Universitat Politècnica de Catalunya
Programa de Doctorat de Matemàtica Aplicada
Departament de Matemàtiques

Elliptic problems: regularity of stable solutions and a nonlocal Weierstrass extremal field theory

by

Iñigo Urtiaga Erneta

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Adviser: Xavier Cabré
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"That we are capable only of being what we are remains our unforgivable sin." G. Wolfe

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Summary

This PhD dissertation deals with qualitative questions from the theory of elliptic Partial Differential Equations (PDE) and integro-differential equations. We are primarily interested in a distinguished class of solutions satisfying appropriate minimality conditions.

The first part of the thesis provides a regularity theory for stable solutions to semilinear problems involving variable coefficients. Here, stability refers to the nonnegativity of the principal eigenvalue of the linearized equation. For variational problems, this amounts to the nonnegativity of the second variation, a necessary condition for minimality. Our main achievement is to show the boundedness of stable solutions in $C^{1,1}$ domains in the optimal range of dimensions $n \leq 9$. This result is new even for the Laplacian, for which a C^3 assumption on the domain was needed.

The second part furnishes natural sufficient conditions for the minimality of critical points in a general nonlocal framework. Namely, we construct a calibration for nonlocal energy functionals, under the assumption that the critical point is embedded in a family of sub/supersolutions whose graphs produce a foliation. As a consequence, we deduce that the solution is a minimizer with respect to competitors taking values in the foliated region. Our result extends, for the first time, the classical Weierstrass extremal field theory in the Calculus of Variations to a nonlocal setting. To find a calibration for the most basic fractional functional, the Gagliardo-Sobolev seminorm, was an important open problem that we have solved.

The contents of the thesis are based off of the following articles and preprints:

[52] I. U. Erneta, *Stable solutions to semilinear elliptic equations for operators with variable coefficients*, Commun. Pure Appl. Anal. **22** (2023), 530–571.

[51] I. U. Erneta, *Energy estimate up to the boundary for stable solutions to semilinear elliptic problems*, preprint available at arXiv:2305.07058.

[50] I. U. Erneta, *Boundary Hölder continuity of stable solutions to semilinear elliptic problems in $C^{1,1}$ domains*, preprint available at arXiv:2305.07062.

[23] X. Cabré, I. U. Erneta, J.C. Felipe-Navarro, *A Weierstrass extremal field theory for the fractional Laplacian*, to appear in Adv. Calc. Var., preprint available at arXiv:2211.16536.

[22] X. Cabré, I. U. Erneta, J.C. Felipe-Navarro, *Null-Lagrangians and Calibrations for general nonlocal functionals and an application to the viscosity theory*, in preparation.

Contents

Introduction

Our quantitative description of reality rests upon a collection of universal principles, expressed in mathematical language, from which all phenomena may be deduced. Such principles or laws often manifest in the form of equations, which are in turn satisfied by functions accounting for the state of the system under consideration.

Depending on the nature of the phenomenon being studied, different types of equations are of interest. Classically, Partial Differential Equations (PDE) have been successfully applied to model diverse phenomena arising from short-range interactions. These equations involve relations between partial derivatives of the solution. To compute such derivatives at a point, it is only necessary to know the function in a vicinity of the point in question, reflecting the *local* nature of PDEs. More recently, integro-differential equations have gained an increasing notoriety in applications. Such equations capture the effect of long-range interactions by integrating differences of the solution in the whole space. Notice that this operation takes into account the value of the solution everywhere and not just at a point. Thanks to this feature, they effectively model *nonlocal* phenomena.

Within these two broad categories of equations, *elliptic* ones hold a distinct position. This family of equations can be thought of as describing equilibrium configurations of evolution processes. As a closed system evolves in time, it reaches a steady state characterized by such an elliptic problem. These equations are typically nonlinear and solutions are by no means uniquely determined. In fact, the natural boundary (or exterior) value problems associated to such equations may admit multiple solutions or even infinitely many of them. Nevertheless, there is a special class of solutions comprising most physical situations of interest, the so called *stable solutions*. They are distinguished by the fulfillment of certain minimality conditions with respect to the problem in question (as we explain later). For instance, local minimizers of energy functionals are stable solutions of the Euler-Lagrange equations. In a sense, these solutions correspond to the observables of Nature, since they do not disappear under small perturbations of the parameters (hence their name "stable").

In this thesis, we are interested in various qualitative aspects of stable solutions to both PDEs and integro-differential equations. We focus primarily on the regularity and minimality properties of these solutions, which play crucial roles in many applications. Next, we briefly comment on our contributions to these fields.

First, we investigate the regularity of stable solutions to nonlinear elliptic PDEs. This turns out to be delicate question which usually depends on the dimension of the space, even for apparently simple semilinear problems. Our principal result shows the boundedness (and hence smoothness) of stable solutions to semilinear equations with variable coefficients, up to the optimal dimension 9. Such a result was only known for the Laplacian in C^3 domains. As a consequence of our analysis, we are able to weaken the regularity of the domain to a $C^{1,1}$ assumption, which is natural in the elliptic estimates for non-divergence form operators.

Secondly, we study sufficient conditions for minimality (and hence stability) of solutions to equations associated to nonlinear elliptic functionals. Before our work, a theory of sufficient conditions was only available for local equations, in the form of the classical Weierstrass extremal field theory. There, the basic result states that if a solution can be embedded in a family of critical points (i.e., solutions) whose graphs produce a foliation, then it must be a minimizer with respect to competitors taking values in the foliated region. Our main result provides the analogue of this theorem in the general nonlocal framework. Namely, under the foliation assumption, we construct a *calibration* for nonlocal functionals. This calibration is a null-Lagrangian satisfying certain additional properties which lead directly to the minimality of the solution. In particular, our construction gives a calibration for the simplest nonlocal functional (the fractional Gagliardo-Sobolev seminorm corresponding to the fractional Laplacian) which was not known to exist prior to this work.

In the following, we further elaborate on the notions that we just mentioned.

Stable solutions

As mentioned above, we are interested in solutions that describe stable configurations. To better explain this concept, we consider the steady state reaction-diffusion equation

$$-\Delta u = f(u) \quad \text{in } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth function. Later, we will further assume f to be nonnegative, nondecreasing, and convex.

There are several equivalent ways to define the stability of a solution to (1), as explained below. Some of these definitions are variational in nature, reflecting a minimality property of u with respect to some functional. Others are intrinsic to the linearization of (1) and can be generalized to other, non-variational situations. In the first part of the thesis, we will be particularly interested in this second case.

A first definition is based on the variational structure of (1). Notice that equation (1) is the first variation of the energy functional[1]

$$\mathcal{E}_{1,F}(u) = \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 - F(u) \right\} \mathrm{d}x, {}^{2} \tag{2}$$

where F is a primitive of the nonlinearity f. Computing the second variation at u

$$\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} \mathcal{E}_{1,F}(u + \varepsilon\xi)\Big|_{\varepsilon=0} = \int_\Omega \left\{ |\nabla\xi|^2 - f'(u)\xi^2 \right\} \mathrm{d}x, \tag{3}$$

where $\xi \in C_c^\infty(\Omega)$, we have the following:

Definition 1. A solution u of (1) is *stable* if the second variation (3) of $\mathcal{E}_{1,F}$ at u is nonnegative. Equivalently, the solution u is stable if

$$\int_\Omega f'(u)\xi^2 \,\mathrm{d}x \le \int_\Omega |\nabla\xi|^2 \,\mathrm{d}x \quad \text{for all } \xi \in C_c^\infty(\Omega).{}^{3} \tag{4}$$

[1]Here, we call functionals "energies" following elliptic PDE terminology. In other contexts, to avoid ambiguity, it might be more appropriate to refer to them as *action functionals* instead.

[2]The subindex 1 is in contrast to the fractional parameters $s \in (0, 1)$ appearing later in the text. As mentioned below, $\mathcal{E}_{1,F}$ can be recovered as the formal limit of a family of nonlocal functionals $\mathcal{E}_{s,F}$ as $s \uparrow 1$.

[3]In the literature, solutions satisfying such a nonnegativity assumption have also been called *semistable*, while the term *stable* is reserved for the strict inequality. Instead, we will employ the terms *stable* and *strictly stable*, respectively.

Notice here that the test functions ξ vanish at the boundary $\partial\Omega$ and hence the competitors $u+\varepsilon\xi$ have the same boundary condition as u. It follows that stability is a necessary condition for u to be a minimizer with respect to small perturbations. However, being stable is not sufficient for minimality. For instance, if f is superlinear and u vanishes on $\partial\Omega$, then, taking rescalings λu as competitors, it is easy to check that $\mathcal{E}_{1,F}$ is unbounded from below. In this case, the functional does not admit global minimizers but it may have stable solutions; see [11, 24, 44, 64]. Moreover, stability does not even guarantee the local minimality of the solution, as we will see in the next paragraphs.

It is useful to note that the second variation can always be written as the quadratic form associated to the linearized equation. More precisely, integrating by parts we have

$$\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\mathcal{E}_{1,F}(u+\varepsilon\xi)\Big|_{\varepsilon=0} = -\int_\Omega (J_u\xi)\,\xi\,\mathrm{d}x \quad \text{for all } \xi\in C^\infty_c(\Omega), \tag{5}$$

where J_u is the linearization of (1) at u, namely,

$$J_u\xi := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big(\Delta(u+\varepsilon\xi)+f(u+\varepsilon\xi)\Big)\Big|_{\varepsilon=0} = \Delta\xi + f'(u)\xi.$$

The operator J_u is also known as the *Jacobi operator* in the literature. We point out that the representation (5) holds for the second variation of general functionals, not only for the ones associated to semilinear equations.

With (5) at hand, we can give another definition of stability. Let $\mu_1[J_u,\Omega]$ denote the principal (smallest) eigenvalue of J_u in Ω with respect to homogeneous Dirichlet conditions. Here and throughout the text, we use the sign convention $J_u\xi = -\mu\,\xi$ for the eigenvalues of J_u. By the variational characterization of $\mu_1[J_u,\Omega]$, we have

$$\mu_1[J_u,\Omega] = \inf_{\xi\in C^\infty_c(\Omega),\xi\neq 0} \frac{-\int_\Omega (J_u\xi)\,\xi\,\mathrm{d}x}{\int_\Omega \xi^2\,\mathrm{d}x}.$$

Hence, we see that Definition 1 above is equivalent to the following:

Definition 2. A solution u of (1) is *stable* if the principal eigenvalue of the linearized operator J_u is nonnegative, that is, if $\mu_1[J_u,\Omega]\geq 0$.

Let us come back to the minimality properties of stable solutions. Given u a stable solution of (1), we consider a principal eigenfunction ϕ_1 of J_u in Ω. Namely, ϕ_1 satisfies

$$\begin{cases} -J_u\phi_1 = \mu_1[J_u,\Omega]\phi_1 & \text{in } \Omega\\ \phi_1 > 0 & \text{in } \Omega\\ \phi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

If $\mu_1[J_u,\Omega]=0$ then, assuming f to be strictly convex, it is not hard to see that the competitor $u+\varepsilon\phi_1$ has less energy than u. This shows that u need not be a local minimizer, as we claimed above. On the other hand, if $\mu_1[J_u,\Omega]>0$, then it can be shown that u is a minimizer with respect to competitors in a small tubular neighborhood around itself.[4] When $\mu_1[J_u,\Omega]>0$, we say that u is a *strictly stable* solution. It is worth mentioning that this last condition is equivalent to the operator J_u satisfying the maximum principle; see [9].

[4]More precisely, we have $\mathcal{E}_{1,F}(u+v)\geq\mathcal{E}_{1,F}(u)$ for all $v\in C^1_c(\Omega)$ with $\|v\|_{L^\infty(\Omega)}<\varepsilon$, for a small $\varepsilon>0$.

Definition 2 offers a crucial advantage over the first one, since it applies to more general problems. The key point is that the principal eigenvalue can be defined for any linear operator, not necessarily a variational one. In particular, it allows us to consider stable solutions of equations involving non self-adjoint operators. Thus, in the first part of the thesis, our main focus will be in stable solutions to semilinear equations of the form

$$-Lu = f(u) \quad \text{in } \Omega, \tag{7}$$

where L is a linear second order elliptic operator with variable coefficients

$$L = a_{ij}(x)\partial^2_{ij} + b_i(x)\partial_i. \tag{8}$$

This class of operators constitutes a natural generalization of the Laplacian. First of all, while Δ models the simplest (homogeneous, isotropic) diffusion process, the operator L in (8) can be used to describe more complex phenomena, such as an arbitrary diffusion in the presence of advection (i.e., a flow). Studying equations of the type (7) is thus crucial in real-world applications where multiple physical phenomena act at the same time. Secondly, under a general change of coordinates, the Laplacian transforms into an operator of the form (8). Such transformations are relevant, for instance, when investigating the boundary regularity of solutions, since the curved boundary can be flattened out by a change of variables. This way, it suffices to study equations involving an operator L in a flat domain.

Notice that the linearization of (7) at u corresponds to the Jacobi operator

$$J_u = L + f'(u) = a_{ij}(x)\partial^2_{ij} + b_i(x)\partial_i + f'(u(x)),$$

where we are still using the same notation as above. To give a more precise definition of stability for the equation (7), it will be worth recalling some fundamental properties of linear elliptic operators such as J_u.

The classic work of Berestycki, Nirenberg, and Varadhan [9] provides a theory of principal eigenvalues for elliptic operators with bounded coefficients. There, one assumes that the coefficient matrix $A(x) = (a_{ij}(x))$ is uniformly elliptic, that is, satisfying

$$c_0|p|^2 \le a_{ij}(x)p_ip_j \le C_0|p|^2 \quad \text{for all } p \in \mathbb{R}^n, \tag{9}$$

for some positive constants c_0 and C_0, as well as the regularity assumptions

$$a_{ij} \in C^0(\Omega), \quad b_i \in L^\infty(\Omega), \tag{10}$$

$$f'(u) \in L^\infty(\Omega).^5 \tag{11}$$

In [9], the authors gave the following definition for the principal eigenvalue of an operator J_u satisfying (9), (10), and (11):

$$\mu_1[J_u, \Omega] := \sup\left\{\mu \in \mathbb{R} \colon \exists \varphi > 0 \in W^{2,n}_{\text{loc}}(\Omega) \text{ satisfying } J_u\varphi + \mu\varphi \le 0 \text{ in } \Omega\right\}. \tag{12}$$

Moreover, without any regularity assumption on Ω, they prove the existence of a principal eigenfunction ϕ_1 in $L^\infty(\Omega) \cap W^{2,p}_{\text{loc}}(\Omega)$, for all $p < \infty$, satisfying (6) above, where the boundary condition must be appropriately interpreted.[6]

[5]In applications, assuming f to be locally Lipschitz, the boundedness of the zero order coefficient $f'(u)$ will follow from the boundedness of the solution u.

[6]One first constructs a solution to $-Lu_0 = 1$ in Ω by approximation, solving the Dirichlet problem with zero boundary conditions in smooth domains that exhaust Ω and taking a converging subsequence. We say that $\phi_1 = 0$ on $\partial\Omega$ if, whenever $u_0(x_k) \to 0$ for some $x_k \in \Omega$ converging to $\partial\Omega$, then $\phi_1(x_k) \to 0$.

It is not hard to see that the variational definition of μ_1 is equivalent to (12). Essentially, μ_1 is a threshold for the existence of positive supersolutions, the principal eigenfunction being the "last" one. From this characterization follows another definition of stability:

Definition 3. A solution u of (7) is *stable* if there is a function $\varphi \in W^{2,n}_{\mathrm{loc}}(\Omega)$ such that

$$\begin{cases} J_u\varphi \le 0 & \text{in } \Omega \\ \varphi > 0 & \text{in } \Omega. \end{cases}$$

By (12), a solution that is stable in the sense of Definition 3 is also stable for Definition 2. For the converse, it suffices to take φ equal to ϕ_1, the principal eigenfunction of J_u above.

Although we will never use it, it is interesting to note that the function φ in Definition 3 can also be taken such that the inequality becomes an equality, i.e., with $J_u\varphi = 0$ in Ω. Indeed, if $\mu_1[J_u,\Omega] > 0$, then J_u satisfies the maximum principle and we can solve the boundary value problem $J_u\varphi = 0$ in Ω, $\varphi = c$ on $\partial\Omega$ for a positive constant $c > 0$; see [9]. Otherwise, if $\mu_1[J_u,\Omega] = 0$, then we simply take the principal eigenfunction.

Regularity of stable solutions

Having introduced the notion of stability in the previous section, we now turn toward the main focus of the first part of this work: the regularity of stable solutions. Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a nonlinearity $f \in C^1(\mathbb{R})$, we consider the boundary value problem

$$\begin{cases} -Lu = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

where L is the linear operator introduced in (8) above. We always assume that L satisfies the uniform ellipticity condition (9) and the regularity assumptions (10).

Since (13) involves a semilinear equation, the question of regularity reduces to showing the boundedness of solutions. Indeed, if u is in $L^\infty(\Omega)$, then $f(u)$ is bounded and by L^p estimates we deduce that u is in $W^{2,p}_{\mathrm{loc}}(\Omega)$, for all $p < \infty$. The latter estimate holds up to the boundary in domains of class $C^{1,1}$ and under the additional assumption that

$$a_{ij} \in C^0(\overline{\Omega}), \tag{14}$$

which allows us to apply the L^p estimates in [63, Theorem 9.14]. For further smoothness properties of the solution, more assumptions on the coefficients and the domain are needed.

If we do not restrict the class of solutions, then the regularity will depend on the growth of the nonlinearity. For the Laplacian $L = \Delta$ and assuming f to grow like a power

$$|f(u)| \le C(1 + |u|^m),$$

if m is subcritical with respect to Sobolev exponent, i.e., $m < \frac{n+2}{n-2} =: m_s$, then every *energy solution* of (13) is bounded. By an energy solution, we mean a weak solution in $W^{1,2}_0(\Omega)$, the natural energy space for the functional $\mathcal{E}_{1,F}$ to be well defined. This boundedness result can be easily proved by the iterated L^p estimates of Crandall and Rabinowitz [44]. Moreover, using the Brezis-Kato trick [13], the same conclusion continues to hold for critical nonlinearities. However, when $n \ge 3$, the singular solution

$$U_m(x) := (m-1)\left(|x|^{-\frac{2}{m-1}} - 1\right)$$

is an energy solution in the supercritical range $m > m_s$. Notice that U_m solves (13) in $\Omega = B_1$ with $f(u) = \lambda_{n,m}(1 + \frac{1}{m-1}u)^m$, where $\lambda_{n,m} := 2\big(n - \frac{2m}{m-1}\big)$.

We are interested in a regularity theory for stable solutions that applies to *all* nonlinearities, independently of their growth. For instance, the stability of U_m amounts to

$$\frac{m}{m-1}\lambda_{n,m}\int_{B_1}\frac{\xi^2}{|x|^2}\,\mathrm{d}x \le \int_{B_1}|\nabla\xi|^2\,\mathrm{d}x \quad \text{for all } \xi \in C_c^\infty(B_1). \tag{15}$$

By Hardy's inequality with best constant, this is equivalent to $\frac{m}{m-1}\lambda_{n,m} \le \frac{(n-2)^2}{4}$ and hence, in the supercritical range $m > m_s$, it can be checked that U_m is stable for $m \ge \frac{n-2\sqrt{n-1}}{n-4-2\sqrt{n-1}}$. This last condition on the exponent is equivalent to the inequality

$$n \ge 2 + 4\sqrt{1 + \frac{1}{m-1}} + 4\left(1 + \frac{1}{m-1}\right). \tag{16}$$

As a consequence, we deduce the existence of singular stable energy solutions for $n > 10$. In fact, letting $m \to \infty$ above, the solutions U_m converge to the logarithm

$$U^\sharp(x) := -2\log|x|, \tag{17}$$

which solves (13) with $L = \Delta$ and the exponential nonlinearity $f(u) = 2(n-2)e^u$, in the weak sense. This singular solution is always in the energy space and is stable for $n \ge 10$ (for instance, by taking the limit in (16) above).

When $\frac{n}{n-2} < m \le m_s$, the function U_m is no longer in the energy space. Nevertheless, it still lies in a weaker Sobolev space and satisfies $f(U_m) \in L^1(B_1)$, which is enough to show that U_m solves (13) in a weak sense. What is more surprising is the fact that U_m satisfies the stability condition (15) in the smaller range $\frac{n}{n-2} < m \le \frac{n+2\sqrt{n-1}}{n-4+2\sqrt{n-1}}$; see [14]. This last example suggests that a satisfactory regularity theory, if any, can only be expected for *stable solutions in the energy class.* Furthermore, even in that case, the smoothness of these solutions must depend on the dimension n of the domain. As we have seen, singularities may appear for $n \ge 10$, and it is therefore natural to ask whether this is the only case.

The Gelfand problem

Before discussing the optimality of dimension 10, we comment on a physical motivation for considering exponential nonlinearities (which led to this dimensional threshold). Such reaction terms arise naturally in combustion theory, this being the main inspiration for the first regularity results.

In the 1930's, Frank-Kamenetskii suggested the use of reaction-diffusion equations to model thermal combustion phenomena. The temperature of the combustible mixture is assumed to satisfy such an equation, with a reaction governed by the Arrhenius Law. Under the simplifying assumption that the fuel consumption time is much larger than the ignition time, the term coming from the Arrhenius Law can be well approximated by an exponential. In this theory, the solvability of the steady-state equation is used to determine whether an explosion will happen. Namely, the nonexistence of solutions is interpreted as the occurrence of an explosion, while the existence, for instance, of stable solutions would mean that the temperature evolves towards a stationary profile.

More precisely, let u denote a suitably normalized temperature of the combustible mixture. Suppose that the mixture is contained in a cylindrical vessel of cross-section $\Omega \subset \mathbb{R}^n$, with u vanishing on the boundary (meaning that the walls of the vessel are at constant temperature). Then, after a sufficiently long time, u solves the *Gelfand problem*

$$\begin{cases} -Lu = \lambda g(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{18}$$

with $L = \Delta$ and $g(u) = e^u$, where $\lambda > 0$ is a nondimensional parameter quantifying the relative strength of the reaction with respect to the diffusion. According to Frank-Kamenetskii, solutions to (18) are not expected to exist for sufficiently large λ; see [57].

In the sixties, Barenblatt [60] introduced problem (18) in the mathematical literature, initiating the rigorous study of explosion phenomena.[7] One natural way of constructing solutions to (18) is by applying the implicit function theorem at $u = 0$, $\lambda = 0$, solving for u in terms of λ and continuing this branch of solutions. Since the linearized operator at the trivial solution is the Laplacian, by continuity, the first portion of the branch consists of stable solutions. Another way of obtaining this same branch is by monotone iteration starting at $u = 0$. Here, one constructs barriers for the problem (18) to ensure that the iteration converges. Furthermore, this procedure shows that solutions are not only stable but also *minimal*, in the sense that they are the smallest positive supersolutions of (18).

Thanks to the robustness of these methods, they can also be applied to more general nonlinearities than exponential or power-type ones. To reproduce the behavior expected by Frank-Kamenetskii, it is natural to consider $g\colon [0, +\infty) \to \mathbb{R}$ satisfying

$$g(0) > 0, \quad g \text{ nondecreasing, convex, and superlinear at } +\infty, \tag{19}$$

where the last condition means that

$$\lim_{u \to +\infty} \frac{g(u)}{u} = +\infty.$$

Under assumptions (19), the fundamental existence theory for (18) can be summarized in the following proposition, which has appeared in various forms in the literature:

Proposition 4 ([8, 11, 38, 43]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let L be an operator of the form* (8) *with coefficients satisfying* (9) *and* (10) *in Ω. Assume that $g \in C^1(\mathbb{R})$ satisfies* (19)*. Then, there exists a constant $\lambda^\star \in (0, +\infty)$ such that:*

(i) *For each $\lambda \in (0, \lambda^\star)$ there is a unique minimal strong solution $u_\lambda \in L^\infty(\Omega) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ of* (18)*. In particular, u_λ is stable and we have $u_\lambda < u_{\lambda'}$ in Ω for $\lambda < \lambda'$.*

(ii) *For $\lambda > \lambda^\star$ there are no strong solutions of* (18) *in $L^\infty(\Omega) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$.*

Assume moreover that Ω is $C^{1,1}$ and that the coefficients of L are smooth up to the boundary.[8] *Then:*

(iii) *For $\lambda = \lambda^\star$ there exists a unique L^1-weak solution $u^\star$ of* (18)*, in the following sense: $u^\star \in L^1(\Omega)$, $g(u^\star)\mathrm{dist}(\cdot, \partial\Omega) \in L^1(\Omega)$, and, letting L^T denote the adjoint of L,*

$$-\int_\Omega u^\star L^T \zeta \, \mathrm{d}x = \lambda^\star \int_\Omega g(u^\star)\zeta \, \mathrm{d}x$$

for all test functions $\zeta \in W^{2,n}(\Omega)$ such that $L^T\zeta \in L^\infty(\Omega)$ and $\zeta|_{\partial\Omega} = 0$. Moreover, $u^\star$ can be obtained as the pointwise limit $u_\lambda \uparrow u^\star$ as $\lambda \uparrow \lambda^\star$.

[7] Barenblatt's work appeared in the volume [60] edited by Gelfand, and the latter's name became associated with the explosion problem.

[8] We will give more precise regularity assumptions in Chapter 3 below.

Recall here that, under minimal regularity assumptions on the coefficients (condition (10) above), the best solutions to (18) that we can expect are *strong solutions* in $L^\infty(\Omega) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$. By this we mean that they solve the equation almost everywhere in Ω and satisfy the boundary condition in an appropriate sense (see footnote 6). In $C^{1,1}$ domains and under the additional assumption (14), bounded solutions are in $W^{2,p}(\Omega)$, for all $p < \infty$ (and hence in $C^{1,\alpha}(\overline{\Omega})$ for all $0 < \alpha < 1$).

Proposition 4 underlines the important role played by stable solutions in the Gelfand problem. Namely, for $\lambda \in (0, \lambda^\star)$ there is a distinguished stable solution u_λ, which is bounded (and hence smooth if the domain and the coefficients of L are smooth), while for $\lambda > \lambda^\star$ there are no solutions (not even in a weak sense, as shown in [12]). As we approach the extremal parameter $\lambda^\star$, the stable solutions u_λ converge to a function $u^\star$ in L^1, which solves (18) in a weak sense. This is the *extremal solution* and it is also the unique solution of (18) for $\lambda = \lambda^\star$, as proved by Martel [73].

A priori, the extremal solution $u^\star$ is only in L^1 and, therefore, it is natural to ask whether it is bounded. In the seventies, Joseph and Lundgren [64] considered the Gelfand problem (18) in the unit ball for exponential and power nonlinearities, with $L = \Delta$. Since positive solutions are radially symmetric in this case, the problem reduces to an ODE. Using phase plane techniques, they showed that the stable branch (and hence $u^\star$) is always bounded when $n \leq 9$, while it can be unbounded for $n \geq 10$. In fact, for $n \geq 10$, the logarithm $U^\sharp$ given by (17) above is the extremal solution for $g(u) = e^u$.[9]

As noted by Crandall and Rabinowitz [44], the boundedness of $u^\star$ in general domains could be deduced from L^∞ a priori estimates for the stable solutions $\{u_\lambda\}_{0<\lambda<\lambda^\star}$. In the seminal work [44], they obtained such a bound for exponential and power-type nonlinearities in the optimal range $n \leq 9$. Their result applies to variational problems involving operators in divergence form with bounded coefficients. Using appropriate test functions in the variational stability inequality, for exponential and power nonlinearities they showed that $\|u_\lambda\|_{L^\infty(\Omega)} \leq C$ for $n \leq 9$ and $\lambda \in (0, \lambda^\star)$, where the constant C is independent of λ. Taking the limit as $\lambda \to \lambda^\star$, it follows that $u^\star$ is bounded in this case.

Optimal dimension for the Laplacian

After the work of Crandall and Rabinowitz, no singular stable solutions were found in low dimensions, which led to the conjecture that they must all be bounded in this case. In the nineties, Brezis [11] asked whether dimension 10 could also be optimal for more general nonlinearities, emphasizing the class satisfying the natural assumptions (19). Along similar lines, Brezis and Vázquez [14] noticed that the known examples of extremal solutions were always in the energy space, independently of the dimension. Consequently, they wondered if such a phenomenon could hold for general nonlinearities as above.

Both of these questions motivated a series of papers trying to establish a priori estimates that could apply to all stable solutions, at least in low dimensions and for the model case $L = \Delta$. All these attempts over the course of a quarter of a century culminated in the recent work of Cabré, Figalli, Ros-Oton, and Serra in [24], where the two questions were answered affirmatively for the Laplacian in general C^3 regular domains. Next, we comment on the previous works leading to this last result, emphasizing the test functions used in the stability condition (4).

[9] This fact follows directly from a useful characterization of singular extremal solutions due to Brezis and Vázquez [14]. Namely, if a stable energy solution is unbounded, then it is necessarily the extremal solution.

- The first positive result for the Laplacian in smooth domains was obtained by Nedev in [78]. By choosing test functions of the form $\xi = h(u)$ in (4), with h depending on the nonlinearity g, he obtained a priori estimates for the minimal solutions u_λ. As a consequence, he was able to show that $u^\star \in L^\infty(\Omega)$ for $n \le 3$ and $u^\star \in W_0^{1,2}(\Omega)$ for $n \le 5$. Under the additional assumption that Ω is convex, he proved that the last inclusion holds in every dimension [79].

- Cabré and Capella [21] treated the radial case ($\Omega = B_1$, $L = \Delta$) for general nonlinearities. They were able to reach the optimal dimension in this case, showing that $u^\star \in L^\infty(B_1)$ for $n \le 9$ and $u^\star \in W_0^{3,2}(B_1)$ for all n. Inspired by the work of Simons on minimal surfaces [90], they considered a test function of the form $\xi = \mathbf{c}\eta$ in the stability inequality, where $\mathbf{c}$ depends on the solution and η is a cut-off. This choice leads to the inequality
$$\int_\Omega (J_u\mathbf{c})\,\mathbf{c}\eta^2 \le \int_\Omega \mathbf{c}^2|\nabla\eta|^2,$$
and the goal is then to choose $\mathbf{c}$ in a way such that $J_u\mathbf{c}$ no longer depends on the nonlinearity. Writing $u_r = \frac{x}{|x|}\cdot\nabla u$ and $r = |x|$, they considered $\mathbf{c} = ru_r$ and $\eta = (r^{-\alpha} - 2^\alpha)_+$ for the exponent $\alpha < 1 + \sqrt{n-1}$. This choice allowed them to control a weighted norm of u_r, which then yields pointwise bounds for the solution. When $n \le 9$, they obtained an L^∞ estimate valid for all stable energy solutions of (13) and for all C^1 nonlinearities f, not only those satisfying (19). Their result gave credibility to the conjecture that 10 was indeed the lowest dimension where singularities could arise.

