{\ell\in V} \Big(\alpha(x,y)G(x,\ell) + \alpha(y,x)G(y,\ell)\Big)\pi^S(\ell)\\
&\quad + \alpha(x,y)\alpha(z,t)G(x,z) + \alpha(x,y)\alpha(t,z)G(x,t)\\
&\quad + \alpha(y,x)\alpha(z,t)G(y,z) + \alpha(y,x)\alpha(t,z)G(y,t)\\
&\quad - \sum_{\{r,s\}\in F} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})c(s,v_{rs})} \left(\varepsilon_{v_{zt}}(v_{rs}) - \omega(v_{zt})\omega(v_{rs})\right)\omega(v_{xy})\\
&\quad - \sum_{\{r,s\}\in F} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})c(s,v_{rs})} \left(c(r,v_{rs})u_{zt}(r)\right)\omega(v_{xy})\\
&\quad - \sum_{\{r,s\}\in F} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})c(s,v_{rs})} \left(c(s,v_{rs})u_{zt}(s)\right)\omega(v_{xy}).
\end{aligned}$$

Now substituting the values of the solution of the Poisson problem on Γ, the expression becomes

$$\begin{aligned}
G^S_{v_{zt}}(v_{xy}) = {} & \frac{\varepsilon_{zt}(v_{xy})c(x,y)}{c(x,v_{xy})c(y,v_{xy})} - \frac{\omega(v_{zt})\omega(v_{xy})c(x,y)}{c(x,v_{xy})c(y,v_{xy})} \\
& - \omega(v_{zt}) \sum_{\ell \in V} \Big(\alpha(x,y)G(x,\ell) + \alpha(y,x)G(y,\ell) \Big) \pi^S(\ell) \\
& + \alpha(x,y)\alpha(z,t)G(x,z) + \alpha(x,y)\alpha(t,z)G(x,t) \\
& + \alpha(y,x)\alpha(z,t)G(y,z) + \alpha(y,x)\alpha(t,z)G(y,t) \\
& - \frac{c(z,t)\omega^S(v_{zt})\omega(v_{xy})}{c(z,v_{zt})c(t,v_{zt})} + \omega(v_{zt})\omega(v_{xy}) \sum_{\{r,s\} \in F} \frac{c(r,s)\omega^S(v_{rs})^2}{c(r,v_{rs})c(s,v_{rs})} \\
& + \sum_{\{r,s\} \in F} \frac{c(r,s)\omega^S(v_{rs})}{c(s,v_{rs})} \left(-\omega(v_{zt}) \sum_{\ell \in V} G(r,\ell)\pi^S(\ell) \right) \omega(v_{xy}) \\
& - \sum_{\{r,s\} \in F} \frac{c(r,s)\omega^S(v_{rs})}{c(s,v_{rs})} \left(\alpha(z,t)G(r,z) + \alpha(t,z)G(r,t) \right) \omega(v_{xy}) \\
& + \sum_{\{r,s\} \in F} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})} \left(-\omega(v_{zt}) \sum_{\ell \in V} G(s,\ell)\pi^S(\ell) \right) \omega(v_{xy}) \\
& - \sum_{\{r,s\} \in F} \frac{c(r,s)\omega^S(v_{rs})}{c(r,v_{rs})} \left(\alpha(z,t)G(s,z) + \alpha(t,z)G(s,t) \right) \omega(v_{xy}).
\end{aligned}$$

A little more development of the expression turns to

$$\begin{aligned}
G^S_{v_{zt}}(v_{xy}) = {} & \frac{\varepsilon_{zt}(v_{xy})c(x,y)}{c(x,v_{xy})c(y,v_{xy})} - \frac{\omega(v_{zt})\omega(v_{xy})c(x,y)}{c(x,v_{xy})c(y,v_{xy})} \\
& - \omega(v_{zt}) \sum_{\ell \in V} \Big(\alpha(x,y)G(x,\ell) + \alpha(y,x)G(y,\ell) \Big) \pi^S(\ell) \\
& + \alpha(x,y)\alpha(z,t)G(x,z) + \alpha(x,y)\alpha(t,z)G(x,t) \\
& + \alpha(y,x)\alpha(z,t)G(y,z) + \alpha(y,x)\alpha(t,z)G(y,t) \\
& + \omega(v_{zt})\omega(v_{xy}) \sum_{\{r,s\} \in F} \frac{c(r,s)\omega^S(v_{rs})^2}{c(r,v_{rs})c(s,v_{rs})} \\
& + \sum_{r,\ell \in V} G(r,\ell)\pi^S(\ell)\pi^S(r)\omega(v_{zt})\omega(v_{xy}) \\
& - \sum_{r \in F} \left(\alpha(z,t)G(r,z) + \alpha(t,z)G(r,t) \right) \pi^S(r)\omega(v_{xy}).
\end{aligned}$$

And finally we complete the proof by writting

$$\begin{aligned} G^S_{v_{zt}}(v_{xy}) = & \frac{\varepsilon_{zt}(v_{xy})c(x,y)}{c(x,v_{xy})c(y,v_{xy})} + \omega^S(v_{zt})\omega(v_{xy})\beta \\ & - \omega^S(v_{zt})\omega(v_{xy})\left(\frac{c(x,y)}{c(x,v_{xy})c(y,v_{xy})} + \frac{c(z,t)}{c(z,v_{zt})c(t,v_{zt})}\right) \\ & - \sum_{\ell\in V}\omega(v_{zt})\Big(\alpha(x,y)G(x,\ell) + \alpha(y,x)G(y,\ell)\Big)\pi^S(\ell) \\ & - \sum_{\ell\in V}\omega(v_{xy})\Big(\alpha(z,t)G(z,\ell) + \alpha(t,z)G(t,\ell)\Big)\pi^S(\ell) \\ & + \alpha(x,y)\alpha(z,t)G(x,z) + \alpha(x,y)\alpha(t,z)G(x,t) \\ & + \alpha(y,x)\alpha(z,t)G(y,z) + \alpha(y,x)\alpha(t,z)G(y,t). \end{aligned}$$

□

If we consider $E_1 = E$; that is, the case of subdivision networks, the above result coincides except for a constant with [26, Proposition 3.1]. The scalar is due to the fact that in the mentioned work, we were considering no weights in the vertex set; *i.e.*, $\omega(x) = 1$ for any $x \in V$ and hence the normalization factor appears.

5

Resistance Distance and Kirchhoff Index.

The concept of distance is a basic one in the whole human experience. In every day life, it usually means some degree of closeness of two physical objects or ideas, *i.e.* length, time interval, gap, rank difference, coolness or remoteness, while the term metric is often used as a standard for a measurement.

The mathematical notions of (i) distance metric (*i.e.* a nonnegative function $d(x, y)$ that vanishes only when $x = y$, symmetric and that fulfills the so called triangular inequality $d(x, y) \leq d(x, z) + d(z, y)$), and of (ii) metric space, were originated a little over a century ago by M. Fréchet [1] and F. Hausdorff[2] even though the triangle inequality above appears in Euclid.

Distances and metrics have become now an essential tool in many areas of mathematics and its applications including geometry, probability, statistics, coding/graph theory, clustering, data analysis, pattern recognition, networks, engineering, computer graphics/vision, astronomy, cosmology, molecular biology, and many others areas of science. Devising the most suitable distance metric and similarities, in order to quantify the proximity between objects, has become a standard task for many researchers. Especially intense ongoing search for such distances occurs in so many and diverse areas as for example computational biology, image analysis, speech recognition and information retrieval, just to mention a few.

Canonical geodesic distance on graphs is probably the most ancient, natural and used notion of distance on discrete structures and has proved to be very suitable for dealing with so many situations and purposes of great interest. But also shows its limitations when the phenomena that is being studied

[1] In 1906, Maurice Fréchet submitted his outstanding thesis Sur Quelques points du Calcul Fonctionnel introducing (within a systematic study of functional operation) the notion of metric space (E-espace, E from écart in french, which means gap).

[2] In 1914, Felix Hausdorff published his famous Grundzüge der Mengen–Lehre where the theory of topological and metric spaces (metrische Räume) was created.

is considered to be transmitted in a quite more sophisticated way as when the framework is that of virus diseases or social media or electrical circuits for instance. Due to the own nature of whatever is propagating, it does not take place by solely using one option (the shortest path) but all possibilities are taken into account. Hence looking for new alternatives to this distance concept is justified.

Klein and Randic's 1993 electrical idea of effective resistances, see [58], is then a different paradigm that has shown its versatility and opportunity to deal with these situations. This is verified not only because of the big amount of bibliographic references that the mathematical community is generating more and more and progressively, but also because, and very interestingly, the same distance metric appeared independently and in the apparently very poorly related framework of social media in Stephenson and Zeller, [70]. It is quite often that a result or an idea is found in different areas, but this happy coincidence reaffirms the importance and opportunity of the discovery. Equivalance between apparentely different distances is stated in [18, 37], for example.

Another concern of major interest in the study of discrete structures is the obtention of topological indices. Graphs and networks do have properties that are very useful when unravelling the information that they contain, so it is very interesting to extract meaningful information that you would not have if the individual components were examined separately with all the complexity of the whole vertices and edges set. In this sense, several measures of network performance, topological indices and classification parameters have been introduced and studied.

The way in which the nodes and edges are arranged within a discrete structure is its particular topology and can help us to identify relevant aspects of it. Topological properties can apply to the network as a whole or to individual nodes and edges, they are of a great nature of origins and some of them, just to put a few examples, are the degree and the centrality of a vertex, the degree distribution (that defines whether a network is scale–free or not), the diameter of the whole structure, the closeness and betweenness concepts, ...

Perhaps the best known and most widely used topological index is the Wiener index which is based on the geodesic distance of vertices. It was defined and used by Wiener in 1947 when comparing the boiling points of some alkane isomers. Since then, over 3000 topological graph indices are registered including Hosoya index, Estrada index, Randič connectivity index, Zabreg group for instance. There is also a rapidly increasing interest in this topic, therefore topological graph indices are researched worldwide in so many scientific diverse areas.

In this sense there are many proposals that have proven their opportunity and *raison d'être* and that are worth analyzing. Among them the Kirchhoff index is a global parameter that is tested day by day as really suitable. The Kirchhoff index topologically analyzes the structure, initially from the point of view of its global connectivity but also, and more lately, for centrality concerns (as those of closeness and betweenness just mentioned) as well. An interpretation that is usually given for the Kirchhoff index is as a generalization of the former Wiener index when resistance distances are considered in a discrete structure instead of the primitive geodesic shortest path distance.

Therefore, we will devote this chapter to these two topics: resistance distances and Kirchhoff index. After a very brief introduction of both concepts, we will develop a more detailed exposition of them, how they are considered and computed, whenever the structure in which they are defined turns more elaborated from time to time, it is for graph, for networks and finally in a Schödinger type operator setting. Finally we obtain resistance distances and Kirchhoff index for subdivision networks in the three different scenarios we are treating in this thesis, relating them to their correspondents on the former discrete structure, the one existing previously to the subdivision procedure. To be honest, all these results are obtained after Green's function for the adequately stated Poisson problem as mere by–products. Even though they are very important results in some specific areas on discrete structures or network science, the power of discrete potential theory allows the obtention of them quite straightforward.

5.1 Resistance distance or effective resistance

Geodesic distance is enough for in some environments, but it is clearly unsatisfactory in many other more, when information, current, whatever is distributed within a discrete structure is more properly assumed to be diffused not only in a "shortest" way, but in all possible ways (as do fluids on the air for instance).

The geodesic distance between two nodes does not consider the actual number of (shortest) paths that lie among the two vertices: two nodes that are separated by a single path are at the same distance than two nodes that are separated by many more paths of the same length. In many applications, however, paths longer than geodesic ones are also relevant, since information or whatever flows on the network does not necessarily choose an optimal path. For instance, in social networks, a fad does not know the optimal route to move among actors, but simply wanders around more or less randomly; moreover, nodes separated by many pathways are often perceived closer than nodes separated by few pathways, even if the paths have all the same length. Communication between nodes is typically enhanced as soon as more routes

are possible, [70].

We give another argument, following [63, 67], insisting in the idea that the use of shortest path has, sometimes, some drawbacks as in many cases, shortest paths form a small subset of all paths between two nodes. It follows then that paths even slightly longer than the shortest one, potentially many more, are not considered at all.

Hence resistance distance, also known as effective resistance as will be clear soon, is an answer to a new paradigm, where the distance between two nodes is sensible to all paths that can be considered between them (and not only the shortest) and also diminishes as soon as more routes are stablished. Therefore is an interesting (and maybe still underestimated) metric on networks. It has a strong mathematical background, and persuasive interpretations in both information and electrical network theory.

One of the fundamental problems in electric circuit theory is the computation of the electrical effective resistance between two vertices of the network. Informally, the effective resistance between two vertices of a network, assuming that a network is seen as an electrical circuit, can be calculated by the well known series and parallel manipulations. Two edges, corresponding to resistors with resistance r_1 and r_2 ohm, in series can be replaced by one edge with effective resistance $r = r_1 + r_2$. If the two edges are connected in parallel, the two edges can be replaced by one edge with effective resistance r such that $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$.

The point, see [58], is to view a network $\Gamma = (V, E, c)$ as a resistor network in which edges $\{x, y\} \in E$ are resistors and the nodes $\{x, y, \ldots\} \in V$ are junctions between resistors. Each edge is possibly assigned with a positive value $c(x, y) \geq 0$ indicating the conductance (the reciprocal of the resistance) of the edge. Hence, the distance between two nodes x and y is defined as the potential difference of nodes x and y when a unit of current is injected in source x and removed from target y; since the current is equal to unity, the potential difference is also the *effective resistance* between nodes x and y. A high resistance (potential difference) between nodes indicates that the two nodes are far away, while low resistance between nodes means that the nodes are close points.

In [50] the authors show that the resistance distance notion satisfies many interesting mathematical properties and it has different intriguing interpretations. For instance, it establishes a metric on the graph and in particular the resistance distance matrix is an euclidean distance matrix. Furthermore, the resistance distance is a monotone decreasing function as well as a convex function of the edge weight (conductance) vector of the graph.

Resistance distance across a pair of nodes is the same as the effective resistance across that pair, treating each edge as a 1 ohm resistance. A special

case of this restricted to only the edges of a graph was studied by Foster [48].

Resistance distance has an easy interpretation in terms of random walks on graphs. The resistance distance between nodes u and v of a graph is proportional to the average commute time of nodes x and y of the Markov chain defined by the graph, which is the average number of steps it takes to return to node x for the first time after starting from x and passing through y. Curiously, resistance distance has been also used in the context of bibliometrics to express a rational version of the popular Erdös number [4].

5.2 Kirchhoff index

Topological indices are mathematical measures corresponding to discrete structures that are invariant under isomorphisms. The significance of these topological indices is usually associated with quantitative structures property relationship (QSPR) and quantitative structure activity relationship (QSAR) (see [67]).

Probably the idea of a topological index, in order to obtain critical information from a discrete structure, appears in a work by Wiener (see [73]) in 1947 when he was working on the boiling point of paraffin. The so called Wiener index of graph Γ, began the theory of topological indices being defined as

$$W(\Gamma) = \frac{1}{2} \sum_{\{x,y\}} d(x,y)$$

where $\{x, y\}$ is any ordered pair of vertices in Γ and $d(x, y)$ is the shortest path $x - y$ geodesic distance (when it exists). More information about the Wiener index can be found in [45, 51, 79].

Thus it was first in the field of organic chemistry, with the very powerful idea of a molecular graph $\Gamma = (V, E)$ representing a given molecule so that vertices are atoms and edges correspond to electron pair bonds, that the idea of having a parameter to represent an object took its roots. After that it has been used in a variety of ways from predicting antibacterial activity in drugs to correlating thermodynamic parameters in physical chemistry and modelling various solid–state phenomena.

The formal definition of the *Kirchhoff index*, or *effective graph resistance* or *total resistance* as well, is the sum of the effective resistances over all pairs of vertices. Then it is

$$\mathsf{k}(\Gamma) = \frac{1}{2} \sum_{x,y \in V} R(x,y).$$

So from these series and parallel manipulations, it follows that it takes into account both the number of (not necessarily disjoint) paths between two

vertices and their length, intuitively measuring the presence and quality of back–up possibilities.

Although derived from the field of electric circuit analysis, it was also introduced in chemistry as a better alternative to other parameters used as a discriminator among different molecules with similar shapes and structures, see the seminal paper [58] again.

Since then a new line of research with a considerable amount of production has been developed and the Kirchhoff index has been computed for some classes of graphs with symmetries and for composite graphs (product, lexicographic product, join, corona, cluster, see for instance [78]). This paper also motivated the study of the Kirchhoff index for graphs derived from a single graph, such as the line graph, the subdivision graph and the total graph of a regular graph in [49]. See [60, 3] and [63] for more examples. Also closed–form formulae for the Kirchhoff index have been developed, for example in [68, 69] following a Foster's formulae approach.

Moreover, Kirchhoff index is deeply related to the Kemeny constant when considering a Markov chain associated with a network, see [54].

The lower the value of $\mathsf{k}(\Gamma)$ for a network (from amongst all possible networks with the same number of nodes and edges), the more compact the embedding, and the more structurally robust the overall network is. For short, the Kirchhoff index provides a geometric measure to rank different networks of comparable sizes.

5.3 Resistance distance and Kirchhoff index

In this section, we will expose some facts related to resistance distances and Kirchhoff index when the discrete structure considered is just a graph, that is, where there is no possibility of differentiating an existing edge from another existing edge on the graph except by the vertices they link.

Using standard methods of the theory of electrical networks, that are the very well known Ohm's and Kirchhoff's law for electrical currents, the effective resistance between two vertices of a graph $\Gamma = (V, E)$ can be calculated, from the excellent seminal paper [58], in terms of some specifically determinate entries of $\mathsf{L}^\dagger$, the Moore–Penrose generalized inverse matrix of the Laplacian matrix associated with a graph. Given a labelling of $V = \{x_1, \ldots, x_n\}$ it is

$$R(x_i, x_j) = l^\dagger_{ii} + l^\dagger_{jj} - l^\dagger_{ij} - l^\dagger_{ji}, \quad \text{for } i \neq j.$$

In this way, many properties of resistance distances were proved using the Laplacian matrix see [43, 74], and many dynamical properties of a graph or network can be obtained from their study.

Then the *resistance matrix* associated with a connected graph G, that will be denoted by $\mathsf{R} = [R_{ij}]$ can be defined as $R_{ij} = R(x_i, x_j)$ if $i \neq j$. In addition we set $R_{ii} = 0$, for all $i = 1, 2, \ldots, n$ without contradicting the previous definition.

When applying the symmetry of $\mathsf{L}^\dagger$, it is $R_{ij} = l^\dagger_{ii} + l^\dagger_{jj} - 2l^\dagger_{ij}$ for $i, j = 1, \ldots, n$.

Let us now denote the eigenvalues of L by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n$, always enumerated in decreasing order and repeated according to their multiplicity. It is very well known, after applying Gerschgorin's disk theorem, that the Laplacian spectrum is contained in the real interval $[0, 2\max_{x \in V} \deg(x)]$, in a non–sharp upper bound. Hence, all eigenvalues for L are positive real numbers; it is always $\mu_n = 0$, because $\mathsf{L1} = \mathsf{0}$, so the lowest is equal to 0, and the biggest as large as desired. Therefore, we conclude that the combinatorial Laplacian matrix is positive semi–definite.

Now we denote the corresponding eigenvectors of L by u_k with $k = 1, \ldots, n$. So the equality

$$\mathsf{L}u_k = \mu_k \; u_k$$

holds for $k = 1, \ldots, n$. In addition we specify that the components of every eigenvector are $u_k = [u_{1k}, u_{2k}, \ldots, u_{nk}]^T$, for $k = 1, \ldots, n$.