- Cabré [16] proved the boundedness of $u^\star$ up to $n \le 4$ in convex domains. Using stability, he first controlled stable energy solutions in an arbitrary domain by their L^∞ norm close to the boundary. Assuming the convexity of the domain, the method of moving planes then yields an estimate of this last quantity in terms of the L^1 norm of the solution. To obtain the first estimate, he again considered a test function of the form $\xi = \mathbf{c}\eta$ in the stability inequality, letting
$$\mathbf{c} = |\nabla u| \quad \text{and} \quad \eta = h(u)$$
for an appropriate h. This choice had already appeared in the works of Sternberg and Zumbrun on phase transitions [91,92] and leads to control certain geometric quantities associated to the level sets of the solution, such as their curvature. Combined with the Sobolev inequality of Michael, Simon, and Allard [2,76] on level sets, the stability of u yields the L^∞ estimate for $n \le 4$.

  Later, Villegas [95] extended this boundedness result to all smooth domains, combining ideas of Nedev [78] and of Cabré [16]. Moreover, he showed that $u^\star$ is in $W_0^{1,2}(\Omega)$ for $n \le 6$.

- Cabré and Ros-Oton [27] showed that $u^\star$ is bounded for $n \le 7$ in convex domains of double revolution (invariant under rotations of the first m and last $n-m$ variables). Here, by symmetry, the solution depends only on the radial variables $s = \sqrt{x_1^2 + \cdots + x_m^2}$ and $t = \sqrt{x_{m+1}^2 + \cdots + x_n^2}$. Similar to the radial case treated in [21], they considered test functions $\xi = \mathbf{c}\eta$, letting $\mathbf{c} = su_s$ and $\eta = s^{-\alpha}\zeta$ first (where ζ is a cut-off), and then $\mathbf{c} = tu_t$ and $\eta = t^{-\beta}\zeta$, for exponents in the range $\alpha < 1 + \sqrt{m-1}$ and $\beta < 1 + \sqrt{n-m-1}$. As in [16], this yields an L^∞ estimate in terms of the solution close to the boundary, which can be further bounded in convex domains. Without this convexity assumption, they were able to show that $u^\star \in W_0^{1,2}(\Omega)$ for $n \le 6$, later improved by Villegas in [95] (as commented above).

As mentioned above, the conjectures of Brezis and Brezis-Vázquez for the Laplacian were eventually solved by Cabré, Figalli, Ros-Oton, and Serra in [24], where they showed that $u^\star \in L^\infty(\Omega)$ for $n \le 9$ and $u^\star \in W_0^{1,2}(\Omega)$ for all n, in domains of class C^3. In that paper, the authors consider several of the test functions that we have already mentioned and they also introduce new ideas to complete the proof. Their strategy is to upgrade the proof of [21] for the radial case, which already yields an optimal result, to make it work in general domains.

The main result in [21] consists of two types of universal a priori estimates for bounded stable solutions. Namely, an energy estimate, valid in all dimensions, and a Hölder estimate in the optimal range $n \le 9$, all in terms of the L^1 norm of the solution. We briefly discuss their proof, which is based on obtaining local interior and boundary analogues of these estimates.

The interior estimates in [24] apply to classical stable solutions of (1) in the unit ball $\Omega = B_1$ (without boundary conditions), and only need the nonnegativity of f. For the energy estimate, they choose a Sternberg-Zumbrun type test function in the stability inequality, letting $\xi = \mathbf{c}\eta$ with $\mathbf{c} = |\nabla u|$ and where η is a cut-off. Applying the divergence theorem, they are then able to control the Dirichlet energy on level surfaces, and by the coarea formula the higher integrability of the gradient follows. In contrast, the interior Hölder estimate uses a variant of the test function for the radial case in [21]. Letting

$$\mathbf{c}(x) = x \cdot \nabla u(x) \quad \text{and} \quad \eta(x) = |x|^{\frac{2-n}{2}}\zeta$$

for a cut-off ζ, when $n \le 9$, the stability condition leads to a control of the weighted scale-invariant quantity $\int_{B_\rho} u_r^2 r^{2-n}$, with $\rho < 1/2$, by the Dirichlet energy in the annulus $B_{2\rho} \setminus B_\rho$. It remains to bound the full gradient in terms of the radial derivative, since the previous estimate would then lead to the algebraic decay of the weighted integral, and hence to a C^α bound. To show this, they argue by compactness, using that there are no nonconstant homogeneous superharmonic functions in an annulus.

On the other hand, the boundary estimates in C^3 domains require that f be nonnegative, nondecreasing, and convex. The test functions are variants of the ones used in the interior estimates, obtained by modifying $\mathbf{c}$ so that it vanishes on the curved boundary $\partial\Omega$. Once again, the crucial point is controlling the full derivative by the radial one. Here, the superharmonicity no longer suffices to have such a property, and one must further use that u is a stable solution to a semilinear equation. The authors of [24] are only able to show this in flat domains, by compactness, which requires a subtle closedness result for stable solutions with convex nonlinearities. To obtain the Hölder estimate in curved domains, they apply a blow-up procedure, and the problem reduces to proving a Liouville theorem in half-spaces. This result follows from the stability inequality with $\mathbf{c} = x \cdot \nabla u$ combined with the estimate for u_r in half-balls.

Notice how the use of compactness arguments was crucial to obtain the optimal Hölder estimate in [24]. While these methods can be quite versatile, they do not provide a means to quantify the constants that appear in the estimates. Recently, Cabré [19, 20] has found a new quantitative proof for the Laplacian in half-balls. Namely, he obtains interior estimates assuming only the nonnegativity of f, and boundary estimates in half-balls for nonnegative, nondecreasing, and convex nonlinearities. This new quantitative proof greatly simplifies the previous boundary argument. The author's approach is to control the L^1 norm of the solution by the L^1 norm of its radial derivative directly, in a quantitative way. For the interior estimates, the key is proving L^1 estimates for the Neumann problem and using the superharmonicity of the solution to carry out a comparison argument. The boundary estimates use the equation satisfied by the solution under rescalings, taking a derivative with respect to the scaling parameter and applying the stability inequality.

Another natural question concerns the optimal regularity when $n \geq 10$. This has been solved by Peng, Zhang, Zhou [81], who show that, for the Laplacian, stable energy solutions belong to the Morrey space $M^{m_n, 4+\frac{2}{m_n-2}}(\Omega)$ with $m_n = \frac{2(n-2\sqrt{n-1}-2)}{n-2\sqrt{n-1}-4}$.[10]

Operators with variable coefficients

While the works mentioned above dealt with the Laplacian in sufficiently smooth domains, the same optimal regularity question can be asked in more general scenarios. Some natural extensions include substituting the Laplacian by an operator with variable coefficients, or weakening the regularity of the domain Ω in the boundary value problem. We will obtain an optimal regularity result for operators in non-divergence form in $C^{1,1}$ domains, as we explain later after first describing the previously known literature.

Concerning operators with variable coefficients, the only optimal results apply to particular nonlinearities. For (self-adjoint) operators in divergence form, Crandall and Rabinowitz [44] showed the boundedness up to dimension 9 for exponential and power-type nonlinearities. This result was extended by Cowan and Ghoussoub [40] to non self-adjoint operators of the form $L = \Delta + b_i(x)\partial_i$ in smooth domains, with $b_i \in C^\infty(\overline{\Omega})$. In this setting, a variational stability inequality is no longer available and one must solely rely on the nonnegativity of the principal eigenvalue $\mu_1[J_u, \Omega]$. To surmount this difficulty, the authors in [40] apply a general Hardy inequality by Cowan [39] to the principal eigenfunction, resulting in an alternative integral inequality. Subsequently, the test functions of [44] lead to the boundedness of $u^\star$ in the optimal range. The Hardy inequality by Cowan uses a Hodge-type decomposition of the vector field $b(x) = (b_i(x))$, which requires it to be smooth. We will get a more general result for bounded b_i, as explained in the last section of this Introduction.

We now discuss the regularity of the domain. Before the optimal dimension for the Laplacian ($n \leq 9$) was reached in [24], all previous works assumed the smoothness of $\partial\Omega$ but did not investigate the possibility of weakening this assumption. On the other hand, the local analysis of [24] around the boundary works precisely in C^3 regular domains.

To prove the Hölder estimate for classical stable solutions in [24], the authors consider a portion of the boundary that is close to flat in C^3 norm. Under this assumption, when applying the blow-up procedure, the domain converges to a half-space and all error terms in their estimates vanish. Moreover, the C^3 regularity assumption ensures that classical stable solutions are in the Sobolev space $W^{3,p}(\Omega)$, for all $p < \infty$. The existence of third weak derivatives is crucial for the estimates not to depend on the nonlinearity. Here, when computing the linearized operator J_u acting on the test functions $\mathbf{c} = |\nabla u|$ and $\mathbf{c} = x \cdot \nabla u$, the nonlinearity cancels out, but this can only happen if u has third derivatives.

Instead of using a blow-up procedure, we will study curved boundaries by flattening them out locally. Under such a transformation, the Laplacian becomes an operator with variables coefficients of the form (8). Our finer estimates for these operators in flat domains

[10]Here, we say that u belongs to the Morrey space $M^{m,\beta}(\Omega)$ if $\sup_{y\in\Omega,\rho>0} \rho^{\beta-n} \int_{\Omega\cap B_\rho(y)} |u|^m < \infty$.

will be applicable to domains that are close to flat in C^2 norm. Moreover, by an approximation argument, such estimates will continue to hold in $C^{1,1}$ regular domains. This is a natural assumption from the point of view of the regularity theory for non-divergence form operators.

Stable solutions to other problems

The regularity of stable solutions can also be studied in other frameworks. Here, we briefly mention some natural extensions obtained by modifying the nonlinearity or the operator in the equation.

An important class of nonlinearities consists in functions blowing up at a point. The Gelfand problem in this case models the behavior of MEMS devices. For a positive, increasing, and convex nonlinearity $f\colon [0,1) \to (0,\infty)$ blowing up at 1, stable solutions u are always bounded by 1, but are only regular if $u < 1$. Here, the optimal dimension for the regularity of stable solutions depends on the blow-up rate of f, and is open in certain general classes; see the monograph [53].

Quasilinear operators have also been considered in the literature. Here, the operator L in the equation $-Lu = f(u)$ is replaced by a quasilinear one. When L is the p-Laplacian, the optimal dimension depends on p. Namely, explicit examples of singular stable energy solutions (in $W^{1,p}$) have only been found for $n \geq p + \frac{4p}{p-1}$. For $p > 2$, Cabré, Miraglio, and Sanchón [25] have shown that this range is optimal. When $1 < p < 2$, they have also proved the boundedness of stable solutions for $n < 5p$, but it is not known whether this can be improved. Mellet and Vovelle [75] have treated the case when L is the mean curvature operator. Here, smoothness does not follow from the boundedness of the solution (in fact, stable solutions are always bounded) and the goal is to show that the gradient is bounded. In [75] the authors show that, in the radial case, stable solutions are regular independently of the dimension. Whether this result holds in general domains is an open problem.

Finally, another interesting scenario arises when substituting the differential operator L by a nonlocal one. The model case here is the fractional Laplacian $(-\Delta)^s$ (on which we comment more in the next section below), and the regularity of stable solutions depends on the fractional parameter $s \in (0,1)$. For this operator, the optimal dimension is believed to be the one given by the exponential nonlinearity in [83]. However, for general nonlinearities, this dimension has not been reached, not even in the radial case; see [29, 85, 86] and the references therein.

Minimizers and calibrations

Up until this point we have focused on the stability of solutions, a necessary condition for minimality. Now, we turn our attention to minimizers of energy functionals. In the second part of this dissertation, we will be particularly interested in the theory of sufficient conditions for nonlocal functionals. Namely, given a solution to a variational equation, we would like to identify auxiliary conditions guaranteeing its minimality. While such a theory has been known for a long time for classical local problems, before this work there were almost no results in the nonlocal setting.

First, we give a brief summary of the classical theory of necessary and sufficient conditions. For this part, we mostly follow the celebrated monograph of Giaquinta and Hildebrandt [61]. Then, we move onto nonlocal energy functionals, focusing on the fractional Gagliardo-Sobolev seminorm.

The classical theory of necessary and sufficient conditions

One of the greatest discoveries in Physics was the realization that the laws of nature could be deduced from variational principles. An early example is Fermat's principle of least time in optics, whereby the path of a light ray traveling between two points is determined by minimizing the traveling time. Such principles motivated the development of the Calculus of Variations, the mathematical discipline dealing with the minimization of functionals.

A great part of the classical theory deals with general energy functionals of the form

$$\mathcal{E}_{\mathrm{L}}(w) = \int_\Omega G_{\mathrm{L}}(x, w(x), \nabla w(x))\,\mathrm{d}x.[11]$$

Here, we assume that the *Lagrangian* function $G_{\mathrm{L}} = G_{\mathrm{L}}(x,\lambda,q)$ is smooth and that the functional $\mathcal{E}_{\mathrm{L}}\colon \mathcal{A}_{\mathrm{L}} \to \mathbb{R}$ acts on a class $\mathcal{A}_{\mathrm{L}}$ of scalar functions $w\colon \overline{\Omega} \to \mathbb{R}$ in $C^1(\overline{\Omega})$, defined in some bounded domain $\Omega \subset \mathbb{R}^n$.[12] We say that $u \in \mathcal{A}_{\mathrm{L}}$ is a minimizer of $\mathcal{E}_{\mathrm{L}}$ if

$$\mathcal{E}_{\mathrm{L}}(u) \le \mathcal{E}_{\mathrm{L}}(w) \quad \text{for all } w \in \mathcal{A}_{\mathrm{L}} \text{ with } w = u \text{ on } \partial\Omega.$$

That is, we always consider minimizers with respect to competitors satisfying the same Dirichlet boundary conditions.

To compute minimizers, the first step is to find necessary conditions for minimality. This is achieved by testing the minimality condition with suitable competitors. Given $u \in \mathcal{A}_{\mathrm{L}}$ a minimizer of $\mathcal{E}_{\mathrm{L}}$, we consider competitors of the form $u + \varepsilon\xi$ with $\xi \in C^\infty_c(\Omega)$ and where $\varepsilon > 0$ is sufficiently small so that $u + \varepsilon\xi \in \mathcal{A}_{\mathrm{L}}$ (which is satisfied in most applications). By minimality of u, the first variation of $\mathcal{E}_{\mathrm{L}}$ at u (i.e., the derivative of $\varepsilon \mapsto \mathcal{E}_{\mathrm{L}}(u+\varepsilon\xi)$ at $\varepsilon = 0$) must vanish. Since ξ is arbitrary, after a simple computation and integrating by parts, it follows that u satisfies the Euler-Lagrange equation

$$-\mathrm{div}\,(\partial_q G_{\mathrm{L}}(x,u,\nabla u)) + \partial_\lambda G_{\mathrm{L}}(x,u,\nabla u) = 0.$$

Solutions to the Euler-Lagrange equation are called *critical points* or *extremals* of the functional $\mathcal{E}_{\mathrm{L}}$.

As discussed above, another necessary condition follows from taking the second variation of $\mathcal{E}_{\mathrm{L}}$ at u, which must be nonnegative and leads to the stability of the critical point. Choosing test functions of the form $\xi = \varepsilon\eta(\cdot/\varepsilon)$ in the stability inequality and letting $\varepsilon \to 0$ leads to Legendre's necessary condition

$$\partial^2_{q_i,q_j} G_{\mathrm{L}}(x,u(x),\nabla u(x)) p_i p_j \ge 0 \quad \text{for all } p \in \mathbb{R}^n \text{ and } x \in \Omega.$$

Notice that the linearized Euler-Lagrange equation at u is the Jacobi operator

$$J_u\xi = \mathrm{div}\left(\partial^2_{q,q} G_{\mathrm{L}}(x,u,\nabla u)\nabla\xi\right) + \left\{\mathrm{div}\left(\partial^2_{q,\lambda} G_{\mathrm{L}}(x,u,\nabla u)\right) - \partial^2_{\lambda,\lambda} G_{\mathrm{L}}(x,u,\nabla u)\right\}\xi.$$

When u is a minimizer, it follows that the principal part of J_u is nonnegative definite, that is, J_u is an elliptic operator in this case.

Until the XIX century, it was believed that extremals of elliptic energy functionals were necessarily minimizers. Weierstrass disproved this belief by producing examples of critical

[11] The subindices L and N will be used throughout the work to denote "local" and "nonlocal" objects, respectively.

[12] Here it is worth mentioning that, historically, vector valued functions of one variable were also crucial in the development of the Calculus of Variations. Minimization problems in that case leads to the study of systems of ODEs, while in this thesis we focus on scalar PDEs.

points which did not minimize the corresponding functional. The claim is only true for convex Lagrangians, that is, such that the joint function $(\lambda, q) \mapsto G_{\mathrm{L}}(x, \lambda, q)$ is convex. However, this convexity assumption is too restrictive in most applications, where only ellipticity is available. These observations motivate the search for sufficient conditions.

Jacobi developed a theory of sufficient conditions for *weak* minimizers, namely, local minimizers in a C^1 neighborhood. More precisely, $u \in \mathcal{A}_{\mathrm{L}}$ is a weak minimizer of $\mathcal{E}_{\mathrm{L}}$ if

$$\mathcal{E}_{\mathrm{L}}(u) \leq \mathcal{E}_{\mathrm{L}}(u+\xi) \quad \text{for all } \xi \in C_c^\infty(\Omega) \text{ with } \|\xi\|_{C^1(\overline{\Omega})} < \varepsilon,$$

for some small $\varepsilon > 0$. Arguing as above, we have that weak minimizers satisfy Legendre's condition. Under a uniform ellipticity assumption on J_u (the *strict Legendre condition*), Jacobi proved that strictly stable solutions are weak minimizers. To see this, by Taylor's expansion, it suffices to show that the second variation at $u + \xi$ is positive. Since G_{L} and all its derivatives are continuous with respect to C^1 perturbations, the positivity at $u+\xi$ follows from the one at u, up to errors of order $\|\nabla\xi\|^2_{L^2(\Omega)}$. These errors can then be controlled using Gårding's inequality, which shows the claim.

As we just saw, Jacobi's sufficient condition (i.e., the strict stability of u) yields local minimality in a very small neighborhood of functions whose slopes are close to those of u. In applications, however, a stronger and more precise notion of minimality is often needed. Following this direction, Weierstrass studied the class of *strong* minimizers: local minimizers with respect to the C^0 topology. Thus, $u \in \mathcal{A}_{\mathrm{L}}$ is a strong minimizer of $\mathcal{E}_{\mathrm{L}}$ if there is a $\varepsilon > 0$ such that

$$\mathcal{E}_{\mathrm{L}}(u) \leq \mathcal{E}_{\mathrm{L}}(u+\xi) \quad \text{for all } \xi \in C_c^\infty(\Omega) \text{ with } \|\xi\|_{C^0(\overline{\Omega})} < \varepsilon.$$

First, Weierstrass obtained a necessary condition for strong minimizers. For this, taking perturbations of the form $\xi(x) = \psi_\varepsilon(x_n)\zeta(x'/\varepsilon)$ (up to translations and rotations), where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, ζ is a cut-off, and

$$\psi_\varepsilon(x_n) := \begin{cases} (x_n + \varepsilon^2)_+ & \text{for } x_n \leq 0 \\ \varepsilon(\varepsilon - x_n)_+ & \text{for } x_n > 0, \end{cases}$$

and letting $\varepsilon \to 0$ in the strong minimality condition, it follows that

$$G_{\mathrm{L}}(x, u(x), \nabla u(x) + p) \geq G_{\mathrm{L}}(x, u(x), \nabla u(x)) + \partial_{q_i} G_{\mathrm{L}}(x, u(x), \nabla u(x))\, p_i$$

for all $p \in \mathbb{R}^n$ and $x \in \Omega$. This is the *Weierstrass necessary condition*, which corresponds to a stronger notion of ellipticity for J_u than Legendre's condition.

Later, Weierstrass found a sufficient condition which gives minimality in a more precise region of space, not just in a small C^0 neighborhood. It states that if a solution can be embedded in a family of extremals, whose graphs produce a foliation, then it is a minimizer in the region foliated by these critical points. In the literature, this is known as the *Weierstrass extremal field theory*.

We now explain this necessary condition in more detail. Let $\{u^t\}_{t\in I}$, where $I \subset \mathbb{R}$ is an interval, be a one-parameter family of critical points whose graphs give a foliation of a certain region $\mathcal{G}_{\mathrm{L}} \subset \overline{\Omega} \times \mathbb{R}$. In particular, the graphs of u^t do not touch. Such a family is called a *field of extremals*,[13] a nomenclature originating in scalar ODE, where the slopes of the u^t would define a vector field in $\mathcal{G}_{\mathrm{L}}$. The extremal field theory asserts the following: if every critical point $\{u^t\}_{t\in I}$ satisfies the Weierstrass necessary condition, then each u^t is a

[13]The term *extremal field* is also often used in the literature, but we find it ambiguous.

minimizer with respect to competitors $w \in \mathcal{A}_{\mathrm{L}}$, taking values in the foliated region $\mathcal{G}_{\mathrm{L}}$, and satisfying $w = u^t$ on $\partial\Omega$. In particular, the field of extremals is made up of minimizers.

This minimality result can be proved directly by constructing an auxiliary functional $\mathcal{C}_{\mathrm{L}} \colon \mathcal{A}_{\mathrm{L}} \to \mathbb{R}$, defined next. Here, the foliation property of the field is crucial. Using that for each point $(x, \lambda) \in \mathcal{G}_{\mathrm{L}} \subset \overline{\Omega} \times \mathbb{R}$ there is exactly one $t = t(x, \lambda) \in I$ such that $u^t(x) = \lambda$, we define $\mathcal{C}_{\mathrm{L}}$ by

$$\mathcal{C}_{\mathrm{L}}(w) := \int_\Omega \Big\{ G_{\mathrm{L}}(x, u^t(x), \nabla u^t(x)) + \partial_q G_{\mathrm{L}}(x, u^t(x), \nabla u^t(x)) \cdot (\nabla w - \nabla u^t) \Big\}\Big|_{t=t(x,w(x))} \,\mathrm{d}x.$$

Notice that $\mathcal{C}_{\mathrm{L}}$ is obtained by letting $p = \nabla w(x) - \nabla u^t(x)\big|_{t=t(x,w(x))}$ in the Weierstrass necessary condition satisfied by $u^{t(x,w(x))}$ and then integrating in $x \in \Omega$. It follows immediately that $\mathcal{C}_{\mathrm{L}}$ touches $\mathcal{E}_{\mathrm{L}}$ by below at each extremal u^t. Moreover, using that $\{u^t\}_{t \in I}$ are critical points, it can be shown that $\mathcal{C}_{\mathrm{L}}$ is also a null-Lagrangian, in the sense that it only depends on the boundary values of the functions. Hence, we have $\mathcal{C}_{\mathrm{L}}(w) = \mathcal{C}_{\mathrm{L}}(\widetilde{w})$ for all w, $\widetilde{w} \in \mathcal{A}_{\mathrm{L}}$ such that $w = \widetilde{w}$ on $\partial\Omega$. From these properties, it is now easy to deduce the minimality of each u^t.

Functionals such as $\mathcal{C}_{\mathrm{L}}$ are called *calibrations* in the Calculus of Variations, and they can be used in many situations to conclude that certain solutions are minimizers. The strategy is to embed such a critical point u in a family of extremals $\{u^t\}_{t \in I}$, so that $u = u^{t_0}$ for some $t_0 \in I$. More generally, assuming that $t \mapsto u^t$ is increasing in $t \in I$, to prove the minimality of u^{t_0} it suffices that the functions u^t above u^{t_0} (with $t \geq t_0$) are supersolutions of the Euler-Lagrange equations, while the ones below (with $t \leq t_0$) are subsolutions. Such approaches have found important applications in the theory of minimal surfaces; see [10, 45, 46, 71].

We conclude this section by mentioning that Jacobi's sufficient condition above yields the existence of an extremal field in a small C^0 neighborhood of the solution. Namely, assuming the strict Legendre condition, if u is strictly stable, then one can construct extremals of the form $u^t = u + \xi^t$ by the Banach fixed point theorem, with ξ^t small. To conclude their minimality in the foliated neighborhood, the Weierstrass necessary condition is still needed.

Nonlocal problems

Recently, there has been an increasing interest in nonlocal problems. Classical functionals such as $\mathcal{E}_{\mathrm{L}}$ cannot capture the long range interactions present in many relevant situations. Minimizing the energy functionals associated to these processes leads to nonlocal Euler-Lagrange equations. Nonlocality refers to the fact that evaluating these equations at a point requires knowledge of the solution everywhere in space.

One of the simplest energy functionals modeling nonlocal phenomena is given by

$$\mathcal{E}_{s,F}(w) = \frac{c_{n,s}}{4} \iint_{Q(\Omega)} \frac{|w(x) - w(y)|^2}{|x - y|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x,$$

where $s \in (0, 1)$, $c_{n,s}$ is a positive normalizing constant, $F \in C^1(\mathbb{R})$, and

$$Q(\Omega) = (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)$$

for a bounded domain $\Omega \subset \mathbb{R}^n$, with $\Omega^c = \mathbb{R}^n \setminus \Omega$. The functional $\mathcal{E}_{s,F}$ is the "fractional" version of $\mathcal{E}_{1,F}$ considered in (2) above. It is obtained substituting the Dirichlet energy by the Gagliardo seminorm [58]. Moreover, one can formally recover $\mathcal{E}_{1,F}$ from $\mathcal{E}_{s,F}$ by letting $s \uparrow 1$; see [47].

Notice that here, in the definition of $\mathcal{E}_{s,F}$, we are considering functions $w \colon \mathbb{R}^n \to \mathbb{R}$ taking values in the whole space, not just in the domain Ω. Thus, the functional accounts for interactions between points that can be very far apart from each other. In this context, it is natural to consider competitors having the same value in the exterior $\Omega^c = \mathbb{R}^n \setminus \Omega$. This is one reason why we do not include $\Omega^c \times \Omega^c$ interactions in $\mathcal{E}_{s,F}$, the other being that such an integral need not be finite, even for smooth and bounded functions. Hence, the exterior datum becomes a nonlocal analogue of the boundary conditions from classical local problems.

Minimizers of $\mathcal{E}_{s,F}$ satisfy the Euler-Lagrange equation

$$(-\Delta)^s u = f(u) \quad \text{in } \Omega,$$

where $(-\Delta)^s$ is the fractional Laplacian defined by

$$(-\Delta)^s u(x) = c_{n,s}\, \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, \mathrm{d}y.$$

Here, P.V. indicates that the integral must be understood in the principal value sense. Once again, in order to compute the fractional Laplacian of u at a point, we need to know the values of the function in the whole $\mathbb{R}^n$. The normalizing constant $c_{n,s}$ is determined by taking the Fourier transform $\mathcal{F}$ and imposing that

$$\mathcal{F}[(-\Delta)^s u](z) = |z|^{2s} \mathcal{F}[u](z) \quad \text{for all } z \in \mathbb{R}^n.$$

In particular, $(-\Delta)^s$ converges to the ordinary Laplacian $-\Delta$ as $s \uparrow 1$, in a suitable sense.

While being a nonlocal operator, the fractional Laplacian can also be studied via the Caffarelli-Silvestre extension [33], an auxiliary local problem in an extended $n+1$ dimensional space. There, writing $(x, y) \in \mathbb{R}^n \times (0, \infty) =: \mathbb{R}^{n+1}_+$, one considers $U \colon \overline{\mathbb{R}^{n+1}_+} \to \mathbb{R}$ to be the solution of the degenerate elliptic problem

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ U = u & \text{on } \partial \mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases}$$

It can then be shown that $(-\Delta)^s u(x) = -\lim_{y \to 0^+} y^{1-2s} \partial_y U(x,y)$, allowing us to recover the fractional Laplacian of u from its harmonic s-extension U. Notice that the semilinear equation in Ω above becomes a Neumann reaction problem for the extension.

A theory of necessary of sufficient conditions as the one described in the previous section does not seem to be available for nonlocal functionals at the moment. Such results would help us to better understand nonlocal variational problems and would also equip us with a valuable toolkit to effectively attack these problems. For instance, in the last section of this Introduction, we give some nontrivial applications of having sufficient conditions.

Concerning necessary conditions, even the notion of ellipticity has not been properly defined in that case. The fractional Laplacian is admitted to be an elliptic operator in some sense, since its Fourier symbol is invertible away from zero (a concept of ellipticity from the theory of pseudo-differential operators) and because it satisfies a maximum principle. While the former condition is only applicable to pseudo-differential operators, the latter condition continues to hold for more general nonlocal operators satisfying some monotonicity assumption, which can be taken as a nonlocal notion of ellipticity (as explained below in the last section of this Introduction).

On the other hand, sufficient conditions for minimality had only been found in the context of nonlocal minimal surfaces. There, one considers functionals $\mathcal{P}_{\mathrm{N}}$ acting on sets $E \subset \mathbb{R}^n$ instead of functions, and which describe a nonlocal analogue of the perimeter of the surface ∂E. These functionals are of the form

$$\mathcal{P}_{\mathrm{N}}(E) = \frac{1}{2} \iint_{Q(\Omega)} |\mathbb{1}_E(x) - \mathbb{1}_E(y)| K(x-y) \, \mathrm{d}x \, \mathrm{d}y,$$

where K is a singular, nonnegative, symmetric kernel. We note in particular that quadratic functionals of the form $\mathcal{E}_{s,F}$ are not covered by this setting. For nonlocal perimeters, the only results are by Cabré [18] and Pagliari [80]. In [18], the author built a calibration for $\mathcal{P}_{\mathrm{N}}$ in the presence of a field of extremals by nonlocal minimal surfaces (i.e., critical points of $\mathcal{P}_{\mathrm{N}}$). This construction suggested that a nonlocal version of the Weierstrass extremal field theory could exist for other functionals, but the method of proof could not be generalized to that framework. As we will see in the last section below, the present dissertation will confirm the validity of such a conjecture. Lastly, [80] gave a calibration for the nonlocal total variation functional

$$\mathcal{E}_{\mathrm{NTV}}(w) := \frac{1}{2} \iint_{Q(\Omega)} |w(x) - w(y)| \, K(x-y) \, \mathrm{d}x \, \mathrm{d}y,$$

(which can be obtained as the integral of the $\mathcal{P}_{\mathrm{N}}$ on level sets) for the particular case of characteristic functions of the half-space. As an application, the author of [80] deduced that halfspaces minimize $\mathcal{P}_{\mathrm{N}}$. While this is the first nonlocal calibration (for the types of functionals that we are interested in) that has appeared in the literature, the proof does not use fields of extremals and only works for the characteristic function of a half-space. Thus, it was not known how it could be generalized to other situations.

Finally, we would like to point out that it is not clear a priori how to use the local extension problem to obtain sufficient conditions for $\mathcal{E}_{s,F}$. As shown in [32], minimality properties of $\mathcal{E}_{s,F}$ can be related to those of a functional acting on functions defined in the extended space $\mathbb{R}^{n+1}_+$. However, applying the local theory to functionals in $\mathbb{R}^{n+1}_+$ leads to intricate conditions that, at first glance, cannot be written on $\mathbb{R}^n$.

Results and outline of the thesis

The thesis is divided in two parts. Part I is devoted to the regularity of stable solutions, while Part II deals with necessary conditions for minimality in a nonlocal setting. First, in Part I, we develop a regularity theory for stable solutions to semilinear elliptic equations with variable coefficients. It consists of Chapters 1–3 and Appendices A–F. Later, in Part II, we present a nonlocal Weierstrass extremal field theory. This portion includes Chapters 4–5 and Appendices G-J.

Part I: Regularity of stable solutions

Below, L denotes a linear elliptic operator in non-divergence form, given by (8), that is,

$$L = a_{ij}(x)\partial_{ij} + b_i(x)\partial_i.$$

As before, we assume that L is uniformly elliptic, i.e., satisfying (9) above or

$$c_0|p|^2 \le a_{ij}(x)p_ip_j \le C_0|p|^2 \quad \text{for all } p \in \mathbb{R}^n, \tag{20}$$

recalled here for convenience. Our regularity theory requires some additional assumptions on the coefficients beyond the basic hypothesis (10). Namely, we always assume that

$$a_{ij} \in C^{0,1}(\overline{\Omega}), \quad b_i \in L^\infty(\Omega). \tag{21}$$

For the boundary regularity, we will also need the interior continuity assumption

$$b_i \in L^\infty(\Omega) \cap C^0(\Omega). \tag{22}$$

Chapter 1 provides interior estimates for stable solutions under the sole assumption that the nonlinearity is nonnegative. Here and in the following, whenever we write $C = C(\ldots)$, we mean that the constant C depends only on the quantities between the parentheses. The main result of that chapter is the following:

Theorem 5 (Theorem 1.1.1)**.** *Let L satisfy conditions* (20) *and* (21) *in $\Omega = B_1$, and let $f \in C^1(\mathbb{R})$ be nonnegative.*

Let $u \in W^{3,p}(B_1)$, for some $p > n$, be a stable solution of $-Lu = f(u)$ in B_1.

Then

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/2})} \leq C\|u\|_{L^1(B_1)},$$

where $\gamma = \gamma(n) > 0$ and $C = C(n, c_0, C_0, \|\nabla a_{ij}\|_{L^\infty(B_1)}, \|b_i\|_{L^\infty(B_1)})$. In addition,

$$\|u\|_{C^\alpha(\overline{B_{1/2}})} \leq C\|u\|_{L^1(B_1)} \quad \text{if } n \leq 9,$$

where $\alpha = \alpha(n, c_0, C_0) > 0$ and $C = C(n, c_0, C_0, \|\nabla a_{ij}\|_{L^\infty(B_1)}, \|b_i\|_{L^\infty(B_1)})$.

This theorem gives two types of a priori estimates: an energy estimate in every dimension and a Hölder estimate in the optimal range $n \leq 9$. Our bounds depend only on the ellipticity constants, as well as the Lipschitz norm of a_{ij} and the L^∞ norm of b_i. In particular, the constants do not depend on the nonlinearity. Previously, such a result was only known to hold for the Laplacian; see [24].

Notice that u is assumed to have three derivatives. This is needed to have a cancellation which removes the nonlinearity in the stability condition, ensuring that the estimates do not depend on f. Assuming the coefficient b_i to be smooth, one can show the existence of third derivatives for bounded strong solutions. However, in our boundary regularity result below, thanks to an approximation argument, we will not need this additional assumption.

In order to establish Theorem 5, we test the stability condition with variants of the test functions in [24]. Since our problem is not variational, the characterization of stability via the second variation is not available, and we must obtain an integral inequality from the pointwise condition in Definition 3. A first approach to the computations leads to bounds depending on the C^2 norm of a_{ij} and the C^1 norm of b_i. The main difficulty in the proofs is to obtain the claimed $C^{0,1}$ and L^∞ dependence, which is crucial for the subsequent boundary estimates to hold in $C^{1,1}$ domains. This forces us to treat certain error integrals involving the Hessian of the solution, which are controlled via a Sternberg-Zumbrun type inequality (see Theorem 1.1.2).

Moreover, thanks to a device of [19] for the Laplacian, all our estimates are quantifiable. Instead, the proof in [24] used contradiction-compactness arguments that did not yield a quantitative control of the constants. Adapting the method from [19] to our setting adds some technical difficulties. In essence, the problem reduces to obtaining estimates for a Neumann problem, which we only know how to achieve by a Moser iteration.

In Chapter 2, we prove an energy estimate up to the boundary in half-balls, valid in every dimension. This result is the boundary analogue of the energy estimate in Theorem 5, and requires the nonlinearity f to be nonnegative and nondecreasing. Its proof uses the interior estimates from Chapter 1 together with a delicate Sternberg-Zumbrun estimate up to the boundary in flat domains (Theorem 2.1.4 below). For this, we again use a variant of the test functions in [24]. The dependence on the constants is through the same coefficient norms appearing in Theorem 5. This chapter serves as an intermediate step between the interior estimates and the curved boundary estimates.

Finally, in Chapter 3, we prove boundary estimates in curved domains of class $C^{1,1}$. These estimates are the main conclusion of Part I, and can be summarized in the following:

Theorem 6 (Theorem 3.1.1). *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain. Let L satisfy conditions* (9), (21), *and* (22) *in Ω, and let $f \in C^1(\mathbb{R})$ be nonnegative, nondecreasing, and convex.*

Let $u \in C^0(\overline{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ be a stable solution of $-Lu = f(u)$ in Ω, with $u = 0$ on $\partial\Omega$. Then,

$$\|\nabla u\|_{L^{2+\gamma}(\Omega)} \leq C\|u\|_{L^1(\Omega)},$$

where $\gamma = \gamma(n) > 0$ and $C = C(\Omega, c_0, C_0, \|\nabla a_{ij}\|_{L^\infty(\Omega)}, \|b_i\|_{L^\infty(\Omega)})$. In addition,

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C\|u\|_{L^1(\Omega)} \qquad \text{if } n \leq 9,$$

where $\alpha = \alpha(n, c_0, C_0) > 0$ and $C = C(\Omega, c_0, C_0, \|\nabla a_{ij}\|_{L^\infty(\Omega)}, \|b_i\|_{L^\infty(\Omega)})$.

Similarly to the interior bounds above, Theorem 6 establishes boundary energy and Hölder estimates in the optimal range of dimensions. These are new even for the Laplacian in [24], a result which required a C^3 regularity assumption on the domain. Moreover, by adapting some ideas of [19] for the Laplacian in half-balls, we give the first quantitative proof in curved domains. It is worth mentioning that in both [24], [19], and in Theorem 6, the nonlinearity is required to be nonnegative, nondecreasing, and convex.

Our proof uses all previous results from Chapters 1 and 2. Here, the strategy is to flatten the boundary locally and then choosing appropriate test functions in the alternative integral stability inequality obtained in Chapter 1.

Notice that our result applies to strong stable solutions. In fact, since we are only assuming conditions (21) and (22) on the coefficients, our solutions need not have third derivatives, while the a priori estimates from the preceding chapters required this assumption. To circumvent this issue, we carry out an approximation argument. Namely, we consider a smooth exhaustion Ω_k of our $C^{1,1}$ domain, and we construct regular stable solutions u_k to an auxiliary problem in each Ω_k. These solutions approximate the original solution in some sense. Applying our estimates to the sequence u_k and passing to the limit then yields the result. For this method to work in $C^{1,1}$ domains, it is crucial that the bounds depend only on the $C^{0,1}$ and L^∞ norms of a_{ij} and b_i, respectively. Indeed, when flattening the boundary by a change of variables, the solution in the new coordinates satisfies an equation of the same form but with different coefficients. Namely, the new a_{ij} coefficients involve first derivatives of the flattening map, while the b_i contain second derivatives of the same map. Thus, the stated dependence on the norms corresponds to a $C^{1,1}$ domain.

We conclude Part I with Appendices A–F, which provide certain accessory results required by the proofs of the main theorems above. In Appendix A we show that the stability condition is not equivalent to the integral inequality obtained in Chapter 1. Appendix B contains an elementary proof of the Sobolev trace inequality in the ball, which is needed in Chapter 1 to carry out the Moser iteration leading to quantitative interior estimates. In Appendix C we recall some useful interpolation inequalities of Cabré [19, 20]. Appendix D serves the purpose of recalling a celebrated lemma by Simon [89] for absorbing errors in large balls. In Appendix E, we explain how to approximate $C^{1,1}$ domains from the interior by smooth sets satisfying uniform bounds, an important fact in the boundary regularity theory of Chapter 3. Finally, Appendix F is devoted to proving the uniqueness of stable solutions for convex nonlinearities, an auxiliary result needed in our approximation argument.

Part II: A nonlocal Weierstrass extremal field theory

In Chapter 4, we extend the classical Weierstrass extremal field theory to the fractional setting. As explained above, here the central theme is the notion of "field", namely, a one-parameter family of functions producing a foliation. If the field is made of extremals (i.e., solutions), then one expects to be able to construct a calibration functional. The existence of such an object proves the minimality of each extremal among competitors taking values in the foliated region.

Given an interval $I \subset \mathbb{R}$ (not necessarily open or bounded), we say that a family $\{u^t\}_{t\in I}$ of functions $u^t\colon \mathbb{R}^n \to \mathbb{R}$ is a C^2 *field in* $\mathbb{R}^n$ if the map $(x,t) \mapsto u^t(x)$ is C^2 in $\mathbb{R}^n \times I$ and for each $x \in \mathbb{R}^n$ the function $t \mapsto u^t(x)$ is increasing in I. In particular, the graphs of $\{u^t\}_{t\in I}$ produce a foliation of a region $\mathcal{G} \subset \mathbb{R}^n \times \mathbb{R}$ and we may define the parameter $t = t(x,\lambda)$ as above (i.e., as the unique $t \in I$ such that $u^t(x) = \lambda$).

Our main result of Chapter 4 is the construction of a calibration for the functional $\mathcal{E}_{s,F}$, the Gagliardo-Sobolev seminorm with a potential term, introduced above. Recall that, given a bounded domain $\Omega \subset \mathbb{R}^n$, we write $Q(\Omega) = (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)$, where $\Omega^c = \mathbb{R}^n \setminus \Omega$. The existence of a calibration for $\mathcal{E}_{s,F}$ was an important open problem that we have solved in the following:

Theorem 7 (Theorem 4.1.3). *Let $I \subset \mathbb{R}$ be an interval, $\Omega \subset \mathbb{R}^n$ a bounded domain, and $s \in (0,1)$. Let $\{u^t\}_{t\in I}$ be a C^2 field in $\mathbb{R}^n$ satisfying*

$$|u^t(x)| + |\partial_t u^t(x)| \le C \quad \text{for all } x \in \mathbb{R}^n \text{ and } t \in I,$$

for some constant C. Consider the admissible functions

$$\mathcal{A}_s = \{w \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \ : \ \operatorname{graph} w \subset \mathcal{G}\},$$

where

$$\mathcal{G} = \{(x,\lambda) \in \mathbb{R}^n \times \mathbb{R} \colon \lambda = u^t(x) \text{ for some } t \in I\}.$$

Given $t_0 \in I$ and $F \in C^1(\mathbb{R})$, let $\mathcal{C}_{s,F}$ be the functional

$$\mathcal{C}_{s,F}(w) := c_{n,s}\,\mathrm{P.V.}\iint_{Q(\Omega)} \int_{u^{t_0}(x)}^{w(x)} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x + \frac{c_{n,s}}{4}\iint_{Q(\Omega)} \frac{|u^{t_0}(x) - u^{t_0}(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y$$

defined for $w \in \mathcal{A}_s$, where $c_{n,s}$ is the positive constant in the definition of $\mathcal{E}_{s,F}$.

If $\{u^t\}_{t\in I}$ is a field of extremals, that is, if

$$(-\Delta)^s u^t = F'(u^t) \quad \text{in } \Omega \quad \text{for all } t \in I,$$

then $\mathcal{C}_{s,F}$ is a calibration for $\mathcal{E}_{s,F}$ and u^{t_0}. More precisely, $\mathcal{C}_{s,F}$ satisfies the three properties

($\mathcal{C}1$) $\mathcal{C}_{s,F}(u^{t_0}) = \mathcal{E}_{s,F}(u^{t_0})$.

($\mathcal{C}2$) $\mathcal{C}_{s,F}(w) \leq \mathcal{E}_{s,F}(w)$ *for all* $w \in \mathcal{A}_s$ *such that* $w \equiv u^{t_0}$ *in* Ω^c.

($\mathcal{C}3$) $\mathcal{C}_{s,F}(w) = \mathcal{C}_{s,F}(\widetilde{w})$ *for all* $w, \widetilde{w} \in \mathcal{A}$ *with* $w \equiv \widetilde{w} \equiv u^{t_0}$ *in* Ω^c.

As a consequence, the function u^{t_0} *minimizes* $\mathcal{E}_{s,F}$ *among functions* w *in* $\mathcal{A}_s$ *such that* $w \equiv u^{t_0}$ *in* Ω^c.

In fact, Theorem 7 is a simplified version of our main result in Theorem 4.1.3 below. There, in addition to fields of extremals, we consider families of super and subsolutions, which suffice to prove the minimality of u^{t_0}. The regularity of the field can also be

Recall that, prior to our result above, a nonlocal calibration had only been found in the setting of nonlocal minimal surfaces, in [18, 80]. Before our work, a calibration or null-Lagrangian structure for fractional quadratic functionals was not known. Thus, it was not even clear whether a calibration such as $\mathcal{C}_{s,F}$ could exist.

We first succeeded in constructing many functionals satisfying properties ($\mathcal{C}1$) and ($\mathcal{C}2$) above, but the null-Lagrangian property ($\mathcal{C}3$) was either not satisfied or seemed too difficult to prove. Coming back to the work [18] on the nonlocal perimeter, we noticed that the calibration there could be written as a sum of two terms: one depending on the nonlocal mean curvature (the first variation of the perimeter) and another depending only on the exterior datum. Inspired by this observation, we searched for such a structure in the classical local theory but, to our surprise, it was never mentioned in the literature. Eventually, we were able to show that the classical calibration $\mathcal{C}_{\mathrm{L}}$ could be written in terms of the Euler-Lagrange equation and the Neumann condition satisfied by the field. In particular, in a field of extremals, such an expression depends only on the values of the function on the boundary. This discovery allowed us to guess a natural calibration candidate for $\mathcal{E}_{s,F}$, obtained by simply considering the fractional Euler-Lagrange and Neumann operators in the previous identity. By construction, the candidate satisfied properties ($\mathcal{C}1$) and ($\mathcal{C}3$), and we later proved that it also satisfied ($\mathcal{C}2$).

Finally, in Chapter 5, we extend the fractional construction from Chapter 4 to more general nonlocal elliptic energy functionals. For this, the idea is to carry out the process explained in the previous paragraph, identifying a natural notion of ellipticity. Our main result applies to energy functionals of the form

$$\mathcal{E}_{\mathrm{N}}(w) = \frac{1}{2} \iint_{Q(\Omega)} G_{\mathrm{N}}(x, y, w(x), w(y))\, \mathrm{d}x\, \mathrm{d}y,$$

where $G_{\mathrm{N}} = G_{\mathrm{N}}(x, y, a, b)$ is a function with $G_{\mathrm{N}}(x, y, a, b) = G_{\mathrm{N}}(y, x, b, a)$ and satisfying the ellipticity condition

$$\partial^2_{ab} G_{\mathrm{N}}(x, y, a, b) \leq 0. \tag{23}$$

Here, we are only interested in the most general class of Lagrangians for which our proof above works and, hence, we do not make any growth and regularity assumptions on the Lagrangian G_{N}. In that sense, our next result is only formal, but can be made fully rigorous (as Theorem 7 above) for appropriate classes of Lagrangians, which would be modeled after each particular nonlinear problem. This is what we mean by "sufficiently regular for G_{N}" in the statement below. We have the following:

Theorem 8. *Let* $I \subset \mathbb{R}$ *be an interval and let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain. Given a smooth function* $G_{\mathrm{N}} = G_{\mathrm{N}}(x, y, a, b)$, *with* $G_{\mathrm{N}}(x, y, a, b) = G_{\mathrm{N}}(y, x, b, a)$, *and satisfying the ellipticity condition* (23), *let* $\{u^t\}_{t \in I}$ *be a field in* $\mathbb{R}^n$ *which is sufficiently regular for* G_{N}.

Given $t_0 \in I$, let $\mathcal{C}_{\mathrm{N}}$ be the functional

$$\mathcal{C}_{\mathrm{N}}(w) := \iint_{Q(\Omega)} \int_{u^{t_0}(x)}^{w(x)} \Big\{ \partial_a G_{\mathrm{N}}(x, y, u^t(x), u^t(y)) \Big\}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x \, \mathrm{d}y + \mathcal{E}_{\mathrm{N}}(u^{t_0})$$

defined in a set $\mathcal{A}_{\mathrm{N}}$ of sufficiently regular admissible functions $w\colon \mathbb{R}^n \to \mathbb{R}$ (for G_{N}) satisfying $\operatorname{graph} w \subset \mathcal{G}$, *where*

$$\mathcal{G} = \big\{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R} \colon \lambda = u^t(x) \quad \textit{for some } t \in I \big\}.$$

If $\{u^t\}_{t \in I}$ is a field of extremals, that is, if

$$\int_{\mathbb{R}^n} \partial_a G_{\mathrm{N}}(x, y, u^t(x), u^t(y)) \, \mathrm{d}y = 0 \quad \textit{for all} \quad x \in \Omega \quad \textit{and} \quad t \in I,$$

then $\mathcal{C}_{\mathrm{N}}$ is a calibration for $\mathcal{E}_{\mathrm{N}}$ and u^{t_0}. As a consequence, the function u^{t_0} minimizes $\mathcal{E}_{\mathrm{N}}$ among functions w in $\mathcal{A}_{\mathrm{N}}$ such that $w \equiv u^{t_0}$ in Ω^c.

As a first application of Theorem 8, we prove that monotone solutions are minimizers. While such a result was known for functionals with an existence and regularity theorem for minimizers, our calibration proof does not need to assume the existence of a minimizer a priori. This allows us to prove the minimality of monotone solutions for more general functionals. Such a question was motivated by the fractional version of a conjecture of De Giorgi conjecture on the symmetry of monotone solutions to the Allen-Cahn equation, see the survey [36].

As a second application, we show that minimizers of $\mathcal{E}_{\mathrm{N}}$ are viscosity solutions. This is perhaps surprising, since here minimality is assumed instead of being concluded. Previous proofs [5, 66, 87] of such a result needed a weak comparison principle for the Euler-Lagrange equation, restricting the class of functionals where they could be applied. However, a calibration argument will yield the viscosity property in a more general setting, without the need of a comparison principle.

Part II includes the following appendices expanding on the theory of nonlocal calibrations. In Appendix G, following Chapter 4, we apply the local theory to the Caffarelli-Silvestre extension to give a calibration for the fractional Laplacian in the extended space. Appendix H enumerates other natural candidates for a calibration associated to the fractional Laplacian; some of them are shown not to be calibrations. In Appendix I we relate the ellipticity condition for the general Lagrangian in Chapter 5 to a strong comparison principle for nonlocal nonlinear operators. Finally, in Appendix J we apply our calibration from Chapter 5 to the nonlocal total variation, relating it with the calibration for the nonlocal perimeter constructed in [18].

Part I

Regularity of stable solutions to semilinear elliptic problems

Chapter 1

Interior regularity

In this chapter, we extend the interior estimates for stable solutions in [24] to operators with variable coefficients. We show that stable solutions to the semilinear elliptic equation $a_{ij}(x)u_{ij} + b_i(x)u_i + f(u) = 0$ are Hölder continuous in the optimal range of dimensions $n \leq 9$. Our bounds are independent of the nonlinearity $f \in C^1$, which we assume to be nonnegative.

The main achievement of our work is to make the constants in our estimates depend on the $C^{0,1}$ norm of a_{ij} and the L^∞ norm of b_i, instead of their C^2 and C^1 norms, respectively, which arise in a first approach to the computations.

1.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f\colon \mathbb{R} \to \mathbb{R}$ a C^1 function. We consider stable solutions (defined below) $u\colon \overline{\Omega} \to \mathbb{R}$ to the semilinear equation

$$-Lu = f(u) \quad \text{in } \Omega, \tag{1.1.1}$$

where L is a second order linear elliptic differential operator of the form

$$Lu = a_{ij}(x)u_{ij} + b_i(x)u_i, \quad a_{ij}(x) = a_{ji}(x). \tag{1.1.2}$$

For simplicity, throughout this chapter we assume that the coefficients are smooth up to the boundary, i.e., we have $a_{ij} \in C^\infty(\overline{\Omega})$ and $b_i \in C^\infty(\overline{\Omega})$. However, our results continue to hold for merely Lipschitz a_{ij} and bounded b_i.

The purpose of this chapter is to extend the recent results of Cabré, Figalli, Ros-Oton, and Serra in [24] and of Cabré in [20] for the Laplacian to the above operators with variable coefficients. In [24], the authors solved a long-standing conjecture concerning the regularity of stable solutions to semilinear problems. They showed that stable solutions are bounded (and hence smooth) in dimension $n \leq 9$. This result is optimal, since there are examples of singular stable solutions for $n \geq 10$.

In the papers [20, 24], the authors obtain universal a priori estimates that do not depend on the nonlinearity f. They prove interior regularity bounds, assuming $f \geq 0$, and boundary regularity estimates on C^3 domains, assuming $f \geq 0$, $f' \geq 0$, and $f'' \geq 0$. The boundary result applies only to solutions vanishing on the boundary.

The first main interest in extending these results to operators with variable coefficients (besides possible future applications to nonlinear problems) is to simplify the boundary regularity arguments, even for the Laplacian. Indeed, starting from a curved boundary, the proof of regularity in [24] requires a delicate blow-up and Liouville theorem argument, which is needed in order to apply a result of theirs only available on a flat boundary. In addition, this proof is by contradiction-compactness and does not allow to quantify the constants in the estimates. Recently in [20], Cabré has given a quantitative proof of this result for the Laplacian in the case of a flat boundary. On the other hand, a curved boundary can be flattened out by a change of variables. Note now that, in the new coordinates, the Laplacian is written as an operator of the form (1.1.2). It is therefore natural to establish quantitative a priori estimates for our family of equations in the half-ball. By extending the techniques in [20] to operators with variable coefficients, we will avoid the intricate blow-up and Liouville theorem result form [24] as well as the compactness part.

An important feature of our estimates is that they depend only on the ellipticity constants and on the norms $\|\nabla a_{ij}\|_{L^\infty}$ and $\|b_i\|_{L^\infty}$ of the coefficients. The main difficulty in our proofs will be to obtain this dependence instead of on $\|a_{ij}\|_{C^2}$ and $\|b_i\|_{C^1}$, which are the norms that appear naturally in a first approach to the computations. This will be especially relevant for boundary regularity (treated in the next chapters), since it will allow us to relax the C^3 regularity requirement of the domain in [24]. The key point here is that, as mentioned before, flattening the boundary transforms the Laplacian into an operator of the form (1.1.2), with coefficients a_{ij} and b_i involving first and second derivatives of the boundary surface, respectively. Thus, a $C^{0,1}$, C^0 bound of the coefficients a_{ij}, b_i would correspond to a C^2 bound of the boundary. In fact, the result holds in weaker domains, as shown in Chapters 2 and 3.

Moreover, our methods will also be useful in a future work where we treat the case of the Laplacian with non-homogeneous boundary conditions. Recall that the previous papers [24] and [20] require strongly that the solutions vanish on the boundary. Flattening the boundary, we will be able to reduce the problem to an equation on the half-space for an operator of the form (1.1.2) with zero boundary conditions and an additional source term.

The study of the regularity of stable solutions was initiated in the seventies by Crandall and Rabinowitz in [44]. There, they showed the boundedness of stable solutions when $n \leq 9$ for exponential and power-type nonlinearities. Their work was motivated by problems in combustion [60], commonly known as "Gelfand-type problems"; for more information on these problems, we refer the reader to the monograph of Dupaigne [48] (see also the Introduction or Chapter 3 of this thesis). Later, in the mid-nineties, Brezis [11] asked for an extension of this regularity result to a larger class of nonlinearities. The boundedness of stable solutions was proven by Nedev [78] for $n \leq 3$, and by Cabré [16] for $n = 4$. The optimal dimension $n \leq 9$ remained open until it was finally reached by Cabré, Figalli, Ros-Oton, and Serra in [24].

1.1.1 The setting. Stability.

We are interested in the class of stable solutions to the semilinear equation (1.1.1). Assume that the domain $\Omega \subset \mathbb{R}^n$ is smooth. We say that u is a *stable* solution of (1.1.1) if there exists a function $\varphi \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{cases} -L\varphi \geq f'(u)\varphi & \text{in } \Omega, \\ \varphi > 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1.3}$$

Equivalently, a solution is stable when the principal Dirichlet eigenvalue of the linearized equation is nonnegative; see [9]. We denote the linearized equation at u by

$$J_u\varphi = L\varphi + f'(u)\varphi, \tag{1.1.4}$$

also known as the *Jacobi operator*.

Since the coefficients of L are smooth, our operator (1.1.2) can be written in divergence form as

$$\mathcal{L}u = \partial_i\left(a_{ij}(x)\partial_j u\right) + d_i(x)\partial_i u \tag{1.1.5}$$

for certain appropriate coefficients d_i. Now, recall that our bounds for non-divergence operators depend on the norms $\|a_{ij}\|_{C^1}$ and $\|b_i\|_{C^0}$ of the coefficients. As a consequence, our results continue to hold for every divergence-form operator $\mathcal{L}$ as in (1.1.5), with constants depending on $\|a_{ij}\|_{C^1}$ and $\|d_i\|_{C^0}$ instead.

We assume that the symmetric coefficient matrix $A(x) = (a_{ij}(x))$ is uniformly elliptic, i.e., there are positive constants c_0, C_0 such that

$$c_0|p|^2 \le a_{ij}(x)p_ip_j \le C_0|p|^2 \quad \text{for all } p \in \mathbb{R}^n.$$

This condition will be written as $c_0 \le A(x) \le C_0$. In particular, the matrix $A(x)$ is positive definite and defines a norm

$$|p|_{A(x)} := \left(a_{ij}(x)p_ip_j\right)^{1/2}$$

on vectors $p \in \mathbb{R}^n$.

For variational equations $-\partial_i\left(a_{ij}(x)\partial_j u\right) = f(u)$ stability is equivalent to the nonnegativity of the second variation of the associated energy functional. This provides the useful integral inequality

$$\int_\Omega f'(u)\xi^2\,\mathrm{d}x \le \int_\Omega |\nabla\xi|^2_{A(x)}\,\mathrm{d}x, \tag{1.1.6}$$

satisfied by all test functions $\xi \in C^1_c(\Omega)$. A key strategy to derive a priori estimates in that setting is to choose appropriate test functions in (1.1.6). When chosen correctly in terms of the Jacobi operator, the test functions allow to get rid of the nonlinearity within the proofs. This is what is done for the Laplacian in [24].

Since our operator L does not have variational structure, (1.1.6) is not available. Nevertheless, we are able to exploit the pointwise stability condition (1.1.3) for φ to obtain a convenient integral inequality which does not involve the function φ. We will use it as replacement of (1.1.6) in our non-variational setting. To derive the integral inequality, we first write the operator L in divergence form as in (1.1.5) with $d_i(x) = \widehat{b}_i(x)$, where $\widehat{b}$ is the vector field given by

$$\widehat{b}_i(x) = b_i(x) - \partial_k a_{ki}(x), \tag{1.1.7}$$

hence $Lu = \partial_i\left(a_{ij}(x)u_j\right) + \widehat{b}_i(x)u_i$. Now, for a test function $\xi \in C^1_c(\Omega)$, multiply (1.1.3) by ξ^2/φ and integrate by parts to obtain

$$\begin{aligned}\int_\Omega f'(u)\xi^2\,\mathrm{d}x &\le \int_\Omega \left(A(x)\nabla\varphi\cdot\nabla\left(\frac{\xi^2}{\varphi}\right) - \widehat{b}(x)\cdot\frac{\xi^2}{\varphi}\nabla\varphi\right)\mathrm{d}x\\ &= \int_\Omega\left(-|\xi\nabla\log\varphi|^2_{A(x)} + 2A(x)\xi\,\nabla\log\varphi\cdot\nabla\xi - \xi\,\widehat{b}(x)\cdot\xi\,\nabla\log\varphi\right)\mathrm{d}x.\end{aligned}$$

Using that

$$2A(x)\xi\,\nabla\log\varphi\cdot\left(\nabla\xi - \tfrac{1}{2}\xi A^{-1}(x)\widehat{b}(x)\right) - |\xi\nabla\log\varphi|^2_{A(x)} \le \left|\nabla\xi - \tfrac{1}{2}\xi A^{-1}(x)\widehat{b}(x)\right|^2_{A(x)},$$

we deduce

$$\int_\Omega f'(u)\xi^2 \,\mathrm{d}x \le \int_\Omega \left|\nabla \xi - \tfrac{1}{2}\xi A^{-1}(x)\widehat{b}(x)\right|^2_{A(x)} \mathrm{d}x \quad \text{ for all } \xi \in C^1_c(\Omega). \tag{1.1.8}$$

We remark that, in general, (1.1.8) is not equivalent to the stability condition (1.1.3). The main reason is that our equation $-Lu = f(u)$ is not variational due to the presence of $\widehat{b}$ when written in the divergence form (1.1.5). In Appendix A, we give an example of the non-equivalence, and, at the same time, we characterize the drifts $\widehat{b}$ for which the equivalence holds.

1.1.2 Main results

This chapter concerns the interior regularity of stable solutions. Boundary regularity results will be treated in Chapters 2 and 3. Therefore, it suffices to consider stable solutions to $-Lu = f(u)$ in the unit ball B_1. A constant depending only on n, c_0, and C_0 will be called *universal*, a terminology that we use throughout the work.