It is always possible to choose the Laplacian eigenvectors to be real, normalized and mutually orthogonal. So the matrix $U = [u_1, u_2, \ldots, u_n]$ is orthogonal, *i.e.* $UU^T = U^T U = \mathsf{I}$,

and as $U^T \mathsf{L} U = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_n)$ we have

$$l_{ij} = \sum_{k=1}^{n} \mu_k u_{ik} u_{jk}.$$

Then, see [52], the entries of the Moore–Penrose generalized inverse for the Laplacian matrix associated with a connected graph Γ can be defined by

$$l^\dagger_{ij} = \sum_{k=1}^{n-1} \frac{1}{\mu_k} u_{ik} u_{jk}.$$

Hence, returning to the resistance matrix, it is

$$R_{ij} = \sum_{k=1}^{n-1} \frac{1}{\mu_k} \left(u_{ik} u_{ik} + u_{jk} u_{jk} - 2u_{ik} u_{jk} \right) = \sum_{k=1}^{n-1} \frac{1}{\mu_k} \left(u_{ik} - u_{jk} \right)^2$$

insisting in the idea of distance as $r(x_i, x_j)$ is the squared euclidean distance between the i-th and the j-th rows of U.

Another way of calculating resistance distances in the case that the subjacent structure is a graph is considering the non–singular matrix $\mathsf{L} + \dfrac{1}{n}\mathsf{J}$, where

J stands for the all ones matrix. When considering its inverse $X = [x_{ij}] = \mathsf{L}^\dagger + \frac{1}{n}\mathsf{J}$, the entries of the resistance matrix may be expressed as, see [5]

$$R_{ij} = x_{ii} + x_{jj} - 2x_{ij}.$$

Now, as for the Laplacian matrix and its generalized inverse it is $\mathsf{L}^2\mathsf{L}^\dagger = \mathsf{L}^\dagger\mathsf{L}^2 = \mathsf{L}$, some properties satisfied by the resistance matrix R can be established such as $\mathsf{LRL} = -2\mathsf{L}$ and also $\mathsf{L}^\dagger\mathsf{RL}^\dagger = -2\left(\mathsf{L}^\dagger\right)^3$. Finally , using both relations a reverting expression can be obtained so that the Moore–Penrose inverse matrix for the Laplacian matrix can be expressed in terms of the resistance matrix

$$\mathsf{L}^\dagger = -\frac{1}{2}\left[\mathsf{R} - \frac{1}{n}(\mathsf{RJ} + \mathsf{JR}) + \frac{1}{n^2}\mathsf{JRJ}\right].$$

The Laplacian matrix of a graph is sometimes called Kirchhoff matrix or admittance matrix (or many other names, see [64] and the references therein).

An interesting property of the resistance distance, which is useful to illuminate its meaning, is that, given any undirected, unweighted, and connected graph with n nodes, the sum of resistance distances between pairs of nodes connected by an edge is $n-1$, independently of the number of edges of the graph (here we assume that each pair defining an undirected edge is considered only once in the sum). Notice that the geodesic distance between two nodes connected by an edge is 1, and hence the sum of geodesic distances on a graph is the number of edges of the graph. Thus if, for instance, the graph is acyclic, so it is a tree, then resistance and geodesic distances coincide, because in a tree there is a unique path between any two nodes, and the resistance distance for two nodes is the length of this path, that is, the geodesic distance. Since a tree of n nodes has $n-1$ edges, we have that the sum of distances on edges is the number of edges, that is $n-1$. However, the presence of cycles in a graph reduces resistance distances in comparison with geodesic counterparts as, in general, more paths are available between pairs of nodes when loops are present in the graph. Hence resistance distance is a thinner distance between vertices than geodesic shortest path. For a graph with the maximum number of cycles, a complete graph in which any pair of nodes is linked by an edge, each of the $n(n-1)/2$ pairs of nodes linked by an edge are distant $2/n$ in the case that resistance distance is considered with a remarkable reduction of their geodesic distance of 1 as soon as $n > 2$ and the difference increasing with n. After summing up all possibilites, we obtain a total of $n-1$ again, much less than the geodesic counterpart which is $2n$. So the denser the graph, the more paths there are between nodes, the smaller are resistance distances compared with geodesic ones.

One of the main topics in the study of the resistance distance is the computation problem: besides some typical techniques used in electrical network

theory various kinds of formulas for computing resistance distances have been developed including algebraic, probabilistic, combinatorial, recursive an so on. For more details see [77] and the references therein.

The resistance distances (and following Kirchhoff index as well) have been extensively studied in chemical literature, [58, 74]. They both appear in several applications: electrical networks, averaging networks, and experiment designs [21, 58]. The results of [50] represent an excellent survey on effective resistances, the authors also consider the problem of obtaining optimal conductances for the effective resistance be minimal given the network structure.

We have already mentioned that the connection between resistance distances and random walks on graphs has also been established and discussed, [54]. In [72] Tetali proved Foster's first theorem using certain results from the theory of Markov chains, then Palacios gave an extension of Foster's second theorem in [69]. Generalization of all of the Foster's theorems are given by some authors, see [15].

A very important result ([51]), in connection with the Kirchhoff index of a graph now, is the next result, where the Kirchhoff index is related to the inverse value of the non–null eigenvalues of the Laplacian matrix, or with the eigenvalues of $\mathsf{L}^{\dagger}$,

$$\mathsf{k}(\Gamma) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i}.$$

For more information we refer to [65], in which Mohar gives a clear survey on the Laplacian, its properties, and its applications; to appendix B.8 of [38] where an extensive review of publications on the Laplacian of graphs is given. Also [3, 69], are very well known references in this field.

5.4 Resistance distances and Kirchhoff index on networks

In this section we are replicating a presentation of the resistance distance concept but now we consider the case when the discrete structure $\Gamma = (V, E, c)$ allows to differentiate edges because of a conductance is defined for every one of them.

In this case some minor observations have to be done in order to calculate pairwise resistance distances because the corresponding Laplacian matrix, now attached to the network, can be defined and considered similarly as in the previous case of a simple graph.

But also a different approach, based on discrete Potential Theory, is applicable to calculate effective resistances, as was demonstrated in [8]. In this

paper the authors relate equilibria measures associated with the combinatorial Laplacian kernel and the corresponding Wiener capacities, a powerful idea when the structure is symmetric. The authors also generalize Palacio's techniques, [69].

As in the former standard setting, the effective resistance between vertices x and y can also be defined through the solution of the Poisson problem $\mathcal{L}(u) = f$ when the data function is the dipole with poles at x and at y; that is $f = \varepsilon_x - \varepsilon_y$. Being data $f \in \mathcal{C}(V)$, so that the corresponding Poisson problem is compatible, the effective resistance is defined as

$$R(x, y) = u(x) - u(y)$$

with $u \in \mathcal{C}(V)$ is any solution, no matter what. We remark that $R(x, y)$ is independent on the chosen solution u.

Again the *Kirchhoff index* or *Total resistance* of a network is

$$\mathsf{k}(\Gamma) = \frac{1}{2} \sum_{x,y \in V} R(x, y).$$

Effective resistances can be used to deduce important properties of electrical networks as can be seen in [50, 58, 74]. Also a couple of good references where some calculations have been developed are [5, 8].

5.5 Resistance distances and Kirchhoff index for Schrödinger operators

In [9] a generalization of the concept of effective resistance with respect to a value $\lambda \geq 0$ and a weight $\omega \in \Omega(V)$ was introduced through a commonly used technique in the context of electrical networks and Markov chains and with the aim of generalizing the Fiedler characterization of irreducible, symmetric and diagonally dominant M–matrices as resistive inverses, see [46], to all irreducible and symmetric M–matrices or equivalently, to all positive semi–definite Schrödinger operators. This generalization is essential to obtain the expression for the Kirchhoff index of a composite network in terms of the Kirchhoff indexes of the factors, see [1, 2].

When associating a positive value to each node of a network and then defining a one parametric family of resistance distances associated with this weight function through a positive semi–definite Schrödinger type operator (for which the parameter and the function are the lowest eigenvalue and the corresponding eigenfunction), the framework of discrete potential theory is applicable to analyze the main properties of these distances.

This generalized effective resistance, with respect to λ and ω, define a distance on the network as in the standard case and hence it can be also used with the same aims, see [14]. Actually we will explain how the effective resistance verifies analogous properties to those that the classical case satisfy. Among them, the relation between the Kirchhoff index with respect to λ and ω and the eigenvalues of the associated Schrödinger operator as well as the relation between the effective resistances with respect to λ and ω and the eigenvalues and eigenfunctions of the mentioned operator.

Specifically, given $\lambda \geq 0$ and $\omega \in \Omega(V)$ so that $q = q_\omega + \lambda$, the ω–dipole function with poles at x and y is defined as $f = \dfrac{1}{\omega}\left(\varepsilon_x - \varepsilon_y\right)$. Clearly $\mathcal{P}(f) = 0$ so the Poisson problem $\mathcal{L}_q(u) = f$ is consistent. Moreover every one of its solutions maximizes the functional

$$\mathcal{I}_{x,y}(u) = 2\Big[\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\Big] - \langle \mathcal{L}_q(u,u)\rangle.$$

Then given $x, y \in V$, the generalization of the effective resistance is defined, with respect to λ and ω so that

$$R_{\lambda,\omega}(x,y) = \max_{u\in\mathcal{C}(V)} \{\mathcal{I}_{x,y}(u)\}.$$

Since the matrix associated with the Schrödinger operator $\mathcal{L}_q$ is an irreducible, symmetric M–matrix, and conversely, every irreducible, symmetric M–matrix appears as associated with a Schrödinger operator, we can assign an effective resistance function to any irreducible, symmetric M–matrix (not necessarily diagonally–dominant).

Hence, the *generalized Kirchhoff index of* Γ *with respect to* λ *and* ω is

$$\mathsf{k}(\lambda,\omega) = \frac{1}{2}\sum_{x,y\in V} R_{\lambda,\omega}(x,y)\omega^2(x)\,\omega^2(y).$$

Observe that, our definitions of effective resistance and Kirchhoff index when ω is constant, differ from the classical ones in a factor of $|V|$, since the weight is always normalized to 1. As we will see for us the Kirchhoff index is basically the trace of the Green function, giving to that index a physical meaning. Also note, that the defined Kirchhoff index can be seen as one half of the energy of the kernel given by the effective resistance matrix applied to the vector ω and finally, $R_{0,1}(x,y) = R(x,y)$.

Nevertheless, the calculation of $R_{\lambda,\omega}(x,y)$ is again in terms of the solution of a Poisson problem with the ω–dipole function as data function, as can be seen in [9]. Moreover, symmetry, non–negativity and relationship with the Green kernel function is also obtained from the fact that, for $u \in \mathcal{C}(V)$ any

solution of the Poisson equation $\mathcal{L}_q(u) = f$, then

$$R_{\lambda,\omega}(x,y) = \langle \mathcal{L}_q(u), u\rangle = \frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}.$$

. From this expression, $R_{\lambda,\omega}$ is symmetric, non–negative and vanishes if and only if $x = y$. In addition,

$$R_{\lambda,\omega}(x,y) = \frac{G_q(x,x)}{\omega^2(x)} + \frac{G_q(y,y)}{\omega^2(y)} - 2\frac{G_q(x,y)}{\omega(x)\omega(y)}.$$

So, the role of the Green function is crucial in order to evaluate effective resistances and the subsequent Kirchhoff index of the network as

$$\mathsf{k}(\lambda,\omega) = \sum_{x\in V} G_q(x,x) - \lambda^{\dagger}.$$

where $\lambda^{\dagger} = \lambda^{-1}$ if $\lambda > 0$ and $\lambda^{\dagger} = 0$ if $\lambda = 0$. We see from the above expression that the Kirchhoff index is the trace of the Green function.

The main properties satisfied by this generalized effective resistance may be are those listed in the next final result, see [24].

Theorem 5.5.1. *If Γ is a connected network, the effective resistance with respect to a parameter and a weight satisfies the following properties:*

1. *The effective resistance $R_{\lambda,\omega}(x,y)$ determines a distance on the network. Moreover, $R_{\lambda,\omega}(x,y) = R_{\lambda,\omega}(x,z) + R_{\lambda,\omega}(z,y)$ if and only if $\lambda = 0$ and z separates x and y.*
2. *For $0 \leq \hat{\lambda} \leq \lambda$ and $\hat{q} = q_\omega + \hat{\lambda}$ then $R_{\lambda,\omega} \leq R_{\hat{\lambda},\omega} \leq R_{0,\omega}$.*
3. *$R_{\lambda,\omega}(x,y) \leq d_{\hat{c}}(x,y)$, where $\hat{c}(x,y) = c(x,y)\omega(x)\omega(y)$, with equality if and only if $\lambda = 0$ and there exists a unique path from x to y.*
4. *$\lim_{\lambda\to+\infty} R_{\lambda,\omega} = 0$ and $\lim_{\lambda\to 0} R_{\lambda,\omega} = R_{0,\omega}$.*

Or in other words, for a fixed weight function ω, the associated effective resistance distances are continuous and monotone decreasing with respect to the parameter λ (the larger the parameter, the lower the resistance) and they are upper bounded by the weighted geodesic distance of the network. Moreover, both distances do coincide if and only if the parameter is null and the network is a tree. In addition, this new generalized resistance distance is graph geodetic if and only if the operator is singular.

Just to end this section we mention that in the case of constant weight, by applying some electrical equivalences it is possible to show that the one–parametric family of effective resistances can be seen as the effective resistance associated with the combinatorial Laplacian of a complete network. The so called *forest distance* and *adjusted forest distance* can be recovered as this particular case, see [30].

5.6 Resistance distances and Kirchhoff index on subdivision networks

After this presentation of both treated concepts and their different interpretation and computation depending on the case we are considering, we are now in a position to present our results when electrical subdivision of a network is considered. Of course we did our job in the three, at this point, well known cases we are considering in this manuscript: standard electrical subdivision for a combinatorial Laplacian operator, also standard electrical subdivision for a normalized Laplacian operator and finally the third a more general case of partial electrical subdivision for a Schrödinger type operator.

What it follows, structured in subsections, are the corresponding results we have obtained in every case. Hence this section mimics previous sections of Chapter 3.

5.6.1 Subdivision for combinatorial Laplacian

We are now concerned with the relation between effective resistances in a base network Γ and the effective resistances, R^S, in a subdivision network Γ^S.

Theorem 5.6.1. *Let $\Gamma = (V, E, c)$ be a network and $\Gamma^S = (V^S, E^S, c)$ its subdivision network, then*

$$\begin{aligned}
R^S(x,y) &= R(x,y),\\
R^S(x,v_{zt}) &= \frac{1}{k(v_{zt})} + \alpha(z,t)R(x,z) + \alpha(t,z)R(x,t) - \alpha(z,t)\alpha(t,z)R(z,t),\\
R^S(v_{xy},v_{zt}) &= \frac{1}{k(v_{xy})} + \frac{1}{k(v_{zt})}\\
&\quad - \alpha(x,y)\alpha(y,x)R(x,y) - \alpha(z,t)\alpha(t,z)R(z,t)\\
&\quad + \alpha(x,y)\alpha(z,t)R(x,z) + \alpha(x,y)\alpha(t,z)R(x,t)\\
&\quad + \alpha(z,t)\alpha(y,x)R(y,z) + \alpha(y,x)\alpha(t,z)R(y,t),\ \textit{for}\ \ v_{xy} \neq v_{zt}.
\end{aligned}$$

Proof. The proof is a direct consequence of Proposition 4.3.3. Let us do the non–trivial case 2. The case 3, can be proved similarly.

$$R^S(x,v_{zt}) = G^S(x,x) + G^S(v_{zt},v_{zt}) - 2G^S(x,v_{zt})$$

$$
\begin{aligned}
R^S(x, v_{zt}) &= G(x,x) - \frac{2}{n+m}\sum_{\ell\in V} G(x,\ell)\pi^S(\ell) \\
&\quad + \alpha(z,t)\Big(\alpha(z,t)G(z,z) + \alpha(t,z)G(t,z)\Big) + \\
&\quad + \alpha(t,z)\Big(\alpha(z,t)G(z,t) + \alpha(t,z)G(t,t)\Big) \\
&\quad - \frac{2}{n+m}\sum_{\ell\in V}\Big[\alpha(z,t)G(z,\ell) + \alpha(t,z)G(t,\ell)\Big]\pi^S(\ell) + \frac{\varepsilon_{v_{zt}}(v_{zt})}{k(v_{zt})} \\
&\quad - \frac{2}{(n+m)k(v_{zt})} - 2\alpha(z,t)G(z,x) - 2\alpha(t,z)G(t,x) \\
&\quad + \frac{2}{n+m}\sum_{\ell\in V}\Big[\alpha(z,t)G(z,\ell) + \alpha(t,z)G(t,\ell) + G(x,\ell)\Big]\pi^S(\ell) \\
&\quad + \frac{2}{(n+m)k(v_{zt})}
\end{aligned}
$$

$$
\begin{aligned}
R^S(x, v_{zt}) &= \frac{1}{k(v_{zt})} + G(x,x) - 2\alpha(z,t)G(z,x) - 2\alpha(t,z)G(t,x) \\
&\quad + \alpha(z,t)\Big(\alpha(z,t)G(z,z) + \alpha(t,z)G(t,z)\Big) \\
&\quad + \alpha(t,z)\Big(\alpha(z,t)G(z,t) + \alpha(t,z)G(t,t)\Big) \\
&= \frac{1}{k(v_{zt})} + \alpha(z,t)\Big[G(x,x) + G(z,z) - 2G(x,z)\Big] \\
&\quad + \alpha(t,z)\Big[G(x,x) + G(t,t) - 2G(x,t)\Big] \\
&\quad - \alpha(t,z)\alpha(z,t)\Big[G(z,z) + G(t,t) - 2G(z,t)\Big],
\end{aligned}
$$

and hence, the result follows. □

Observe that the effective resistance between vertices of the original network remains unchanged, as expected. In particular for the standard subdivision graph we get the following result, which coincides with the obtained in [32, 71, 76], up to the factor 2 due to our (electrically compatible)–choice of the conductances.

Corollary 5.6.2. *Let $\Gamma = (V, E, c)$ be a network and $\Gamma^S = (V^S, E^S, c)$ its standard subdivision network, then*

$$
\begin{aligned}
R^S(x,y) &= R(x,y), \\
R^S(x, v_{zt}) &= \frac{1 + 2R(x,z) + 2R(x,t) - R(z,t)}{4}, \\
R^S(v_{xy}, v_{zt}) &= \frac{2 - R(x,y) - R(z,t) + R(x,z) + R(x,t) + R(y,z) + R(y,t)}{4}, \\
&\quad \textit{for any } \ v_{xy} \neq v_{zt}.
\end{aligned}
$$

Next we obtain an expression for the Kirchhoff index of the subdivision network, $\mathsf{k}(\Gamma^S)$, in terms of the Kirchhoff index, k, of the base network and other parameters.