The following is our main result of this chapter, which provides interior a priori estimates for stable solutions: a Hölder bound when $n \le 9$, and a $W^{1,2+\gamma}$ estimate in every dimension. The only requirement for the nonlinearity is $f \ge 0$, as in [24]. An important accomplishment in our estimates is that they involve the norms $\|\nabla a_{ij}\|_{L^\infty}$ and $\|b_i\|_{L^\infty}$, while a first approach to the problem leads to computations including second derivatives of a_{ij} and first derivatives of b_i. On the other hand, our bounds are independent of f. Here and throughout the work, when we write $C = C(\dots)$ we mean that the constant C depends only on the quantities inside the parentheses.

Theorem 1.1.1. *Let $u \in C^\infty(\overline{B}_1)$ be a stable solution of $-Lu = f(u)$ in $B_1 \subset \mathbb{R}^n$, for some nonnegative function $f \in C^1(\mathbb{R})$.*

Then

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/2})} \le C\|u\|_{L^1(B_1)}, \tag{1.1.9}$$

where $\gamma = \gamma(n) > 0$ and $C = C(n, c_0, C_0, \|\nabla a_{ij}\|_{L^\infty(B_1)}, \|b_i\|_{L^\infty(B_1)})$. In addition,

$$\|u\|_{C^\alpha(\overline{B}_{1/2})} \le C\|u\|_{L^1(B_1)} \quad \text{if } n \le 9, \tag{1.1.10}$$

where $\alpha = \alpha(n, c_0, C_0) > 0$ and $C = C(n, c_0, C_0, \|\nabla a_{ij}\|_{L^\infty(B_1)}, \|b_i\|_{L^\infty(B_1)})$.

For applications, it may be useful to point out that the result only needs a_{ij} to be Lipschitz and b_i to be bounded. Our direct computations within the proofs assume $a_{ij} \in C^1$ and $b_i \in C^1$ in order to evaluate certain identities pointwise. However, we only need these to be meaningful in a weaker sense; see Remarks 1.3.1 and 1.5.2. Similarly, we only need u to be C^2 and to have weak third derivatives. These last conditions seem to require more regularity of the drift b; see Remark 1.3.1. When we treat the boundary regularity in Chapters 2 and 3, we will make weaker, more precise, regularity assumptions on the data.

The proof of Theorem 1.1.1 will rely on our second main result, Theorem 1.1.2 below, and its consequences. It consists of two types of Hessian estimates. The first one, (1.1.11), is an extension of the geometric stability condition due to Sternberg and Zumbrun [91] to operators with variable coefficients. The second one, (1.1.13)-(1.1.14), controls the L^1 norm of the "Hessian times the gradient", $|D^2u||\nabla u|$, in balls and annuli by the L^2 norm squared of the gradient whenever the lower order coefficients are small.

Theorem 1.1.2. *Let $u \in C^\infty(\overline{B}_1)$ be a stable solution of $-Lu = f(u)$ in B_1, for some function $f \in C^1(\mathbb{R})$. Assume that*

$$\|DA\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)} \le \varepsilon$$

for some $\varepsilon > 0$.

Then

$$\begin{aligned}\int_{B_1} \mathcal{A}^2\eta^2 \,\mathrm{d}x \le \int_{B_1} |\nabla u|^2_{A(0)} \left(|\nabla\eta|^2_{A(x)} + C\varepsilon|\nabla(\eta^2)| + C\varepsilon^2\eta^2\right) \mathrm{d}x \\ + C\varepsilon \int_{B_1} |D^2u||\nabla u|\eta^2 \,\mathrm{d}x\end{aligned} \tag{1.1.11}$$

for all $\eta \in C_c^\infty(B_1)$, where C is a universal constant and

$$\mathcal{A} := \begin{cases} \left(\operatorname{tr}(A(x)D^2uA(0)D^2u) - |\nabla u|^{-2}_{A(0)}|D^2uA(0)\nabla u|^2_{A(x)}\right)^{1/2} & \text{if } \nabla u \ne 0 \\ 0 & \text{if } \nabla u = 0. \end{cases} \tag{1.1.12}$$

Assume in addition that $f \ge 0$. If $\varepsilon \le \varepsilon_0$, then

$$\|\, |\nabla u|\, D^2u \,\|_{L^1(B_{3/4})} \le C\|\nabla u\|^2_{L^2(B_1)} \tag{1.1.13}$$

and

$$\|\, |\nabla u|\, D^2u \,\|_{L^1(B_{1/2}\setminus B_{1/4})} \le C\|\nabla u\|^2_{L^2(B_1\setminus B_{1/8})}, \tag{1.1.14}$$

where $\varepsilon_0 > 0$ and C are universal constants.

Our first inequality (1.1.11) generalizes the Sternberg-Zumbrun estimate for the Laplacian, which corresponds to the case $\varepsilon = 0$ and $A(0) = I$. The peculiar form of the function $\mathcal{A}$ in (1.1.12) (the coefficients of A are evaluated both at x and 0) will guarantee that the direct computations within our proofs give dependence on the norm $\|a_{ij}\|_{C^1}$, instead of $\|a_{ij}\|_{C^2}$ for other choices of $\mathcal{A}$. In this direction, it is worth noting that the classical Sternberg-Zumbrun result and the function $\mathcal{A}$ have a Riemannian analogue (found by Farina, Sire, and Valdinoci [56]) which can be related to our Euclidean setting with variable coefficients. The estimate from the Riemannian framework leads to bounds depending on $\|a_{ij}\|_{C^2}$. We elaborate on these topics further in Remarks 1.3.2 and 1.3.3.

The "Hessian times the gradient" estimates (1.1.13)-(1.1.14) rely on the inequality (1.1.11) with sufficiently small errors ε, and will require the assumption $f \ge 0$. While the bound on annuli (1.1.14) can be deduced from the one in balls (1.1.13) by a standard scaling and covering argument, we include it in the statement since it will be crucial in the proof of the Hölder estimate in Theorem 1.1.1.

1.1.3 Structure of the proof

By a scaling and covering argument, it suffices to obtain the a priori estimates from Theorem 1.1.1 in small balls. There, the problem can be written as an equation in the unit ball involving an operator L that is close to the Laplacian, i.e., whose coefficients satisfy $A(0) = I$ and $\|\nabla a_{ij}\|_{L^\infty(B_1)} + \|b_i\|_{L^\infty(B_1)} \le \varepsilon$, with ε small. We explain this in more detail in Section 1.2 below.

The key estimates leading to Theorem 1.1.1 are contained in Propositions 1.1.3, 1.1.4, and 1.1.5 below. Our proofs are all quantitative as in the paper [20] and avoid the compactness argument from the previous work [24]. The proofs of the first two propositions use the Hessian estimates of Theorem 1.1.2 above. In particular, this forces us to prove the Sternberg-Zumbrun inequality before the crucial weighted L^2 estimate for the radial derivative (Proposition 1.1.4). It is worth noting that, for the Laplacian, these two results are independent from each other (and hence can be obtained in any order, as in the works [24] and [20]), while this is no longer the case for operators with variable coefficients.

In the first proposition, we control the L^2 norm of the gradient by the L^1 norm of the solution under a smallness condition on the coefficients, namely, when the error ε is sufficiently small. This is a direct consequence of Theorem 1.1.2 and the interpolation inequalities of Cabré in [20]. We recall these inequalities in Appendix C.

Proposition 1.1.3. *Let $u \in C^\infty\big(\overline{B}_1\big)$ be a stable solution of $-Lu = f(u)$ in B_1, for some nonnegative function $f \in C^1(\mathbb{R})$. Assume that*

$$\|DA\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)} \le \varepsilon$$

for some $\varepsilon > 0$.

If $\varepsilon \le \varepsilon_0$, then

$$\|\nabla u\|_{L^2(B_{1/2})} \le C\|u\|_{L^1(B_1)}, \tag{1.1.15}$$

where $\varepsilon_0 > 0$ and C are universal constants.

The second proposition is a weighted L^2 estimate for the radial derivative in a ball by the full gradient in an annulus. It is here that we need $n \le 9$. Again, we will assume that the coefficient error ε is small and that the nonlinearity is nonnegative $f \ge 0$. Here and throughout the paper we use the notation

$$r = |x|, \quad u_r = \frac{x}{|x|} \cdot \nabla u$$

for the radial derivative.

Proposition 1.1.4. *Let $u \in C^\infty\big(\overline{B}_1\big)$ be a stable solution of $-Lu = f(u)$ in B_1, for some nonnegative function $f \in C^1(\mathbb{R})$. Assume that*

$$\|DA\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)} \le \varepsilon$$

for some $\varepsilon > 0$.

If $3 \le n \le 9$ and $\varepsilon \le \varepsilon_0$, then

$$\int_{B_\rho} r^{2-n} u_r^2 \,\mathrm{d}x \le C \int_{B_{2\rho}\setminus B_\rho} r^{2-n}|\nabla u|^2 \,\mathrm{d}x + C\varepsilon \int_{B_{4\rho}} r^{3-n}|\nabla u|^2 \,\mathrm{d}x \tag{1.1.16}$$

for all $\rho \le 1/4$, where $\varepsilon_0 > 0$ and C are universal constants.

Notice that this result requires $n \ge 3$. However, adding superfluous variables to the solution, we can also use it when $n \le 2$.

Our inequality (1.1.16) in Proposition 1.1.4 is an analogue of Lemma 2.1 in [24], where the authors obtain a similar bound for the Laplacian ($\varepsilon = 0$ and $A(0) = I$) without the nonnegativity assumption on f. Recall that this assumption is needed in the Hessian estimates (1.1.13)-(1.1.14) in Theorem 1.1.2 above, which will allow us to treat a weighted $|D^2u||\nabla u|$ error term which does not appear for the Laplacian. We will be able to control this error by writing it as an infinite sum on dyadic annuli, pulling the weight out of the integral in each annulus, and applying the bound (1.1.14).

Finally in the third proposition we show that, under the assumption that $A(0) = I$, (generalized) superharmonic functions are controlled by the radial derivative plus an error involving the full gradient in L^1. This is an extension of Lemma 4.1 in Cabré [20] to operators with variable coefficients.

Proposition 1.1.5. *Let $u \in C^\infty(\overline{B}_1)$ be superharmonic in the sense that $Lu \le 0$ in B_1. Assume that*

$$A(0) = I \quad \text{and} \quad \|DA\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)} \le \varepsilon$$

for some $\varepsilon > 0$.

Then there exists a constant t, which depends on u, such that

$$\|u - t\|_{L^1(B_1 \setminus B_{1/8})} \le C\|u_r\|_{L^1(B_1 \setminus B_{1/8})} + C\varepsilon\|\nabla u\|_{L^1(B_1)},$$

where C is a constant depending only on n and c_0.

Our proof of Proposition 1.1.5 is by comparison with harmonic functions. The required L^1 estimates for harmonic functions will follow by duality from the L^∞ bounds of a Neumann problem. Such L^∞ estimates are the most technical part of the argument, where we use a Moser iteration in the spirit of Winkert [96] to deduce the uniform bounds.

The $W^{1,2+\gamma}$ result (1.1.9) in Theorem 1.1.1 will follow from the Hessian estimate on balls (1.1.13) in Theorem 1.1.2 together with Proposition 1.1.3. To show it, we first control the L^2 norm of the gradient uniformly on level sets by the Dirichlet integral in a ball, and hence by the L^1 norm of the solution. A device from [24] will then allow us to deduce the higher integrability.

To prove the Hölder estimate (1.1.10) in Theorem 1.1.1, we will show that the scale-invariant weighted integral $\int_{B_\rho} r^{2-n}|\nabla u|^2$ decays algebraically. In the previous works [24] and [20], the authors proved the decay of the weighted radial derivative instead. They could later deduce the C^α estimate by either averaging or applying a version of Morrey's embedding for radial derivatives. Here we will obtain the decay of the full gradient directly for the first time. For this, combining Propositions 1.1.3 and 1.1.5, we are able to bound the full gradient by the radial derivative on annuli in L^2. This, together with the dyadic decomposition explained above, allows us to control the weighted integral of the gradient by that of the radial derivative (up to gradient errors). Now, Proposition 1.1.4 will yield a control of the weighted integral of the gradient in the ball by the same quantity on an annulus. A standard iteration then leads to the decay.

Our integral stability inequality (1.1.8) will be crucial in the proofs of both Theorem 1.1.2 and Proposition 1.1.4. These will follow from (1.1.8), with $\Omega = B_1$, by choosing appropriate test functions in terms of the Jacobi operator, as we explain next. Taking a test function of the form $\xi = \mathbf{c}\eta$, where $\mathbf{c}$ and η are smooth and $\operatorname{supp}\eta \subset B_1$, the integrand on the right-hand side of (1.1.8) becomes

$$\begin{aligned} \left|\nabla\xi - \tfrac{1}{2}\xi A^{-1}(x)\widehat{b}(x)\right|^2_{A(x)} &= \left|\eta\nabla\mathbf{c} + \mathbf{c}\nabla\eta - \tfrac{1}{2}\mathbf{c}\eta A^{-1}(x)\widehat{b}(x)\right|^2_{A(x)} \\ &= |\eta\nabla\mathbf{c}|^2_{A(x)} + 2A(x)\eta\nabla\mathbf{c}\cdot\mathbf{c}\nabla\eta - \eta^2\mathbf{c}\,\widehat{b}(x)\cdot\nabla\mathbf{c} + \mathbf{c}^2\left|\nabla\eta - \tfrac{1}{2}\eta A^{-1}(x)\widehat{b}(x)\right|^2_{A(x)}. \end{aligned} \tag{1.1.17}$$

Integrating in B_1, the first term $\int_{B_1}|\eta\nabla\mathbf{c}|^2_{A(x)}\,\mathrm{d}x$ in (1.1.17) can be integrated by parts as

$$\int_{B_1}|\eta\nabla\mathbf{c}|^2_{A(x)}\,\mathrm{d}x = \int_{B_1}\Big(-\operatorname{div}(A(x)\nabla\mathbf{c})\ \mathbf{c}\eta^2 - 2A(x)\eta\nabla\mathbf{c}\cdot\mathbf{c}\nabla\eta\Big)\,\mathrm{d}x$$

and hence, by (1.1.17), (1.1.8), and rearranging terms, it follows that

$$\int_{B_1} (J_u \mathbf{c})\, \mathbf{c}\eta^2 \, \mathrm{d}x \le \int_{B_1} \mathbf{c}^2 \left|\nabla\eta - \tfrac{1}{2}\eta A^{-1}(x)\widehat{b}(x)\right|^2_{A(x)} \mathrm{d}x. \tag{1.1.18}$$

The key idea now is to choose the $\mathbf{c}$ function in a way such that $J_u\mathbf{c}$ becomes independent of the nonlinearity. This will yield universal a priori estimates for stable solutions. Since u solves the equation $-Lu = f(u)$, taking a derivative, we have that $f'(u)\nabla u = -\nabla Lu$ and hence $J_u\nabla u = L\nabla u - \nabla Lu$ no longer involves f. This computation suggests that we choose $\mathbf{c}$ as a function of the gradient of u. Thus, to prove the estimate for $\mathcal{A}$ in Theorem 1.1.2, we will make the choice

$$\mathbf{c}(x) = |\nabla u(x)|_{A(0)} = \big(a_{ij}(0)u_iu_j\big)^{1/2}.$$

On the other hand, the weighted L^2 bound in Proposition 1.1.4 will also require to choose the auxiliary function η above carefully. The test functions leading to this estimate are

$$\mathbf{c}(x) = x\cdot\nabla u = ru_r \quad \text{and} \quad \eta(x) = |x|^{\frac{2-n}{2}}_{A^{-1}(0)}\zeta,$$

where $\zeta \in C^\infty_c(B_1)$ is a cut-off.

We note that our test functions above are the ones used in the paper [24] under the linear transformation $x \mapsto A^{1/2}(0)x$, where $A^{1/2}(0)$ is the positive square root of the matrix $A(0)$. These seem to be the simplest functions leading to a priori estimates in the variable coefficients framework. Moreover, thanks to the particular form of these functions, all direct computations within our proofs will only involve first derivatives of the coefficients A and b, while other choices of functions require two derivatives of A. A suitable integration by parts will yield bounds in terms of the norms $\|A\|_{C^1}$ and $\|b\|_{C^0}$, as in the results mentioned above.

1.1.4 Outline of the chapter

In Section 1.2 we briefly comment on the invariance of stability under affine transformations. Section 1.3 is devoted to proving Theorem 1.1.2 and Proposition 1.1.3. In Section 1.4 we prove the $W^{1,2+\gamma}$ bound (1.1.9) from Theorem 1.1.1. Section 1.5 contains the proof of Proposition 1.1.4. In Section 1.6 we prove Proposition 1.1.5. Finally, in Section 1.7 we prove the Hölder bound (1.1.10) in Theorem 1.1.1.

1.2 Preliminaries: Invariance under affine transformations

To prove Theorem 1.1.1, we will analyze the semilinear equation $-Lu = f(u)$ in small balls. Since the class of stable solutions is invariant under affine transformations, the question reduces to studying an equation in the unit ball involving an operator that is close to the Laplacian. After proving the necessary estimates in this setting, the theorem will follow from a scaling and covering argument. It is worth mentioning that the nonnegativity of the nonlinearity, which is required in our main results, is preserved under these transformations.

We now explain this invariance with more detail in different particular situations. First we study the equation under translations and scalings. These simple yet important transformations will be used several times throughout the paper. Secondly, we consider the equation under general linear transformations. These allow us to reduce ourselves to the case where the coefficient matrix is the identity at the origin. Notice that this is only required in Proposition 1.1.5, but will be crucial in the proof of the C^α bound (1.1.10) in Theorem 1.1.1 given in Section 1.7.

As mentioned in the Introduction, the bounds in our a priori estimates depend only on the ellipticity constants c_0 and C_0 and on the quantity

$$\|DA\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)} \leq \varepsilon \tag{1.2.1}$$

involving the coefficients. As we will see now, the two norms in (1.2.1) have the same scaling. It is therefore natural to state our results in terms of this quantity.

(i) **Translation and scale invariance**. If u is a stable solution of $-Lu = f(u)$ in a ball $B_\rho(y)$, then the function $u^{y,\rho} := u(y + \rho\,\cdot)$ is a solution of $-L^{y,\rho}u^{y,\rho} = \rho^2 f(u^{y,\rho})$ in B_1, where $L^{y,\rho}$ is the linear operator

$$L^{y,\rho}v = \operatorname{tr}\left(A^{y,\rho}(x)D^2 v\right) + b^{y,\rho}(x)\cdot\nabla v$$

with coefficients

$$A^{y,\rho}(x) = A(y+\rho\, x) \quad \text{ and } \quad b^{y,\rho}(x) = \rho\, b(y+\rho\, x).$$

The stability condition (1.1.3) in $B_\rho(y)$ becomes $-L^{y,\rho}\varphi^{y,\rho} \leq \rho^2 f'(u^{y,\rho})\varphi^{y,\rho}$ in B_1, where $\varphi^{y,\rho} = \varphi(y+\rho\,\cdot)$ satisfies the assumptions in (1.1.3), and hence $u^{y,\rho}$ is stable.

Since the coefficients satisfy the bounds

$$\|DA^{y,\rho}\|_{L^\infty(B_1)} \leq \rho\|DA\|_{L^\infty(B_\rho(y))} \quad \text{ and } \quad \|b^{y,\rho}\|_{L^\infty(B_1)} \leq \rho\|b\|_{L^\infty(B_\rho(y))},$$

whenever $B_\rho(y) \subset B_R$ for some $R > 0$ and L is defined in this larger ball, we have

$$\|DA^{y,\rho}\|_{L^\infty(B_1)} + \|b^{y,\rho}\|_{L^\infty(B_1)} \leq \rho\left(\|DA\|_{L^\infty(B_R)} + \|b\|_{L^\infty(B_R)}\right), \tag{1.2.2}$$

which can be made small for ρ small. In particular, assuming that L is close to the Laplacian as in the statements of our propositions and by (1.2.2), we deduce the following property: if $\|DA\|_{L^\infty(B_R)} + \|b\|_{L^\infty(B_R)} \leq \varepsilon$, then

$$\|DA^{y,\rho}\|_{L^\infty(B_1)} + \|b^{y,\rho}\|_{L^\infty(B_1)} \leq \rho\varepsilon \quad \text{ for all } B_\rho(y) \subset B_R.$$

This elementary observation will be used throughout the paper.

(ii) **Invariance under linear transformations**. Given a symmetric positive definite matrix $M \in \mathbb{R}^{n\times n}$, if u is a stable solution of $-Lu = f(u)$ in the unit ball B_1, then the function $u^M := u(M\,\cdot)$ is a solution of $-L^M u^M = f(u^M)$ in $M^{-1}(B_1)$, where L^M is the operator

$$L^M v = \operatorname{tr}\left(A^M(x)D^2 v\right) + b^M(x)\cdot\nabla v,$$

with coefficients

$$A^M(x) = M^{-1}A(Mx)M^{-1} \quad \text{ and } \quad b^M(x) = M^{-1}b(Mx).$$

As above, the stability condition (1.1.3) in B_1 becomes $-L^M\varphi^M \leq f'(u^M)\varphi^M$ in $M^{-1}(B_1)$, where $\varphi^M = \varphi(M\cdot)$, and hence u^M is a stable solution.

If M satisfies $\sqrt{c_0} \le M \le \sqrt{C_0}$, then the new coefficients A^M are uniformly elliptic with $c_0/C_0 \le A^M(x) \le C_0/c_0$. Moreover, we have the bounds

$$|DA^M(x)| \le \frac{\sqrt{C_0}}{c_0}|DA(Mx)| \quad \text{and} \quad |b^M(x)| \le \frac{1}{\sqrt{c_0}}|b(Mx)|,$$

and taking the supremum in $x \in M^{-1}(B_1)$, using that $B_{1/\sqrt{C_0}} \subset M^{-1}(B_1)$, we deduce

$$\|DA^M\|_{L^\infty(B_{1/\sqrt{C_0}})} + \|b^M\|_{L^\infty(B_{1/\sqrt{C_0}})} \le \frac{\sqrt{C_0}}{c_0}\left(\|DA\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)}\right). \tag{1.2.3}$$

In particular, taking $M = A^{1/2}(0)$ as the unique positive square root of $A(0)$, we see that $u^{A^{1/2}(0)}$ solves an elliptic equation in the ball $B_{1/\sqrt{C_0}}$ with coefficients $(A^{A^{1/2}(0)})(x)$ satisfying $(A^{A^{1/2}(0)})(0) = I$, i.e., equal to the identity at the origin. By the monotonicity of the principal eigenvalue with respect to the domain, it follows that $u^{A^{1/2}(0)}$ is also a stable solution in this ball.

It is now easy to combine these transformations with the ones given in the first part (i). For each ball $B_\rho(y) \subset B_1$, the function $\widetilde{u} = u\big(y + \frac{\rho}{\sqrt{C_0}}A^{1/2}(y)\,\cdot\big)$ is a stable solution of an elliptic equation $-\widetilde{L}\widetilde{u} = \widetilde{f}(\widetilde{u})$ in B_1. Here, $\widetilde{f}$ is the nonlinearity $\widetilde{f} = \frac{\rho^2}{C_0}f$, while $\widetilde{L}$ is an operator of the form (1.1.2) with coefficients

$$\widetilde{A}(x) = A^{-1/2}(y)A\big(y + \tfrac{\rho}{\sqrt{C_0}}A^{1/2}(y)x\big)A^{-1/2}(y)$$

and

$$\widetilde{b}(x) = \tfrac{\rho}{\sqrt{C_0}}A^{-1/2}(y)b\big(y + \tfrac{\rho}{\sqrt{C_0}}A^{1/2}(y)x\big).$$

Notice that the matrix $\widetilde{A}(x)$ is uniformly elliptic with $c_0/C_0 \le \widetilde{A}(x) \le C_0/c_0$ and is equal to the identity at the origin. Furthermore, combining (1.2.3) and (1.2.2), the coefficients can be bounded by

$$\|D\widetilde{A}\|_{L^\infty(B_1)} + \|\widetilde{b}\|_{L^\infty(B_1)} \le \frac{\rho}{c_0}\left(\|DA\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)}\right).$$

As mentioned above, this observation will be important in the proof of the Hölder estimate 1.1.10 in Section 1.7 below.

1.3 Hessian and $W^{1,2}$ estimates

The goal of this section is to prove Theorem 1.1.2 and the energy estimate Proposition 1.1.3.

Recall the function $\mathcal{A}\colon \overline{B}_1 \to \mathbb{R}$ introduced in (1.1.12) in the statement of Theorem 1.1.2. This function can also be written as

$$\mathcal{A} = \begin{cases} \left(\|A^{1/2}(x)D^2uA^{1/2}(0)\|_{\mathrm{HS}}^2 - |A^{1/2}(x)D^2uA^{1/2}(0)\mathbf{n}(x)|^2\right)^{1/2} & \text{if } \nabla u \ne 0 \\ 0 & \text{if } \nabla u = 0, \end{cases} \tag{1.3.1}$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Euclidean Hilbert-Schmidt norm for matrices[1] and $\mathbf{n}(x)$ is the unit vector field $\mathbf{n}\colon \overline{B}_1 \cap \{\nabla u \ne 0\} \to \mathbb{R}$ given by

$$\mathbf{n}(x) := |\nabla u|_{A(0)}^{-1}A^{1/2}(0)\nabla u(x). \tag{1.3.2}$$

[1]Recall that, for a matrix $M \in \mathbb{R}^{n\times n}$, this norm squared is $\|M\|_{\mathrm{HS}}^2 = \mathrm{tr}(M^TM) = \sum_{i,j=1}^n M_{ij}^2$.

The equivalence between the expressions (1.1.12) and (1.3.1) follows from the identities

$$\|A^{1/2}(x)D^2uA^{1/2}(0)\|^2_{\mathrm{HS}} = \operatorname{tr}\left(A(x)D^2uA(0)D^2u\right)$$

and

$$|A^{1/2}(x)D^2uA^{1/2}(0)\mathbf{n}(x)|^2 = |\nabla u|^{-2}_{A(0)}|D^2uA(0)\nabla u|^2_{A(x)},$$

which are easy to check.

We start by proving the bound (1.1.11) in Theorem 1.1.2, which is a generalization of the geometric stability inequality due to Sternberg and Zumbrun [91] for stable solutions to $-\Delta u = f(u)$. For this, we will test the integral stability inequality (1.1.18) with the function

$$\mathbf{c}(x) = |\nabla u|_{A(0)}$$

and a cut-off η. The proof of the remaining estimates in Theorem 1.1.2 will rely on this preliminary inequality.

Two comments are in order. First, with this choice of $\mathbf{c}$, our result requires an appropriate integration by parts to allow dependence of the bounds on only $\|b\|_{C^0}$. Secondly, after the proof, in Remarks 1.3.2 and 1.3.3 we will comment on alternative choices of $\mathbf{c}$ and of the function $\mathcal{A}$.

We originally took $|\nabla u|_{A(x)}$ as our $\mathbf{c}$ function, a choice that required the regularity $A \in C^2$ and $b \in C^1$ when computing $J_u\mathbf{c}$ in the stability inequality (1.1.18). With that choice, a further integration by parts was needed to obtain bounds depending only on $\|A\|_{C^1}$. Instead, our function $|\nabla u|_{A(0)}$ only needs $A \in C^1$ and $b \in C^1$. Moreover, the proof with our choice is easier and we only need an integration by parts to get rid of the first derivatives of b. Note that the function $|\nabla u|_{A(x)}$ is motivated by geometric considerations and had already appeared in the Riemannian analogue of the Sternberg-Zumbrun estimates, as explained in Remark 1.3.3.

Proof of (1.1.11) *in Theorem 1.1.2.* Since $|\nabla u|_{A(0)}$ is not necessarily smooth when $\nabla u = 0$, we consider the smooth function

$$\mathbf{c}_\delta := \sqrt{|\nabla u|^2_{A(0)} + \delta^2}$$

instead. We will apply the integral stability inequality (1.1.18) with $\mathbf{c} = \mathbf{c}_\delta$. In the end we will let $\delta \to 0$, which will yield the claim. Throughout this proof, the letter C denotes a generic universal constant.

By the stability inequality (1.1.18), we have the upper bound

$$\begin{aligned}\int_{B_1} \mathbf{c}_\delta\, J_u\mathbf{c}_\delta\, \eta^2 \,\mathrm{d}x &\le \int_{B_1} \left(|\nabla u|^2_{A(0)} + \delta^2\right) \left|\nabla\eta - \tfrac{1}{2}\eta A^{-1}(x)\widehat{b}(x)\right|^2_{A(x)} \mathrm{d}x \\ &\le \int_{B_1} \left(|\nabla u|^2_{A(0)} + \delta^2\right) \left(|\nabla\eta|^2_{A(x)} + C\varepsilon|\nabla(\eta^2)| + C\varepsilon^2\eta^2\right) \mathrm{d}x,\end{aligned} \tag{1.3.3}$$

where in the last line we have expanded the quadratic expression and applied the bounds of the coefficients.