Theorem 5.6.3. *Let $\Gamma = (V, E, c)$ be a network and $\Gamma^S = (V^S, E^S, c)$ its subdivision network, then*

$$\begin{aligned}\mathsf{k}(\Gamma^S) = \frac{n+m}{n}\mathsf{k}(\Gamma) + (n+m)\sum_{x\in V} G(x,x)\pi^S(x) - \sum_{x,y\in V} G(x,y)\pi^S(x)\pi^S(y)\\ - (n+m)\sum_{x\sim y}\alpha(x,y)\alpha(y,x)R(x,y) + (n+m-1)\sum_{x\sim y}\frac{1}{k(v_{xy})}.\end{aligned}$$

Proof.

$$\begin{aligned}\mathsf{k}(\Gamma^S) &= (n+m)\sum_{x\in V} G^S(x,x) + (n+m)\sum_{v_{xy}\in V'} G^S(v_{xy},v_{xy})\\
&= \frac{(n+m)}{n}\,\mathsf{k}(\Gamma) - 2\sum_{x\in V}\sum_{\ell\in V} G(x,\ell)\pi^S(\ell)\\
&+ (n+m)\sum_{v_{xy}\in V'}\Big(\alpha(x,y)^2G(x,x)+2\alpha(x,y)\alpha(y,x)G(y,x)+\alpha(y,x)^2G(y,y)\Big)\\
&- 2\sum_{v_{xy}\in V'}\sum_{\ell\in V}\Big[\alpha(x,y)G(x,\ell)+\alpha(y,x)G(y,\ell)\Big]\pi^S(\ell)\\
&+ \sum_{x,y\in V} G(x,y)\pi^S(x)\pi^S(y) + (n+m-1)\sum_{x\sim y}\frac{1}{k(v_{xy})}\end{aligned}$$

$$\begin{aligned}&= \frac{n+m}{n}\,\mathsf{k}(\Gamma) + (n+m)\sum_{x,y\in V}\Big(\alpha(x,y)^2G(x,x)+\alpha(x,y)\alpha(y,x)G(y,x)\Big)\\
&- \sum_{x,y\in V} G(x,y)\pi^S(x)\pi^S(y) + (n+m-1)\sum_{x\sim y}\frac{1}{k(v_{xy})}\\
&= \frac{n+m}{n}\mathsf{k}(\Gamma) + (n+m)\sum_{x\in V} G(x,x)\pi^S(x) - (n+m)\sum_{x\sim y}\alpha(x,y)\alpha(y,x)R(y,x)\\
&- \sum_{x,y\in V} G(x,y)\pi^S(x)\pi^S(y) + (n+m-1)\sum_{x\sim y}\frac{1}{k(v_{xy})}.\quad\square\end{aligned}$$

In particular, the Kirchhoff index of the standard subdivision graph has the following expression which, coincides with [71, Th 3.1]. In the case of k–regular graph the result coincides with [49, Th 3.5].

Corollary 5.6.4. *Let Γ^S be the standard subdivision network of a graph Γ; then*

$$\begin{aligned}\mathsf{k}(\Gamma^S) = \frac{n+m}{n}\mathsf{k}(\Gamma) + (n+m)\sum_{x\in V} G(x,x)\pi^S(x) - \sum_{x,y\in V} G(x,y)\pi^S(x)\pi^S(y)\\ + \frac{m^2-n^2+n}{4}.\end{aligned}$$

In particular, if Γ *is* k*–regular*

$$\mathsf{k}(\Gamma^S) = \frac{(k+2)^2}{4}\mathsf{k}(\Gamma) + \frac{(k^2-4)n^2+4n}{16}.$$

5.6.2 Subdivision for the normalized Laplacian

From Theorem 4.2.3, we easily calculate the values of the effective resistances for the subdivision network. Moreover, we also give the expression of its Kirchhoff index.

Theorem 5.6.5. *Let* Γ^S *be the subdivision network of* Γ*, then for any* $x, z \in V$ *and* $v_{xy}, v_{zt} \in V'$*, the effective resistances between vertices of* Γ^S *are given by*

$$\begin{aligned}
R^S(x,z) &= R(x,z),\\
R^S(v_{xy},z) &= \frac{1}{4}\frac{1}{c(x,y)} + \frac{1}{2}R(x,z) + \frac{1}{2}R(y,z) - \frac{1}{4}R(x,y),\\
R^S(v_{xy},v_{zt}) &= \frac{1}{4}\left(\frac{1}{c(x,y)} + \frac{1}{c(z,t)}\right)\\
&\quad + \frac{1}{4}\big(R(x,z) + R(x,t) + R(y,z) + R(y,t) - R(x,y) - R(z,t)\big),
\end{aligned}$$

for any $v_{xy} \neq v_{zt}$.

Moreover, the Kirchhoff index of the subdivision network is

$$\mathsf{k}(\Gamma^S) = 16\,\mathsf{k}(\Gamma) + 2\,\mathrm{vol}(\Gamma)(2m-2n+1).$$

Proof. The expressions for the effective resistance follow directly from Theorem 4.2.3 and Proposition 4.3.3. On the other hand, from the definition of Kirchhoff index we get

$$\begin{aligned}
\mathsf{k}(\Gamma^S) =& 8\mathsf{k}(\Gamma) + 2\mathrm{vol}(\Gamma)(2m-1)\\
&+ 4\mathrm{vol}(\Gamma)\sum_{x\sim y} c(x,y)\left(\frac{G(x,x)}{k(x)} + 2\frac{G(x,y)}{\sqrt{k(x)k(y)}} + \frac{G(y,y)}{k(y)}\right)\\
=& 8\mathsf{k}(\Gamma) + 2\mathrm{vol}(\Gamma)(2m-1)\\
&+ 4\mathrm{vol}(\Gamma)\sum_{x\in V}\frac{G(x,x)}{k(x)}\sum_{y\sim x} c(x,y)\\
&+ 4\mathrm{vol}(\Gamma)\sum_{x\in V}\frac{1}{\sqrt{k(x)}}\sum_{y\sim x} c(x,y)\frac{G(x,y)}{\sqrt{k(y)}}\\
=& 12\mathsf{k}(\Gamma) + 2\mathrm{vol}(\Gamma)(2m-1) + 4\mathrm{vol}(\Gamma)\sum_{x\in V}\frac{G(x,x)}{k(x)}\sum_{y\sim x} c(x,y)\\
&+ 4\mathrm{vol}(\Gamma)\sum_{x\in V}\frac{1}{\sqrt{k(x)}}\sum_{y\sim x} c(x,y)\left(\frac{G(x,y)}{\sqrt{k(y)}} - \frac{G(x,x)}{\sqrt{k(x)}}\right)\\
=& 16\mathsf{k}(\Gamma) + 2\mathrm{vol}(\Gamma)(2m-1) - 4\mathrm{vol}(\Gamma)\sum_{x\in V}\left(1 - \frac{\sqrt{k(x)}}{\mathrm{vol}(\Gamma)}\sqrt{k(x)}\right)\\
=& 16\mathsf{k}(\Gamma) + 2\mathrm{vol}(\Gamma)(2m-2n+1). \quad \square
\end{aligned}$$

5.6.3 Partial Subdivision for Schrödinger operator

In this section we aim at obtaining the expression for the effective resistances on a partial subdivision network of a given network Γ. The expression will follow by taking into account the expression for the effective resistances in terms of Green's function.

Proposition 5.6.6. *Let Γ^S be a partial subdivision network of Γ, then for any $x, z \in V$ and $v_{xy}, v_{zt} \in V'$, the effective resistances of Γ^S are given by*

$$
\begin{aligned}
R^S_{\omega^S}(x,z) &= \frac{1}{\alpha^2} R_\omega(x,y),\\
R^S_{\omega^S}(v_{zt},x) &= \frac{c(z,t)}{\alpha^2 c(z,v_{zt})c(t,v_{zt})\omega(v_{zt})^2}\\
&\quad + \frac{\omega(z)\omega(t)}{\alpha^2\omega(v_{zt})}\left(\frac{\alpha(z,t)R_\omega(x,z)}{\omega(t)} + \frac{\alpha(t,z)R_\omega(x,t)}{\omega(z)}\right.\\
&\qquad \left. - \frac{\alpha(t,z)\alpha(z,t)R_\omega(z,t)}{\omega(v_{zt})}\right),\\
R^S_{\omega^S}(v_{xy},v_{zt}) &= \frac{c(x,y)}{\alpha^2 c(x,v_{xy})c(y,v_{xy})\omega(v_{xy})^2} + \frac{c(z,t)}{\alpha^2 c(z,v_{zt})c(t,v_{zt})\omega(v_{zt})^2}\\
&\quad + \frac{1}{\alpha^2\omega(v_{xy})\omega(v_{zt})}\Big[\alpha(x,y)\alpha(z,t)\omega(x)\omega(z)R_\omega(x,z)\\
&\qquad + \alpha(x,y)\alpha(t,z)\omega(x)\omega(t)R_\omega(x,t)\\
&\qquad + \alpha(y,x)\alpha(z,t)\omega(y)\omega(z)R_\omega(y,z)\\
&\qquad + \alpha(y,x)\alpha(t,z)\omega(y)\omega(t)R_\omega(y,t)\Big]\\
&\quad - \frac{\alpha(x,y)\alpha(y,x)\omega(x)\omega(y)}{\alpha^2\omega(v_{xy})^2}R_\omega(x,y)\\
&\quad - \frac{\alpha(z,t)\alpha(t,z)\omega(z)\omega(t)}{\alpha^2\omega(v_{zt})^2}R_\omega(z,t),\ \textit{for any } v_{xy}\neq v_{zt}.
\end{aligned}
$$

Proof. Suppose that $x, z \in V$, then

$$
\begin{aligned}
R^S_{\omega^S}(x,z) &= \frac{G^S_{q'}(x,x)}{[\omega^S(x)]^2} + \frac{G^S_{q'}(z,z)}{[\omega^S(z)]^2} - 2\frac{G^S_{q'}(x,z)}{\omega^S(x)\omega^S(z)}\\
&= \frac{1}{[\omega^S(x)]^2}\Big(G_{q_\omega}(x,x) - 2\omega^S(x)\sum_{\ell\in V}G_{q_\omega}(x,\ell)\pi^S(\ell)\Big)\\
&\quad + \frac{1}{[\omega^S(z)]^2}\Big(G_{q_\omega}(z,z) - 2\omega^S(z)\sum_{\ell\in V}G_{q_\omega}(z,\ell)\pi^S(\ell)\Big)\\
&\quad - \frac{2}{\omega^S(x)\omega^S(z)}\Big(G_{q_\omega}(x,z) - \sum_{\ell\in V}\Big(\omega^S(x)G_{q_\omega}(z,\ell) + \omega^S(z)G_{q_\omega}(x,\ell)\Big)\pi^S(\ell)\Big)\\
&= \frac{G_{q_\omega}(x,x)}{[\omega^S(x)]^2} + \frac{G_{q_\omega}(z,z)}{[\omega^S(z)]^2} - 2\frac{G_{q_\omega}(x,z)}{\omega^S(x)\omega^S(z)} = \frac{1}{\alpha^2}R_\omega(x,z).
\end{aligned}
$$

Moreover, if $x \in V$ and $v_{zt} \in V'$, then

$$R^S_{q'}(x, v_{zt}) = \frac{G^S_{q'}(x,x)}{[\omega^S(x)]^2} + \frac{G^S_{q'}(v_{zt}, v_{zt})}{[\omega^S(v_{zt})]^2} - 2\frac{G^S_{q'}(x, v_{zt})}{\omega^S(x)\omega^S(v_{zt})},$$

where from Proposition 4.3.3

$$\begin{aligned}
\frac{G^S_{q'}(x,x)}{\omega^S(x)^2} &= \frac{G_{q_\omega}(x,x)}{\omega^S(x)^2} - \frac{2}{\omega^S(x)}\sum_{\ell\in V} G_{q_\omega}(x,\ell)\pi^S(\ell) + \beta,\\
\frac{G^S_{q'}(v_{zt}, v_{zt})}{\omega^S(v_{zt})^2} &= \frac{c(z,t)}{c(z,v_{zt})c(t,v_{zt})\omega^S(v_{zt})^2} - 2\frac{c(z,t)}{c(z,v_{zt})c(t,v_{zt})}\\
&\quad - \frac{2}{\omega^S(v_{zt})}\sum_{\ell\in V}\Big(\alpha(z,t)G_{q_\omega}(z,\ell) + \alpha(t,z)G_{q_\omega}(t,\ell)\Big)\pi^S(\ell)\\
&\quad + \frac{\alpha(z,t)^2 G_{q_\omega}(z,z) + 2\alpha(z,t)\alpha(t,z)G_{q_\omega}(z,t) + \alpha(t,z)^2 G_{q_\omega}(t,t)}{\omega^S(v_{zt})^2}\\
&\quad + \beta\\
\frac{G^S_{q'}(v_{zt}, x)}{\omega^S(v_{zt})\omega^S(x)} &= \frac{\alpha(z,t)G_{q_\omega}(z,x) + \alpha(t,z)G_{q_\omega}(t,x)}{\omega^S(v_{zt})\omega^S(x)} - \frac{c(z,t)}{c(z,v_{zt})c(t,v_{zt})} + \beta\\
&\quad - \sum_{\ell\in V}\Big(\frac{\alpha(z,t)G_{q_\omega}(z,\ell)}{\omega^S(v_{zt})} + \frac{\alpha(t,z)G_{q_\omega}(t,\ell)}{\omega^S(v_{zt})} + \frac{G_{q_\omega}(x,\ell)}{\omega^S(x)}\Big)\pi^S(\ell).
\end{aligned}$$

Summing up

$$\begin{aligned}
R^S_{\omega^S}(x, v_{zt}) &= \frac{c(z,t)}{c(z,v_{zt})c(t,v_{zt})\omega^S(v_{zt})^2} + \frac{G_{q_\omega}(x,x)}{\omega^S(x)^2}\\
&\quad + \frac{\alpha(z,t)^2 G_{q_\omega}(z,z)}{\omega^S(v_{zt})^2} + \frac{\alpha(t,z)^2 G_{q_\omega}(t,t)}{\omega^S(v_{zt})^2}\\
&\quad + \frac{2\alpha(z,t)\alpha(t,z)G_{q_\omega}(z,t)}{\omega^S(v_{zt})^2} - 2\frac{\alpha(z,t)G_{q_\omega}(z,x) + \alpha(t,z)G_{q_\omega}(t,x)}{\omega^S(v_{zt})\omega^S(x)}\\
&= \frac{c(z,t)}{c(z,v_{zt})c(t,v_{zt})\omega^S(v_{zt})^2}\\
&\quad + \frac{G_{q_\omega}(x,x)}{\omega^S(x)^2}\left(\frac{\alpha(z,t)\omega(z)}{\omega(v_{zt})} + \frac{\alpha(t,z)\omega(t)}{\omega(v_{zt})}\right)\\
&\quad + \frac{G_{q_\omega}(z,z)}{\omega^S(z)^2}\left(\frac{\alpha(z,t)\omega(z)}{\omega(v_{zt})} - \frac{\alpha(t,z)\alpha(z,t)\omega(z)\omega(t)}{\omega(v_{zt})^2}\right)\\
&\quad + \frac{G_{q_\omega}(t,t)}{\omega^S(t)^2}\left(\frac{\alpha(t,z)\omega(t)}{\omega(v_{zt})} - \frac{\alpha(t,z)\alpha(z,t)\omega(z)\omega(t)}{\omega(v_{zt})^2}\right)\\
&\quad + \frac{2\alpha(z,t)\alpha(t,z)\omega(z)\omega(t)}{\omega^S(v_{zt})^2}\frac{G_{q_\omega}(z,t)}{\omega(z)\omega(t)}\\
&\quad - 2\frac{\alpha(z,t)\omega(z)}{\omega^S(v_{zt})}\frac{G_{q_\omega}(z,x)}{\omega^S(x)\omega(z)} - 2\frac{\alpha(t,z)\omega(t)}{\omega^S(v_{zt})}\frac{G_{q_\omega}(t,x)}{\omega^S(x)\omega(t)}\\
&= \frac{c(z,t)}{c(z,v_{zt})c(t,v_{zt})\omega^S(v_{zt})^2} + \frac{\alpha(z,t)\omega(z)}{\alpha^2\omega(v_{zt})}R_\omega(x,z)\\
&\quad + \frac{\alpha(t,z)\omega(t)}{\alpha^2\omega(v_{zt})}R_\omega(x,t) - \frac{\alpha(t,z)\alpha(z,t)\omega(z)\omega(t)}{\alpha^2\omega(v_{zt})^2}R_\omega(z,t).
\end{aligned}$$

For the last case, take into account also that

$$\begin{aligned}\frac{G^S_{q'}(v_{xy},v_{zt})}{\omega^S(v_{xy})\omega^S(v_{zt})} &= \frac{\varepsilon_{zt}(v_{xy})c(x,y)}{c(x,v_{xy})c(y,v_{xy})\omega^S(v_{xy})\omega^S(v_{zt})}\\ &\quad - \left(\frac{c(x,y)}{c(x,v_{xy})c(y,v_{xy})}+\frac{c(z,t)}{c(z,v_{zt})c(t,v_{zt})}\right)\\ &\quad - \sum_{\ell\in V}\left[\frac{\Big(\alpha(x,y)G(x,\ell)+\alpha(y,x)G(y,\ell)\Big)}{\omega^S(v_{xy})}\right.\\ &\quad \left.+\frac{\Big(\alpha(z,t)G(z,\ell)+\alpha(t,z)G(t,\ell)\Big)}{\omega^S(v_{zt})}\right]\pi^S(\ell)\\ &\quad + \frac{\alpha(x,y)\alpha(z,t)G(x,z)+\alpha(x,y)\alpha(t,z)G(x,t)}{\omega^S(v_{xy})\omega^S(v_{zt})}\\ &\quad + \frac{\alpha(y,x)\alpha(z,t)G(y,z)+\alpha(y,x)\alpha(t,z)G(y,t)}{\omega^S(v_{xy})\omega^S(v_{zt})}.\end{aligned}$$

Hence the expression for $R^S_{\omega^s}(v_{xy},v_{zt})$ is

$$\begin{aligned}R^S_{\omega^s}(v_{xy},v_{zt}) &= \frac{c(x,y)}{c(x,v_{xy})c(y,v_{xy})\omega^S(v_{xy})^2}+\frac{c(z,t)}{c(x,v_{zt})c(t,v_{zt})\omega^S(v_{zt})^2}\\ &\quad + \frac{\alpha(x,y)^2G(x,x)+2\alpha(x,y)\alpha(y,x)G(x,y)+\alpha(y,x)^2G(y,y)}{\omega^S(v_{xy})^2}\\ &\quad + \frac{\alpha(z,t)^2G(z,z)+2\alpha(z,t)\alpha(t,z)G(z,t)+\alpha(t,z)^2G(t,t)}{\omega^S(v_{zt})^2}\\ &\quad - 2\frac{\alpha(x,y)\alpha(z,t)G(x,z)+\alpha(x,y)\alpha(t,z)G(x,t)}{\omega^S(v_{xy})\omega^S(v_{zt})}\\ &\quad - 2\frac{\alpha(y,x)\alpha(z,t)G(y,z)+\alpha(y,x)\alpha(t,z)G(y,t)}{\omega^S(v_{xy})\omega^S(v_{zt})}\\ &= \frac{c(x,y)}{c(x,v_{xy})c(y,v_{xy})\omega^S(v_{xy})^2}+\frac{c(z,t)}{c(x,v_{zt})c(t,v_{zt})\omega^S(v_{zt})^2}\\ &\quad + \frac{1}{\alpha^2\omega^S(v_{xy})\omega^S(v_{zt})}\\ &\qquad \Big[\alpha(x,y)\alpha(z,t)\omega(x)\omega(z)R(x,z)+\alpha(x,y)\alpha(t,z)\omega(x)\omega(t)R(x,t)\\ &\qquad +\alpha(y,x)\alpha(z,t)\omega(y)\omega(z)R(y,z)+\alpha(y,x)\alpha(t,z)\omega(y)\omega(t)R(y,t)\Big]\\ &\quad - \frac{\alpha(x,y)\alpha(y,x)\omega(x)\omega(y)}{\alpha^2\omega^S(v_{xy})^2}R(x,y)-\frac{\alpha(x,t)\alpha(t,z)\omega(z)\omega(t)}{\alpha^2\omega^S(v_{zt})^2}R(z,t). \quad \square\end{aligned}$$

Proposition 5.6.7. *Let Γ^S be the partial subdivision network of Γ, then the Kirchhoff index of Γ^S is given by*

$$\begin{aligned}\mathsf{k}(\omega^S) &= \mathsf{k}(\omega)+\sum_{x\in V}\frac{\pi^S(x)}{\omega^S(x)}G_{q_\omega}(x,x)+\sum_{\{x,y\}\in E_1}\frac{c(x,y)}{c(x,v_{xy})c(y,v_{xy})}-\beta\\ &\quad + \sum_{\{x,y\}\in E_1}\alpha(x,y)\alpha(y,x)\omega(x)\omega(y)R_\omega(x,y).\end{aligned}$$

Proof. It is enough to compute the trace of Green's function.

$$
\begin{aligned}
\mathsf{k}(\omega^S) &= \sum_{x\in V}\Big[G_{q_\omega}(x,x) - 2\omega^S(x)\sum_{\ell\in V} G_{q_\omega}(x,\ell)\pi^S(\ell) + \beta\omega^S(x)^2\Big] \\
&+ \sum_{\{x,y\}\in E_1}\Big[\frac{c(x,y)}{c(x,v_{xy})c(y,v_{xy})} - \omega^S(v_{xy})^2\left(2\frac{c(x,y)}{c(x,v_{xy})c(y,v_{xy})} - \beta\right) \\
&- 2\omega(v_{xy})\sum_{\ell\in V}\Big(\alpha(x,y)G_{q_\omega}(x,\ell) + \alpha(y,x)G_{q_\omega}(y,\ell)\Big)\pi^S(\ell) \\
&+ \alpha(x,y)^2 G_{q_\omega}(x,x) + 2\alpha(x,y)\alpha(y,x)G_{q_\omega}(x,y) + \alpha(y,x)^2 G_{q_\omega}(y,y)\Big] \\
&= \mathsf{k}(\omega) + \beta + \textstyle\sum_{\{x,y\}\in E_1}\Big[\frac{c(x,y)}{c(x,v_{xy})c(y,v_{xy})} - 2\frac{c(x,y)\omega^S(v_{xy})^2}{c(x,v_{xy})c(y,v_{xy})}\Big] \\
&- 2\sum_{x,\ell\in V} G_{q_\omega}(x,\ell)\pi^S(x)\pi^S(\ell) \\
&+ \sum_{\{x,y\}\in E_1}\Big[\alpha(x,y)^2 G_{q_\omega}(x,x) + 2\alpha(x,y)\alpha(y,x)G_{q_\omega}(x,y) + \alpha(y,x)^2 G_{q_\omega}(y,y)\Big] \\
&= \mathsf{k}(\omega) + \sum_{x\in V} G_{q_\omega}(x,x)\frac{\pi^S(x)}{\omega^S(x)} - \sum_{\{x,y\}\in E_1}\alpha(x,y)\alpha(y,x)\omega(x)\omega(y)R_\omega(x,y) \\
&+ \sum_{\{x,y\}\in E_1}\frac{c(x,y)}{c(x,v_{xy})c(y,v_{xy})} - \beta. \quad \square
\end{aligned}
$$

6
The case of Star and Wheel networks.

In this last chapter we consider two examples, namely Star and Wheel networks. We perform the subdivision operation to obtain Green's functions, effective resistances and Kirchhoff indexes on the new obtained networks for the different operators we have been considered, see [26, 27, 28].

6.1 Partial subdivision on star networks

Let us consider $S_{2n} = (V, E, c)$, the Star network with vertex set $V = \{x_0, x_1, \ldots, x_{2n}\}$ and positive conductances

$$c(x_0, x_{2j}) = a_j, \qquad c(x_0, x_{2j-1}) = a'_j$$

for $j = 1, \ldots n$ and $c(x_i, x_j) = 0$ otherwise. We define $S^S_{2n} = (V \cup V', c^S)$, the partial subdivision network of S_{2n} where $V' = \{x_{0\,2i}, |i = 1, \ldots, n\}$ and

$$c^S(x_{0\,2i}, x_0) = c_{0\,i} > 0, \qquad c^S(x_{0\,2i}, x_{2i}) = c_{i\,0} > 0$$

for $i = 1, \ldots, n$ and $c^S(x_i, x_j) = c(x_i, x_j)$ otherwise. See, Figure 6.1.

Let $\omega : V \to \mathbb{R}^+$ be a weight on S_{2n} and we define an extension of this weight function $\omega^S : V \cup V' \to \mathbb{R}^+$ as mentioned in Section 4.3. In order to simplify the notation, the weight function S^S_{2n} will be denoted

$$\omega^S(x_j) = \omega_j, \qquad \omega^S(x_{0\,2i}) = \omega_{0\,2i}$$

for any $j = 0, \ldots, 2n, \quad i = 1, \ldots, n$. The compatibility condition (4.4) reads

$$\frac{\omega_{0\,2i}}{a_i} = \frac{\omega_{2i}}{c_{0\,i}} + \frac{\omega_0}{c_{i\,0}}.$$

$$\begin{aligned}
\alpha(x_0, x_{2i}) &= \frac{c_{0\,i}\,\omega_{0\,2i}}{c_{0\,i}\omega_0 + c_{i\,0}\omega_{2i}} = \frac{a_i}{c_{0\,i}}, \\
\pi^S(x_0) &= \sum_{i=1}^{n} \frac{c_{0\,i}\,\omega^2_{0\,2i}}{c_{0\,i}\omega_0 + c_{i\,0}\omega_{2i}} = \sum_{i=1}^{n} \frac{a_i\,\omega_{0\,2i}}{c_{i\,0}}, \\
\pi^S(x_{2i}) &= \frac{a_i\,\omega_{0\,2i}}{c_{0\,i}}.
\end{aligned}$$

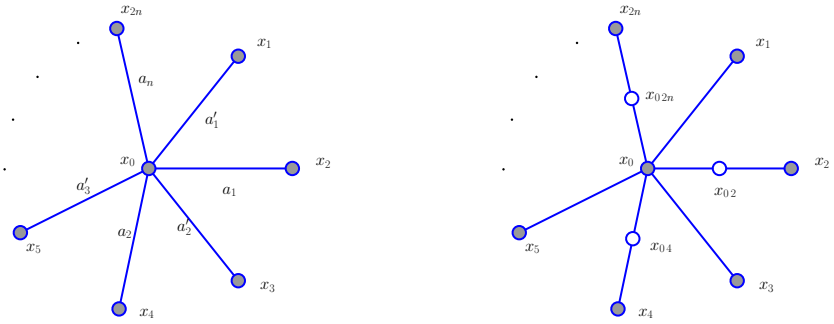


Figure 6.1 The Star network (left) and a partial subdivision (right)

Let us assume constant weight for the Star network $\omega(x_i) = \dfrac{1}{\sqrt{2n+1}}$, for any $i = 0, \ldots, 2n$, and we assign a positive constant γ to the weight of the new vertices. Thus,

$$\omega^S(x_i) = \omega_i = \frac{\alpha}{\sqrt{2n+1}}, \qquad \omega^S(x_{0\,2i}) = \omega_{0\,2i} = \alpha\gamma.$$

In this case, $\alpha^2 = \dfrac{1}{1+n\gamma^2}$. Moreover, we assume equal conductance for each pair of new edges, this is, $c_{i\,0} = c_{0\,i}$ for any $i = 1, \ldots, n$. Then, according to expression (4.4), the following equality holds

$$\frac{a_i}{c_{0\,i}} = \frac{a_i}{c_{i\,0}} = \frac{1}{2}\gamma\sqrt{2n+1}.$$

Moreover, the parameters are

$$\begin{aligned}
\alpha(x_0, x_{2i}) &= \frac{c_{0\,i}\,\omega_{0\,2i}}{c_{0\,i}\omega_0 + c_{i\,0}\omega_{2i}} = \frac{a_i}{c_{0\,i}} = \frac{1}{2}\gamma\sqrt{2n+1},\\
\pi^S(x_0) &= \sum_{i=1}^{n} \frac{c_{0\,i}\,\omega_{0\,2i}^2}{c_{0\,i}\omega_0 + c_{i\,0}\omega_{2i}} = \sum_{i=1}^{n} \frac{a_i\,\omega_{0\,2i}}{c_{i\,0}} = \frac{n}{2}\alpha\gamma^2\sqrt{2n+1},\\
\pi^S(x_{2i}) &= \frac{a_i\,\omega_{0\,2i}}{c_{0\,i}} = \frac{1}{2}\alpha\gamma^2\sqrt{2n+1}.
\end{aligned}$$

If we denote

$$Q = \frac{1}{2n+1}\sum_{j=1}^{n}\left(\frac{1}{a_j} + \frac{1}{a'_j}\right),$$

the expression for the Green function and the effective resistance of the Star network

is, see [23]

$$
\begin{aligned}
G(x_0,x_0) &= \omega_0^2 Q = \frac{1}{2n+1}Q = \frac{1}{(2n+1)^2}\sum_{j=1}^{n}\left(\frac{1}{a_j}+\frac{1}{a'_j}\right),\\
G(x_0,x_{2i}) &= \omega_{2i}\left(\omega_0 Q - \frac{\omega_{2i}}{a_i}\right) = \frac{1}{2n+1}(Q-\frac{1}{a_i}),\\
G(x_0,x_{2i+1}) &= \omega_{2i-1}\left(\omega_0 Q - \frac{\omega_{2i+1}}{a'_i}\right) = \frac{1}{2n+1}(Q-\frac{1}{a'_i}),\\
G(x_i,x_k) &= \frac{\omega_i\omega_k}{\omega_0}\left(\omega_0 Q - \frac{\omega_i}{a_i}-\frac{\omega_k}{a_k}\right) = \frac{1}{2n+1}(Q-\frac{1}{a_i}-\frac{1}{a_j}),\\
G(x_i,x_i) &= \frac{\omega_i^2}{\omega_0}\left(\omega_0 Q - 2\frac{\omega_i}{a_i}\right)+\frac{\omega_i}{a_i\omega_0} = \frac{1}{2n+1}(Q-\frac{2}{a_i})+\frac{1}{a_i}
\end{aligned}
$$

and

$$
R(x_0,x_k) = \frac{2n+1}{c(x_0,x_k)}, \qquad R(x_\ell,x_k) = (2n+1)\left(\frac{1}{c(x_0,x_\ell)}+\frac{1}{c(x_0,x_k)}\right),
$$

where $\ell \neq k$ and $\ell, k = 1,\ldots,2n$.

Moreover, under the previous assumptions, the expression for the Kirchhoff index is

$$
\mathsf{k}(S_{2n}) = 2nQ.
$$

In order to obtain the Green function for the partial subdivision network of the Star, we first compute β, that in this case is

$$
\begin{aligned}
\beta &= G(x_0,x_0)\pi^S(x_0)^2 + 2\sum_{i=1}^{n} G(x_0,x_{2i})\pi^S(x_0)\pi^S(x_{2i})\\
&+ \sum_{i,j=1}^{n} G(x_{2i},x_{2j})\pi^S(x_{2i})\pi^S(x_{2j}) + \sum_{i=1}^{n}\frac{a_i\omega_{0\,2i}^2}{c_{0\,i}c_{i\,0}}\\
&= \frac{1}{4}\,n^2\alpha^2\gamma^4\,Q + \frac{2}{4}n^2\alpha^2\gamma^4\,Q - \frac{2n}{4}\alpha^2\gamma^4\sum_{i=1}^{n}\frac{1}{a_i} + \frac{2}{4}\alpha^2\gamma^4\sum_{i<j,i=1}^{n}\left[Q-\frac{1}{a_i}-\frac{1}{a_j}\right]\\
&+ \frac{1}{4}\alpha^2\gamma^4\sum_{i=1}^{n}\left[Q-\frac{2}{a_i}\right] + \frac{(2n+1)}{4}\alpha^2\gamma^2\sum_{i=1}^{n}\frac{1}{a_i} + \sum_{i=1}^{n}\frac{a_i\omega_{0\,2i}^2}{c_{0\,i}c_{i\,0}}\\
&= n^2\alpha^2\gamma^4\,Q + \frac{\alpha^2\gamma^4}{2}\sum_{i=1}^{n}\frac{1}{a_i}.
\end{aligned}
$$

Proposition 6.1.1. *The Green function for the partial subdivision network of the Star network S_{2n} has the following expression according to the different vertices*

involved

$$G^S_{\omega^S}(x_0,x_0)=\frac{\alpha^4}{2n+1}\,Q+\alpha^2\gamma^2\mathrm{f}_{\mathrm{ev}}\sum_{i=1}^{n}\frac{1}{a_i},$$

$$G^S_{\omega^S}(x_0,x_{2i})=G^S_{\omega^S}(x_0,x_0)-\mathrm{f}_{\mathrm{ev}}\,\frac{1}{a_i},\quad G^S_{\omega^S}(x_0,x_{2i-1})=G^S_{\omega^S}(x_0,x_0)-\mathrm{f}_{\mathrm{odd}}\,\frac{1}{a'_i},$$

$$G^S_{\omega^S}(x_{2i},x_{2j})=G^S_{\omega^S}(x_0,x_0)+\varepsilon_{x_{2i}}(x_{2j})\,\frac{1}{a_i}-\mathrm{f}_{\mathrm{ev}}\,\left(\frac{1}{a_i}+\frac{1}{a_j}\right),$$

$$G^S_{\omega^S}(x_{2i},x_{2j-1})=G^S_{\omega^S}(x_0,x_0)-\mathrm{f}_{\mathrm{ev}}\,\frac{1}{a_i}-\mathrm{f}_{\mathrm{odd}}\,\frac{1}{a'_j},$$

$$G^S_{\omega^S}(x_{2i-1},x_{2j-1})=G^S_{\omega^S}(x_0,x_0)+\varepsilon_{x_{2i}}(x_{2j})\,\frac{1}{a'_i}-\mathrm{f}_{\mathrm{odd}}\,\big(\frac{1}{a'_i}+\frac{1}{a'_j}\big),$$

$$G^S_{\omega^S}(x_0,x_{0\,2i})=\gamma\sqrt{2n+1}\Big(G^S_{\omega^S}(x_0,x_0)-\mathrm{f}_{\mathrm{sub}}\,\frac{1}{a_i}\Big),$$

$$G^S_{\omega^S}(x_{2i-1},x_{0\,2j})=\gamma\sqrt{2n+1}\Big(G^S_{\omega^S}(x_0,x_0)-\mathrm{f}_{\mathrm{odd}}\,\frac{1}{a'_i}-\mathrm{f}_{\mathrm{sub}}\,\frac{1}{a_j}\Big),$$

$$G^S_{\omega^S}(x_{2i},x_{0\,2j})=\gamma\sqrt{2n+1}\Big(G^S_{\omega^S}(x_0,x_0)+\frac{1}{2}\varepsilon_{x_{2i}}(x_{2j})\,\frac{1}{a_j}-\mathrm{f}_{\mathrm{ev}}\,\frac{1}{a_i}-\mathrm{f}_{\mathrm{sub}}\,\frac{1}{a_j}\Big),$$

$$G^S_{\omega^S}(x_{0\,2i},x_{0\,2j})=\gamma^2(2n+1)\Big(G^S_{\omega^S}(x_0,x_0)+\frac{1}{2}\varepsilon_{x_{2i}}(x_{2j})\,\frac{1}{a_j}-\mathrm{f}_{\mathrm{sub}}\,\big(\frac{1}{a_i}+\frac{1}{a_j}\big)\Big),$$

where

$$\mathrm{f}_{\mathrm{odd}}=\frac{\alpha^2}{2n+1},\qquad \mathrm{f}_{\mathrm{ev}}=\frac{2+\alpha^2\gamma^2}{2(2n+1)}\qquad \textit{and}\qquad \mathrm{f}_{\mathrm{sub}}=\frac{1+(n+1)\alpha^2\gamma^2}{2(2n+1)}.$$

Proof. We compute only some cases by using the results given in Proposition 4.3.3.

$$\begin{aligned}G^S_{\omega^S}(x_0,x_0)&=G(x_0,x_0)-2\sum_{\ell\in V}\omega_0G(x_0,x_{2i})\pi^S(\ell)+\beta\omega_0^2\\&=\frac{\alpha^4}{2n+1}\,Q+\frac{\alpha^2\gamma^2(2+\alpha^2\gamma^2)}{2(2n+1)}\sum_{i=1}^{n}\frac{1}{a_i}.\end{aligned}$$

$$\begin{aligned}G^S_{\omega^S}(x_0,x_{2i})&=G(x_0,x_{2i})-\omega^S(x_{2i})G(x_0,x_0)\pi^S(x_0)\\&\quad-\omega^S(x_0)G(x_{2i},x_0)\pi^S(x_0)+\beta\omega^S(x_0)\omega^S(x_{2i})\\&\quad-\sum_{j=1}^{n}\Big[\omega^S(x_{2i})G(x_0,x_{2j})+\omega^S(x_0)G(x_{2i},x_{2j})\Big]\pi^S(x_{2j})\\&=\frac{1}{2n+1}\left(Q-\frac{1}{a_i}\right)-\frac{n\alpha^2\gamma^2}{2(2n+1)}\left(2Q-\frac{1}{a_i}\right)\end{aligned}$$

$$
\begin{aligned}
&- \frac{\alpha^2\gamma^2}{2(2n+1)} \sum_{j=1}^{n} \left(Q - \frac{1}{a_j} \right) - \frac{\alpha^2\gamma^2}{2(2n+1)} \sum_{j=1}^{n} \left(Q - \frac{1}{a_i} - \frac{1}{a_j} \right) \\
&- \frac{\alpha^2\gamma^2}{2} \frac{1}{a_i} + \left(n^2\alpha^2\gamma^4\, Q + \frac{\alpha^2\gamma^4}{2} \sum_{i=1}^{n} \frac{1}{a_i} \right) \frac{\alpha^2}{2n+1} \\
&= \frac{\alpha^4}{2n+1}\, Q + \left(-\frac{1}{2n+1} + \frac{n\alpha^2\gamma^2}{2n+1} - \frac{\alpha^2\gamma^2}{2} \right) \frac{1}{a_i} \\
&+ \frac{\alpha^2\gamma^2(2+\alpha^2\gamma^2)}{2n+1} \sum_{j=1}^{n} \frac{1}{a_j} \\
&= \frac{\alpha^4}{2n+1}\, Q - \frac{2+\alpha^2\gamma^2}{2(2n+1)} \frac{1}{a_i} + \frac{\alpha^2\gamma^2(2+\alpha^2\gamma^2)}{2(2n+1)} \sum_{j=1}^{n} \frac{1}{a_j}.
\end{aligned}
$$

The expression for the Green function for positions involving new vertices is

$$
\begin{aligned}
G^S_{\omega^S}(x_0, x_{0\,2i}) &= \frac{1}{2}\gamma\sqrt{2n+1}\big(G(x_0,x_0) + G(x_0,x_{2i})\big) \\
&- \frac{n\sqrt{2n+1}\alpha^2\gamma^3}{4}\Big(3G(x_0,x_0) + G(x_{2i},x_0)\Big) \\
&- \frac{\sqrt{2n+1}\alpha^2\gamma^3}{4} \sum_{j=1}^{n} \Big(3G(x_0,x_{2j}) + G(x_{2i},x_{2j})\Big) \\
&+ \left(n^2\alpha^4\gamma^4\, Q + \frac{\alpha^2\gamma^4}{2} \sum_{j=1}^{n} \frac{1}{a_j} - \frac{\gamma^2(2n+1)}{4} \frac{1}{a_i} \right) \frac{\alpha^2\gamma}{\sqrt{2n+1}} \\
&= \frac{\alpha^4\gamma}{\sqrt{2n+1}} Q - \frac{\gamma(1+(n+1)\alpha^2\gamma^2)}{2\sqrt{2n+1}\, a_i} + \frac{\alpha^2\gamma^3(2+\alpha^2\gamma^2)}{2\sqrt{2n+1}} \sum_{j=1}^{n} \frac{1}{a_j}.
\end{aligned}
$$