We will bound the expression $\mathbf{c}_\delta\, J_u\mathbf{c}_\delta = \mathbf{c}_\delta\, L\mathbf{c}_\delta + f'(u)\mathbf{c}_\delta^2$ from below. Since

$$\partial_i\mathbf{c}_\delta = \mathbf{c}_\delta^{-1}\, u_{ik}a_{kl}(0)u_l$$

and

$$\partial^2_{ij}\mathbf{c}_\delta = \mathbf{c}_\delta^{-1}\, u_{ijk}a_{kl}(0)u_l + \mathbf{c}_\delta^{-1}u_{ik}a_{kl}(0)u_{jl} - \mathbf{c}_\delta^{-3}\, u_{ik}a_{kl}(0)u_l\, u_{jp}a_{pq}(0)u_q,$$

we deduce

$$\begin{aligned}\mathbf{c}_\delta\, a_{ij}(x)\partial^2_{ij}\mathbf{c}_\delta &= a_{kl}(0)u_l\, a_{ij}(x)u_{ijk}\\ &\qquad + \|A^{1/2}(x)D^2uA^{1/2}(0)\|^2_{\mathrm{HS}} - \tfrac{|\nabla u|^2_{A(0)}}{|\nabla u|^2_{A(0)}+\delta^2}|A^{1/2}(x)D^2uA^{1/2}(0)\mathbf{n}|^2 \\ &\geq a_{kl}(0)u_l\, a_{ij}(x)u_{ijk} + \mathcal{A}^2\end{aligned} \tag{1.3.4}$$

and

$$\mathbf{c}_\delta\, b_i(x)\partial_i\mathbf{c}_\delta = a_{kl}(0)u_l\, b_i(x)u_{ik}. \tag{1.3.5}$$

Adding (1.3.4) and (1.3.5), we obtain

$$\mathbf{c}_\delta\, J_u\mathbf{c}_\delta \geq a_{kl}(0)u_l\, Lu_k + \mathcal{A}^2 + f'(u)\mathbf{c}_\delta^2. \tag{1.3.6}$$

Differentiating the equation $-Lu = f(u)$ in the direction of $A(0)\nabla u$, we have

$$A(0)\nabla(Lu)\cdot\nabla u = -f'(u)|\nabla u|^2_{A(0)}. \tag{1.3.7}$$

The first term on the right-hand side of (1.3.6) can be written in terms of this derivative as

$$a_{kl}(0)u_l\, Lu_k = A(0)\nabla(Lu)\cdot\nabla u - a_{kl}(0)u_l\partial_k a_{ij}(x)u_{ij} - a_{kl}(0)u_l\partial_k b_i(x)u_i,$$

hence, by (1.3.7) and the coefficient estimates, we can bound this expression from below as

$$a_{kl}(0)u_l\, Lu_k \geq -f'(u)|\nabla u|^2_{A(0)} - C\varepsilon|D^2u||\nabla u| - a_{kl}(0)u_l\partial_k b_i(x)u_i. \tag{1.3.8}$$

Applying (1.3.8) in (1.3.6), since $\mathbf{c}_\delta^2 - |\nabla u|^2_{A(0)} = \delta^2$, we obtain

$$\mathbf{c}_\delta\, J_u\mathbf{c}_\delta \geq \mathcal{A}^2 + \delta^2 f'(u) - C\varepsilon|D^2u||\nabla u| - a_{kl}(0)u_l\partial_k b_i(x)u_i. \tag{1.3.9}$$

Multiplying (1.3.9) by η^2 and integrating, the last term $-\int_{B_1} a_{kl}(0)u_l\partial_k b_i(x)u_i\eta^2\,\mathrm{d}x$ can be integrated by parts as

$$\begin{aligned}\left|-\int_{B_1} a_{kl}(0)u_l\partial_k b_i(x)u_i\,\eta^2\,\mathrm{d}x\right| &= \left|\int_{B_1} b_i(x)\partial_k(a_{kl}(0)u_lu_i\,\eta^2)\,\mathrm{d}x\right|\\ &\leq C\varepsilon\int_{B_1}|D^2u||\nabla u|\eta^2\,\mathrm{d}x + C\varepsilon\int_{B_1}|\nabla u|^2|\nabla(\eta^2)|\,\mathrm{d}x.\end{aligned} \tag{1.3.10}$$

Combining (1.3.9), (1.3.10), and (1.3.3), rearranging terms, we obtain

$$\begin{aligned}&\int_{B_1}\left(\mathcal{A}^2 + \delta f'(u)\right)\eta^2\,\mathrm{d}x\\ &\quad\leq \int_{B_1}\left(|\nabla u|^2_{A(0)}+\delta^2\right)\left(|\nabla\eta|^2_{A(x)} + C\varepsilon|\nabla(\eta^2)| + C\varepsilon^2\eta^2\right)\mathrm{d}x\\ &\qquad\qquad + C\varepsilon\int_{B_1}|D^2u||\nabla u|\eta^2\,\mathrm{d}x,\end{aligned}$$

and letting $\delta\to 0$ yields the claim. □

Several remarks are in order:

Remark 1.3.1. In (1.3.7) above we took a derivative of the equation in the direction $A(0)\nabla u$ to get rid of the dependence on the nonlinearity. Instead, we could have multiplied the equation by the test function $\xi = \operatorname{div}(A(0)\nabla u\, \eta^2)$ and integrated by parts. Notice that this avoids having to take any derivatives of b, since the term involving it can be bounded directly.

In the argument above, we need u to have three (weak) derivatives, otherwise we cannot compute $L\mathbf{c}$ (or rather $L\mathbf{c}_\delta$). In [24], the authors only need to assume $u \in C^2(B_1)$ to deduce the analogue estimate for the Laplacian, since this already gives $u \in W^{3,p}_{\mathrm{loc}}$ for all $p < \infty$. Indeed, differentiating the equation, $-\Delta u_i = f'(u)u_i \in L^p_{\mathrm{loc}}$ and by L^p estimates they deduce $u_i \in W^{2,p}_{\mathrm{loc}}$, hence $u \in W^{3,p}_{\mathrm{loc}}$ for all $p < \infty$. This fact allows them to carry out a similar argument to the one explained above.[2]

For an operator with variable coefficients L, the regularity of the solution depends on that of the coefficients. Assuming $a_{ij} \in C^{0,1}(B_1)$, $b_i \in L^\infty(B_1)$, and u bounded, applying L^p estimates to the equation $-Lu = f(u) \in L^p_{\mathrm{loc}}$, we deduce $u \in W^{2,p}_{\mathrm{loc}}$ for all $p < \infty$ (and hence in $C^{1,\alpha}$ for all $0 < \alpha < 1$). Now, for u to be in $C^2 \cap W^{3,p}_{\mathrm{loc}}$ we need more regularity on the drift b. To see this, taking a derivative of the equation we have $-Lu_k = \partial_k a_{ij}(x)u_{ij} + \partial_k b_i(x)u_i + f'(u)u_k$ and the right hand side is in L^p for $\partial_k a_{ij} \in L^\infty$ (i.e., a_{ij} Lipschitz) and $\partial_k b_i \in L^p$. In particular, if $b \in W^{1,p}(B_1)$ with $p > n$, we deduce $u \in C^2(B_1) \cap W^{3,p}_{\mathrm{loc}}(B_1)$. This is somewhat surprising, since our estimates do not involve any derivatives of b.

Remark 1.3.2. The following comments concern the form of the function $\mathcal{A}$ in our a priori estimate (1.1.11) in Theorem 1.1.2. Recall that $\mathcal{A}$ quantifies a part of the "mixed", non-symmetric matrix $A^{1/2}(x)D^2uA^{1/2}(0)$, which includes both the variable coefficients $A^{1/2}(x)$ and the constants $A^{1/2}(0)$. We are led naturally to it from the choice of test function $\mathbf{c} = |\nabla u|_{A(0)}$ in the integral stability inequality, which is the function used by Cabré, Figalli, Ros-Oton, and Serra in [24] after a linear transformation.

We could have also given an estimate for the function

$$\mathcal{A}_x := \left(\|A^{1/2}(x)D^2uA^{1/2}(x)\|^2_{\mathrm{HS}} - \left|A^{1/2}(x)D^2uA^{1/2}(x)\mathbf{n}_A(x)\right|^2 \right)^{1/2} \tag{1.3.11}$$

involving the symmetric matrix $A^{1/2}(x)D^2uA^{1/2}(x)$, where $\mathbf{n}_A$ is the vector field $\mathbf{n}_A(x) := |\nabla u|^{-1}_{A(x)} A^{1/2}(x)\nabla u$, by choosing the test function $\mathbf{c} = |\nabla u|_{A(x)}$ instead. However, in this case, the proof of the analogue of Theorem 1.1.2 is more involved. This is why we prefer our choice of $\mathcal{A}$. On the other hand, the choice $\mathbf{c} = |\nabla u|_{A(x)}$ is related to an existing Riemannian version of the Sternberg and Zumbrun inequality, which we explain next in Remark 1.3.3.

This discussion leads to the question of whether a similar estimate exists for $\mathcal{A}_0$, the natural part of the simpler symmetric matrix $A^{1/2}(0)D^2uA^{1/2}(0)$, which only involves the constant coefficients $A^{1/2}(0)$. There does not seem to be a direct way to derive such an estimate from the stability inequality, since it is not clear which $\mathbf{c}$ function could lead to it. Nevertheless, when the parameter ε is small, thanks to (1.3.15) below, it can be shown that $\mathcal{A}_0$ is comparable to $\mathcal{A}$. Hence, for ε small, we can deduce the desired bound for $\mathcal{A}_0$ from our result (1.1.11) for $\mathcal{A}$. We will need this fact in the proof of the Hessian estimates in Theorem 1.1.2, as explained below.

[2] In fact, they are able to deduce the estimate without computing $L\mathbf{c}$ directly, but they still need to have three derivatives of the solution; see the proof of Lemma 2.1 in [24].

Remark 1.3.3. Our result (1.1.11) is related to a Riemannian analogue of the Sternberg and Zumbrun estimate found by Farina, Sire, and Valdinoci in [56]. It states that stable solutions to the equation $-\Delta_{\mathrm{LB}} u = f(u)$ in a Riemannian manifold (M, g), where Δ_{LB} is the Laplace-Beltrami operator, satisfy the inequality

$$\int_M \mathcal{A}_{\mathrm{Riem}}^2 \eta^2 + \int_M \mathrm{Ric}(\nabla_{\mathrm{g}} \mathrm{u}, \nabla_{\mathrm{g}} \mathrm{u}) \, \eta^2 \leq \int_{\mathrm{M}} |\nabla_{\mathrm{g}} \mathrm{u}|_{\mathrm{g}}^2 |\nabla_{\mathrm{g}} \eta|_{\mathrm{g}}^2. \tag{1.3.12}$$

Here, $\mathcal{A}_{\mathrm{Riem}}$ (given by (1.3.13)) is a Riemannian analogue of the function $\mathcal{A}$ in Theorem 1.1.2, Ric denotes the Ricci tensor, and all the norms, gradients, and integrals are intrinsic to the metric g.

When expressed in coordinates, these Riemannian quantities fit within our Euclidean setting with variable coefficients. For instance, the operator Δ_{LB} can be written in coordinates as $Lu = \mathrm{div}\,(A(x)\nabla u) + \widehat{b}(x) \cdot \nabla u$. Here $A(x) = (a_{ij}(x)) = (g^{ij}(x))$ is the inverse of the metric and $\widehat{b}_i(x) = \frac{1}{2} g^{ij}(x) \partial_j \log |g|$ involves the volume density $|g| = \det(g_{ij}(x))$. Moreover, with our notation for matrices, the function $\mathcal{A}_{\mathrm{Riem}}$ in (1.3.12) can be written locally in $\{\nabla_{\mathrm{g}} \mathrm{u} \neq 0\}$ as

$$\mathcal{A}_{\mathrm{Riem}} = \left(\|A^{1/2}(x) H u A^{1/2}(x)\|_{\mathrm{HS}}^2 - \left|A^{1/2}(x) H u A^{1/2}(x) \mathbf{n}_A(x)\right|^2 \right)^{1/2}, \tag{1.3.13}$$

where $\mathbf{n}_A(x) = |\nabla u|_{A(x)}^{-1} A^{1/2}(x) \nabla u$ has appeared in the definition (1.3.11) of $\mathcal{A}_x$ in Remark 1.3.2 and $Hu = ((Hu)_{ij})$ is the Riemannian Hessian matrix given by $(Hu)_{ij} = u_{ij} - \Gamma_{ij}^k u_k$, where Γ_{ij}^k are the Christoffel symbols of the metric.

By this identification of Δ_{LB} with L, applying the Riemannian result in [56], collecting all lower order terms, and estimating the derivatives of the metric, we are led to an a priori bound for the function $\mathcal{A}_x$ in (1.3.11) which involves errors of the same type as in (1.1.11). Due to the presence of the Ricci tensor in (1.3.12), this estimate derived from the Riemannian inequality (1.3.12) depends on the norm $\|A\|_{C^2(\overline{B}_1)}$, i.e., it requires two derivatives of the metric. Nevertheless, integrating the unwanted coefficient derivatives by parts as we did in our proof of (1.1.11), we could deduce an estimate depending only on $\|A\|_{C^1(\overline{B}_1)}$.

The authors in [56] obtain (1.3.12) by choosing the test function $\mathbf{c} = |\nabla_{\mathrm{g}} \mathrm{u}|_{\mathrm{g}}$ in their stability inequality. In our coordinates, this function reads as $\mathbf{c}(x) = |\nabla u|_{A(x)}$. As explained in Remark 1.3.2, this choice of $\mathbf{c}$ and our integral stability inequality (1.1.18) lead to a similar estimate for $\mathcal{A}_x$ by using the ideas from the proof of (1.1.11) above.

We emphasize that both approaches (the Riemannian one and ours) give an estimate for $\mathcal{A}_x$ which contains an error term involving the product $|D^2 u||\nabla u|$. This error arises from the interaction between the second and first order terms in the Riemannian Hessian Hu when squaring $\mathcal{A}_{\mathrm{Riem}}$, and thus squaring $Hu = D^2 u - \Gamma \nabla u = D^2 u + O(\varepsilon |\nabla u|)$.

Next, we prove the "Hessian times the gradient" estimates (1.1.13) and (1.1.14) in Theorem 1.1.2. For this, we will need to consider the auxiliary function

$$\mathcal{A}_0 := \begin{cases} \left(\|A^{1/2}(0) D^2 u A^{1/2}(0)\|_{\mathrm{HS}}^2 - |A^{1/2}(0) D^2 u A^{1/2}(0) \mathbf{n}(x)|^2 \right)^{1/2} & \text{if } \nabla u \neq 0 \\ 0 & \text{if } \nabla u = 0, \end{cases} \tag{1.3.14}$$

where $\mathbf{n}(x) = |\nabla u|_{A^{-1}(0)}^{-1} A^{1/2}(0) \nabla u$ is again the vector field in the definition of $\mathcal{A}$ in (1.3.1). Notice that (1.3.14) is precisely the definition of $\mathcal{A}$ in (1.3.1) with the matrix $A^{1/2}(x)$

replaced by $A^{1/2}(0)$; see Remark 1.3.2. The greatest advantage of the function $\mathcal{A}_0$ over $\mathcal{A}$ is the symmetry of the matrix $A^{1/2}(0)D^2uA^{1/2}(0)$ in the definition above. This will allow us to bound the Hessian of the solution by $\mathcal{A}_0$, with the exception of the $\mathbf{n}\otimes\mathbf{n}$ component, which can be treated separately thanks to the nonnegativity assumption on the nonlinearity.

We will also need the a priori estimate (1.1.11) proved above, which gives a bound for the L^2 norm of the function $\mathcal{A}$. In the proof below, for $\|DA\|_{L^\infty(B_1)} \le \varepsilon$, we will see that

$$|\mathcal{A}^2 - \mathcal{A}_0^2| \le C\varepsilon|x|\mathcal{A}_0^2 \quad \text{in } B_1, \tag{1.3.15}$$

where C is a universal constant. In particular, for ε small, the functions are comparable and (1.1.11) allows us to bound the L^2 norm of $\mathcal{A}_0$ as well.

Proof of (1.1.13) *and* (1.1.14) *in Theorem 1.1.2.* Throughout the proof, C denotes a generic universal constant. The proof is divided into four steps.

Step 1: *We prove that*

$$|D^2u| \le -C\operatorname{tr}\big(A(0)D^2u\big) + C\mathcal{A}_0 + C\varepsilon|x||D^2u| + C\varepsilon|\nabla u| \quad \textit{a.e. in } B_1, \tag{1.3.16}$$

where $C>0$ is universal.

First we bound the full Hessian of u almost everywhere by the function $\mathcal{A}_0$ and the $\mathbf{n}\otimes\mathbf{n}$ component of the matrix $A^{1/2}(0)D^2uA^{1/2}(0)$. If $\nabla u(x)\neq 0$, then, extending $\mathbf{n}(x)$ to an orthonormal basis of $\mathbb{R}^n$, it is easy to see[3] that

$$\|A^{1/2}(0)D^2uA^{1/2}(0)\|_{\mathrm{HS}}^2 \le 2\mathcal{A}_0^2 + \big|(A^{1/2}(0)D^2uA^{1/2}(0))\mathbf{n}(x)\cdot\mathbf{n}(x)\big|^2. \tag{1.3.17}$$

Moreover, by Stampacchia's result, $|D^2u| = 0$ a.e. in $\nabla u = 0$ (see [69, Theorem 6.19]), and the inequality (1.3.17) holds almost everywhere in B_1. By uniform ellipticity we also have $|D^2u| \le C|A^{1/2}(0)D^2uA^{1/2}(0)| \le C\|A^{1/2}(0)D^2uA^{1/2}(0)\|_{\mathrm{HS}}$ and hence

$$|D^2u| \le C\mathcal{A}_0 + C\big|(A^{1/2}(0)D^2uA^{1/2}(0))\mathbf{n}(x)\cdot\mathbf{n}(x)\big| \quad \text{a.e. in } B_1. \tag{1.3.18}$$

Next we use that the nonlinearity is nonnegative to bound the $\mathbf{n}\otimes\mathbf{n}$ component $(A^{1/2}(0)D^2uA^{1/2}(0))\mathbf{n}(x)\cdot\mathbf{n}(x)$ in (1.3.18) in terms of the equation, the function $\mathcal{A}_0$, and lower order terms.

Since $0 \ge -f(u) = Lu = \operatorname{tr}(A(x)D^2u) + b(x)\cdot\nabla u$, we have

$$\begin{aligned}\big|\operatorname{tr}(A(x)D^2u)\big| &= |Lu - b(x)\cdot\nabla u|\\ &\le -Lu + |b(x)\cdot\nabla u| = -\operatorname{tr}(A(x)D^2u) - b(x)\cdot\nabla u + |b(x)\cdot\nabla u|\\ &\le -\operatorname{tr}(A(x)D^2u) + 2\varepsilon|\nabla u|.\end{aligned} \tag{1.3.19}$$

By the mean value theorem we have $|A(x)-A(0)| \le \varepsilon|x|$, and hence by (1.3.19)

$$\begin{aligned}\big|\operatorname{tr}(A(0)D^2u)\big| &\le \big|\operatorname{tr}(A(x)D^2u)\big| + C\varepsilon|x||D^2u|\\ &\le -\operatorname{tr}(A(x)D^2u) + C\varepsilon|x||D^2u| + C\varepsilon|\nabla u|\\ &\le -\operatorname{tr}(A(0)D^2u) + C\varepsilon|x||D^2u| + C\varepsilon|\nabla u|.\end{aligned} \tag{1.3.20}$$

[3]This follows immediately from the fact that, for any symmetric matrix $M\in\mathbb{R}^{n\times n}$, we have $\|M\|_{\mathrm{HS}}^2 = \sum_{i,j=1}^{n-1} M_{ij}^2 + 2\sum_{i=1}^{n-1} M_{in}^2 + M_{nn}^2$ and $\|M\|_{\mathrm{HS}}^2 - |Me_n|^2 = \sum_{i,j=1}^{n-1} M_{ij}^2 + \sum_{i=1}^{n-1} M_{in}^2$.

By the same argument to deduce (1.3.17) above, it is easy to see that[4]

$$\big|A^{1/2}(0)D^2uA^{1/2}(0)-\big[(A^{1/2}(0)D^2uA^{1/2}(0))\mathbf{n}\cdot\mathbf{n}\big]\mathbf{n}\otimes\mathbf{n}\big|\le C\mathcal{A}_0\quad \text{a.e. in } B_1. \tag{1.3.21}$$

Writing the $\mathbf{n}\otimes\mathbf{n}$ component of $A^{1/2}(0)D^2uA^{1/2}(0)$ as

$$\begin{aligned}&(A^{1/2}(0)D^2uA^{1/2}(0))\mathbf{n}\cdot\mathbf{n}\\ &\quad=\operatorname{tr}\Big(A^{1/2}(0)D^2uA^{1/2}(0)\Big)\\ &\qquad-\operatorname{tr}\Big(A^{1/2}(0)D^2uA^{1/2}(0)-\big[(A^{1/2}(0)D^2uA^{1/2}(0))\mathbf{n}\cdot\mathbf{n}\big]\mathbf{n}\otimes\mathbf{n}\Big),\end{aligned}$$

from (1.3.20) and (1.3.21), it follows that

$$\begin{aligned}\big|(A^{1/2}(0)D^2uA^{1/2}(0))\mathbf{n}\cdot\mathbf{n}\big|&\le\big|\operatorname{tr}\big(A^{1/2}(0)D^2uA^{1/2}(0)\big)\big|+C\mathcal{A}_0\\ &\le-\operatorname{tr}(A(0)D^2u)+C\mathcal{A}_0+C\varepsilon|x||D^2u|+C\varepsilon|\nabla u|\end{aligned}\tag{1.3.22}$$

a.e. in B_1. Combining (1.3.22) and (1.3.18) yields the claimed inequality (1.3.16).

Step 2: *We prove that there is a universal $\varepsilon_0>0$ such that, if $\varepsilon\le\varepsilon_0$, then*

$$\int_{B_1}|D^2u||\nabla u|\eta^2\,\mathrm{d}x\le C\int_{B_1}\mathcal{A}|\nabla u|\,\eta^2\,\mathrm{d}x+C\int_{B_1}|\nabla u|^2\big(|\nabla(\eta^2)|+\varepsilon\eta^2\big)\,\mathrm{d}x$$

for all $\eta\in C_c^\infty(B_1)$, where C is universal.

By uniform ellipticity, it suffices to estimate the integral $\int_{B_1}|D^2u||\nabla u|_{A(0)}\eta^2\,\mathrm{d}x$. Multiplying (1.3.16) in Step 1 by $|\nabla u|_{A(0)}\eta^2$ and integrating in B_1, by uniform ellipticity we have

$$\begin{aligned}&\int_{B_1}|D^2u||\nabla u|_{A(0)}\eta^2\,\mathrm{d}x\\ &\quad\le-C\int_{B_1}|\nabla u|_{A(0)}\operatorname{tr}\big(A(0)D^2u\big)\,\eta^2\,\mathrm{d}x+C\int_{B_1}\mathcal{A}_0|\nabla u|_{A(0)}\,\eta^2\,\mathrm{d}x\\ &\qquad+C\varepsilon\int_{B_1}|x||D^2u||\nabla u|_{A(0)}\eta^2\,\mathrm{d}x+C\varepsilon\int_{B_1}|\nabla u|^2\,\eta^2\,\mathrm{d}x.\end{aligned}\tag{1.3.23}$$

The only delicate term in the right-hand side of (1.3.23) is the first one, which can be treated as follows.

We write the product $-|\nabla u|_{A(0)}\operatorname{tr}(A(0)D^2u)$ in $\{\nabla u\neq0\}$ as

$$\begin{aligned}-2|\nabla u|_{A(0)}\operatorname{tr}\big(A(0)D^2u\big)&=-|\nabla u|_{A(0)}\operatorname{tr}(A(0)D^2u)-\operatorname{div}\big(|\nabla u|_{A(0)}A(0)\nabla u\big)\\ &\qquad+\nabla|\nabla u|_{A(0)}\cdot A(0)\nabla u.\end{aligned}\tag{1.3.24}$$

Since

$$\begin{aligned}\nabla|\nabla u|_{A(0)}\cdot A(0)\nabla u&=|\nabla u|_{A(0)}^{-1}\,D^2uA(0)\nabla u\cdot A(0)\nabla u\\ &=|\nabla u|_{A(0)}(A^{1/2}(0)D^2uA^{1/2}(0))\mathbf{n}\cdot\mathbf{n},\end{aligned}$$

[4] Follows from the fact that, for any symmetric matrix $M\in\mathbb{R}^{n\times n}$, we have $|M|\le n\|M\|_{\mathrm{HS}}$ and $\|M-M_{nn}e_n\otimes e_n\|_{\mathrm{HS}}^2\le2\left(\|M\|_{\mathrm{HS}}^2-|Me_n|^2\right)$.

by (1.3.24) and using that $\mathbf{n}$ is unitary, it follows that

$$\begin{aligned}
&-2|\nabla u|_{A(0)}\mathrm{tr}\big(A(0)D^2u\big)\\
&= -|\nabla u|_{A(0)}\mathrm{tr}\Big(A^{1/2}(0)D^2uA^{1/2}(0) - \big[\big(A^{1/2}(0)D^2uA^{1/2}(0)\big)\mathbf{n}\cdot\mathbf{n}\big]\mathbf{n}\otimes\mathbf{n}\Big)\\
&\qquad - \mathrm{div}\big(|\nabla u|_{A(0)}A(0)\nabla u\big)
\end{aligned} \tag{1.3.25}$$

a.e. in B_1. By the bound (1.3.21) in the proof of Step 1 above, it follows that

$$\Big|\mathrm{tr}\Big(A^{1/2}(0)D^2uA^{1/2}(0) - \big[\big(A^{1/2}(0)D^2uA^{1/2}(0)\big)\mathbf{n}\cdot\mathbf{n}\big]\mathbf{n}\otimes\mathbf{n}\Big)\Big| \le C\mathcal{A}_0$$

a.e. in B_1, and hence from (1.3.25) we deduce

$$-2|\nabla u|_{A(0)}\mathrm{tr}\big(A(0)D^2u\big) \le -\mathrm{div}\big(|\nabla u|_{A(0)}A(0)\nabla u\big) + C\mathcal{A}_0|\nabla u|_{A(0)} \quad \text{a.e. in } B_1. \tag{1.3.26}$$

Substituting (1.3.26) in (1.3.23) leads to

$$\begin{aligned}
&\int_{B_1}|D^2u||\nabla u|_{A(0)}\eta^2\,\mathrm{d}x\\
&\qquad\le -C\int_{B_1}\mathrm{div}\big(|\nabla u|_{A(0)}A(0)\nabla u\big)\,\eta^2\,\mathrm{d}x + C\int_{B_1}\mathcal{A}_0|\nabla u|_{A(0)}\,\eta^2\,\mathrm{d}x\\
&\qquad\qquad + C\varepsilon\int_{B_1}|x||D^2u||\nabla u|_{A(0)}\,\eta^2\,\mathrm{d}x + C\varepsilon\int_{B_1}|\nabla u|^2\,\eta^2\,\mathrm{d}x,
\end{aligned}$$

and integrating by parts the divergence term, we obtain the inequality

$$\begin{aligned}
\int_{B_1}|D^2u||\nabla u|_{A(0)}\eta^2\,\mathrm{d}x \le C\int_{B_1}|\nabla u|^2\,\big(|\nabla\eta^2| + \varepsilon\eta^2\big)\,\mathrm{d}x + C\int_{B_1}\mathcal{A}_0|\nabla u|_{A(0)}\,\eta^2\,\mathrm{d}x\\
+ C\varepsilon\int_{B_1}|x||D^2u||\nabla u|_{A(0)}\,\eta^2\,\mathrm{d}x.
\end{aligned} \tag{1.3.27}$$

Since $|x| \le 1$ in B_1, choosing $\varepsilon_0 > 0$ universal small such that $C\varepsilon_0 = 1/2$, we can absorb the "Hessian times the gradient" error in (1.3.27) into the left-hand side to obtain

$$\int_{B_1}|D^2u||\nabla u|_{A(0)}\eta^2\,\mathrm{d}x \le C\int_{B_1}|\nabla u|^2\,\big(|\nabla\eta^2| + \varepsilon\eta^2\big)\,\mathrm{d}x + C\int_{B_1}\mathcal{A}_0|\nabla u|_{A(0)}\,\eta^2\,\mathrm{d}x. \tag{1.3.28}$$

To conclude the argument, let us show that $\mathcal{A}$ and $\mathcal{A}_0$ are comparable for ε small. Letting $E(x) = A(x) - A(0)$ and $M(x) = D^2u(x)A^{1/2}(0)$, it is easy to check that

$$\mathcal{A}^2 = \mathcal{A}_0^2 + \mathrm{tr}\,\big(M(x)^TE(x)M(x)\big) - (M(x)^TE(x)M(x))\mathbf{n}\cdot\mathbf{n} \quad \text{in } \{\nabla u \ne 0\},$$

and for $x \in \{\nabla u \ne 0\}$, extending $\mathbf{n} = \mathbf{n}(x)$ to an orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n = \mathbf{n}$ of $\mathbb{R}^n$, we can rewrite this identity as

$$\mathcal{A}^2 = \mathcal{A}_0^2 + \sum_{i=1}^{n-1}E(x)M(x)\mathbf{e}_i\cdot M(x)\mathbf{e}_i \quad \text{in } \{\nabla u \ne 0\}. \tag{1.3.29}$$

By the mean value theorem we can bound the error by $|E(x)| \le \varepsilon|x|$, and hence, by uniform ellipticity,

$$\left|\sum_{i=1}^{n-1}E(x)M(x)\mathbf{e}_i\cdot M(x)\mathbf{e}_i.\right| \le \varepsilon|x|\sum_{i=1}^{n-1}|M(x)\mathbf{e}_i|^2 \le C\varepsilon|x|\sum_{i=1}^{n-1}|M(x)\mathbf{e}_i|^2_{A(0)}. \tag{1.3.30}$$

Since $|M(x)\mathbf{e}_i|^2_{A(0)} = |A^{1/2}(0)D^2u(x)A^{1/2}(0)\mathbf{e}_i|$, by (1.3.21) above, the sum in right-hand side of (1.3.30) can be further bounded by

$$\sum_{i=1}^{n-1} |M(x)\mathbf{e}_i|^2_{A(0)} \le C\mathcal{A}_0 \quad \text{a.e. in } B_1. \tag{1.3.31}$$

Combining (1.3.30) and (1.3.31), from (1.3.29) we conclude that

$$(1 - C\varepsilon|x|)\,\mathcal{A}_0^2 \le \mathcal{A}^2 \le (1 + C\varepsilon|x|)\,\mathcal{A}_0^2 \quad \text{in } B_1,$$

which was the inequality (1.3.15) mentioned before the proof. Choosing ε_0 smaller if necessary, we may assume that $\mathcal{A}_0 \le 2\mathcal{A}$, which applied in (1.3.27) yields the claim.

Step 3: *We prove that, if $\varepsilon \le \varepsilon_0$, with $\varepsilon_0 > 0$ as in Step 1, then*

$$\int_{B_1} \mathcal{A}^2\eta^2 \,\mathrm{d}x \le C \int_{B_1} |\nabla u|^2\big(|\nabla\eta|^2 + \varepsilon^2\eta^2\big)\,\mathrm{d}x$$

for all $\eta \in C_c^\infty(B_1)$, where C is a universal constant.

Combining (1.1.11) in Theorem 1.1.2 and Step 2, we have

$$\begin{aligned}\int_{B_1} \mathcal{A}^2\eta^2 \,\mathrm{d}x \le C\varepsilon \int_{B_1} \mathcal{A}|\nabla u|\eta^2 \,\mathrm{d}x + \int_{B_1} |\nabla u|^2_{A(0)}|\nabla\eta|^2_{A(x)}\,\mathrm{d}x \\ + C\varepsilon \int_{B_1} |\nabla u|^2\big(|\nabla(\eta^2)| + \varepsilon\eta^2\big)\,\mathrm{d}x.\end{aligned} \tag{1.3.32}$$

By Young's inequality, the first term on the right-hand side of (1.3.32) can be bounded by

$$C\varepsilon \int_{B_1} \mathcal{A}|\nabla u|\eta^2 \,\mathrm{d}x \le \frac{1}{2}\int_{B_1} \mathcal{A}^2\eta^2\,\mathrm{d}x + C\varepsilon^2 \int_{B_1} |\nabla u|^2\eta^2\,\mathrm{d}x,$$

and the $\mathcal{A}^2\eta^2$ integral can be absorbed into the left-hand side. By uniform ellipticity and the bound $\varepsilon|\nabla(\eta^2)| \le |\nabla\eta|^2 + \varepsilon^2\eta^2$, we deduce the claim.