For $i \neq j$,

$$
\begin{aligned}
G^S_{\omega^S}(x_{2i}, x_{0\,2j}) &= \frac{1}{2}\gamma\sqrt{2n+1}\big(G(x_0,x_{2i}) + G(x_{2j},x_{2i})\big) \\
&- \frac{n\alpha^2\gamma^3\sqrt{2n+1}}{4}\Big(G(x_0,x_0) + G(x_{2j},x_0) + 2G(x_{2i},x_0)\Big) \\
&- \frac{\alpha^2\gamma^3\sqrt{2n+1}}{4} \sum_{k=1}^{n} \Big(G(x_0,x_{2k}) + G(x_{2j},x_{2k}) + 2G(x_{2i},x_{2j})\Big) \\
&+ \left(n^2\alpha^4\gamma^4\, Q + \frac{\alpha^2\gamma^4}{2} \sum_{k=1}^{n} \frac{1}{a_k} - \frac{\gamma^2(2n+1)}{4} \frac{1}{a_j} \right) \frac{\alpha^2\gamma}{\sqrt{2n+1}} \\
&= \frac{\alpha^4\gamma}{\sqrt{2n+1}}\, Q - \frac{\gamma(2+\alpha^2\gamma^2)}{2\sqrt{2n+1}} \frac{1}{a_i} - \frac{\gamma(1+(n+1)\alpha^2\gamma^2)}{2\sqrt{2n+1}} \frac{1}{a_j} \\
&+ \frac{\alpha^2\gamma^3(2+\alpha^2\gamma^2)}{2\sqrt{2n+1}} \sum_{k=1}^{n} \frac{1}{a_k}.
\end{aligned}
$$

When $i = j$ we get a similar expression

$$\begin{aligned}
G^S_{\omega^S}(x_{2i}, x_{0\,2i}) &= \frac{1}{2}\gamma\sqrt{2n+1}\big(G(x_0, x_{2i}) + G(x_{2i}, x_{2i})\big) \\
&- \frac{n\alpha^2\gamma^3\sqrt{2n+1}}{4}\Big(G(x_0, x_0) + G(x_{2i}, x_0) + 2G(x_{2i}, x_0)\Big) \\
&- \frac{\alpha^2\gamma^3\sqrt{2n+1}}{4}\sum_{k=1}^{n}\Big(G(x_0, x_{2k}) + G(x_{2i}, x_{2k}) + 2G(x_{2i}, x_{2k})\Big) \\
&+ \left(n^2\alpha^4\gamma^4\,Q + \frac{\alpha^2\gamma^4}{2}\sum_{k=1}^{n}\frac{1}{a_k} - \frac{\gamma^2(2n+1)}{4}\,\frac{1}{a_i}\right)\frac{\alpha^2\gamma}{\sqrt{2n+1}} \\
&= \frac{\gamma}{\sqrt{2n+1}}\,Q + \frac{\gamma(n-1)}{\sqrt{2n+1}}\,\frac{1}{a_i} - \frac{n\gamma^3\alpha^2}{\sqrt{2n+1}}\,Q + \frac{3n\gamma^3\alpha^2}{4\sqrt{2n+1}}\,\frac{1}{a_i} \\
&- \frac{n\gamma^3\alpha^2}{\sqrt{2n+1}}\,Q + \frac{\gamma^3\alpha^2}{\sqrt{2n+1}}\sum_{k=1}^{n}\frac{1}{a_k} - \frac{3(n+1)\gamma^3\alpha^2}{4\sqrt{2n+1}}\frac{1}{a_i} \\
&+ \left(n^2\alpha^4\gamma^4\,Q + \frac{\alpha^2\gamma^4}{2}\sum_{k=1}^{n}\frac{1}{a_k} - \frac{\gamma^2(2n+1)}{4}\,\frac{1}{a_i}\right)\frac{\alpha^2\gamma}{\sqrt{2n+1}} \\
&= \frac{\alpha^4\gamma}{\sqrt{2n+1}}\,Q + \frac{\gamma(4(n-1) - 2(n+2)\alpha^2\gamma^2)}{4\sqrt{2n+1}}\,\frac{1}{a_i} \\
&+ \frac{\alpha^2\gamma^3(2+\alpha^2\gamma^2)}{2\sqrt{2n+1}}\sum_{j=1}^{n}\frac{1}{a_j}.
\end{aligned}$$

For positions that involve only new vertices, we get

$$\begin{aligned}
G^S_{\omega^S}(x_{0\,2i}, x_{0\,2j}) &= -\frac{\alpha^2\gamma^4(2n+1)}{4}\left(\frac{1}{a_i} + \frac{1}{a_j}\right) + \beta\alpha^2\gamma^2 \\
&- \frac{n\alpha^2\gamma^4(2n+1)}{4}\left(2G(x_0, x_0) + G(x_{2i}, x_0) + G(x_{2j}, x_0)\right) \\
&- \frac{\alpha^2\gamma^4(2n+1)}{4}\sum_{k=1}^{n}\left(2G(x_0, x_{2k}) + G(x_{2i}, x_{2k}) + G(x_{2j}, x_{2k})\right) \\
&+ \frac{\gamma^2(2n+1)}{4}\left(G(x_0, x_0) + G(x_0, x_{2j}) + G(x_{2i}, x_0) + G(x_{2i}, x_{2j})\right) \\
&= -\frac{\alpha^2\gamma^4(2n+1)}{4}\left(\frac{1}{a_i} + \frac{1}{a_j}\right) + n^2\alpha^4\gamma^6\,Q + \frac{\alpha^4\gamma^6}{2}\sum_{k=1}^{n}\frac{1}{a_k} \\
&- \frac{n\alpha^2\gamma^4}{4}\left(4Q - \frac{1}{a_i} - \frac{1}{a_i}\right) - \frac{\alpha^2\gamma^4}{4}\sum_{k=1}^{n}\left(4Q - \frac{4}{a_k} - \frac{1}{a_i} - \frac{1}{a_j}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\alpha^2\gamma^4(2n+1)}{4}\left(\frac{1}{a_i}+\frac{1}{a_j}\right)+\frac{\gamma^2}{4}\left(4Q-\frac{1}{a_i}-\frac{1}{a_j}\right)\\
&=\alpha^4\gamma^2\,Q-\frac{\gamma^2\big(1+(n+1)\alpha^2\gamma^2\big)}{2}\left(\frac{1}{a_i}+\frac{1}{a_j}\right)\\
&+\frac{\alpha^2\gamma^4(2+\alpha^2\gamma^2)}{2}\sum_{k=1}^{n}\frac{1}{a_k}.
\end{aligned}
$$

$$
\begin{aligned}
G^S_{\omega^S}(x_{0\,2i},x_{0\,2i}) &= \frac{\gamma^2(2n+1)}{4}\,\frac{1}{a_i}-\frac{\alpha^2\gamma^4(2n+1)}{2}\frac{1}{a_i}+\beta\alpha^2\gamma^2\\
&- \frac{n\alpha^2\gamma^4(2n+1)}{2}\left(G(x_0,x_0)+G(x_{2i},x_0)\right)\\
&- \frac{\alpha^2\gamma^4(2n+1)}{2}\sum_{k=1}^{n}\left(G(x_0,x_{2k})+G(x_{2i},x_{2k})\right)\\
&+ \frac{\gamma^2(2n+1)}{4}\left(G(x_0,x_0)+2G(x_0,x_{2i})+G(x_{2i},x_{2i})\right)\\
&= \frac{\gamma^2(2n+1)}{4}\,\frac{1}{a_i}-\frac{\alpha^2\gamma^4(2n+1)}{2}\frac{1}{a_i}+n^2\alpha^4\gamma^6\,Q+\frac{\alpha^4\gamma^6}{2}\sum_{k=1}^{n}\frac{1}{a_k}\\
&- \frac{n\alpha^2\gamma^4}{4}\left(4Q-\frac{2}{a_i}\right)\\
&- \frac{\alpha^2\gamma^4}{4}\sum_{k=1}^{n}\left(4Q-\frac{4}{a_k}-\frac{2}{a_i}\right)-\frac{\alpha^2\gamma^4(2n+1)}{2a_i}\\
&+ \frac{\gamma^2}{4}\left(4Q-\frac{4}{a_i}\right)+\frac{\gamma^2(2n+1)}{4}\frac{1}{a_i}\\
&= \alpha^4\gamma^2\,Q-\frac{\gamma^2\big(2n-1-2(n+1)\alpha^2\gamma^2\big)}{2a_i}+\frac{\alpha^2\gamma^4(2+\alpha^2\gamma^2)}{2}\sum_{k=1}^{n}\frac{1}{a_k}.
\end{aligned}
$$

The remaining cases will follow by performing similar computations. □

Finally, we obtain the Kirchhoff index for the partial subdivision of the Star.

Proposition 6.1.2. *Let S^S_{2n} be the partial subdivision network of the star S_{2n}, then the Kirchhoff index of S^S_{2n} is given by*

$$
\mathsf{k}^S_{\omega^S}(S^S_{2n}) = n(2+\alpha^2\gamma^2)\,Q+\frac{\gamma^2}{2}\big(2n-1-\alpha^2\gamma^2\big)\sum_{i=1}^{n}\frac{1}{a_i}.
$$

Proof. Taking into account Proposition 5.6.7

$$
\begin{aligned}
\mathsf{k}^S_{\omega^S}(S^S_{2n}) &= \mathsf{k}(S_{2n})+\sum_{x\in V}G(x,x)\frac{\pi^S(x)}{\omega^S(x)}\\
&\quad -\textstyle\sum_{x,y\in F}\alpha(x,y)\alpha(y,x)\omega^S(x)\omega^S(y)R(x,y)\\
&\quad +\sum_{\{x,y\}\in E_1}\frac{c(x,y)}{c(x,v_{xy})c(y,v_{xy})}-\beta
\end{aligned}
$$

$$\begin{aligned}\mathsf{k}^S_{\omega^S}(S^S_{2n}) &= 2nQ + \frac{n\gamma^2}{2}Q + \frac{n\gamma^2}{2}\,Q - \gamma^2\sum_{i=1}^{n}\frac{1}{a_i} + \frac{\gamma^2}{2}(2n+1)\sum_{i=1}^{n}\frac{1}{a_i}\\ &\quad - n^2\alpha^2\gamma^4\,Q - \frac{\alpha^2\gamma^4}{2}\sum_{i=1}^{n}\frac{1}{a_i}\\ &= \left(2n + n\gamma^2 - n^2\alpha^2\gamma^4\right)Q + \frac{\gamma^2}{2}(2n - 1 - \alpha^2\gamma^2)\sum_{i=1}^{n}\frac{1}{a_i}. \quad \square\end{aligned}$$

As expected, the Kirchhoff Index of the partial subdivided Star takes the minimum value for γ approaching zero; this can be interpreted as no subdivision has been performed in the initial network S_{2n}. Actually, $\mathsf{k}^S_{\omega^S}(S^S_{2n})$ attains a minimum for $\gamma = 0$.

6.2 Subdivision on Star networks for the normalized Laplacian

In this section we add results corresponding to the case of the normalized Laplacian for subdivision on the n–Star, see [28]. Moreover, notice that in this case we do the subdivision process on every edge of the network.

Firstly we consider the n–Star network, see Figure 6.2 (left), that has $n+1$ vertices, $\{x_0, x_1, \ldots, x_n\}$, and constant conductance $a > 0$, *i.e.*, $c(x_0, x_i) = a$, for $i = 1, \ldots, n$, and zero otherwise. Thus, the degree function is $k(x_0) = na$ and $k(x_i) = a$ for $i = 1, \ldots, n$, while $\mathrm{vol}(\Gamma) = 2na$. Hence, the subdivision of the n–Star has n new inserted vertices, those white in Fig. 6.2 (right), that we denote as $v_{x_0x_i}$. Accordingly to the definition of the conductances, the degree function of the subdivision n–Star network is $k^S(x_0) = 2na, k^S(x_i) = 2a$, and $k^S(v_{x_0x_i}) = 4a$, $i = 1, \ldots, n$.

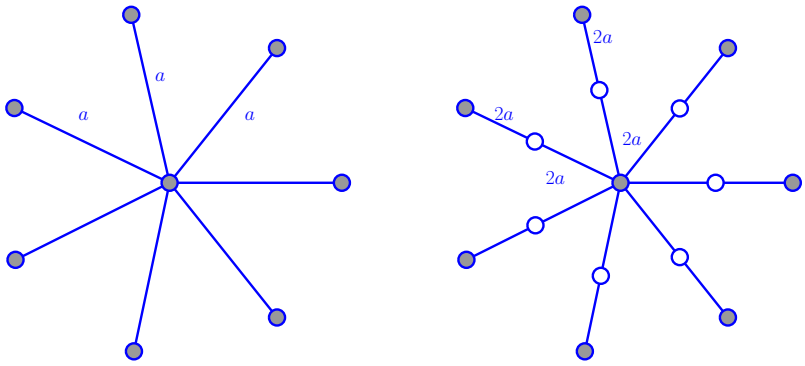


Figure 6.2 The Star network and its subdivision network

Normalized Laplacian matrices of the former network and its subdivision are, re-

spectively

$$\mathsf{L} = \begin{bmatrix} 1 & -\frac{1}{\sqrt{n}}1^T \\ -\frac{1}{\sqrt{n}}1 & \mathsf{I} \end{bmatrix} \quad \text{and} \quad \mathsf{L}_S = \begin{bmatrix} 1 & 0^T & -\frac{1}{\sqrt{2n}}1^T \\ 0 & \mathsf{I} & -\frac{1}{\sqrt{2}}\mathsf{I} \\ -\frac{1}{\sqrt{2n}}1 & -\frac{1}{\sqrt{2}}\mathsf{I} & \mathsf{I} \end{bmatrix}$$

being 0 and 1, n entries vectors (all zeros, all ones) and I the $n \times n$ identity matrix. It can be proved, see [23] for instance, that the group inverse matrix for such an n-Star network is

$$\mathsf{G} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4\sqrt{n}}1^T \\ -\frac{1}{4\sqrt{n}}1 & \mathsf{A} \end{bmatrix}$$

being A an $n \times n$ matrix whose values are $1 - \frac{3}{4n}$ on the diagonal, and $-\frac{3}{4n}$ otherwise.

For the n–Star network, effective resistances are

$$R(x_0, x_i) = \frac{1}{a}, \qquad R(x_i, x_j) = \frac{2}{a},$$

while its corresponding Kirchhoff index is $\mathsf{k}(\Gamma) = n(2n-1)a$.

Hence, using Proposition 4.3.3, we calculate

$$\mathsf{G}^{\mathsf{S}} = \begin{bmatrix} \frac{5}{8} & -\frac{3}{8\sqrt{n}}1^T & -\frac{\sqrt{2}}{8\sqrt{n}}1^T \\ -\frac{3}{8\sqrt{n}}1 & \mathsf{A}_1 & \mathsf{A}_2 \\ \frac{\sqrt{2}}{8\sqrt{n}}1 & \mathsf{A}_2^T & \mathsf{A}_3 \end{bmatrix}$$

where matrices $\mathsf{A}_1, \mathsf{A}_2$ and A_3 have all the same "shape"; that is, a constant value on the diagonal and a different one off the diagonal, so they can be expressed in terms of the identity matrix and J the all ones matrix as

$$\mathsf{A}_1 = 2\mathsf{I} - \frac{11}{8n}\mathsf{J}, \quad \mathsf{A}_2 = \sqrt{2}\mathsf{I} - \frac{9\sqrt{2}}{8n}\mathsf{J} \quad \text{and} \quad \mathsf{A}_3 = 2\mathsf{I} - \frac{7}{4n}\mathsf{J}.$$

After using Proposition 5.6.6, effective resistances for the subdivision network of n–Star network are to be

$$R^S(x_0, x_i) = R^S(v_{x_0x_i, x_0x_j}) = \frac{1}{a},$$
$$R^S(x_0, v_{x_0x_i}) = R^S(x_i, v_{x_0x_i}) = \frac{1}{2a},$$
$$R^S(x_i, x_j) = \frac{2}{a} \quad \text{and} \quad R^S(x_i, v_{x_0x_j}) = \frac{3}{2a},$$

for $i, j = 1, \ldots, n, \quad i \neq j$. Please note that as the subdivided star is a tree, the values of the effective resistances do agree with those obtained by direct application of simple electrical properties.

And also, the Kirchhoff index for the subdivision network of the n–Star network is,

$$\mathsf{k}(\Gamma^S) = 4na(8n-5).$$

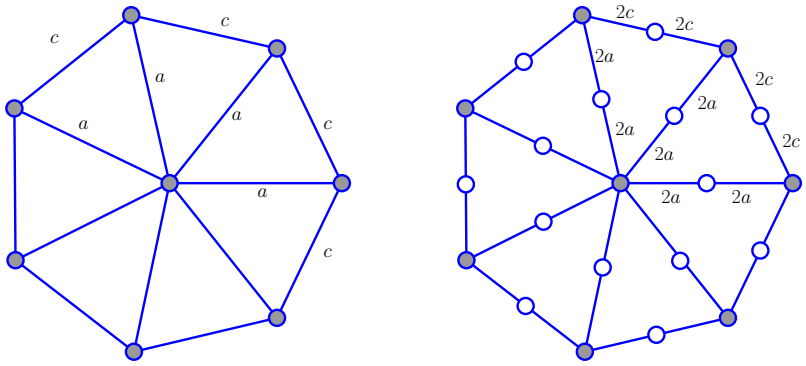


Figure 6.3 The Wheel network and its subdivision network

6.3 Subdivision on Wheel networks for the normalized Laplacian

In our second example, we consider the n–Wheel network, see Figure 6.3 (left), W_n, that has $n+1$ vertices labelled $\{x_0, x_1, \ldots, x_n\}$. The results in this section can be found in [25]. The only non null conductances are $c(x_0, x_i) = a > 0$ and $c(x_i, x_{i+1}) = c > 0$, for $i = 1, \ldots, n$, assuming $x_{n+1} = x_1$. Thus, the degree function is defined as $k(x_0) = na$ and $k(x_i) = a + 2c$ for $i = 1, \ldots, n$. In addition, $\mathrm{vol}(W_n) = 2n(a+c)$.

As the normalized Laplacian operator on a network can be seen as a particular Schrödinger operator, we use the results in [23] again to obtain the Green function of the normalized Laplacian for the n–Wheel network. And it is

$$\begin{aligned}
G(x_0, x_0) &= \frac{(a+2c)^2}{(2(a+c))^2} \\
G(x_0, x_i) &= -\frac{(a+2c)\sqrt{na(a+2c)}}{n(2(a+c))^2}, \quad i = 1, \ldots, n \\
G(x_i, x_j) &= -\frac{(a+2c)^2}{2na(a+c)}\Big(\frac{a}{2(a+c)} + 1\Big) + p\frac{U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p)}{T_n(p) - 1}
\end{aligned}$$

$i, j = 1, \ldots, n$, where $p = 1 + \dfrac{a}{2c}$ and $T_k(p)$ and $U_k(p)$ are the first and the second kind Chebyshev polynomials respectively.

Then the effective resistances are

$$R(x_0, x_i) = \frac{n(a+c)}{c} \frac{U_{n-1}(p)}{(T_n(p)-1)}$$
$$R(x_i, x_j) = 2\frac{n(a+c)}{c} \frac{\Big(U_{n-1}(p) + U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p)\Big)}{(T_n(p)-1)},$$

$i, j = 1, \ldots, n, i \neq j$ and the Kirchhoff index is

$$\begin{aligned} \mathsf{k}(W_n) &= 2n(a+c)\left(-\frac{(a+2c)^2}{2a(a+c)} + \frac{n(a+2c)}{2c}\frac{U_{n-1}(p)}{T_n(p)-1}\right) \\ &= 2n(a+c)\left(\frac{p^2}{(2p-1)(1-p)} + \sum_{j=0}^{n-1}\frac{p}{p-\cos(\frac{2\pi j}{n})}\right), \end{aligned}$$

taking into account that $\dfrac{nU_{n-1}(p)}{T_n(p)-1} = \displaystyle\sum_{j=0}^{n-1}\frac{1}{p-\cos(\frac{2\pi j}{n})}$.

Let us now consider the subdivision network of the n–Wheel network. We denote the new white vertices in Figure 6.3 (right), by $v_{x_0x_i}$ and $v_{x_ix_{i+1}}$, $i = 1, \ldots, n$ provided $x_{n+1} = x_1$, as before. According to the notation, the degree of the vertices in the subdivision network of the n–Wheel network are $k^S(x_0) = 2na$, $k^S(x_i) = 2(a+2c)$, $k^S(v_{x_0x_i}) = 4a$ and $k^S(v_{x_ix_{i+1}}) = 4c$.

From Proposition 4.3.3, we obtain the expression of the values of the Green function for the subdivision of the Wheel network case by case.

Initially when only former vertices are concerned

$$\begin{aligned} G^S(x_0, x_0) &= \frac{5a^2+17ac+16c^2}{8(a+c)^2}, \\ G^S(x_0, x_i) &= -\frac{\sqrt{na(a+2c)}}{8n(a+c)^2}(3a+7c), \quad i = 1, \ldots, n; \\ G^S(x_i, x_j) &= -\frac{(a+2c)^2}{8an(a+c)^2}(11a^2+31ac+16c^2), \\ &\quad + \frac{2p\Big(U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p)\Big)}{(T_n(p)-1)}, \quad i, j = 1, \ldots, n. \end{aligned}$$

Secondly when both, former and later, kinds of vertices are involved for $i, j =$

$1, \ldots, n$

$$
\begin{aligned}
G^S(x_0, v_{x_0x_i}) &= \frac{\sqrt{2}}{8\sqrt{n}} \frac{8c^2 + 3ac - a^2}{(a+c)^2};\\
G^S(x_0, v_{x_ix_{i+1}}) &= \frac{\sqrt{2nac}}{8n(a+c)^2}(-5a - 9c) \text{ assuming } x_{n+1} = x_1;\\
G^S(x_j, v_{x_0x_i}) &= -\frac{\sqrt{2a(a+2c)}}{8na(a+c)^2}(9a^2 + 21ac + 8c^2)\\
&\quad + \frac{p\sqrt{2a}}{\sqrt{a+2c}} \frac{\Big(U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p)\Big)}{(T_n(p) - 1)};\\
G^S(x_j, v_{x_ix_{i+1}}) &= \frac{\sqrt{2(a+2c)c}}{8an(a+c)^2}(-11a^2 - 33ac - 16c^2) \text{ with } x_{n+1} = x_1.
\end{aligned}
$$

And finally when only new vertices are taken into account

$$
\begin{aligned}
G^S(v_{x_0x_i}, v_{x_0x_j}) &= -\frac{a(7a+11c)}{4n(a+c)^2} + \frac{a}{2c} \frac{\Big(U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p)\Big)}{(T_n(p) - 1)}\\
&\quad + \varepsilon_{v_{x_0x_i}}(v_{x_0x_j}), \quad i, j = 1, \ldots, n;\\
G^S(v_{x_0x_i}, v_{x_jx_{j+1}}) &= -\sqrt{ac}\frac{(11a^2 + 23ac + 8c^2)}{4an(a+c)^2}\\
&\quad + \frac{\sqrt{ac}}{2c} \frac{\Big(U_{n-1-|i-j|}(p) + U_{n-1-|i-j-1|}(p)\Big)}{T_n(p) - 1}\\
&\quad + \frac{\sqrt{ac}}{2c} \frac{\Big(U_{|i-j|-1}(p) + U_{|i-j-1|-1}(p)\Big)}{T_n(p) - 1},\\
G^S(v_{x_ix_{i+1}}, v_{x_jx_{j+1}}) &= -\frac{c(15a^2 + 35ac + 16c^2)}{4an(a+c)^2}\\
&\quad + (1+p)\frac{U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p)}{T_n(p) - 1} + \varepsilon_{x_i}(x_j),
\end{aligned}
$$

with $i, j = 1, \ldots, n$ in every case and, as usual with $x_{n+1} = x_1$ when required.

Applying Proposition 5.6.6 we can obtain the effective resistances for the subdivision network of the n–Wheel. In what follows we compute just some of them as examples.

$$
\begin{aligned}
R^S(x_i, x_j) &= R(x_i, x_j), \quad i, j = 0, ..., n \quad i \neq j;\\
R^S(x_0, v_{x_0x_i}) &= \frac{1}{4a} + \frac{n(a+c)}{c} \frac{U_{n-1}(p)}{T_n(p) - 1}, \quad i = 1, \ldots, n;\\
R^S(x_j, v_{x_0x_i}) &= \frac{1}{4a} + \frac{5}{4}\frac{n(a+c)}{c} \frac{U_{n-1}(p)}{T_n(p) - 1}\\
&\quad + \frac{n(a+c)}{c} \frac{U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p)\Big)}{T_n(p) - 1}, \quad i, j = 1, \ldots, n;
\end{aligned}
$$

$$R^S(v_{x_ix_{i+1}},x_j)=\frac{1}{2c}+\frac{n(a+c)}{c}\frac{U_{n-1}(p)}{T_n(p)-1}-\frac{n(a+c)}{c}\frac{\Big(U_{n-2}(p)+1\Big)}{T_n(p)-1}$$
$$+\frac{n(a+c)}{c}\frac{\Big(U_{n-1-|i-j|}(p)+U_{|i-j|-1}(p)\Big)}{T_n(p)-1}$$
$$+\frac{n(a+c)}{c}\frac{\Big(U_{n-1}(p)+U_{n-1-|i+1-j|}(p)+U_{|i+1-j|-1}(p)\Big)}{T_n(p)-1},$$

with $i,j=1,\dots,n$ and assuming $x_{n+1}=x_1$ once more.

In addition, the Kirchhoff index for the subdivision network of the n–Wheel is

$$\mathsf{k}^S(W_n^S)=n+4np\frac{U_{n-1}(p)}{(T_n(p)-1)}-\frac{5a^2+17ac+16c^2}{2a(a+c)}.$$

6.4 Subdivision on Wheel networks for the Laplacian

We consider the wheel network with constant conductances and a subdivision of it for the combinatorial Laplacian. The result of this section can be found in [25]. Let W_n be the wheel network with vertex set $V=\{x_0,x_1,\dots,x_n\}$, where x_0 has degree n, and conductances $c(x_0,x_i)=a>0$ for any $i=1,\dots,n$, $c=c(x_i,x_{i+1})$ if $i=1,\dots,n-1$ and $c=c(x_n,x_1)$, as can be seen in Figure 6.4. For the sake of simplicity we consider that $x_{n+1}=x_1$.

It is known, see for instance [23], that the Green function of W_n is

$$G(x_0,x_0)=\frac{n}{a(n+1)^2},$$
$$G(x_0,x_i)=\frac{-1}{a(n+1)^2},\quad i=1,\dots,n,$$
$$G(x_i,x_j)=-\frac{n+2}{a(n+1)^2}+\frac{U_{n-1-|i-j|}(p)+U_{|i-j|-1}(p)}{2c\big(T_n(p)-1\big)},\quad i,j=1,\dots,n,$$

where $p=1+\frac{a}{2c}$ and $U_\ell(x),T_\ell(x)$ are the Chebyshev polynomials of 1st and 2nd order defined by the recurrence $P_m(x)=2xP_{m-1}(x)-P_{m-2}(x)$ $m\geq 0$ provided that $U_0(x)=1,\ U_1(x)=x$ and $T_{-2}(x)=-1,\ T_{-1}(x)=0$, respectively.

Let us now define the standard subdivision of the wheel network. The new vertices are $y_i=v_{x_0x_i}$ and $z_i=v_{x_ix_{i+1}}$ if $i=1,\dots,n$. The conductances for the new edges are $2a=c(x_0,y_i)$ and $2c=c(x_i,z_i)$ for $i=1,\dots,n$. Whereas, the conductance of the remaining edges follows taking into account relation (4.1).

Observe that $k(y_i)=4a$ and $k(z_i)=4c$ for $i=1,\dots,n$. Moreover, $\alpha(x,y)=\frac{1}{2}$, for every pair of adjacent vertices and $\pi^S(x_0)=\frac{n}{2}$ and $\pi^S(x_i)=\frac{3}{2},i=1,\dots,n$. Then, the expression of the Green kernel for the subdivision network is given next.

Proposition 6.4.1. *Let W_n^S be the subdivision network of W_n, and for any $i,j=1,\dots,n$ consider*

$$g_{ij}(p)=\frac{U_{n-1-|i-j|}(p)+U_{|i-j|-1}(p)}{2c\big(T_n(p)-1\big)}.$$

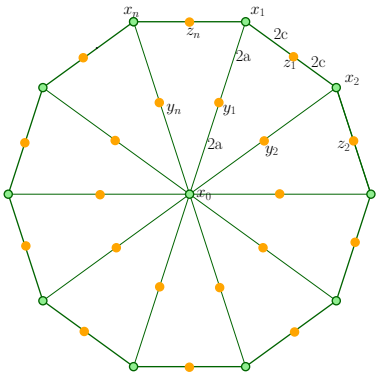


Figure 6.4 Subdivision network of a wheel of $n+1$ vertices

Then, the Green kernel for W_n^S is given by

$$
\begin{aligned}
G^S(x_0,x_0) &= \frac{n(a+26c)}{4ac(3n+1)^2},\\
G^S(x_0,x_i) &= \frac{1}{4ac(3n+1)}\left(\frac{n(a+26c)}{3n+1}-10c\right),\\
G^S(x_0,y_i) &= \frac{1}{4ac(3n+1)}\Big(\frac{n(a+26c)}{3n+1}-6c\Big),\\
G^S(x_0,z_i) &= \frac{1}{4ac(3n+1)}\Big(\frac{n(a+26c)}{3n+1}-(a+10c)\Big),\\
G^S(x_i,x_j) &= g_{ij}(p)+\frac{n(a-34c)-20c}{4ac(3n+1)^2},\\
G^S(x_i,y_j) &= \frac{1}{2}g_{ij}(p)+\frac{n(a-34c)-20c}{4ac(3n+1)^2}+\frac{1}{a(3n+1)},\\
G^S(x_i,z_j) &= \frac{1}{2}\big(g_{ij}(p)+g_{i\,j+1}(p)\big)+\frac{n(a-34c)-20c}{4ac(3n+1)^2}-\frac{1}{4c(3n+1)},\\
G^S(y_i,y_j) &= \frac{1}{4}g_{ij}(p)+\frac{\varepsilon_{y_i}(y_j)}{4a}+\frac{n(a-34c)-20c}{4ac(3n+1)^2}+\frac{2}{a(3n+1)},\\
G^S(y_i,z_j) &= \frac{1}{4}\big(g_{ij}(p)+g_{i\,j+1}(p)\big)+\frac{n(a-34c)-20c}{4ac(3n+1)^2}-\frac{a-4c}{4ac(3n+1)},\\
G^S(z_i,z_j) &= \frac{p+1}{2}g_{ij}(p)+\frac{n(a-34c)-20c}{4ac(3n+1)^2}-\frac{1}{2c(3n+1)}+\frac{\varepsilon_{z_i}(z_j)}{4c}.
\end{aligned}
$$

Proof. The expressions given in the proposition follow from the expression for the Green kernel obtained in Proposition 4.3.3. We compute one of the cases in order to illustrate the methodology.

Firstly, we compute the constant

$$\begin{aligned}\beta &= \frac{1}{(n+m)^2}\sum_{s,r\in V} G(s,r)\pi^S(r)\pi^S(s) + \frac{1}{(n+m)^2}\sum_{r\sim s}\frac{1}{k(v_{rs})}\\ &= \frac{n}{4a(3n+1)^2}\left[\left(\frac{n-3}{n+1}\right)^2 + \frac{a}{c} + 1\right],\end{aligned}$$

where we have taken into account that $\sum_{r\in V} G(s,r) = 0$ and hence

$$\sum_{r\in V} G(s,r)\pi^S(r) = \frac{n-3}{2}G(s,x_0).$$

Consider $z_i = v_{x_i x_{i+1}}$ and $z_j = v_{x_j x_{j+1}}$, then

$$\begin{aligned}G^S(z_i,z_j) &= \frac{1}{4}\Big(G(x_i,x_j) + G(x_{i+1},x_j) + G(x_i,x_{j+1}) + G(x_{i+1},x_{j+1})\Big)\\ &\quad - \tfrac{1}{2(3n+1)}\sum_{\ell=0}^{n}\big(G(x_j,x_\ell) + G(x_{j+1},x_\ell) + G(x_j,x_\ell) + G(x_{j+1},x_\ell)\big)\pi^S(x_\ell)\\ &\quad + \frac{\varepsilon_{z_j}(z_i)}{k(z_i)} - \frac{1}{(3n+1)k(z_i)} - \frac{1}{(3n+1)k(z_j)} + \beta\\ &= -\frac{n+2}{a(n+1)^2} + \frac{2U_{|i-j|-1}(p) + U_{|i+1-j|-1}(p) + U_{|i-j-1|-1}(p)}{8c\big(T_n(p)-1\big)}\\ &\quad + \frac{2U_{n-1-|i-j|}(p) + U_{n-1-|i+1-j|}(p) + U_{n-1-|i-j-1|}(p)}{8c\big(T_n(p)-1\big)}\\ &\quad + \frac{n-3}{(3n+1)(n+1)^2} + \frac{\varepsilon_{z_j}(z_i)}{4c} - \frac{2}{(3n+1)4c} + \beta\\ &= \frac{(a+4c)\Big(U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p)\Big)}{8c^2\big(T_n(p)-1\big)} - \frac{n(5a+34c)+2a+20c}{4ac(3n+1)^2}\\ &\quad + \frac{\varepsilon_{z_i}(z_j)}{4c}. \quad \square\end{aligned}$$

To end up the section we compute the Kirchhoff index of the standard subdivision graph associated with the wheel W_n.

Corollary 6.4.2. *The Kirchhoff index of the standard subdivision network of W_n^S is*

$$\mathsf{k}^S(W_n^S) = \frac{3n^2(a+c)-25cn}{4ac} + \frac{n(3n+1)\big(7U_{n-1}(p)+2U_{n-2}(p)+2\big)}{8c\big(T_n(p)-1\big)}.$$

7

Future Work

At this very particular moment, July 2022, we have in mind two exciting work in progress that evidently must be considered in a chapter untitled Future Work of this PhD thesis memory.

The second one of them, it can be understood as a natural extension of the mathematical path we have followed and have explained so far. Presenting what we have called electrical path–subdivision of a network that is accomplished by substituting only some initial edges by general paths (of length greater than two and eventually of different lengths each of them). We face the obtention of explicit expression of Green's operator and Green's function for a non–singular Schrödinger type operator, as it is the more general linear difference second order operator that can be related to diffusion problems. In this work we will take profit of a very important result we are to obtain in the first work we are about to introduce right now.

And the first one is a generalized inverse matrix problem so, in some sense, related with all of our previous work as well. We expect two obtain two different publications from it. It is conceived from a slightly different point of view, as the result is established directly in a matrix version framework but is obtained also by considering modified discrete structures from a given initial one and taking advantage of related boundary value problems posed on the related networks. In this work there is no references to the subdivision procedure, hence being settled in a more general framework, and the main goal is that of figuring out an explicit expression for an inverse matrix of a reduced dimension matrix in terms of the Moore–Penrose inverse matrix of a higher dimensional inverse matrix that is initially given. With such a result we obtain a generalization of the well known bottleneck matrices and also a second important result (which is the one used in the other work in progress) that is an explicit expression for the Green's function on a path for Scödinger type operators on networks for Dirichlet boundary value problems. This, we think, is also a rather important result by itself that deserves its own publication aside. Hence we expect, hopefully, for two more publications from this work once we conclude it.

Hence, we proceed now to explain these two works that are quite developed and also some other ideas that we will take into consideration once this thesis processus ends.

7.1 Inverse matrix of a submatrix of a singular M–matrix

Given L, a symmetric singular positive semi–definite M–matrix, we have explained that it can be related with a network $\Gamma = (V, E, c)$ where a singular Schrödinger type operator $\mathcal{L}_{q_\omega}$ for some potential associated to a weight, hence to some positive and unitary function $\omega \in \Omega(V)$, is considered. In this context the Moore–Penrose inverse matrix of the initially given matrix is an expression of the kernel associated to Green's operator (again, associated to $\mathcal{L}_{q_\omega}$), that is $\mathsf{L}^\dagger = \mathsf{G}_{\mathsf{q}_\omega}$.

Our goal is to find out an explicit expression to compute, entrywise, the inverse matrix of any submatrix of L that we want to consider, so the title of this section, by using mainly the contents of $\mathsf{G}_{\mathsf{q}_\omega}$.

Of course that we put the problem in a networks setting where discrete potential theory is going to be applied. As the entries of the chosen submatrix can be related to some of the vertices of the initial given network, we take into account a subset $F \subset V$ and pose convenient Poisson but also Dirichlet problems in carefully chosen discrete structures that are related to the former one, Γ, and the one defined upon the vertex set F, noted as Γ^F.

As Dirichlet problems are always compatible, the inverse operator of $\mathcal{L}_{q_\omega}$ on $\mathcal{C}(F)$ is called *Green's operator for* F and denoted by $\mathcal{G}^F_{q_\omega}$. The associated function $G^F_{q_\omega} : F \times F \longrightarrow \mathbb{R}$ defined for any $y \in F$ as $G^F_q(\cdot, y) = \mathcal{G}^F_q(\varepsilon_y)$, is called the *Green function for* F. It is symmetric, can be found by solving homogeneous Dirichlet problems en F and satisfies that $\mathcal{G}^F_q(f)(x) = \sum_{y \in F} G^F_q(x, y) f(y)$, $x \in F$, for every $f \in \mathcal{C}(F)$. Hence our efforts are devoted to find out which is the kernel $G^F_{q_\omega}$ so as to define the inverse operator $\mathcal{G}^F_{q_\omega}$ on the subnetwork.

7.1.1 Host networks, null–extensions, contractions, bordered and host operators

At this point we define some technical linear operators that, having no physical meaning (at least we do not care if they have any), are suitable for our purposes.

In first place we define new discrete structures by adding a new vertex. Let $\hat{x} \notin V$ and suppose that both ω and G_{q_ω}, are extended by 0 to $\widehat{V} = V \cup \{\hat{x}\}$ and $\widehat{V} \times \widehat{V}$ respectively; that is, let us assume that $\omega(\hat{x}) = 0$ and $G_{q_\omega}(\hat{x}, x) = G_{q_\omega}(x, \hat{x}) = G_{q_\omega}(\hat{x}, \hat{x}) = 0$ for any $x \in V$. We then take into account $\mathcal{G}_{q_\omega, \widehat{V}}$, the *null–extension* of $\mathcal{G}_{q_\omega}$ to $\mathcal{C}(\widehat{V})$ defined as $\mathcal{G}_{q_\omega, \widehat{V}} \colon \mathcal{C}(\widehat{V}) \to \mathcal{C}(\widehat{V})$

$$\mathcal{G}_{q_\omega, \widehat{V}}(u)(x) = \sum_{y \in \widehat{V}} G_{q_\omega}(x, y) u(y)$$

Therefore, for a proper $F \subset V$, let us consider $\widehat{F} = F \cup \{\hat{x}\}$ and let us define not only the *contraction to* F *of the Green operator* $\mathcal{G}_{q_\omega}$, denoted $\mathcal{G}_{q_\omega, F} \colon \mathcal{C}(F) \to \mathcal{C}(F)$ but also its *null–extension to* $\widehat{F}$ noted as $\mathcal{G}_{q_\omega, \widehat{F}} \colon \mathcal{C}(\widehat{F}) \to \mathcal{C}(\widehat{F})$, analogously.