Step 4: *Conclusion.*

Combining Steps 2 and 3, for $\varepsilon \le \varepsilon_0$ as above and by Cauchy-Schwarz, we obtain

$$\begin{aligned}&\int_{B_1} |D^2u||\nabla u|\eta^2\,\mathrm{d}x \\ &\quad\le C\left(\int_{B_1} \mathcal{A}^2\eta^2\,\mathrm{d}x\right)^{1/2}\left(\int_{B_1}|\nabla u|^2\eta^2\,\mathrm{d}x\right)^{1/2} + C\int_{B_1}|\nabla u|^2\big(|\nabla(\eta^2)| + \varepsilon\eta^2\big)\,\mathrm{d}x \\ &\quad\le C\left(\int_{B_1}|\nabla u|^2\big(|\nabla\eta|^2 + \varepsilon^2\eta^2\big)\,\mathrm{d}x\right)^{1/2}\left(\int_{B_1}|\nabla u|^2\eta^2\,\mathrm{d}x\right)^{1/2} \\ &\qquad + C\int_{B_1}|\nabla u|^2\big(|\nabla(\eta^2)| + \varepsilon\eta^2\big)\,\mathrm{d}x.\end{aligned} \tag{1.3.33}$$

The inequalities (1.1.13) and (1.1.14) in Theorem 1.1.2 follow easily from (1.3.33) by choosing appropriate cut-off functions and using that ε is bounded by a universal constant ε_0. Choosing $\eta \in C_c^\infty(B_1)$ such that $\eta = 1$ in $B_{3/4}$ and $0 \le \eta \le 1$ in B_1 leads to the estimate in balls (1.1.13). The second estimate in annuli (1.1.14) follows by choosing $\eta \in C_c^\infty(B_1 \setminus \overline{B}_{1/8})$ with $\eta = 1$ in $B_{1/2} \setminus B_{1/4}$ and $0 \le \eta \le 1$ in B_1. □

Remark 1.3.4. By the proof above, we can also deduce an interior a priori estimate for the L^1 norm of the Hessian. For this, assuming ε to be small, recalling that $\mathcal{A}_0 \le C\mathcal{A}$ by (1.3.15), and absorbing the Hessian term in Step 1, we obtain

$$|D^2u| \le -C\mathrm{div}\,(A(0)\nabla u) + C\mathcal{A} + C|\nabla u| \quad \text{a.e. in } B_1.$$

Multiplying this inequality by a cut-off function and integrating by parts, using the bound for $\mathcal{A}$ in Step 3 and applying Cauchy-Schwarz, we deduce an estimate for the L^1 norm of the Hessian in terms of the L^2 norm of the gradient in a larger ball.

We conclude this section by proving Proposition 1.1.3. To show that the L^2 norm of the gradient is controlled by the L^1 norm of the function in a larger ball, we use the interpolation inequalities of Cabré [20] combined with the Hessian estimates from Theorem 1.1.2. The errors in larger balls can then be absorbed thanks to a well-known lemma of Simon [89]. We recall the interpolation inequalities of Cabré in Appendix C and Simon's lemma in Appendix D.

Proof of Proposition 1.1.3. We cover $B_{1/2}$ (except for a set of measure zero) with a family of disjoint open cubes Q_j of the same side-length and small enough so that $Q_j \subset B_{3/4}$. The side-length and the number of cubes depend only on n. Combining the interpolation inequalities of Propositions C.1 and C.3, rescaled from the unit cube to Q_j, with $\tilde{\delta} = \delta^{3/2}$ for a given $\delta \in (0,1)$, we obtain

$$\int_{Q_j} |\nabla u|^2 dx \le C\delta \int_{Q_j} |D^2u||\nabla u|\, dx + C\delta \int_{Q_j} |\nabla u|^2 dx + C\delta^{-2-\frac{3n}{2}} \left(\int_{Q_j} |u|\, dx \right)^2.$$

Since $Q_j \subset B_{3/4}$, applying (1.1.13) from Theorem 1.1.2, for $\varepsilon \le \varepsilon_0$ we have

$$\int_{Q_j} |\nabla u|^2 dx \le C\delta \int_{B_1} |\nabla u|^2 dx + C\delta^{-2-\frac{3n}{2}} \left(\int_{B_1} |u|\, dx \right)^2.$$

Adding up these inequalities, we obtain

$$\|\nabla u\|^2_{L^2(B_{1/2})} \le C\delta \|\nabla u\|^2_{L^2(B_1)} + C\delta^{-2-\frac{3n}{2}} \|u\|^2_{L^1(B_1)} \qquad \text{for } \delta \in (0,1) \text{ and } \varepsilon \le \varepsilon_0. \quad (1.3.34)$$

As explained in Section 1.2, for $B_\rho(y) \subset B_1$, the function $u^{y,\rho} := u(y + \rho\,\cdot)$ is a stable solution to a semilinear equation with coefficients $A^{y,\rho} = A(y+\rho\,\cdot)$ and $b^{y,\rho} = \rho\, b(y+\rho\,\cdot)$. In particular, since $\rho \le 1$, for $\varepsilon \le \varepsilon_0$ we have that

$$\|DA^{y,\rho}\|_{L^\infty(B_1)} + \|b^{y,\rho}\|_{L^\infty(B_1)} \le \rho\varepsilon \le \varepsilon_0,$$

and we can apply (1.3.34) to $u^{y,\rho}$, which yields

$$\begin{aligned} \rho^{n+2} \int_{B_{\rho/2}(y)} |\nabla u|^2\, dx &\le C\delta\rho^{n+2} \int_{B_\rho(y)} |\nabla u|^2\, dx + C\delta^{-2-\frac{3n}{2}} \left(\int_{B_\rho(y)} |u|\, dx \right)^2 \\ &\le C\delta\rho^{n+2} \int_{B_\rho(y)} |\nabla u|^2\, dx + C\delta^{-2-\frac{3n}{2}} \left(\int_{B_1} |u|\, dx \right)^2. \end{aligned}$$

By Lemma D.1 with $\sigma(B) := \|\nabla u\|^2_{L^2(B)}$, the claim follows. $\square$

1.4 The $W^{1,2+\gamma}$ estimate

Here we prove the higher integrability estimate (1.1.9) in Theorem 1.1.1. The strategy of proof is the same as for the Laplacian in [24]. First we bound the $L^{2+\gamma}$ norm in terms of the L^2 norm of the gradient when the coefficients are small. This will follow from a uniform estimate of the Dirichlet norm on level sets, which relies on the Hessian estimates in Theorem 1.1.2.

Lemma 1.4.1. *Let $u \in C^\infty(\overline{B}_1)$ be a stable solution of $-Lu = f(u)$ in B_1, for some nonnegative function $f \in C^1(\mathbb{R})$. Assume that*

$$\|DA\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)} \le \varepsilon$$

for some $\varepsilon > 0$.

If $\varepsilon \le \varepsilon_0$, then

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/2})} \le C\|\nabla u\|_{L^2(B_1)},$$

where $\gamma > 0$ depends only on n, and $\varepsilon_0 > 0$ and C are universal constants.

Proof. The proof is divided in two steps.

Step 1: *We prove that, if $\varepsilon \le \varepsilon_0$, then for a.e. $t \in \mathbb{R}$ we have*

$$\int_{\{u=t\}\cap B_{1/2}} |\nabla u|^2 \, \mathrm{d}\mathcal{H}^{n-1} \le C\|\nabla u\|^2_{L^2(B_1)},$$

where $\varepsilon_0 > 0$ and C are universal.

Since $\left|\operatorname{div}\big(|\nabla u|\nabla u\big)\right| \le C|D^2 u||\nabla u|$, by (1.1.13) in Theorem 1.1.2, for $\varepsilon \le \varepsilon_0$ we have

$$\left\|\operatorname{div}\big(|\nabla u|\nabla u\big)\right\|_{L^1(B_{3/4})} \le C\|\nabla u\|^2_{L^2(B_1)}. \tag{1.4.1}$$

Consider a cut-off function $\eta \in C^\infty_c(B_{3/4})$ with $\eta = 1$ in $B_{1/2}$ and $0 \le \eta \le 1$. By the divergence theorem, for a.e. $t \in \mathbb{R}$ we have

$$\begin{aligned}
\int_{\{u=t\}\cap B_{1/2}} |\nabla u|^2 \, \mathrm{d}\mathcal{H}^{n-1} &\le \int_{\{u=t\}\cap B_1 \cap\{\nabla u\neq 0\}} |\nabla u|^2\eta^2 \, \mathrm{d}\mathcal{H}^{n-1} \\
&= -\int_{\{u>t\}\cap B_1\cap\{\nabla u \neq 0\}} \operatorname{div}\big(|\nabla u|\nabla u\, \eta^2\big) \, \mathrm{d}x \\
&\le \int_{B_1} |\nabla u|^2 |\nabla(\eta^2)| \, \mathrm{d}x + \int_{B_1} \left|\operatorname{div}\big(|\nabla u|\nabla u\big)\right|\eta^2 \, \mathrm{d}x
\end{aligned}$$

and applying (1.4.1) we obtain the claim.

Step 2: *Conclusion.*

Let

$$v := \frac{u - (u)_{B_1}}{\|\nabla u\|_{L^2(B_1)}},$$

where $(u)_{B_1} := \dfrac{1}{|B_1|}\displaystyle\int_{B_1} u \, \mathrm{d}x$. In particular $\|\nabla v\|_{L^2(B_1)} = 1$ and by the Sobolev-Poincaré inequality, for some dimensional $p > 2$, we have

$$\left(\int_{B_1} |v|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \le C\left(\int_{B_1} |\nabla v|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} = C. \tag{1.4.2}$$

By the coarea formula and (1.4.2), we have

$$\int_{\mathbb{R}} \mathrm{d}t \int_{\{v=t\}\cap\{|\nabla v|\neq 0\}} |t|^p |\nabla v|^{-1} \,\mathrm{d}\mathcal{H}^{n-1} = \int_{B_1\cap\{|\nabla v|\neq 0\}} |v|^p \,\mathrm{d}x \leq C. \tag{1.4.3}$$

Since $p > 2$, we can choose dimensional constants $q > 1$ and $\theta \in (0, 1/3)$ such that $p/q = (1-\theta)/\theta$. We define

$$h(t) := \max\{1, |t|\}.$$

Using the coarea formula and the Hölder inequality (note that $p\theta - q(1-\theta) = 0$), we obtain

$$\begin{aligned}
\int_{B_{1/2}} |\nabla v|^{3-3\theta} \,\mathrm{d}x &= \int_{\mathbb{R}} \mathrm{d}t \int_{\{v=t\}\cap B_{1/2}\cap\{|\nabla v|\neq 0\}} h(t)^{p\theta - q(1-\theta)} |\nabla v|^{-\theta+2(1-\theta)} \,\mathrm{d}x \\
&\leq \left(\int_{\mathbb{R}} \mathrm{d}t \int_{\{v=t\}\cap B_1\cap\{|\nabla v|\neq 0\}} h(t)^p |\nabla v|^{-1} \,\mathrm{d}\mathcal{H}^{n-1} \right)^{\theta} \cdot \\
&\qquad \cdot \left(\int_{\mathbb{R}} \mathrm{d}t \int_{\{v=t\}\cap B_{1/2}} h(t)^{-q} |\nabla v|^2 \,\mathrm{d}\mathcal{H}^{n-1} \right)^{1-\theta}.
\end{aligned}$$

Thanks to (1.4.3) and the definition of $h(t)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}t \int_{\{v=t\}\cap B_1\cap\{|\nabla v|\neq 0\}} h(t)^p |\nabla v|^{-1} \,\mathrm{d}\mathcal{H}^{n-1} \\
&\leq \int_{-1}^{1} \mathrm{d}t \int_{\{v=t\}\cap B_1\cap\{|\nabla v|\neq 0\}} |\nabla v|^{-1} \,\mathrm{d}\mathcal{H}^{n-1} + C \leq |B_1| + C \leq C.
\end{aligned}$$

Since $q > 1$, it follows that $\int_{\mathbb{R}} h(t)^{-q}\,\mathrm{d}t$ is finite and by Step 1, for $\varepsilon \leq \varepsilon_0$, we have

$$\int_{\mathbb{R}} \mathrm{d}t\, h(t)^{-q} \int_{\{v=t\}\cap B_{1/2}} |\nabla v|^2 \,\mathrm{d}\mathcal{H}^{n-1} \leq C.$$

Finally, we obtain

$$\int_{B_{1/2}} |\nabla v|^{3-3\theta} \,\mathrm{d}x \leq C$$

which gives the claim, since $\nabla v \equiv \nabla u / \|\nabla u\|_{L^2(B_1)}$. □

To deduce the $L^{2+\gamma}$ estimate (1.1.9) in Theorem 1.1.1, we will combine Proposition 1.1.3 with Lemma 1.4.1, and apply a scaling and covering argument.

Proof of (1.1.9) *in Theorem 1.1.1.* Combining Proposition 1.1.3 and Lemma 1.4.1, applied to $u(\cdot/2)$, we deduce that there is a universal $\varepsilon_0 > 0$ such that, if $\varepsilon \leq \varepsilon_0$, then

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/4})} \leq C \|u\|_{L^1(B_1)}, \tag{1.4.4}$$

where $\gamma > 0$ depends only on n, and C is universal.

Now (1.1.9) will follow easily from (1.4.4) by a scaling and covering argument. Let $\rho \in (0, 1)$ to be chosen later. We cover the ball $B_{1/2}$ by a finite number of balls $B_{\rho/4}(y_j)$ with $B_\rho(y_j) \subset B_1$. The number balls depends only on n and ρ. As explained in Section 1.2, the functions $u^{y_j,\rho} = u(y_j + \rho\,\cdot)$ are stable solutions to a semilinear equation with coefficients

$A^{y_j,\rho} = A(y_j+\rho\,\cdot)$ and $b^{y_j,\rho} = \rho\, b(y_j+\rho\,\cdot)$. Choosing ρ small enough so that $\rho\big(\|DA\|_{L^\infty(B_1)}+\|b\|_{L^\infty(B_1)} \le \varepsilon\big) \le \varepsilon_0$, it follows that

$$\|DA^{y_j,\rho}\|_{L^\infty(B_1)} + \|b^{y_j,\rho}\|_{L^\infty(B_1)} \le \varepsilon_0,$$

and we can apply (1.4.4) to each $u^{y_j,\rho}$, which yields

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/2})} \le \sum_j \|\nabla u\|_{L^{2+\gamma}(B_{\rho/4}(y_j))} \le C\sum_j \|u\|_{L^1(B_\rho(y_j))} \le C\|u\|_{L^1(B_1)},$$

for some C depending only on n, c_0, C_0, and ρ. Since ρ depends only on $\|DA\|_{L^\infty(B_1)}$, $\|b\|_{L^\infty(B_1)}$, and ε_0, which is universal, this concludes the proof. $\square$

1.5 The weighted L^2 estimate for radial derivatives

Our goal in this section is to prove Proposition 1.1.4, where we bound the weighted L^2 norm of the radial derivative in balls by the L^2 norm of the full gradient in annuli. To prove the estimate, we will first apply the integral stability inequality with the test functions

$$\mathbf{c}(x) = x\cdot\nabla u \quad \text{ and } \quad \eta = |x|_{A^{-1}(0)}^{\frac{2-n}{2}}\zeta,$$

where ζ is a cut-off. Our choice will yield the desired bound with an additional error term involving a weighted integral of the "Hessian times the gradient", which we will be able to treat thanks to the a priori estimates on annuli from Theorem 1.1.2.

We start by choosing $\mathbf{c} = x\cdot\nabla u$ and a generic test function η in the integral stability inequality:

Lemma 1.5.1. *Let $u \in C^\infty\big(\overline{B}_1\big)$ be a stable solution of $-Lu = f(u)$ in B_1, for some function $f \in C^1(\mathbb{R})$. Assume that*

$$\|DA\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)} \le \varepsilon$$

for some $\varepsilon > 0$.

Then

$$\begin{aligned}
&\int_{B_1} |\nabla u|^2_{A(x)}\Big((n-2)\eta^2 + x\cdot\nabla(\eta^2)\Big)\,\mathrm{d}x\\
&\quad+\int_{B_1}\Big(-2(x\cdot\nabla u)A(x)\nabla u\cdot\nabla(\eta^2) - |x\cdot\nabla u|^2|\nabla\eta|^2_{A(x)}\Big)\,\mathrm{d}x\\
&\qquad\qquad\le C\varepsilon\int_{B_1}|D^2u||\nabla u||x|^2\eta^2\,\mathrm{d}x\\
&\qquad\qquad\qquad+C\varepsilon\int_{B_1}|\nabla u|^2\Big(|x|^2|\nabla(\eta^2)| + \big(|x|+|x|^2\varepsilon\big)\,\eta^2\Big)\,\mathrm{d}x
\end{aligned}$$

for all $\eta \in C^\infty_c(B_1)$, where C is a universal constant.

Proof. Throughout the proof, C denotes a generic universal constant. Testing the integral stability inequality (1.1.18) with η and $\mathbf{c} = x\cdot\nabla u$, we deduce

$$\int_{B_1}(x\cdot\nabla u)J_u(x\cdot\nabla u)\,\eta^2\,\mathrm{d}x \le \int_{B_1}|x\cdot\nabla u|^2\left|\nabla\eta - \tfrac{1}{2}\eta A^{-1}(x)\widehat{b}(x)\right|^2_{A(x)}\mathrm{d}x. \tag{1.5.1}$$

The quadratic term on the right-hand side of (1.5.1) can be bounded by

$$\left|\nabla\eta - \tfrac{1}{2}\eta A^{-1}(x)\widehat{b}(x)\right|^2_{A(x)} \le |\nabla\eta|^2_{A(x)} + C\varepsilon|\nabla(\eta^2)| + C\varepsilon^2\eta^2,$$

and hence

$$\begin{aligned}\int_{B_1}(x\cdot\nabla u)J_u(x\cdot\nabla u)\,\eta^2\,\mathrm{d}x \le \int_{B_1}&|x\cdot\nabla u|^2|\nabla\eta|^2_{A(x)}\,\mathrm{d}x\\ &+C\varepsilon\int_{B_1}|\nabla u|^2|x|^2\big(|\nabla(\eta^2)|+\varepsilon\eta^2\big)\,\mathrm{d}x.\end{aligned}\tag{1.5.2}$$

To compute the Jacobi operator $J_u(x\cdot\nabla u) = L(x\cdot\nabla u) + f'(u)(x\cdot\nabla u)$, we differentiate the equation $-Lu = f(u)$ in the direction of x, which yields

$$-\,x\cdot\nabla(Lu) = f'(u)\,(x\cdot\nabla u)\,,\tag{1.5.3}$$

and hence

$$\begin{aligned}J_u(x\cdot\nabla u) &= L\,(x\cdot\nabla u) - x\cdot\nabla\,(Lu)\\ &= 2a_{ij}(x)u_{ij} - x_k\partial_k a_{ij}(x)u_{ij} + b_i(x)u_i - x_j\partial_j b_i(x)u_i.\end{aligned}\tag{1.5.4}$$

From (1.5.4), by the coefficient bounds, it follows that

$$\begin{aligned}(x\cdot\nabla u)J_u(x\cdot\nabla u) \ge 2x_ku_ka_{ij}(x)u_{ij} &- x_ku_kx_j\partial_jb_i(x)u_i\\ &- C\varepsilon|x|^2|D^2u||\nabla u| - C\varepsilon|x||\nabla u|^2.\end{aligned}\tag{1.5.5}$$

The idea now is to integrate by parts to get rid of the highest order terms on the left-hand side of (1.5.2). For this we must rewrite the term $2x_ku_k\,a_{ij}(x)u_{ij}$ in (1.5.5) in divergence form. By the chain rule, we have

$$x_ku_k\,a_{ij}(x)u_{ij} = \partial_i\big(x_ku_k\,a_{ij}(x)u_j\big) - x_ku_k\partial_ia_{ij}(x)u_j - a_{ij}(x)u_iu_j - x_ka_{ij}(x)u_{ik}u_j.\tag{1.5.6}$$

Using that $a_{ij}(x) = a_{ji}(x)$, the last term in (1.5.6) can be written as

$$x_ka_{ij}(x)u_{ik}u_j = \frac{1}{2}\partial_k\big(a_{ij}(x)u_iu_jx_k\big) - \frac{n}{2}a_{ij}(x)u_iu_j - \frac{1}{2}x_k\partial_ka_{ij}(x)u_iu_j,$$

and hence

$$\begin{aligned}2x_ku_k\,a_{ij}(x)u_{ij} &= \partial_i\Big(2x_ku_k\,a_{ij}(x)u_j - a_{jk}(x)u_ju_kx_i\Big) + (n-2)a_{ij}(x)u_iu_j\\ &\quad - 2x_ku_k\partial_ia_{ij}(x)u_j + x_k\partial_ka_{ij}(x)u_iu_j\\ &\ge \operatorname{div}\big(2(x\cdot\nabla u)A(x)\nabla u - |\nabla u|^2_{A(x)}x\big) + (n-2)|\nabla u|^2_{A(x)}\\ &\quad - C\varepsilon|x||\nabla u|^2,\end{aligned}\tag{1.5.7}$$

where in the last inequality we have used the estimates for the coefficients. Combining (1.5.7) and (1.5.5), we obtain

$$\begin{aligned}&(x\cdot\nabla u)J_u(x\cdot\nabla u)\\ &\quad\ge \operatorname{div}\big(2(x\cdot\nabla u)A(x)\nabla u - |\nabla u|^2_{A(x)}x\big) + (n-2)|\nabla u|^2_{A(x)}\\ &\qquad - x_ku_kx_j\partial_jb_i(x)u_i - C\varepsilon|x|^2|D^2u||\nabla u| - C\varepsilon|x||\nabla u|^2.\end{aligned}\tag{1.5.8}$$

Multiplying (1.5.8) by η^2 and integrating, the term $-\int_{B_1} x_k u_k x_j \partial_j b_i(x) u_i \eta^2 \,\mathrm{d}x$ can be integrated by parts and estimated by

$$\begin{aligned}\left| -\int_{B_1} x_k u_k x_j \partial_j b_i(x) u_i \eta^2 \,\mathrm{d}x \right| &= \left| \int_{B_1} b_i(x) \partial_j (x_k u_k x_j u_i \eta^2) \,\mathrm{d}x \right| \\ &\le C\varepsilon \int_{B_1} |D^2 u||\nabla u||x|^2 \eta^2 \,\mathrm{d}x \\ &\quad + C\varepsilon \int_{B_1} |\nabla u|^2 \left(|x|^2 |\nabla(\eta^2)| + |x|\eta^2 \right) \mathrm{d}x. \end{aligned} \tag{1.5.9}$$

Substituting (1.5.8) in the inequality (1.5.2), rearranging terms and by the error bound (1.5.9), it follows that

$$\begin{aligned}&\int_{B_1} \left((n-2)|\nabla u|^2_{A(x)} \eta^2 - |x \cdot \nabla u|^2 |\nabla \eta|^2_{A(x)} \right) \mathrm{d}x \\ &+ \int_{B_1} \operatorname{div}\Big(2(x \cdot \nabla u) A(x) \nabla u - |\nabla u|^2_{A(x)} x \Big) \eta^2 \,\mathrm{d}x \\ &\le C\varepsilon \int_{B_1} |D^2 u||\nabla u||x|^2 \eta^2 \,\mathrm{d}x + C\varepsilon \int_{B_1} |\nabla u|^2 \Big(|x|^2 |\nabla(\eta^2)| + (|x| + \varepsilon |x|^2)\eta^2 \Big) \,\mathrm{d}x. \end{aligned} \tag{1.5.10}$$

Integrating by parts the divergence term on the left-hand side of (1.5.10) yields the claim. □

Remark 1.5.2. In (1.5.3) we took a derivative of the equation in the x direction to get rid of the dependence on the nonlinearity. Instead, we could have multiplied the equation by the test function $\xi = \operatorname{div}(x\,(x \cdot \nabla u)\eta^2)$ and integrated by parts. Thanks to this, we avoid having to take any derivatives of b, since the term involving it can be bounded directly. Notice also that we need u to have three derivatives to be able to compute $L\mathbf{c}$. This is the same phenomenon as in the proof of Theorem 1.1.2; see the discussion in Remark 1.3.1.

Remark 1.5.3. Since $|A(x) - A(0)| \le C\varepsilon|x|$, the inequality in Lemma 1.5.1 also holds if we replace $A(x)$ by the constant matrix $A(0)$ and we add an additional error term $C\varepsilon \int_{B_1} |\nabla u|^2 |x|^3 |\nabla \eta|^2 \,\mathrm{d}x$ on the right-hand side. For future use, the final estimate involving $A(0)$ instead of $A(x)$ reads as

$$\begin{aligned}&\int_{B_1} |\nabla u|^2_{A(0)} \Big((n-2)\eta^2 + x \cdot \nabla(\eta^2) \Big) \,\mathrm{d}x \\ &+ \int_{B_1} \Big(-2(x \cdot \nabla u) A(0) \nabla u \cdot \nabla(\eta^2) - |x \cdot \nabla u|^2 |\nabla \eta|^2_{A(0)} \Big) \,\mathrm{d}x \\ &\qquad \le C\varepsilon \int_{B_1} |D^2 u||\nabla u||x|^2 \eta^2 \,\mathrm{d}x \\ &\qquad\quad + C\varepsilon \int_{B_1} |\nabla u|^2 \Big(|x|^3 |\nabla \eta|^2 + (|x| + \varepsilon|x|^2)\eta^2 \Big) \,\mathrm{d}x, \end{aligned} \tag{1.5.11}$$

where we have used that $|x|^2 |\nabla(\eta^2)| \le |x|^3 |\nabla \eta|^2 + |x|\eta^2$.

Next, we choose the singular test function $\eta = |x|^{-a/2}_{A^{-1}(0)} \zeta$ in Lemma 1.5.1, where the exponent $a \ge 0$ will satisfy $a \le n-2$ when $n \le 9$. Recall our notation for the modulus of the position vector and the radial derivative

$$r = |x| \quad \text{and} \quad u_r = \frac{x}{|x|} \cdot \nabla u.$$

Lemma 1.5.4. *Let $u \in C^\infty(\overline{B}_1)$ be a stable solution of $-Lu = f(u)$ in B_1, for some function $f \in C^1(\mathbb{R})$. Assume that*

$$\|DA\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)} \le \varepsilon$$

for some $\varepsilon > 0$.

If

$$0 \le a \le \min\{10, n\} - 2, \tag{1.5.12}$$

then

$$\begin{aligned}(n-2-a)\int_{B_\rho} r^{-a}|\nabla u|^2\,\mathrm{d}x + \frac{a(8-a)}{4}\int_{B_\rho} r^{-a}u_r^2\,\mathrm{d}x \\ \le C\int_{B_{2\rho}\setminus B_\rho} r^{-a}|\nabla u|^2\,\mathrm{d}x + C\varepsilon\int_{B_{2\rho}} r^{2-a}|D^2u||\nabla u|\,\mathrm{d}x \\ + C\varepsilon\int_{B_{2\rho}}\big(r^{1-a}+\varepsilon r^{2-a}\big)|\nabla u|^2\,\mathrm{d}x.\end{aligned} \tag{1.5.13}$$

for all $\rho \le 1/2$, where C is a universal constant.

Proof. By approximation, the inequality in Lemma 1.5.1 is valid for Lipschitz test functions $\eta \in C_c^{0,1}(B_1)$. Moreover, this inequality also holds for the singular test function

$$\eta := |x|_{A^{-1}(0)}^{-a/2}\zeta,$$

where $\zeta \in C_c^{0,1}(B_1)$ is a cut-off. To see this, for $\delta > 0$ consider the $C_c^{0,1}$ approximation

$$\eta_\delta = \min\{|x|_{A^{-1}(0)}^{-a/2}, \delta^{-a/2}\}\zeta$$

and apply dominated convergence to take the limit as $\delta \to 0$.

By Remark 1.5.3, it suffices to compute the left-hand side of the inequality in Lemma 1.5.1 with $A(0)$ in place of $A(x)$. Since

$$\nabla(\eta^2) = -a\zeta^2|x|_{A^{-1}(0)}^{-(a+2)}A^{-1}(0)x + |x|_{A^{-1}(0)}^{-a}\nabla(\zeta^2),$$

the first integrand in (1.5.11) is equal to

$$\begin{aligned}&|\nabla u|_{A(0)}^2\Big((n-2)\eta^2 + x\cdot\nabla(\eta^2)\Big) \\ &\quad = (n-2-a)|x|_{A^{-1}(0)}^{-a}|\nabla u|_{A(0)}^2\zeta^2 + |x|_{A^{-1}(0)}^{-a}|\nabla u|_{A(0)}^2(x\cdot\nabla(\zeta^2)).\end{aligned} \tag{1.5.14}$$

Moreover, since

$$|\nabla\eta|_{A(0)}^2 = \frac{a^2}{4}|x|_{A^{-1}(0)}^{-(a+2)}\zeta^2 - \frac{a}{2}|x|_{A^{-1}(0)}^{-(a+2)}\big(x\cdot\nabla(\zeta^2)\big) + |x|_{A^{-1}(0)}^{-a}|\nabla\zeta|_{A(0)}^2,$$

the second integrand is

$$\begin{aligned}&-2(x\cdot\nabla u)A(0)\nabla u\cdot\nabla(\eta^2) - |x\cdot\nabla u|^2|\nabla\eta|_{A(0)}^2 \\ &\quad = \frac{a(8-a)}{4}|x|_{A^{-1}(0)}^{-(a+2)}|x\cdot\nabla u|^2\zeta^2 - 2|x|_{A^{-1}(0)}^{-a}(x\cdot\nabla u)A(0)\nabla u\cdot\nabla(\zeta^2) \\ &\qquad - |x\cdot\nabla u|^2|x|_{A^{-1}(0)}^{-a}|\nabla\zeta|_{A(0)}^2 + \frac{a}{2}|x\cdot\nabla u|^2|x|_{A^{-1}(0)}^{-a-2}\big(x\cdot\nabla(\zeta^2)\big).\end{aligned} \tag{1.5.15}$$

From the identities (1.5.14) and (1.5.15), by (1.5.11), it follows that

$$
\begin{aligned}
(n-2-a)\int_{B_1} |x|^{-a}_{A^{-1}(0)}|\nabla u|^2_{A(0)}\zeta^2\,\mathrm{d}x + \frac{a(8-a)}{4}\int_{B_1}|x|^{-a-2}_{A^{-1}(0)}|x\cdot\nabla u|^2\zeta^2\,\mathrm{d}x \\
\le C\int_{B_1}\big(r^{2-a}|\nabla\zeta|^2 + r^{1-a}|\nabla(\zeta^2)|\big)|\nabla u|^2\,\mathrm{d}x + C\varepsilon\int_{B_1} r^{2-a}|D^2u||\nabla u|\zeta^2\,\mathrm{d}x \\
+ C\varepsilon\int_{B_1}\big(r^{3-a}|\nabla\zeta|^2 + r^{2-a}|\nabla(\zeta^2)|\big)|\nabla u|^2\,\mathrm{d}x \\
+ C\varepsilon\int_{B_1}\big(r^{1-a}+\varepsilon r^{2-a}\big)|\nabla u|^2\zeta^2\,\mathrm{d}x,
\end{aligned}
\tag{1.5.16}
$$

for some universal constant C, where we have controlled the remainder terms thanks to the uniform ellipticity and the fact that a is bounded by a dimensional constant.

For $0<\rho\le 1/2$ as in the statement, we consider a Lipschitz function ζ, with $0\le\zeta\le 1$, such that $\zeta|_{B_\rho}=1$, $\operatorname{supp}\zeta\subset\overline{B}_{2\rho}$, and $|\nabla\zeta|\le C/\rho$. Plugging this cutoff function in (1.5.16), using that r is comparable with ρ inside $\operatorname{supp}\nabla\zeta\subset\overline{B}_{2\rho}\setminus B_\rho$, we deduce that

$$
\begin{aligned}
(n-2-a)\int_{B_1} |x|^{-a}_{A^{-1}(0)}|\nabla u|^2_{A(0)}\zeta^2\,\mathrm{d}x + \frac{a(8-a)}{4}\int_{B_1}|x|^{-a-2}_{A^{-1}(0)}|x\cdot\nabla u|^2\zeta^2\,\mathrm{d}x \\
\le C\int_{B_{2\rho}\setminus B_\rho} r^{-a}|\nabla u|^2\,\mathrm{d}x + C\varepsilon\int_{B_{2\rho}} r^{2-a}|D^2u||\nabla u|\,\mathrm{d}x \\
+ C\varepsilon\int_{B_{2\rho}}\big(r^{1-a}+\varepsilon r^{2-a}\big)|\nabla u|^2\,\mathrm{d}x.
\end{aligned}
\tag{1.5.17}
$$

Since a is in the range (1.5.12), the constants in the left-hand side of (1.5.17) are nonnegative. Moreover, by uniform ellipticity we have $|x|_{A^{-1}(0)}\le c_0^{-1/2}|x|$ and $|\nabla u|^2_{A(0)}\ge c_0|\nabla u|^2$, hence, since $\zeta|_{B_\rho}=1$ and $\zeta\ge 0$, it follows that

$$
\begin{aligned}
\lambda^{a/2+1}\left((n-2-a)\int_{B_\rho} r^{-a}|\nabla u|^2\,\mathrm{d}x + \frac{a(8-a)}{4}\int_{B_\rho} r^{-a}u_r^2\,\mathrm{d}x\right) \\
\le (n-2-a)\int_{B_1} |x|^{-a}_{A^{-1}(0)}|\nabla u|^2_{A(0)}\zeta^2\,\mathrm{d}x + \frac{a(8-a)}{4}\int_{B_1}|x|^{-a-2}_{A^{-1}(0)}|x\cdot\nabla u|^2\zeta^2\,\mathrm{d}x.
\end{aligned}
\tag{1.5.18}
$$

Using (1.5.18) in (1.5.17) and multiplying by $\lambda^{-a/2-1}$ now yields the claim. □

We can finally prove Proposition 1.1.4. For this, we will apply Lemma 1.5.4 with the exponent $a=n-2$. The key point in the proof will be to control the weighted L^1 norm of $|D^2u||\nabla u|$ in the right-hand side of (1.5.13) by a weighted L^2 norm of the gradient. We obtain this bound by writing the integral as an infinite sum on dyadic annuli and by using that the weight in each annulus can be pulled out of the integral. This allows us to apply the non-weighted a priori estimate for the “Hessian times the gradient” (1.1.14) from Theorem 1.1.2.

Proof of Proposition 1.1.4. Since $3\le n\le 9$, we have that $\min\{10,n\}-2=n-2$ and we may choose the exponent $a = n - 2$ in Lemma 1.5.4, which yields the inequality

$$
\begin{aligned}
&\frac{(n-2)(10-n)}{4} \int_{B_\rho} r^{2-n} u_r^2 \, \mathrm{d}x \\
&\qquad \le C \int_{B_{2\rho} \setminus B_\rho} r^{2-n} |\nabla u|^2 \, \mathrm{d}x + C\varepsilon \int_{B_{2\rho}} r^{4-n} |D^2 u| |\nabla u| \, \mathrm{d}x \\
&\qquad\qquad + C\varepsilon \int_{B_{2\rho}} \big(r^{3-n} + \varepsilon r^{4-n}\big) |\nabla u|^2 \, \mathrm{d}x.
\end{aligned} \tag{1.5.19}
$$

To bound the Hessian term $\int_{B_{2\rho}} r^{4-n} |D^2 u||\nabla u| \, \mathrm{d}x$ on the right-hand side of (1.5.19), we will apply the a priori estimate on annuli (1.1.14) from Theorem 1.1.2 at different scales.

Let $r_j := 2^{-j}$ with $j \ge 0$. As explained in Section 1.2, the functions $u(r_j \cdot)$ are stable solutions to a semilinear equation with coefficients $A(r_j \cdot)$ and $r_j b(r_j \cdot)$. In particular, since $\|DA\|_{C^0(\overline{B}_1)} + \|b\|_{C^0(\overline{B}_1)} \le \varepsilon$, we also have $\|DA(r_j \cdot)\|_{L^\infty(B_1)} + \|r_j b(r_j \cdot)\|_{L^\infty(B_1)} \le \varepsilon r_j \le \varepsilon$. Hence, by (1.1.14) in Theorem 1.1.2 applied to $u(r_j \cdot)$, there is a universal $\varepsilon_0 > 0$ with the following property: if $\varepsilon \le \varepsilon_0$, then

$$
\int_{B_{r_{j+1}} \setminus B_{r_{j+2}}} |D^2 u||\nabla u| \, \mathrm{d}x \le C r_j^{-1} \int_{B_{r_j} \setminus B_{r_{j+3}}} |\nabla u|^2 \, \mathrm{d}x \quad \text{for all } j \ge 0, \tag{1.5.20}
$$

where C is a universal constant.

Writing the weighted integral as an infinite sum on annuli, we have

$$
\begin{aligned}
\int_{B_{1/2}} r^{4-n} |D^2 u||\nabla u| \, \mathrm{d}x &= \sum_{j=0}^{\infty} \int_{B_{r_{j+1}} \setminus B_{r_{j+2}}} r^{4-n} |D^2 u||\nabla u| \, \mathrm{d}x \\
&\le C \sum_{j=0}^{\infty} r_j^{4-n} \int_{B_{r_{j+1}} \setminus B_{r_{j+2}}} |D^2 u||\nabla u| \, \mathrm{d}x,
\end{aligned} \tag{1.5.21}
$$

where in the last line we have used that $r^{4-n} \le C r_j^{4-n}$ in $B_{r_{j+1}} \setminus B_{r_{j+2}}$, with C universal. Multiplying (1.5.20) by r_j^{4-n} and summing in j, the right-hand side in (1.5.21) can be bounded by

$$
\begin{aligned}
\sum_{j=0}^{\infty} r_j^{4-n} \int_{B_{r_{j+1}} \setminus B_{r_{j+2}}} |D^2 u||\nabla u| \, \mathrm{d}x &\le C \sum_{j=0}^{\infty} r_j^{3-n} \int_{B_{r_j} \setminus B_{r_{j+3}}} |\nabla u|^2 \, \mathrm{d}x \\
&\le C \sum_{j=0}^{\infty} \int_{B_{r_j} \setminus B_{r_{j+3}}} r^{3-n} |\nabla u|^2 \, \mathrm{d}x \\
&\le C \int_{B_1} r^{3-n} |\nabla u|^2 \, \mathrm{d}x.
\end{aligned} \tag{1.5.22}
$$

Combining (1.5.21) and (1.5.22), we deduce

$$
\int_{B_{1/2}} r^{4-n} |D^2 u||\nabla u| \, \mathrm{d}x \le C \int_{B_1} r^{3-n} |\nabla u|^2 \, \mathrm{d}x, \tag{1.5.23}
$$

where C is universal. Applying (1.5.23) to the stable solutions $u(4\rho \cdot)$, there is a universal $\varepsilon_0 > 0$ with the following property: if $\varepsilon \le \varepsilon_0$, then

$$
\int_{B_{2\rho}} r^{4-n} |D^2 u||\nabla u| \, \mathrm{d}x \le C \int_{B_{4\rho}} r^{3-n} |\nabla u|^2 \, \mathrm{d}x \quad \text{for all } \rho \le 1/4, \tag{1.5.24}
$$

where C is a universal constant.

Applying (1.5.24) in (1.5.19), we deduce the key estimate

$$\begin{aligned}\frac{(n-2)(10-n)}{4}&\int_{B_\rho} r^{2-n}u_r^2\,\mathrm{d}x\\ &\le C\int_{B_{2\rho}\setminus B_\rho} r^{2-n}|\nabla u|^2\,\mathrm{d}x + C\varepsilon\int_{B_{4\rho}} r^{3-n}|\nabla u|^2\,\mathrm{d}x \\ &\qquad\qquad\qquad\qquad\text{for } \rho\le 1/4 \text{ and } \varepsilon\le\varepsilon_0,\end{aligned} \tag{1.5.25}$$

where we have additionally bounded the last integrand in (1.5.19) by $\varepsilon r^{4-n}\le Cr^{3-n}$. Finally, since $(n-2)(10-n)>0$, absorbing this constant on the right-hand side of (1.5.25) yields the claim. □

Remark 1.5.5. Our proof in Section 1.7 of the C^α bound (1.1.10) from Theorem 1.1.1 controls the weighted integral $\int_{B_\rho} r^{2-n}|\nabla u|^2\,\mathrm{d}x$. It will require a delicate estimate proven in Section 1.6. As a consequence, we will also obtain a bound for the less singular error terms $\int_{B_\rho} r^{3-n}|\nabla u|^2\,\mathrm{d}x$. Here we point out that this last quantity can be estimated directly from our previous Lemma 1.5.4, without the use of Section 1.6. This is done as follows.

Letting $a=n-3$ in Lemma 1.5.4, since $a(8-a)=(n-3)(11-n)\ge 0$ for $3\le n\le 11$, we can drop the radial term in (1.5.13) and the left-hand side becomes $\int_{B_\rho} r^{3-n}|\nabla u|^2\,\mathrm{d}x$. The right-hand side now includes an error term $\varepsilon\int_{B_{2\rho}} r^{5-n}|D^2u||\nabla u|\,\mathrm{d}x$, which can be bounded by $\varepsilon\int_{B_{4\rho}} r^{4-n}|\nabla u|^2\,\mathrm{d}x$ for $\varepsilon\le\varepsilon_0$ as in the proof of Proposition 1.1.4. Hence, we obtain

$$\int_{B_\rho} r^{3-n}|\nabla u|^2\,\mathrm{d}x \le C\int_{B_{4\rho}\setminus B_\rho} r^{3-n}|\nabla u|^2\,\mathrm{d}x + C\varepsilon\int_{B_\rho} r^{4-n}|\nabla u|^2\,\mathrm{d}x$$

for $\rho\le 1/4$. Making ε_0 smaller if necessary, since $r^{4-n}\le r^{3-n}$ in B_ρ, we can absorb the rightmost term into the left-hand side. This leads to an expression which can be hole-filled, and by a standard iteration argument it is easy to deduce the decay estimate

$$\int_{B_\rho} r^{3-n}|\nabla u|^2\,\mathrm{d}x \le C\rho^{2\alpha}\|\nabla u\|^2_{L^2(B_1)} \quad \text{for all } \rho\le 1/4, \tag{1.5.26}$$

where $\alpha>0$ and C are universal constants. We note that, adding superfluous variables, the decay (1.5.26) is satisfied when $n\le 11$.

In general, this strategy allows to prove the decay of weighted integrals of the form $\int_{B_\rho} r^{-a}|\nabla u|^2\,\mathrm{d}x$ for exponents $a<n-2$ and $a\le 8$. As an application, we could extend the optimal regularity result of Peng, Zhang, and Zhou [81] for stable solutions in dimensions $n\ge 10$ to our setting of variable coefficients. The key point in [81] is to find an a a priori estimate of the form $\rho^{-a_n}\int_{B_\rho}|\nabla u|^2\,\mathrm{d}x\le C\|\nabla u\|^2_{L^2(B_1)}$, where $a_n=2(1+\sqrt{n-1})$ is a critical exponent. When considering variable coefficients, one has to deal with error terms $\varepsilon\int_{B_\rho} r^{1-a_n}|\nabla u|^2\,\mathrm{d}x$. Our method above gives the decay of these error terms whenever $10\le n\le 13$. For this, we choose $a=a_n-1$ in (1.5.13), and notice that $a_n-1<n-2$ and $a_n-1\le 8$ in this dimension range. The case $n\ge 14$ can be treated similarly using that $a=a_n-1\ge 8$ and $u_r^2\le|\nabla u|^2$ in (1.5.13).

Remark 1.5.6. The key estimate in the proof of Proposition 1.1.4 is an inequality for weighted integrals,

$$\int_{B_{2\rho}} r^{4-n}|D^2u||\nabla u|\,\mathrm{d}x \le C\int_{B_{4\rho}} r^{3-n}|\nabla u|^2\,\mathrm{d}x, \tag{1.5.27}$$

which has been proved decomposing the integral in dyadic annuli. There is a way to prove a weaker inequality than (1.5.27), namely,

$$\begin{aligned}\int_{B_{2\rho}} r^{4-n}|D^2u||\nabla u|\,\mathrm{d}x \le C\left(\int_{B_{4\rho}} r^{2-n}|\nabla u|^2\,\mathrm{d}x\right)^{1/2}\left(\int_{B_{4\rho}} r^{4-n}|\nabla u|^2\,\mathrm{d}x\right)^{1/2}\\ + C\int_{B_{4\rho}} r^{3-n}|\nabla u|^2\,\mathrm{d}x,\end{aligned} \tag{1.5.28}$$

which avoids the use of annuli and which involves a very recent test function of Cabré [17] and Peng, Zhang, and Zhou [82].

To obtain (1.5.28), one uses the inequality (1.3.33) from the proof of Theorem 1.1.2 with the singular test function $\eta = r^{-n/2}\zeta$, where $\zeta \in C_c^\infty(B_1)$ is a cut-off. It is worth noting that this inequality relies on the Sternberg-Zumbrun estimate for $\int_{B_1}\mathcal{A}^2\eta^2\,\mathrm{d}x$, which comes from choosing the test function $\xi = |\nabla u|\eta = |\nabla u|r^{-n/2}\zeta$ in the integral stability inequality (1.1.8). A function of this form had already appeared in [17], where the author considered $\xi = |\nabla u|r^{-(n+\varepsilon)/2}\zeta$ to prove the boundedness of stable solutions for unsigned nonlinearities when $n \le 4$. Interestingly, our choice $\xi = |\nabla u|r^{-n/2}$ coincides with the test function used by the authors in [82], where they obtained an a priori Hölder estimate for stable solutions when $n \le 5$. While their strategy involves integrating by parts an expression that is already quadratic in the gradient, expressing the new "coupled" Hessian errors $D^2u\nabla u$ as the uncoupled expression $\Delta u\nabla u$ and an error term related to $\mathcal{A}_0$, we do not integrate by parts.

By Young's inequality, the coefficient in front of $\int_{B_{4\rho}} r^{2-n}|\nabla u|^2\,\mathrm{d}x$, the leading order term in (1.5.28), can be made arbitrarily small. This error in a larger ball can still be treated with our methods to yield the Hölder estimate from Theorem 1.1.1.

1.6 The radial derivative controls the function in L^1

The goal of this section is to prove Proposition 1.1.5, where we control the L^1 norm of (generalized) superharmonic functions $Lu \le 0$ by the L^1 norm of the radial derivative on annuli. By a comparison argument, it will suffice to derive the analogue L^1 estimate on spheres for harmonic functions $Lv = 0$, which are obtained by duality from the L^∞ estimates of a Neumann problem.

Let $g \in C^\infty(\partial B_1)$, and consider the Neumann problem in divergence form

$$\begin{cases} \operatorname{div}(A(x)\nabla\varphi) = 0 & \text{in } B_1\\ A(x)\nabla\varphi\cdot\nu = g & \text{on } \partial B_1,\end{cases} \tag{1.6.1}$$

which admits solutions if and only if $\int_{\partial B_1} g\,\mathrm{d}\mathcal{H}^{n-1} = 0$. Recall that the solutions of (1.6.1) are unique up to addition of constants. We will derive an L^∞ a priori estimate for the zero mean solutions of (1.6.1) in terms of the conormal derivative

$$N\varphi := A(x)\nabla\varphi\cdot\frac{x}{|x|}. \tag{1.6.2}$$

This is achieved by a Moser iteration based on the following Sobolev trace inequality

$$\|u\|^2_{L^{2^\star}(\partial B_1)} \le C\big(\|\nabla u\|^2_{L^2(B_1)} + \|u\|^2_{L^2(\partial B_1)}\big) \tag{1.6.3}$$

for $u \in W^{1,2}(B_1)$, where C depends only on n and

$$2^\star := \frac{n-1}{n-2}2 \tag{1.6.4}$$

is the Sobolev trace exponent. We give a short proof of this inequality in Appendix B.

Our proof by Moser iteration is inspired by the one of Winkert in [96], where he obtains L^∞ estimates for general quasilinear Neumann problems. While the author employs certain technical interpolation and trace inequalities from the theory of Besov and Lizorkin-Triebel spaces, we only need the basic trace inequality (1.6.3), for which we give an elementary proof.

Lemma 1.6.1. *Let φ be the unique solution of* (1.6.1) *with $\int_{B_1} \varphi \, \mathrm{d}x = 0$. Then*

$$\|\varphi\|_{L^\infty(B_1)} \le C\|g\|_{L^\infty(\partial B_1)},$$

where C depends only on n and c_0.

Proof. Dividing φ by the norm $\|g\|_{L^\infty(\partial B_1)}$, we may assume $\|g\|_{L^\infty(\partial B_1)} = 1$. By the maximum principle, it suffices to bound the L^∞ norm of φ on the sphere ∂B_1. In this proof, C always denotes a generic constant depending only on n and c_0.

First, we obtain a basic energy estimate. Multiplying the equation (1.6.1) by φ and integrating by parts, we have

$$\int_{B_1} |\nabla \varphi|^2_{A(x)} \, \mathrm{d}x = \int_{\partial B_1} g\varphi \, \mathrm{d}\mathcal{H}^{n-1}. \tag{1.6.5}$$

Combining the standard trace inequality $\|u\|^2_{L^2(\partial B_1)} \le C\big(\|\nabla u\|^2_{L^2(B_1)} + \|u\|^2_{L^2(B_2)}\big)$ with the Poincaré inequality in the ball (recall that $\int_{B_1} \varphi = 0$), we also have

$$\|\varphi\|_{L^2(\partial B_1)} \le C\|\nabla \varphi\|_{L^2(B_1)}. \tag{1.6.6}$$

Hence, applying Cauchy-Schwarz in (1.6.5), by (1.6.6) we obtain

$$\int_{B_1} |\nabla \varphi|^2_{A(x)} \, \mathrm{d}x \le \|g\|_{L^2(\partial B_1)}\|\varphi\|_{L^2(\partial B_1)} \le C\|g\|_{L^2(\partial B_1)}\|\nabla\varphi\|_{L^2(B_1)}. \tag{1.6.7}$$

By uniform ellipticity and the bound $\|g\|_{L^\infty(B_1)} = 1$, from (1.6.7) it follows that

$$\|\nabla \varphi\|_{L^2(B_1)} \le C. \tag{1.6.8}$$

Therefore, by (1.6.3), (1.6.6), and (1.6.8) we deduce the a priori estimate

$$\|\varphi\|_{L^{2^\star}(\partial B_1)} \le C. \tag{1.6.9}$$

Next, we derive an L^∞ bound for the positive part of the solution by Moser iteration. Let $m \ge 2$. Multiplying the equation by the power $(\varphi^+)^{m-1}$ and integrating by parts

$$(m-1)\int_{B_1} (\varphi^+)^{m-2}|\nabla \varphi^+|^2_{A(x)} \, \mathrm{d}x = \int_{\partial B_1} (\varphi^+)^{m-1} g \, \mathrm{d}\mathcal{H}^{n-1}. \tag{1.6.10}$$

Since

$$|\nabla(\varphi^+)^{\frac{m}{2}}|^2_{A(x)} = \frac{m^2}{4}(\varphi^+)^{m-2}|\nabla(\varphi^+)|^2_{A(x)},$$

by (1.6.10) and the uniform ellipticity, using that $\frac{m}{m-1} \leq 2$ for $m \geq 2$, we have

$$\int_{B_1} |\nabla(\varphi^+)^{\frac{m}{2}}|^2 \,\mathrm{d}x \leq Cm \int_{\partial B_1} (\varphi^+)^{m-1} \,\mathrm{d}\mathcal{H}^{n-1}. \tag{1.6.11}$$

Adding the integral $\int_{\partial B_1} (\varphi^+)^m \,\mathrm{d}\mathcal{H}^{n-1}$ to both sides of (1.6.11), we have

$$\|\nabla(\varphi^+)^{\frac{m}{2}}\|^2_{L^2(B_1)} + \|(\varphi^+)^{\frac{m}{2}}\|^2_{L^2(\partial B_1)} \leq Cm\|(\varphi^+)^{m-1}\|_{L^1(\partial B_1)} + \|(\varphi^+)^{\frac{m}{2}}\|^2_{L^2(\partial B_1)}$$

and applying the Sobolev trace inequality (1.6.3) on the left-hand side yields

$$\|\varphi^+\|^m_{L^{\frac{2^\star}{2}m}(\partial B_1)} \leq Cm\|(\varphi^+)^{m-1}\|_{L^1(\partial B_1)} + C\|\varphi^+\|^m_{L^m(\partial B_1)}. \tag{1.6.12}$$

By Hölder and since $m \geq 2$, the L^{m-1} norm in (1.6.12) can be bounded by

$$\|(\varphi^+)^{m-1}\|_{L^1(\partial B_1)} \leq |\partial B_1|^{\frac{1}{m}} \|\varphi^+\|^{m-1}_{L^m(\partial B_1)} \leq C\|\varphi^+\|^{m-1}_{L^m(\partial B_1)},$$

and hence

$$\|\varphi^+\|^m_{L^{\frac{2^\star}{2}m}(\partial B_1)} \leq Cm\|\varphi^+\|^{m-1}_{L^m(\partial B_1)} + C\|\varphi^+\|^m_{L^m(\partial B_1)}. \tag{1.6.13}$$

Since $\|\varphi^+\|^{m-1}_{L^m(\partial B_1)} \leq \max\{1, \|\varphi^+\|_{L^m(\partial B_1)}\}^m$, from (1.6.13) it follows that

$$\|\varphi^+\|_{L^{\frac{2^\star}{2}m}(\partial B_1)} \leq C^{\frac{1}{m}} m^{\frac{1}{m}} \max\{1, \|\varphi^+\|_{L^m(\partial B_1)}\}. \tag{1.6.14}$$

We wish to iterate (1.6.14). Let $m_0 := 2^\star$ and, for $k \in \mathbb{N}$, let

$$m_k := \left(\frac{2^\star}{2}\right)^k m_0.$$

By (1.6.14) and the definition of m_k, we have

$$\begin{aligned}\|\varphi^+\|_{L^{m_k}(\partial B_1)} &\leq C^{\frac{1}{m_{k-1}}} m_{k-1}^{\frac{1}{m_{k-1}}} \max\{1, \|\varphi^+\|_{L^{m_{k-1}}(\partial B_1)}\} \\ &= (Cm_0)^{\frac{1}{m_0}\left(\frac{2}{2^\star}\right)^{k-1}} \left(\frac{2^\star}{2}\right)^{\frac{k-1}{m_0}\left(\frac{2}{2^\star}\right)^{k-1}} \max\{1, \|\varphi^+\|_{L^{m_{k-1}}(\partial B_1)}\}.\end{aligned} \tag{1.6.15}$$

We have exactly one of the following three cases:

- Case 1:
$$\|\varphi^+\|_{L^{m_k}(\partial B_1)} \leq 1 \tag{1.6.16}$$

- Case 2: there is an $l \in \{1, 2, \ldots, k-1\}$ such that
$$\|\varphi^+\|_{L^{m_{k+1-i}}(\partial B_1)} > 1 \quad \text{for } 1 \leq i \leq l, \quad \text{and} \quad \|\varphi^+\|_{L^{m_{k-l}}(\partial B_1)} \leq 1. \tag{1.6.17}$$

- Case 3:
$$\|\varphi^+\|_{L^{m_{k+1-i}}(\partial B_1)} > 1 \quad \text{for } 1 \leq i \leq k. \tag{1.6.18}$$

Case 1 already yields a uniform bound for $\|\varphi^+\|_{L^{m_k}(B_1)}$. If Case 2 holds then, iterating (1.6.15) $l-1$ times, we arrive at

$$\|\varphi^+\|_{L^{m_k}(\partial B_1)} \le (Cm_0)^{\frac{1}{m_0}\sum_{j=k-l}^{k-1}\left(\frac{2}{2^\star}\right)^j}\left(\frac{2^\star}{2}\right)^{\frac{1}{m_0}\sum_{j=k-l}^{k-1} j\left(\frac{2}{2^\star}\right)^j}. \tag{1.6.19}$$

The right-hand side of (1.6.19) is nondecreasing in l for (say) $C \ge 1$, which we can always assume. Finally, if Case 3 holds then, iterating (1.6.15), we obtain

$$\|\varphi^+\|_{L^{m_k}(\partial B_1)} \le (Cm_0)^{\frac{1}{m_0}\sum_{j=0}^{k-1}\left(\frac{2}{2^\star}\right)^j}\left(\frac{2^\star}{2}\right)^{\frac{1}{m_0}\sum_{j=0}^{k-1} j\left(\frac{2}{2^\star}\right)^j}\|\varphi^+\|_{L^{m_0}(\partial B_1)}. \tag{1.6.20}$$

By the monotonicity of (1.6.19) in l and using the a priori estimate (1.6.9) for $\|\varphi\|_{L^{m_0}(\partial B_1)}$ in (1.6.20), we see that in all three cases above we have

$$\|\varphi^+\|_{L^{m_k}(\partial B_1)} \le C(Cm_0)^{\frac{1}{m_0}\sum_{j=0}^{k-1}\left(\frac{2}{2^\star}\right)^j}\left(\frac{2^\star}{2}\right)^{\frac{1}{m_0}\sum_{j=0}^{k-1} j\left(\frac{2}{2^\star}\right)^j},$$

and since the exponent on the right-hand side is uniformly bounded, we deduce

$$\|\varphi^+\|_{L^{m_k}(\partial B_1)} \le C. \tag{1.6.21}$$

Taking the limit as $k \to \infty$ in (1.6.21) now yields

$$\|\varphi^+\|_{L^\infty(\partial B_1)} \le C,$$

which is the desired L^∞ estimate for the positive part of the solutions. The same argument gives an a priori estimate for the negative part φ^- and yields the claim. □

By duality, from the L^∞ estimate in Lemma 1.6.1 we deduce an L^1 bound for the elliptic problem with a source:

Lemma 1.6.2. *Given $h \in C^\infty(\overline{B_1})$, let $v \in C^\infty(\overline{B}_1)$ satisfy*

$$\operatorname{div}(A(x)\nabla v) + h(x) = 0 \quad \text{in } B_1.$$

Then

$$\|v-t\|_{L^1(\partial B_1)} \le C\|Nv\|_{L^1(\partial B_1)} + C\|h\|_{L^1(B_1)},$$

where $t := \inf\{\bar t : |\{v > \bar t\} \cap \partial B_1| \le |\partial B_1|/2\}$ and C depends only on n and c_0.

Proof. Replacing v by $v-t$ we may assume that $t = 0$, therefore $|\{v>0\}\cap\partial B_1| \le |\partial B_1|/2$ and $|\{v<0\}\cap\partial B_1| \le |\partial B_1|/2$. The function $\operatorname{sgn}(v) = v/|v|$ in $v \ne 0$ can then be extended to $\{v = 0\}\cap\partial B_1$, taking values ±1 and in such a way that $\int_{\partial B_1}\operatorname{sgn}(v)\,d\mathcal{H}^{n-1} = 0$. In particular, $|v| = v\operatorname{sgn}(v)$ on ∂B_1.

We define the convolutions on ∂B_1

$$g_k := \operatorname{sgn}(v) \star \eta_k,$$

where $\{\eta_k\}$ is a sequence of smooth mollifiers on ∂B_1. We have $g \in C^\infty(\partial B_1)$, $|g_k| \le 1$, and $\int_{\partial B_1} g_k\,d\mathcal{H}^{n-1} = 0$ since $\operatorname{sgn}(v)$ has zero average on ∂B_1. Moreover, it holds that

$$\int_{\partial B_1}|v|\,d\mathcal{H}^{n-1} = \lim_k \int_{\partial B_1} v g_k\,d\mathcal{H}^{n-1}. \tag{1.6.22}$$

Since g_k has zero average on ∂B_1, we can uniquely solve the Neumann problem

$$\begin{cases} \operatorname{div}(A(x)\nabla\varphi_k) = 0 & \text{in } B_1 \\ N\varphi_k = g_k & \text{on } \partial B_1 \end{cases}$$

imposing additionally that $\int_{B_1} \varphi_k \, \mathrm{d}x = 0$. By Lemma 1.6.1, we deduce

$$\|\varphi_k\|_{L^\infty(\partial B_1)} \le C, \tag{1.6.23}$$

where C depends only on n and c_0. Notice that, integrating by parts, we have

$$\begin{aligned} \int_{\partial B_1} v g_k \, \mathrm{d}\mathcal{H}^{n-1} &= \int_{\partial B_1} v N\varphi_k \, \mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{\partial B_1} (Nv)\varphi_k \, \mathrm{d}\mathcal{H}^{n-1} - \int_{B_1} \operatorname{div}(A(x)\nabla v)\varphi_k \, \mathrm{d}x \\ &= \int_{\partial B_1} (Nv)\, \varphi_k \, \mathrm{d}\mathcal{H}^{n-1} + \int_{B_1} h\, \varphi_k \, \mathrm{d}x, \end{aligned}$$

where in the last equality we have used the equation satisfied by v. Hence, by (1.6.23)

$$\left| \int_{\partial B_1} v g_k \, \mathrm{d}\mathcal{H}^{n-1} \right| \le C \int_{\partial B_1} |Nv| \, \mathrm{d}\mathcal{H}^{n-1} + C \int_{B_1} |h| \, \mathrm{d}x,$$

and the claim follows by (1.6.22). □

We now use the previous estimates on spheres to obtain L^1 bounds on annuli for a divergence-form operator with drift. The drift term will be treated as a source, which will appear as an error in the right-hand side of the estimate. If the coefficient matrix $A(x)$ is close to the identity, then the conormal derivative Nu is close to the radial derivative u_r. Hence, we will obtain Proposition 1.1.5 as a corollary of the following:

Proposition 1.6.3. *Let $u \in C^\infty(\overline{B}_1)$ be a supersolution $\mathcal{L}u \le 0$ in B_1, where $\mathcal{L}$ is the operator $\mathcal{L}u = \operatorname{div}(A(x)\nabla u) + d(x) \cdot \nabla u$. Assume that*

$$\|d\|_{C^0(\overline{B}_1)} \le \varepsilon$$

for some $\varepsilon > 0$.