A critical role in our main result is devoted to two additional operators (and their respective kernels) that are related to the operators we have just defined. The

Bordered operator for $\widehat{F}$ is the linear operator $\mathcal{B}_{\widehat{F}}\colon \mathcal{C}(\widehat{F}) \longrightarrow \mathcal{C}(\widehat{F})$ defined for any $u \in \mathcal{C}(\widehat{F})$ as

$$\begin{cases} \mathcal{B}_{\widehat{F}}(u)(x) = \mathcal{G}_{q_\omega,\widehat{F}}(u)(x) + u(\hat{x})\omega(x) & x \in F, \\ \mathcal{B}_{\widehat{F}}(u)(\hat{x}) = \langle u, \omega \rangle_{\widehat{F}} \end{cases}$$

It is a non–singular, self–adjoint operator and its corresponding kernel, called *bordered kernel for* $\widehat{F}$, and denoted by $B_{\widehat{F}}$ is strongly related with the Green function defined on the initial whole vertex set V. Next, we take into account the so called *host operator for* $\widehat{F}$ as the linear endomorphism on $\mathcal{C}(\widehat{F})$ that is the inverse of the previous bordered operator for $\widehat{F}$. Thus the linear operator $\mathcal{H}_{\widehat{F}} = \left(\mathcal{B}_{\widehat{F}}\right)^{-1}$ has an associated kernel $H_{\widehat{F}}$ that we name *host kernel for* $\widehat{F}$ that is symmetric and can be characterized by some identities.

All previous kernels can be interpreted matricially once a labelling on the vertex set is provided. So let us suppose that $F \subset V$ is such that $|F| = m < n = |V|$ being $F = \{x_1, \ldots, x_m\}$ while $V = \{x_1, \ldots, x_m, \ldots, x_n\}$. Then, if the matrix associated with G_{q_ω} is denoted by $\mathsf{G}_{q_\omega} \in \mathcal{M}_{n\times n}$, we identify the kernel $G_{q_\omega,F}$ with its submatrix

$$\mathsf{G}_{q_\omega,F} = \begin{bmatrix} G_{q_\omega}(x_1,x_1) & \cdots & G_{q_\omega}(x_1,x_m) \\ \vdots & \ddots & \vdots \\ G_{q_\omega}(x_m,x_1) & \cdots & G_{q_\omega}(x_m,x_m) \end{bmatrix} \in \mathcal{M}_{m\times m}.$$

Therefore, the matrices associated with the bordered kernel and the host kernel are in $\mathcal{M}_{(m+1)\times(m+1)}$ and can be denoted respectively by

$$\mathsf{B}_{\widehat{F}} = \begin{bmatrix} \mathsf{G}_{q_\omega,F} & \mathsf{w} \\ \mathsf{w}^\top & 0 \end{bmatrix} \text{ and } \mathsf{H}_{\widehat{F}} = \begin{bmatrix} H_{\widehat{F}}(x_1,x_1) & \cdots & H_{\widehat{F}}(x_1,x_m) & H_{\widehat{F}}(x_1,\hat{x}) \\ \vdots & \ddots & \vdots & \vdots \\ H_{\widehat{F}}(x_m,x_1) & \cdots & H_{\widehat{F}}(x_m,x_m) & H_{\widehat{F}}(x_m,\hat{x}) \\ H_{\widehat{F}}(\hat{x},x_1) & \cdots & H_{\widehat{F}}(\hat{x},x_m) & H_{\widehat{F}}(\hat{x},\hat{x}) \end{bmatrix}$$

with $\mathsf{w} = \left[\omega(x_1), \ldots, \omega(x_m)\right]^\top \in \mathcal{C}(F)$ being the restriction to F of the given initial weight function $\omega \in \mathcal{C}(V)$. We will call both matrices *Bordered matrix for* $\widehat{F}$ and *Host matrix for* $\widehat{F}$, respectively and our target is to find an explicit expression of the entries of $(\mathsf{G}_{q_\omega}, F)^{-1}$.

Since $\mathsf{G}_{q_\omega,F}$ is positive definite, we have that $a = -\langle (\mathsf{G}_q^F)^{-1}\mathsf{w}, \mathsf{w}\rangle < 0$. Therefore, after a quite straightforward blockwise calculation,

$\mathsf{H}_{\widehat{F}}$ can be expressed also as

$$\mathsf{H}_{\widehat{F}} = \frac{1}{a}\begin{bmatrix} a(\mathsf{G}_{q_\omega,F})^{-1} + (\mathsf{G}_{q_\omega,F})^{-1}\mathsf{w}\mathsf{w}^\top(\mathsf{G}_{q_\omega,F})^{-1} & -(\mathsf{G}_{q_\omega,F})^{-1}\mathsf{w} \\ -\mathsf{w}^\top(\mathsf{G}_{q_\omega,F})^{-1} & 1 \end{bmatrix}.$$

7.1.2 A Poisson problem in connection with a Dirichlet problem

The core of the work comes when considering Discrete Potential theory tools. We succeed in establishing a relationship between the solution of a Poisson equation

posed on the whole V, with the solution of a related homogeneous Dirichlet problem stated on the proper $F \subset V$. Applying this idea to suitable Poisson equations and their related Dirichlet problems we are capable to relate their respective Green kernels.

In order to do this, let us first consider a data function $h_F \in \mathcal{C}(F)$, that can be extended by zero to be considered as $h_F \in \mathcal{C}(V)$ too.

For a moment we denote by $\mathcal{L}^F_{q_\omega}$ the automorphism that results from restricting to $\mathcal{C}(F)$ the positive semidefinite Schrödinger operator $\mathcal{L}_{q_\omega}$. Hence the Dirichlet problem

$$\mathcal{L}^F_{q_\omega}(u_F) = h_F, \quad \text{on } F$$

has a unique solution, say $u_F \in \mathcal{C}(F)$ that can be extended by zero so as to be considered in $\mathcal{C}(V)$. In this way $\mathcal{L}_{q_\omega}(u_F) \in \mathcal{C}(V)$ makes sense. We denote $h_{F^c} = \mathcal{L}_{q_\omega}(u_F)_{|_{F^c}} \in \mathcal{C}(F^c)$, that is $h_{F^c}(x) = \mathcal{L}_{q_\omega}(u_F)(x)$ only when $x \in F^c$. As the corresponding null–extensions h_F and h_{F^c} do have supports that do not intersect, the equation

$$\mathcal{L}_{q_\omega}(u_F)(x) = h_F(x) + h_{F^c}(x), \quad \text{for every } x \in V,$$

makes sense. So here it is a Poisson problem on the whole V related with the initial Dirichlet problem defined only on F.

Certainly $h_F + h_{F^c} \in \mathcal{C}(V)$ is a compatible data function for the Poisson equation so the correspondent Green's operator gives us the orthogonal to ω solution of it. Therefore, it must be some $\alpha \in \mathbb{R}$ such that

$$u_F(x) = \mathcal{G}_{q_\omega}(h_F + h_{F^c})(x) + \alpha\omega(x) = \mathcal{G}_{q_\omega}(h_F)(x) + \mathcal{G}_{q_\omega}(h_{F^c})(x) + \alpha\omega(x) \quad (7.1)$$

for every $x \in V$.

7.1.3 Main result and two very important particular cases

After this value for α is figured out, our main result is expressed as

$$\begin{aligned} G^F_{q_\omega}(x,y) = G_{q_\omega}(x,y) &- \sum_{z,t\in F^c} G_{q_\omega}(x,z) H_{\widehat{F^c}}(z,t) G_{q_\omega}(t,y) \\ &- \omega(y) \sum_{z\in F^c} H_{\widehat{F^c}}(\hat{x},z) G_{q_\omega}(z,x) - \omega(x) \sum_{z\in F^c} H_{\widehat{F^c}}(\hat{x},z) G_{q_\omega} v(z,y) \\ &- \omega(x)\omega(y) H_{\widehat{F^c}}(\hat{x},\hat{x}). \end{aligned}$$

There are two immediate corollaries of this wonderful result that are worth to mention: when $F = V \setminus \{z\}$ (all the given initial vertex set except one vertex) and when $F = V \setminus \{z_1, z_2\}$ (that is we are subtracting two of the initial given vertices).

Bottleneck matrix

In the first case we recover the so called bottleneck matrix defined for the Laplacian operator and here extended to the case of singular Schrödinger type operators, see [56].

When ω is constant, Schrödinger type operators turn out to be combinatorial Laplacians and the correspondent kernels are called Laplacian matrices. A very well known tool related with a Laplacian matrix is the so called bottleneck matrix, obtained by erasing the row and column corresponding to a particular vertex of the network. Many authors, see for example [56, 66], have investigated on bottleneck matrices and many important results in relation with the connectivity of the structure for instance can be derived from bottleneck matrices.

In our framework, the idea of a bottleneck matrix can be extended to networks by solving Dirichlet problems when F, the proper subset, is V but just one vertex, lets say z, so that $F^c = \{z\}$. In this particular case, Dirichlet problems are stated as

$$\mathcal{L}_{q_\omega}(u) - \lambda \langle u, \omega \rangle_{|_V} \omega = f \text{ on } F \text{ and } u(z) = 0$$

and then we obtain

$$G_q^F(x,y) = G_q(x,y) - \Big[\frac{G_q(x,z)}{\omega(x)\omega(z)} + \frac{G_q(y,z)}{\omega(y)\omega(z)} - \frac{G_q(z,z)}{\omega(z)^2}\Big]\omega(x)\omega(y),$$

for every $x, y \in F$.

Green's function for a Dirichlet problem on a path

The second particular case turns to be a very important result. As far as we know it is unknown up to today and it is important because it can be related to second order difference equations. The result can be stated in terms of giving a closed expression for the Green's function for a Dirichlet problem when a singular Schrödinger operator is taking into account on a path.

Let $P = \{x = x_1 \sim x_2 \sim \cdots \sim x_n = y\}$ denote a n–vertex path joining vertices x and y equipped with a conductance function $c(x_i, x_j)$ non–null whenever $i, j = 1, \ldots, n$ are consecutive. Moreover there exists $\omega \in \mathcal{C}(P)$ a weight function defined on the vertex set (that will be also denoted by P). We shall use $C_i = \omega(x_i)c(x_i, x_{i+1})\omega(x_{i+1})$, $i = 1, \ldots, n-1$ so that the corresponding Schrödinger type non–singular linear operator from $\mathcal{C}(P)$ to itself is defined as

$$\mathcal{L}_{q_\omega}u(x_i) = \begin{cases} \dfrac{C_1}{\omega(x_1)}\left(\dfrac{u(x_1)}{\omega(x_1)} - \dfrac{u(x_2)}{\omega(x_2)}\right), & i = 1, \\ \dfrac{C_{i-1}}{\omega(x_i)}\left(\dfrac{u(x_i)}{\omega(x_i)} - \dfrac{u(x_{i-1})}{\omega(x_{i-1})}\right) + \dfrac{C_i}{\omega(x_i)}\left(\dfrac{u(x_i)}{\omega(x_i)} - \dfrac{u(x_{i+1})}{\omega(x_{i+1})}\right) & i = 2, \ldots, n-1, \\ \dfrac{C_{n-1}}{\omega(x_n)}\left(\dfrac{u(x_n)}{\omega(x_n)} - \dfrac{u(x_{n-1})}{\omega(x_{n-1})}\right), & i = n, \end{cases} \tag{7.2}$$

for every $u \in \mathcal{C}(P)$.

Within the framework of this second future work, we can consider that $F = \{x_2, \ldots, x_{n-1}\}$ that is the hole vertex set but except for the two end points $x = x_1$ and $y = x_n$, apply the main result obtained and previously presented, figure out

the correspondent expression of the Green kernel and, after some calculation to get resistance distances involved in the cited expression obtaining a beautiful result

$$\frac{G_q^F(x_i, x_j)}{\omega(x_i)\omega(x_j)} = \frac{R(x_1, x_{min\{i,j\}})R(x_1, x_{max\{i,j\}})}{R(x_1, x_n)}$$

for every $i, j = 1, \ldots, n$.

7.2 Path–subdivision

In this second work we have two targets to be achieved: from a first point of view we complete our study of the so called electrical subdivision of a network by considering a generalization that models the most general case that has sense, or it is applicable, for facing diffusion problems. But, and from another point of view, as we consider the case of non-singular Schrödinger operators, we get a contribution to the computation of generalized inverse matrices of large non singular positive semi–definite M–matrices from given generalized inverse matrices of smaller dimension and obviously related to the previous ones.

In these second work, from a technical point of view, we also change of paradigm as we obtain the solutions of boundary value problem that do include Dirichlet boundary value problems as well. Thus a different situation, more ellaborated and that imposes a longer algorithm has to be treated.

7.2.1 Electrically compatible path–partial subdivision

Graphs and networks can be modified in order to obtain other graphs or networks. Vertices and/or edges can be removed or added in an extremely vaste variety of ways in order to obtain new discrete structures that are related with the former ones. Correspondingly the conductance function may be modified too. There are so many operations that transform a given discrete structure into others, simpler or much more complicated.

Up to now we have extensively treated an electrically compatible subdivision procedure of all the edges of a given discrete structure and also a somehow partial–generalization of it, that can be understood in both senses; as it do not forces all edges to be subdivided thus allowing some degrees of freedom it should be clearly considered a generalization, but also it consists in a particular case of the idea, being applied to just one edge or to a few of them, so using not the hole transformation. By the time of baptism we had many doubts and quite a few controversial.

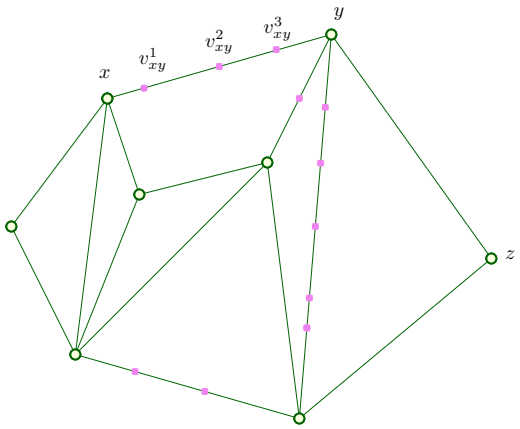
But a really significative generalization of the initial idea came to our minds sometime: using subdivision of just a precise subset of the initially given edge set of the discrete structure but allowing the possibility of every edge to be subdivided in a different way, in the sense of being substituted by a path of different length. For sure we are still installed in the electric circuit setting (by now it is clear that this framework is clearly overcame) but we again force the substituting path to be equivalent to the former edge.

Specifically what we discussed to name generalized electrical subdivision of networks but finally propose to be called *electrical compatible path–partial subdivision of*

networks, consists in, given a network $\Gamma = (V, E, c)$ replacing only some former edges $\{x, y\} \in E$ by a path

$$P_{xy} = \{x = v_{xy}^0 \sim v_{xy}^1 \sim \cdots \sim v_{xy}^{\ell xy} \sim v_{xy}^{\ell xy+1} = y\},$$

not just of length two, but of length $\ell_{xy} + 1$ instead, and the conductance function to be defined on the new vertices introduced in this way such that the electrical condition of a generalized series connection is fulfilled.



Thus a brand new $\Gamma^{pS} = (V^{pS}, E^{pS}, c^{pS})$ is defined as follows. Given a two sets partition of the former edge set E, say $E_1, E_2 \subset E$ such that $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, suppose that the edges in E_1 are those to be electrically path–subdivided while the edges corresponding to E_2 are those devoted to stay. Then the new vertex set V^{pS} is the former set V to which the new vertices included in the collection of paths has been added. Hence some new $v_{xy}^1, v_{xy}^2, \ldots, v_{xy}^{\ell_{xy}}$ vertices are defined for every single former edge $\{x, y\} \in E_1$ to be erased. Thus, being ℓ_{xy} the number of new vertices in a substituting path, easily $|V^{pS}| = |V| + \sum_{\{x,y\} \in E_1} \ell_{xy}$. The new edges set E^{pS} is composed by the initially given edges on Γ not to be replaced, that is $E_2 \subset E$ and the set of new edges that are defined in the paths that substitute the former edges in E_1. In this way it turns out that $|E^{pS}| = |E_2| + \sum_{\{x,y\} \in E_1} (\ell_{xy} + 1) = |E| + \sum_{\{x,y\} \in E_1} \ell_{xy}$. Finally the conductance function $c^{pS} \colon V^{pS} \times V^{pS} \to [0, +\infty)$ is defined almost freely on the new created edges (v_{xy}^i, v_{xy}^{i+1}) for $i = 0, \ldots, \ell_{xy}$ for every P_{xy} adjoined path. We will denote them as $c_{xy}(v_{xy}^i, v_{xy}^{i+1})$ as though there is no possible confusion with former existing conductances and accept an absolutely arbitrary choice of values if that fulfills the electrically compatible series connection condition, which is (for a non–singular Schödinger type operator)

$$\frac{1}{\omega(x)c(x,y)\omega(y)} = \frac{1}{\omega(x)c_{xy}(x, v_{xy}^1)\omega(v_{xy}^1)} + \cdots + \frac{1}{\omega(v_{xy}^{\ell_{xy}-1})c_{xy}(v_{xy}^{\ell_{xy}-1}, y)\omega(y)} \tag{7.3}$$

and is not altered whenever $\{x, y\} \in E_2$.

Remark: we would like to point out that this quite bizarre notation cannot be simplified for our purposes. Even though, the names we have chosen for the vertices avoid any kind of confusion so, in the sequel, we are not using c^{pS} anymore in an obvious abuse of notation. Moreover, as Γ^{pS} is clearly non directed (as a graph) we must take into account that for every $\{x, y\} \in E_1$ paths P_{xy} and P_{yx} consists in the same vertex and edges sets, thus we write $P_{xy} = P_{yx}$. Also $\ell_{xy} = \ell_{yx}$ and $v_{xy}^i = v_{yx}^{\ell_{yx}+1-i}$ for $i = 1, \ldots, \ell_{xy}$. Also we consider $x = v_{xy}^0 = v_{yx}^{\ell_{yx}+1}$ and $y = v_{yx}^0 = v_{xy}^{\ell_{xy}+1}$ in a certain abuse of notation.

Also it is very important for us to notice that this definition of the electrical subdivision of a network has a very clear physical inspiration and sense as it models substitution of a wire in an electric circuit by equivalent more complicated branches. And allows the possibility of replacing only some of the components of an initial given electric circuit. Also we want to point out that physically, we are simulating putting rheostats in any point of the wire. This is a crucial difference with respect to the standard graph subdivision that can be found in the literature because it obliges to consider two substituting wires that are equal and this new electrical subdivision allows substitutions that are globally equivalent but with no other restriction, in particular new wires can behave differently one from each others.

The definition of c_{xy} cannot be misunderstood as all the edges in E_{xy} have both vertices in V_{xy}. Hence, by the sake of simplicity, it will be denoted as c. Moreover for each subdivided edge, there exist infinitely many different choices of conductances fulfilling (4.1), so that different choices will lead to different subdivision networks.

With this definition of the modified structure then, If $\mathcal{L}_{q_\omega}^S$ denotes the positive semi–definite Schrödinger operator related to the weight $\omega \in \mathcal{C}(V^S)$ of Γ^S, then for any $u \in \mathcal{C}(V^S)$ we have that the so called Doob transform expression of $\mathcal{L}_{q_\omega}^S$ is

$$\begin{aligned}\mathcal{L}_{q_\omega}^S(u)(x) = &\sum_{\{x,y\}\in E_1} c(x, v_{xy}^1)\omega(v_{xy}^1)\left(\frac{u(x)}{\omega(x)} - \frac{u(v_{xy}^1)}{\omega(v_{xy}^1)}\right)\\ &+ \sum_{\{x,y\}\in E_2} c(x, y)\omega(y)\left(\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right),\end{aligned}$$

when $x \in V$ (is an old vertex of the former network), whereas

$$\begin{aligned}\mathcal{L}_{q_\omega}^S(u)(v_{xy}^i) = &\, c(v_{xy}^{i-1}, v_{xy}^i)\omega(v_{xy}^{i-1})\left(\frac{u(v_{xy}^i)}{\omega(v_{xy}^i)} - \frac{u(v_{xy}^{i-1})}{\omega(v_{xy}^{i-1})}\right)\\ &+ c(v_{xy}^i, v_{xy}^{i+1})\omega(v_{xy}^{i+1})\left(\frac{u(v_{xy}^i)}{\omega(v_{xy}^i)} - \frac{u(v_{xy}^{i+1})}{\omega(v_{xy}^{i+1})}\right)\end{aligned}$$

when $v_{xy}^i, i = 1, \ldots, \ell_{xy}$ is a new inserted vertex of some of the attached paths.

7.2.2 Overall approach

Our goal is, given $f \in \mathcal{C}(V^{sP})$, to obtain a solution of

$$\mathcal{L}_{q_\omega}^S(u) = f, \quad \text{on } V^{sP} \tag{7.4}$$

that is $u \in \mathcal{C}(V^{sP})$ with the help of the solution of two other auxiliary problems: a Dirichlet boundary value problem on every attached path, and a conveniently stated Poisson problem defined on the base network Γ, that is on former V.

On every P_{xy} inserted we will pose a Dirichlet boundary value problem. As these problems are always compatible, their solution will establish de values of the overall solution u on the new inserted vertices v_{xy}^i for $i = 1, \ldots, \ell_{xy}$ and every $\{x, y\} \in E_1$. These values will be defined explicitly depending on the unknown values $u(x)$ and $u(y)$ (the ends of each path, vertices of the former network) for every path P_{xy}.

Once all the battery of Dirichlet boundary value problems will be solved, a rather well known technique of contracting the data, solving a Poisson problem on Γ for this contracted data and finally extension of the accomplished solution to Γ^S that we have used in all our previous works will lead to a solution of the initial problem.

Finally, after obtaining the solution of (7.4) we will be ready for obtaining the correspondent Green's function.

7.2.3 Dirichlet boundary value problems

Now we recover from Subsection 3.5.1 what a Dirichlet boundary value problem is and we explain some inquiries of the way we solve this situation.

Generally speaking a *Dirichlet problem* consists in, given a proper connected subset $W \subsetneq V$, $f \in \mathcal{C}(W)$ and $g \in \mathcal{C}(\overline{W})$, finding $u \in \mathcal{C}(\overline{W})$ such that

$$\begin{aligned} \mathcal{L}_{q_\omega}(u) = f \quad &\text{on} \quad W; \\ u = g \quad &\text{on} \quad \partial W. \end{aligned} \tag{7.5}$$

Such a boundary value problem has a unique solution (see [7]) for any data $f \in \mathcal{C}(W)$ and $g \in \mathcal{C}(\partial W)$ that can be found equivalently by solving

$$\begin{aligned} \mathcal{L}_{q_\omega}(v) = f - \mathcal{L}_{q_\omega}(g) \quad &\text{on} \quad W; \\ v = 0 \quad &\text{on} \quad \partial W, \end{aligned} \tag{7.6}$$

as $u = v + g$ on $\overline{W}$ easily, once using that a function is extended by zero outside the domain where it is canonically defined.

7.2.4 The Green's function of a path

We apply the previous general setting when $W = \overset{\circ}{P}_{xy}$ (hence $\partial P_{xy} = \{x, y\}$) for every $\{x, y\} \in E_1$. Thus there will be some Dirichlet boundary value problems to be solved, just a few or quite a lot of them depending on $|E_1|$. Remarkably all the solutions of these problems do not share support, or, better expressed, for every vertex v_{xy}^i the obtained value $\overline{u}(v_{xy}^i)$ follows from one specific Dirichlet problem, hence it is well defined. Specifically for every $\{x, y\} \in E_1$ substituted by P_{xy} we write $\mathcal{L}_{q_\omega}^S = \mathcal{L}_{q_\omega}^{P_{xy}}$ the boundary Dirichlet value problem on $P_{xy} \subsetneq V^S$ is established as

$$\mathcal{L}_{q_\omega}^{P_{xy}}(\overline{u})(v_{xy}^i) = f(v_{xy}^i) \quad i = 1, \ldots, \ell_{xy}$$

$$\overline{u}(v_{xy}^0) = u(x)$$

$$\overline{u}(v_{xy}^{\ell_{xy}+1}) = u(y)$$

where $u(x)$ and $u(y)$ are the values that $u \in \mathcal{C}(V^{sP})$ the required solution from (7.4) will take. By now there are presumed and treated as known data.

This required $\overline{u} \in \mathcal{C}(\overline{P_{xy}})$ can be obtained, after (7.6) and some calculation, by solving

$$\begin{aligned} \mathcal{L}_{q_\omega}^{P_{xy}}(\overline{v}) &= f + \left(u(x)c(x, v_{xy}^1)\varepsilon_{v_{xy}^1} + u(y)c(v_{xy}^{\ell_{xy}}, y)\varepsilon_{v_{xy}^{\ell_{xy}}}\right) \text{ on } \overset{\circ}{P}_{xy} \\ \overline{v} &= 0 \text{ on } \{x, y\} = \partial P_{xy} \end{aligned} \tag{7.7}$$

and then

$$\overline{u} = \overline{v} + u(x)\varepsilon_x + u(y)\varepsilon_y$$

by using Green's operator and Green's function obtained from the first on progress work that we have just exposed in the previous subsection.

As we have just said, this solution of a Dirichlet problem can be considered for every edge $\{x, y\} \in E_1$ that has been replaced by a path in Γ^S. Then we denote all these functions as $\overline{u}_{xy}$ and remark that they attain their corresponding values within the paths, while coinciding with u, a function defined on V at the boundary of the paths, that is, at x and y.

At this point we are set to obtain the expression of the Green's function for a path in the case of a Schrödinger type operator, a result by itself that worths a publication as we have mention previously

7.2.5 Related Poisson problem on V

The aim of this section is to obtain a solution of the Poisson problem in Γ^S in terms of the solution of an appropriate Poisson problem on Γ. So we now apply our familiar procedure of contracting data to V, solving a convenient Poisson problem on Γ and finally extending its solution to V^{sP} as this extension is the function we were looking for from the very beginning.

Once the auxiliary Dirichlet problems over the different attached paths have been solved, we use their solutions to define a Poisson problem on the basis network Γ The start of this last stage begins with defining, for each $h \in \mathcal{C}(V^{sP})$ the *contraction* of h to $V, \underline{h} \in \mathcal{C}(V)$, as

$$\underline{h}(x) = h(x) + \sum_{\{x,y\} \in E_1} c(x, v_{xy}^1)(\mathcal{G}_{q_\omega}^{P_{xy}})(v_{xy}^1), \qquad x \in V, \tag{7.8}$$

hence establishing the appropriate Poisson problem

$$\mathcal{L}_{q_\omega}^{S}(u)(x) = \underline{f}(x)$$

for $x \in V$ and $f \in \mathcal{C}(V^{sP})$ from the initial given problem 7.4.

This expression turns to be a Poisson problem for a Schrödinger type operator is some compatibility conditions are fulfilled. After a little calculation the resulting conditions to be imposed for conductances and weights can be written such that

$$\frac{1}{c(x, y)} = \frac{1}{c(x, v_{xy}^1)} \frac{1}{G_{q_\omega}^{P_{xy}}(v_{xy}^1, v_{xy}^{\ell_{xy}})} \frac{1}{c(v_{xy}^{\ell_{xy}}, y)} \tag{7.9}$$

So, with the help of the Green's function for the previous Schrödinger operator we obtain $u \in \mathcal{C}(V)$ that finally is extended to the desired solution of (7.4) by using the *extension of u to* V^S *with respect to* f, $u^f \in \mathcal{C}(V^{sP})$, that is defined as

$$\begin{aligned} u^f(x) &= u(x); \\ u^f(v_{xy}^i) &= \mathcal{G}_{q_\omega}^{P_{xy}}\big(f\big)(v_{xy}^i) + c(x, v_{xy}^1) G_{q_\omega}^{P_{xy}}(v_{xy}^i, v_{xy}^1) u(x) \\ &\quad + c(v_{xy}^{\ell_{xy}}, y) G_{q_\omega}^{P_{xy}}(v_{xy}^i, v_{xy}^{\ell_{xy}}) u(y) \end{aligned} \tag{7.10}$$

for all $x \in V$, $i = 1, \ldots, \ell_{xy}$ and every $\{x, y\} \in F$.

And once the boundary value problem is solved, the computation of the corresponding Green's function follows.

7.3 Some other ideas

To end this chapter we introduce a few ideas that have come to our mind sometime. They are not only attractive as they represent a more or less natural way of developing our research task from our current point, but they are also in agreement with the interests of the research group Mapthe, to which I belong.

1. $\lambda > 0$

   The very first idea, in order to continue with our present research, probably should be attempting the solution of the case of a positive definite Schrödinger operator in a generalized subdivision network. This is a natural prolongation of our works and we are now initiated in the use of host networks. Embedding a given network into a suitable host network should probably be definitive for us in order to extend our result to the case where $\lambda > 0$.

   This technique, see [39, 40] is commonly used in the context of electrical networks and Markov chains. Provided $\mathcal{L}_q$ a positive definite Schrödinger operator on Γ, that is $q = q_\omega + \lambda$ for some $\omega \in \Omega(\Gamma)$ and $\lambda > 0$, we can consider a new network constructed by adding to Γ a brand new vertex, that will represent an absorbing state, that is joined with every each vertex in Γ through edges whose conductances are the, so called, *diagonal excess* after the use of the Doob transform.

   After [9], given $\lambda > 0, \omega \in \Omega(V)$ and $\hat{x} \notin V$, we should consider the network $\Gamma_{\lambda,\omega} = (V \cup \{\hat{x}\}, c_{\lambda,\omega})$ with $c_{\lambda,\omega}(x, y) = c(x, y)$ for $x, y \in V$ but $c_{\lambda,\omega}(x, \hat{x}) = \lambda\omega(x)$ for all $x \in V$. If we denote the combinatorial Laplacian for the new network $\Gamma_{\lambda,\omega}$ as $\mathcal{L}^{\lambda,\omega}$ and define a weight function on the new network, $\hat{\omega} \in \Omega(V \cup \hat{x})$, such that $\hat{\omega}(\hat{x}) = \dfrac{\sqrt{2}}{2}$ and $\hat{\omega}(x) = \dfrac{\sqrt{2}}{2}\omega(x)$ for the rest vertices $x \in V$, it turns out that there exists a simple relation between the positive definite Schrödinger operator $\mathcal{L}_q$ on Γ and a new positive semi–definite Schrödinger operator on $\Gamma_{\lambda,\omega}$.

   $$\mathcal{L}_{\hat{q}}^{\lambda,\omega}(u) = \mathcal{L}_q(u_{|_V}) - \lambda\omega u(\hat{x}) \text{ on } V.$$

   that will permit us to extend our results as we mentioned earlier.

2. Puncturing a net between non adjacent vertices.

   In all the works we have intended, in relation with networks and their subdivision procedures, we have always considered the substitution of edges by, eventually an arbitrary length path so that the remaining discrete structure is electrically equivalent to the given initial one. But we have never tried a puncture of the network between two non–adjacent vertices. As our closed expressions for the Green's function that we have found do not need adjacency as a necessary condition, we think there probably might be a way of joining paths (or much more elaborated networks) to a given network so as to keep electrical equivalence.

3. Spectral study of Schrödinger operators.

   Some members of our research group have occasionally devoted their attention punctually to the so called Kron–reduction operation on networks. The Kron reduction process is ubiquitous in classic circuit theory and in other related theories such as electrical impedance tomography, smart grid monitoring, analysis and simulation of induction motors, for example. The Kron reduction process in used to obtain lower dimensional electrically–equivalent networks. We (the research group) strongly believe that we have the tools to encompass the problem of studying the spectrum of Schrödinger type operators in this situation, so this could be one line of research in which we could collaborate, if the group take this direction.

4. Probabilistic interpretation for Schödinger operators

   Finally another alternative of developing our work in the very soon future to come is concerned with the fact that some of the concepts above mentioned, have a well–known probabilistic counterpart. For instance the effective resistance is related with the escape probability for a reversible Markov chain. Therefore, the effective resistance with respect to a non–negative value and a weight will correspond to a generalization of the escape probability. In turns, the equilibrium measure with respect to λ and ω can be seen as a generalization of the hitting time.

   As a by–product of the expression for the effective resistance with respect to a non–negative value and a weight, Bendito et al. obtained in [10] a full generalization of Foster's formulae; see [8, 57, 69, 72] for the standard case. To get the formulae they introduce a generalization of the transition probability matrix that takes into account the probability of transitioning from one state to another in a single step or remaining in the same state.

   The probability laws governing the evolution of the chain are given by the (*one step*) *transition probability kernel with respect to* λ *and* ω, $\mathcal{P}_{\lambda,\omega} \in \mathcal{C}(V \times V)$, that is defined for any $x, y \in V$ as

$$\mathcal{P}_{\lambda,\omega}(x,y) = \frac{\Big(c(x,y) + \lambda\,\omega(x)\omega(y)\Big)\omega(y)}{\Big(k(x) + q(x)\Big)\omega(x)}. \tag{7.11}$$

   As we can see, the main novelty in our definition is the consideration of a non–negative probability of remaining at vertex x given by the term $\dfrac{\lambda\omega^2(x)}{k(x)+q(x)}$.

The future work we will try to develop should consist in interpreting all the results obtained for Schrödinger operators in probabilistic terms. For instance, we can expect for a formula for the mean first passage times for generalized Markov chains in terms of the group inverse of the corresponding Schrödinger operator; see [56]. This is also related with the inverse M–matrix problem.

8
Conclusions

In this last chapter we would like to include some brief conclusions and remarks that we have reached and find interesting to expose, after devoting our efforts to the work we have explained so far.

1. On discrete potential theory

   The first point we would like to highlight is the framework within we have developed our research. What we call discrete potential theory is not only a set of technicalities that mimick potential theory from the continuum, and that are perfectly constructed. We conceived functions, linear operators, inverse operators, ... in intimately connection with the network where they are applied. Of course we obtain a gain with the non–coordinate notation that we use, treating all vertices as they are the same, at the same time that we see them different one from the others. But the main characteristic of the way we work, is a kind of a play with the structure. Because we apply changes on it, we modify it, sometimes annexing vertices, sometimes erasing edges,... In this way it becomes a very succesful manner to obtain our desired results in the solution of discrete boundary value problems.

   Moreover we have the feeling now that we have to have friends we have never heard of them. And we say this because we have recently discovered, [42, 62], that the notation and vocabulary that we use also appears in publications devoted to topics that are quite far away from our scope, at least apparently. Social networks, pattern recognition, digital image processing, machine learning are worlds where discrete potential theory also applies. Without any doubt this is not a coincidence. We give to this fact a very positive interpretation, of course.

2. On electrical subdivision

   Surprisingly we found very little work devoted to the subdivision of a graph procedure and an electric circuit implemention of it, even though the obvious relationship. We know understand what at that moment we valued as simply incredible. The topic for sure deserved attention, but graphs are not capable to face it.

   A more developed structure had to be used. And fortunately we were in touch with a tool likely to be used to adress the situation. From an electric circuit point of view, when a series connection is considered an equivalent

simpler circuit can be taken into account. This scenario was our starting point and we manage to find a discrete structure operation that do perfectly describe the physical situation. Moreover it modelled all possibilities, as if a rheostat is connected in any point within a wire.

Thus when we started our first research work, conceiving electrical subdivision and trying to solve a Poisson problem posed for the combinatorial Laplacian, we were aware that we were attacking a particular situation that seemed uninteresting to anyone.

But today we have the feeling that the idea opens the door to a new paradigm in diffusion problems as it offers a possibility of manipulating networks so as to achieve whatever our objectives are. In other words, we think that electrical subdivision is a natural discrete structure operation suitable for diffusion problems, as if everything that propagates could be explained and analyzed with the help of it.

3. On Schrödinger type operators

   In a graph theory framework, we have lately witnessed the discovery of the Laplacian matrix as an important tool to work with. Of course the topic is known from old times, but just a few years ago many publications and talks offered at meetings treated mainly the adjacency matrix case. We think then that it is just a matter of time that the communitiy will turn its attention to Schrödinger type operators as well. Graphs, as collection of vertices that are connected or not, are discrete structures of enormous success. But graphs have their limitations too. And Schrödinger operators constitute a tool for modelling discrete structures were not only connections between vertices can be much better explained but also vertexs in the discrete structure can be assigned different roles by its own nature. In this sense our conclusion is that an increasing recognition for the operator can be expected, even though the inherent difficulties of dealing with it.

4. On the normalized Laplacian and electrical subdivision

   From the results we have obtained, the normalized Laplacian, with the very particular role that associates to the vertices of a network, is not an elliptical operator particularly compatible with the concept of electrical subdivision. From this point of view there is something that we miss and we think it would be well worth paying attention to try to clarify this situation, for sure.

   Pels resultats que hem obtingut el Laplacià normalitzat, amb el seu particular paper associat als vèrtexs, no és un operador ellíptic particularment compatible amb el concepte de subdivisió elèctrica. Des d'aquest punt de vista hi ha alguna cosa que se'ns escapa i de ben segur que valdria molt la pena dedicar-hi atenció per mirar d'aclarir-ho.

5. On Green's function

   From our point of George Green's ideas deserve much more recognition from both mathematical and technological environements than they have. It is quite straightforward to see Google's references in the topic to have an approximated idea. In our opinion a rather restricted number of mathematicians are in touch with his ideas and almost all of them because their concerns and interests are located in potential theory but for the continuum. And, without any doubt, it is also a discrete–algebraic topic that should have

much more popularity. It is usually said that linear algebra is too far from applications to really catch students or researchers interest, apart of those whose expertise is in the field. But Green operator and Green's function can be seen as universal solvers, thus very interesting. And that's why we support the idea that it is necessary to make the subject much more attractive to a large part of the mathematical community, at least.

6. On the concept of conductance as a distance.

   The simple fact of considering conductances, a value on each edge, in addition to being used to discriminate between them, has a very interesting geometric lecture that has also been the subject of study and analysis in our work. The value assigned to a given edge can be read as the distance that separates their two adjacent vertices.

   In the case of electrical circuits the value of the conductance of a segment, or of its reciprocal value the resistance, clearly has the meaning to express the facility or difficulty for the electrons to accede from a place to another one (the lower the conductance or the higher the resistance is, the more the cost is). This interpretation can obviously be generalized and applied everywhere. Actually, ordinary people we do already use this idea extensively as for instance, when a locomotive of a train is changed by a new faster model, the train itself runs faster on the same a track. Hence, in some sense, cities linked by the train track are now closer, even though the spatial distance between them remains the same. So its not only the track, for an edge in the network, that matters!

   As far as we know, it seems that very little attention is paid to this conception of what a distance can be in the boundary value problems field. Obviously there are numerical schemas that are used to solve approximately partial differential equations and that are related, for instance, to the (combinatorial) Laplacian operator. But in so many of them, when that conception turns into a matricial setting, the idea of the distance that separates points is always the classical spatial distance. We wonder if a lot of gain would be obtained by including the conductance distance in these numerical algorithms.

7. On effective resistances distance and difusion problems.

   It is clear that diffusion, conceived as the local distribution of a physical entity that ends up having a global reach, has to be treated more in a effective resistance distance scenario that in the classical shortest–path distance one.

   Throughout our research we have discovered many examples corresponding to so many different frameworks where the possibilities of this almost thirty years old distance conception are explored. Usually a process of diffusion takes place taking advantage of all the possibilities, it takes advantage of all the forms in which it can take place, it propagates using all possible ways. Hence this idea that, takes into account all connecting possibilities between to vertices and not only one (the shortest–path), seems to us to be a better alternative to work with in diffusion problems.

   Clearly this point is in close connection with the previous one devoted to conductances, but we would like to emphasize the possibilities of this thinner than geodesic distance approch to be used in applications. And our conclusion is that, in the near future, we will see an increasing amount of research based on this conception.

Moreover, we have realised lately that this idea of distance is been used also as a new tool to establish analysis of robustness, connectivity and other features that are interesting to know given a discrete structure. This is another argument to augur an esplendid future for the idea.

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