Then there exists a constant t, which depends on u, such that

$$\|u - t\|_{L^1(B_1 \setminus B_{1/8})} \le C\|u_r\|_{L^1(B_1 \setminus B_{1/8})} + C\|Nu\|_{L^1(B_1 \setminus B_{1/8})} + C\varepsilon\|\nabla u\|_{L^1(B_1)},$$

where C is a constant depending only on n and c_0.

Proof. Since $\|Nu\|_{L^1(B_1 \setminus B_{1/8})} = \int_{1/8}^1 \mathrm{d}r \int_{\partial B_r} |Nu| \, \mathrm{d}\mathcal{H}^{n-1}$, by the mean value theorem

$$\|Nu\|_{L^1(B_1 \setminus B_{1/8})} = \frac{7}{8} \int_{\partial B_\rho} |Nu| \, \mathrm{d}\mathcal{H}^{n-1} \tag{1.6.24}$$

for some $\rho \in [1/8, 1]$. Let v be the unique solution of the boundary value problem

$$\begin{cases} \operatorname{div}(A(x)\nabla v) + d(x) \cdot \nabla u = 0 & \text{in } B_\rho \\ v = u & \text{on } \partial B_\rho. \end{cases}$$

Since $\operatorname{div}(A(x)\nabla(u-v)) \le 0$ in B_ρ, by the comparison principle $u \ge v$ in B_ρ. Moreover, using that $u = v$ on ∂B_ρ, we deduce that $Nu \le Nv$ on ∂B_ρ. In particular, this gives

$$(Nv)^- \le (Nu)^- \text{ on } \partial B_\rho. \tag{1.6.25}$$

Notice also that, integrating the equation $\operatorname{div}(A(x)\nabla v) = -d(x)\cdot\nabla u$ in B_ρ, by the divergence theorem we have

$$\left|\int_{\partial B_\rho} Nv\,\mathrm{d}\mathcal{H}^{n-1}\right| = \left|\int_{B_\rho} \operatorname{div}\left(A(x)\nabla v\right)\mathrm{d}x\right| = \left|-\int_{B_\rho} d(x)\cdot\nabla u\,\mathrm{d}x\right| \le \varepsilon\|\nabla u\|_{L^1(B_\rho)}$$

and since $\int_{\partial B_\rho} Nv\,\mathrm{d}\mathcal{H}^{n-1} = \|(Nv)^+\|_{L^1(B_\rho)} - \|(Nv)^-\|_{L^1(B_\rho)}$, we deduce

$$\|(Nv)^+\|_{L^1(B_\rho)} \le \|(Nv)^-\|_{L^1(B_\rho)} + \varepsilon\|\nabla u\|_{L^1(B_\rho)}. \tag{1.6.26}$$

Using (1.6.26) and (1.6.25) we obtain

$$\begin{aligned}\|Nv\|_{L^1(\partial B_\rho)} &= \|(Nv)^-\|_{L^1(\partial B_\rho)} + \|(Nv)^+\|_{L^1(\partial B_\rho)} \\ &\le 2\|(Nv)^-\|_{L^1(\partial B_\rho)} + \varepsilon\|\nabla u\|_{L^1(B_\rho)} \\ &\le 2\|Nu\|_{L^1(\partial B_\rho)} + \varepsilon\|\nabla u\|_{L^1(B_\rho)}.\end{aligned} \tag{1.6.27}$$

Applying Lemma 1.6.2 with coefficients $A(\rho\,\cdot)$ and source $h(x) = \rho^2 d(\rho x)\cdot\nabla u(\rho x)$ to the function $v(\rho\,\cdot)$ yields the estimate

$$\|v-t\|_{L^1(\partial B_\rho)} \le C\rho\|Nv\|_{L^1(\partial B_\rho)} + C\rho\varepsilon\|\nabla u\|_{L^1(B_\rho)}. \tag{1.6.28}$$

Since $u - t = v - t$ on ∂B_ρ, combining (1.6.28) and (1.6.27), we obtain

$$\|u-t\|_{L^1(\partial B_\rho)} \le C\rho\|Nu\|_{L^1(\partial B_\rho)} + C\rho\varepsilon\|\nabla u\|_{L^1(B_\rho)},$$

and since $\rho \in [1/8, 1]$, by (1.6.24), we deduce that

$$\|u-t\|_{L^1(\partial B_\rho)} \le C\|Nu\|_{L^1(B_1\setminus B_{1/8})} + C\varepsilon\|\nabla u\|_{L^1(B_1)}. \tag{1.6.29}$$

To conclude the proof, it suffices to show that

$$\|u-t\|_{L^1(B_1\setminus B_{1/8})} \le C\|u-t\|_{L^1(\partial B_\rho)} + C\|u_r\|_{L^1(B_1\setminus B_{1/8})}. \tag{1.6.30}$$

Since $(u-t)(s\sigma) = (u-t)(\rho\sigma) - \int_s^\rho u_r(r\sigma)\,dr$ for every $s \in (1/8, 1)$ and $\sigma \in \partial B_1$, we have

$$s^{n-1}|(u-t)(s\sigma)| \le 8^{n-1}\rho^{n-1}|(u-t)(\rho\sigma)| + 8^{n-1}\int_{1/8}^1 r^{n-1}|u_r(r\sigma)|\,dr.$$

Integrating in $\sigma \in \partial B_1$, and then in $s \in (1/8, 1)$, we deduce (1.6.30). Combining (1.6.29) and (1.6.30) yields the claim. □

Proof of Proposition 1.1.5. We consider the operator $\mathcal{L}$ with $d(x) = \widehat{b}(x)$ given by (1.1.7), so that $\mathcal{L}u = Lu = \operatorname{div}(A(x)\nabla u) + \widehat{b}(x)\cdot\nabla u$. Since $A(0) = I$, writing the conormal derivative (1.6.2) as $Nu = u_r + (A(x)-I)\nabla u\cdot\frac{x}{|x|}$ and by the mean value theorem, we have $|Nu| \le |u_r| + C\varepsilon|\nabla u|$ in B_1. Applying Proposition 1.6.3 now, the conormal term on the right-hand side of the estimate can be bounded by $C\|u_r\|_{L^1(B_1\setminus B_{1/8})} + C\varepsilon\|\nabla u\|_{L^1(B_1)}$, hence the claim. □

1.7 Proof of the C^α estimate

This section is devoted to proving the Hölder regularity estimate (1.1.10) in Theorem 1.1.1. The main goal will be to show that the scale-invariant weighted integral $\int_{B_\rho} r^{2-n}|\nabla u|^2$ decays like a power $\rho^{2\alpha}$, since this implies a C^α bound of the original function. We will show this property under the additional assumption that the operator L is close to the Laplacian, i.e., assuming $A(0) = I$ and $\|DA\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)} \le \varepsilon$ with ε sufficiently small. An affine transformation will then lead to an estimate that is valid for all operators, with bounds depending on the norms of the coefficients.

The key idea is to write the weighted integral of the gradient as an infinite sum on dyadic annuli, pulling out the weights, and applying Propositions 1.1.3 and 1.1.5 in each annulus. This allows to control the weighted L^2 norm of the gradient by a weighted L^2 norm of the radial derivative. Once we have this bound, Proposition 1.1.4 will lead directly to the decay by a standard iteration argument. This will yield a bound of the C^α norm in terms of the L^2 norm of the gradient, which can be controlled by the L^1 norm of the solutions thanks to Proposition 1.1.3.

Proof of the Hölder estimate (1.1.10) *in Theorem 1.1.1.* We may assume that $3 \le n \le 9$. Indeed, when $n = 2$, we recover the estimate by applying Theorem 1.1.1 to the function $\widetilde{u}(x_1, x_2, x_3) := u(x_1, x_2)$, which is a stable solution to the elliptic equation $L\widetilde{u} + c_0\widetilde{u}_{x_3x_3} = f(\widetilde{u})$ in $B_1 \subset \mathbb{R}^3$. Similarly, when $n = 1$, one considers the function $\widetilde{u}(x_1, x_2, x_3) := u(x_1)$.

Throughout the proof, C denotes a generic universal constant unless stated otherwise. The proof is divided in three steps.

Step 1: *Under the assumption that*

$$A(0) = I \quad \text{and} \quad \|DA\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)} \le \varepsilon,$$

we prove that it $\varepsilon \le \varepsilon_0$, then

$$\int_{B_\rho} r^{2-n}|\nabla u|^2 \, \mathrm{d}x \le C\|\nabla u\|^2_{L^2(B_1)}\rho^{2\alpha} \quad \text{for all } \rho \le 1/8, \tag{1.7.1}$$

where $\alpha > 0$, $\varepsilon_0 > 0$, and C are universal constants.

As explained before, we will write the weighted Dirichlet integral as an infinite sum on dyadic annuli, similarly to what we did for the weighted Hessian estimates in the proof of Proposition 1.1.4. We treat the case $\rho = 1/2$ first, and then apply the scaling of the problem.

Let $r_j := 2^{-j}$ with $j \ge 0$. We have

$$\begin{aligned} \int_{B_{1/2}} r^{2-n}|\nabla u|^2 \, \mathrm{d}x &= \sum_{j=0}^{\infty} \int_{B_{r_{j+1}} \setminus B_{r_{j+2}}} r^{2-n}|\nabla u|^2 \, \mathrm{d}x \\ &\le C \sum_{j=0}^{\infty} r_j^{2-n} \int_{B_{r_{j+1}} \setminus B_{r_{j+2}}} |\nabla u|^2 \, \mathrm{d}x. \end{aligned} \tag{1.7.2}$$

We want to apply Proposition 1.1.3 on annuli to control the Dirichlet integrals in (1.7.2) by the L^1 norm of the solution, and then Proposition 1.1.5 to obtain bounds in terms of the radial derivative.

We cover the annulus $B_{1/2} \setminus B_{1/4}$ by a finite number of balls $B_{d/2}(y_j)$, where $d = d(n)$ is small enough so that $B_d(y_j) \subset B_1 \setminus B_{1/8}$. The number of balls depends only on n.

As explained in Section 1.2, the functions $u(y_j + d\cdot)$ are stable solutions to a semilinear equation with coefficients $A(y_j + d\cdot)$ and $d\,b(y_j. + d\cdot)$. Applying Proposition 1.1.3 to each $u(y_j + d\cdot)$, there is a universal $\varepsilon_0 > 0$ such that, for $\varepsilon \le \varepsilon_0$, we have

$$\begin{aligned}\|\nabla u\|^2_{L^2(B_{1/2}\setminus B_{1/4})} &\le \sum_j \|\nabla u\|^2_{L^2(B_{d/2}(y_j))} \le C\sum_j \|u\|^2_{L^1(B_{2d}(y_j))} \\ &\le C\|u\|^2_{L^1(B_1\setminus B_{1/8})}.\end{aligned} \tag{1.7.3}$$

For each $t \in \mathbb{R}$, the function $u - t$ is a stable solution to $-L\widetilde{u} = f(\widetilde{u} + t)$ in B_1. Hence, by (1.7.3), it follows that

$$\|\nabla u\|_{L^2(B_{1/2}\setminus B_{1/4})} \le C\|u - t\|_{L^1(B_1\setminus B_{1/8})} \quad \text{for all } t \in \mathbb{R} \text{ and } \varepsilon \le \varepsilon_0. \tag{1.7.4}$$

Since $A(0) = I$, we can choose t in (1.7.4) to be the constant in the conclusion of Proposition 1.1.5, and by this result we deduce

$$\|\nabla u\|_{L^2(B_{1/2}\setminus B_{1/4})} \le C\|u_r\|_{L^1(B_1\setminus B_{1/8})} + C\varepsilon\|\nabla u\|_{L^1(B_1)} \quad \text{for all } \varepsilon \le \varepsilon_0. \tag{1.7.5}$$

Squaring (1.7.5) and by Cauchy-Schwarz, we also have the weaker

$$\|\nabla u\|^2_{L^2(B_{1/2}\setminus B_{1/4})} \le C\|u_r\|^2_{L^2(B_1\setminus B_{1/8})} + C\varepsilon^2\|\nabla u\|^2_{L^2(B_1)} \quad \text{for all } \varepsilon \le \varepsilon_0. \tag{1.7.6}$$

Now we apply (1.7.6) to the rescaled functions $u(r_j\cdot)$, which gives (see the comments in (i) in Section 1.2)

$$\int_{B_{r_{j+1}}\setminus B_{r_{j+2}}} |\nabla u|^2\,\mathrm{d}x \le C\int_{B_{r_j}\setminus B_{r_{j+3}}} u_r^2\,\mathrm{d}x + C\varepsilon^2 r_j^2 \int_{B_{r_j}} |\nabla u|^2\,\mathrm{d}x \quad \text{for all } \varepsilon \le \varepsilon_0. \tag{1.7.7}$$

Hence, multiplying (1.7.7) by r_j^{2-n} and summing in j

$$\begin{aligned}&\sum_{j=0}^\infty r_j^{2-n}\int_{B_{r_{j+1}}\setminus B_{r_{j+2}}} |\nabla u|^2\,\mathrm{d}x \\ &\quad\le C\sum_{j=0}^\infty r_j^{2-n}\int_{B_{r_j}\setminus B_{r_{j+3}}} u_r^2\,\mathrm{d}x + C\varepsilon^2\sum_{j=0}^\infty r_j^{4-n}\int_{B_{r_j}} |\nabla u|^2\,\mathrm{d}x \\ &\quad\le C\sum_{j=0}^\infty \int_{B_{r_j}\setminus B_{r_{j+3}}} r^{2-n}u_r^2\,\mathrm{d}x + C\varepsilon^2\sum_{j=0}^\infty r_j\int_{B_{r_j}} r^{3-n}|\nabla u|^2\,\mathrm{d}x \quad \text{for all } \varepsilon \le \varepsilon_0,\end{aligned} \tag{1.7.8}$$

where in the last line we have used that $r_j^{3-n} \le r^{3-n}$ in B_{r_j} for $n \ge 3$. Since $r_j = 2^{-j}$, splitting the annuli into $B_{r_j}\setminus B_{r_{j+3}} = (B_{r_j}\setminus B_{r_{j+1}}) \cup (B_{r_{j+1}}\setminus B_{r_{j+2}}) \cup (B_{r_{j+2}}\setminus B_{r_{j+3}})$, we see that the first integral in the right-hand side of (1.7.8) is bounded by

$$\sum_{j=0}^\infty \int_{B_{r_j}\setminus B_{r_{j+3}}} r^{2-n}u_r^2 \le 3\int_{B_1} r^{2-n}u_r^2,$$

while the second can be bounded by

$$\sum_{j=0}^\infty r_j\int_{B_{r_j}} r^{3-n}|\nabla u|^2 \le \left(\sum_{j=0}^\infty r_j\right)\int_{B_1} r^{3-n}|\nabla u|^2 = 2\int_{B_1} r^{3-n}|\nabla u|^2.$$

From this, it follows that

$$\begin{aligned}\sum_{j=0}^{\infty} r_j^{2-n} \int_{B_{r_{j+1}}\setminus B_{r_{j+2}}} &|\nabla u|^2 \,\mathrm{d}x \\ &\le C\int_{B_1} r^{2-n}u_r^2 \,\mathrm{d}x + C\varepsilon^2 \int_{B_1} r^{3-n}|\nabla u|^2 \,\mathrm{d}x \quad \text{for all } \varepsilon \le \varepsilon_0.\end{aligned} \tag{1.7.9}$$

Combining (1.7.2) and (1.7.9) now yields

$$\int_{B_{1/2}} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \le C\int_{B_1} r^{2-n}u_r^2 \,\mathrm{d}x + C\varepsilon^2 \int_{B_1} r^{3-n}|\nabla u|^2 \,\mathrm{d}x \quad \text{for all } \varepsilon \le \varepsilon_0, \tag{1.7.10}$$

and applying (1.7.10) to rescaled functions $u(2\rho\cdot)$, we deduce

$$\begin{aligned}\int_{B_\rho} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \le C\int_{B_{2\rho}} r^{2-n}u_r^2 \,\mathrm{d}x + C\varepsilon^2\rho \int_{B_{2\rho}} r^{3-n}|\nabla u|^2 \,\mathrm{d}x \\ \text{for all } \rho \le 1/2 \text{ and } \varepsilon \le \varepsilon_0.\end{aligned} \tag{1.7.11}$$

Next, we apply the radial estimate (1.1.16) from Proposition 1.1.4 (with 2ρ) to bound the right-hand side of (1.7.11), which gives

$$\begin{aligned}\int_{B_\rho} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \le C\int_{B_{4\rho}\setminus B_{2\rho}} r^{2-n}|\nabla u|^2 \,\mathrm{d}x + C\varepsilon(1+\varepsilon\rho) \int_{B_{8\rho}} r^{3-n}|\nabla u|^2 \,\mathrm{d}x \\ \text{for all } \rho \le 1/8 \text{ and } \varepsilon \le \varepsilon_0.\end{aligned} \tag{1.7.12}$$

Hence, using that the bounds $\rho \le 1/8$ and $\varepsilon \le \varepsilon_0$ are universal, splitting the last integral into $B_{8\rho} = (B_{8\rho}\setminus B_\rho)\cup B_\rho$, and by $r^{3-n} \le r^{2-n}$, from (1.7.12) we deduce

$$\begin{aligned}\int_{B_\rho} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \le C\int_{B_{8\rho}\setminus B_\rho} r^{2-n}|\nabla u|^2 \,\mathrm{d}x + C\varepsilon \int_{B_\rho} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \\ \text{for all } \rho \le 1/8 \text{ and } \varepsilon \le \varepsilon_0.\end{aligned} \tag{1.7.13}$$

Taking $\varepsilon_0 > 0$ universal smaller if necessary, we can absorb the last integral into the left-hand side and obtain

$$\int_{B_\rho} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \le C\int_{B_{8\rho}\setminus B_\rho} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \quad \text{for all } \rho \le 1/8 \text{ and } \varepsilon \le \varepsilon_0. \tag{1.7.14}$$

Hole-filling (1.7.14), we also have

$$\int_{B_\rho} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \le \theta\int_{B_{8\rho}} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \quad \text{for all } \rho \le 1/8 \text{ and } \varepsilon \le \varepsilon_0, \tag{1.7.15}$$

where $\theta = \frac{C}{1+C} \in (0,1)$ is universal. Iterating (1.7.15), for $8^{-(k+1)} < \rho \le 8^{-k}$ we deduce

$$\int_{B_\rho} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \le \theta^k \int_{B_{8^k\rho}} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \le \frac{1}{\theta}\rho^{2\alpha}\int_{B_1} r^{2-n}|\nabla u|^2 \,\mathrm{d}x,$$

where $\alpha = -\frac{1}{2}\log_8\theta > 0$, and hence

$$\int_{B_\rho} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \le C\rho^{2\alpha}\int_{B_1} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \quad \text{for all } \rho \le 1/8 \text{ and } \varepsilon \le \varepsilon_0. \tag{1.7.16}$$

Finally, we can estimate the integral in the right-hand side of (1.7.16) by splitting $B_1 = (B_1 \setminus B_{1/8}) \cup B_{1/8}$ and applying (1.7.14) with $\rho = 1/8$ to bound the term in the annulus. This yields the claim.

Step 2: *Assuming*

$$\|DA\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)} \le \varepsilon,$$

we prove that if $\varepsilon \le \varepsilon_0$, then

$$\|u\|_{C^\alpha(\overline{B}_\theta)} \le C\|u\|_{L^1(B_1)}, \tag{1.7.17}$$

where $\alpha > 0$, $\theta > 0$, $\varepsilon_0 > 0$, and C are universal.

As explained in Section 1.2, for each ball $B_d(y) \subset B_1$, by uniform ellipticity, the function $u^{y,d}(x) := u(y + \frac{d}{\sqrt{C_0}} A^{1/2}(y)\, x)$ is a stable solution of an equation in B_1 with coefficients

$$A^{y,d}(x) := A^{-1/2}(y) A\big(y + \tfrac{d}{\sqrt{C_0}} A^{1/2}(y)x\big) A^{-1/2}(y)$$

and

$$b^{y,d}(x) := \tfrac{d}{\sqrt{C_0}} A^{-1/2}(y) b\big(y + \tfrac{d}{\sqrt{C_0}} A^{1/2}(y)x\big).$$

Notice that the matrix $A^{y,d}$ satisfies $A^{y,d}(0) = I$ and the coefficients can be bounded by $\|DA^{y,d}\|_{L^\infty(B_1)} + \|b^{y,d}\|_{L^\infty(B_1)} \le Cd\left(\|DA\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)}\right) \le \varepsilon$. Choosing $d > 0$ universal sufficiently small so that $Cd \le 1$, we have

$$\|DA^{y,d}\|_{L^\infty(B_1)} + \|b^{y,d}\|_{L^\infty(B_1)} \le \varepsilon \quad \text{for all } y \in B_{1-d}.$$

Hence, for $\varepsilon \le \varepsilon_0$ with the $\varepsilon_0 > 0$ from Step 1, by (1.7.1) we deduce

$$\int_{B_\rho} r^{2-n} |\nabla u^{y,d}|^2 \,\mathrm{d}x \le C\|\nabla u^{y,d}\|^2_{L^2(B_1)} \rho^{2\alpha} \quad \text{for } y \in B_{1-d} \text{ and } \rho \le 1/8,$$

and since $\int_{B_\rho} r^{2-n} |\nabla u^{y,d}|^2 \,\mathrm{d}x \ge \rho^{2-n} \int_{B_\rho} |\nabla u^{y,d}|^2 \,\mathrm{d}x$, we also have

$$\int_{B_\rho} |\nabla u^{y,d}|^2 \,\mathrm{d}x \le C\|\nabla u^{y,d}\|^2_{L^2(B_1)} \rho^{2\alpha+n-2} \quad \text{for } y \in B_{1-d} \text{ and } \rho \le 1/8. \tag{1.7.18}$$

For the remaining part of the proof of Step 2, we assume that $\varepsilon \le \varepsilon_0$.

Now we express (1.7.18) in terms of the original function u. By the change of variables $z = y + \frac{d}{\sqrt{C_0}} A^{1/2}(y)x$ and by uniform ellipticity, using that $B_{\sqrt{c_0}\rho} \subset A^{1/2}(y)(B_\rho)$, we have

$$\begin{aligned}\int_{B_\rho} |\nabla u^{y,d}|^2 \,\mathrm{d}x &= \frac{d^{2-n}}{C_0^{1-n/2}} \det(A(y))^{-1/2} \int_{y + \frac{d}{\sqrt{C_0}} A^{1/2}(y)(B_\rho)} |\nabla u|^2_{A(y)} \,\mathrm{d}z \\ &\ge c\, d^{2-n} \int_{B_{d\sqrt{\frac{c_0}{C_0}}\rho}(y)} |\nabla u|^2 \,\mathrm{d}z\end{aligned}$$

for some universal $c > 0$. Similarly, we also have $\|\nabla u^{y,d}\| \le Cd^{2-n}\|\nabla u\|_{L^2(B_1)}$ and, therefore, from (1.7.18) we deduce

$$\int_{B_{d\sqrt{\frac{c_0}{C_0}}\rho}(y)} |\nabla u|^2 \,\mathrm{d}z \le C\|\nabla u\|^2_{L^2(B_1)} \rho^{n-2+2\alpha} \quad \text{for } y \in B_{1-d} \text{ and } \rho \le 1/8. \tag{1.7.19}$$

Dividing ρ by $d\sqrt{\frac{c_0}{C_0}}$ in (1.7.19) and letting $\theta := \frac{d}{16}\sqrt{\frac{c_0}{C_0}}$, since d is universal, we obtain

$$\int_{B_\rho(y)} |\nabla u|^2 \,\mathrm{d}z \le C\|\nabla u\|^2_{L^2(B_1)}\rho^{n-2+2\alpha} \quad \text{for } y \in B_{1-d} \text{ and } \rho \le 2\theta,$$

and by Cauchy-Schwarz we also have the weaker

$$\int_{B_\rho(y)} |\nabla u| \,\mathrm{d}z \le C\|\nabla u\|_{L^2(B_1)}\rho^{n-1+\alpha} \quad \text{for } y \in B_{1-d} \text{ and } \rho \le 2\theta. \tag{1.7.20}$$

Taking d smaller if necessary, we may assume that $B_{2\theta} \subset B_{1-d}$. Hence, from (1.7.20) it follows that

$$\int_{B_\rho(y)} |\nabla u| \,\mathrm{d}z \le C\|\nabla u\|_{L^2(B_1)}\rho^{n-1+\alpha} \quad \text{for all balls } B_\rho(y) \subset B_{2\theta}. \tag{1.7.21}$$

Applying [63, Theorem 7.19] with $\Omega = B_{2\theta}$, we deduce the Hölder estimate

$$\|u\|_{C^\alpha(B_{2\theta})} \le C\|\nabla u\|_{L^2(B_1)}. \tag{1.7.22}$$

To obtain the final bound (1.7.17) in terms of the L^1 norm, apply (1.7.22) to the rescaled function $u(\cdot/2)$ first, and then Proposition 1.1.3 (taking ε_0 smaller if necessary).

Step 3: *Conclusion. Scaling and covering argument.*

We cover $\overline{B}_{1/2}$ by balls $B_{\theta\rho}(y_j)$, where θ is the universal constant in Step 2 above and ρ is small so that $B_\rho(y_j) \subset B_1$. The number of balls depends only on n, ρ, and $\theta = \theta(n, c_0, C_0)$. We choose ρ smaller still so that

$$\left(\|DA\|_{L^\infty(B_1)} + \|b\|_{L^\infty(B_1)}\right)\rho \le \varepsilon_0, \tag{1.7.23}$$

with $\varepsilon_0 > 0$ the universal constant in Step 2. Thus $\rho = \rho(n, c_0, C_0, \|DA\|_{L^\infty(B_1)}, \|b\|_{L^\infty(B_1)})$. The functions $u(y_j + \rho\cdot)$ are stable solutions of an elliptic equation with coefficients $A^{y_j,\rho} = A(y_j+\rho\cdot)$ and $b^{y_j,\rho} = \rho b(y_j+\rho\cdot)$. Since $B_\rho(y_j) \subset B_1$ and by (1.7.23), the coefficients satisfy the bounds

$$\|DA^{y_j,\rho}\|_{L^\infty(B_1)} + \|b^{y_j,\rho}\|_{L^\infty(B_1)} \le \left(\|DA\|_{L^\infty(B_\rho(y_j))} + \|b\|_{L^\infty(B_\rho(y_j))}\right)\rho \le \varepsilon_0,$$

therefore, we can apply Step 2 to deduce

$$\|u\|_{C^\alpha(\overline{B}_{1/2})} \le \sum_j \|u\|_{C^\alpha(\overline{B}_{\theta\rho}(y_j))} \le C\sum_j \|u\|_{L^1(B_\rho(y_j))} \le C\|u\|_{L^1(B_1)},$$

where $C = C(n, c_0, C_0, \|DA\|_{L^\infty(B_1)}, \|b\|_{L^\infty(B_1)})$. This concludes the proof of the theorem. $\square$

Chapter 2

Energy estimate up to the boundary

In this chapter, we obtain a universal energy estimate up to the boundary for stable solutions of semilinear equations with variable coefficients. Namely, we consider solutions to $-Lu = f(u)$, where L is a linear uniformly elliptic operator and f is C^1, such that the linearized equation $-L - f'(u)$ has nonnegative principal eigenvalue. Our main result is an estimate for the $L^{2+\gamma}$ norm of the gradient of stable solutions vanishing on the flat part of a half-ball, for *any* nonnegative and nondecreasing f. This bound only requires the elliptic coefficients to be Lipschitz. As a consequence, our estimate continues to hold in general $C^{1,1}$ domains if we further assume the nonlinearity f to be convex. This result is new even for the Laplacian, for which a C^3 regularity assumption on the domain was needed.

2.1 Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a function $f \in C^1(\mathbb{R})$, we consider stable solutions $u\colon \overline{\Omega} \to \mathbb{R}$ to the semilinear boundary value problem

$$\begin{cases} -Lu = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1.1}$$

Throughout the text, L denotes a uniformly elliptic operator of the form

$$L = a_{ij}(x)\partial_{ij} + b_i(x)\partial_i, \quad a_{ij}(x) = a_{ji}(x). \tag{2.1.2}$$

A solution u of (2.1.1) is called *stable* if the principal eigenvalue (with respect to Dirichlet conditions) of the linearized equation $J_u := L + f'(u)$ is nonnegative.[1] When the problem is variational, this amounts to the nonnegativity of the second variation, a necessary condition for the minimality of u.

The goal of the present article is to obtain a universal energy estimate for stable solutions to (2.1.1) in the spirit of the pioneering work of Cabré, Figalli, Ros-Oton, and Serra [24] for the Laplacian. In [24], the authors proved two types of a priori bounds for classical stable solutions when $L = \Delta$. Namely, a control of the $L^{2+\gamma}$ norm of the gradient (for some $\gamma > 0$) by the L^1 norm of the function, valid in all dimensions, and an estimate of the Hölder norm of the solution when $n \le 9$. The latter result is optimal, since there are examples of singular (unbounded) stable solutions in dimensions $n \ge 10$. A notable feature of these estimates is that they do not depend on the nonlinearity, which is assumed to be nonnegative, nondecreasing, and convex. Thanks to this, the paper [24]

[1] Here we adopt the sign convention $J_u\varphi = -\mu\varphi$ for the eigenvalues μ of J_u.

answered positively two long-standing open questions of Brezis and Vázquez [14] and of Brezis [11] concerning the regularity of extremal solutions (which are L^1 limits of classical stable solutions), recalled briefly below.

Here we will be interested in extending the $L^{2+\gamma}$ energy estimate to operators with variable coefficients as in (2.1.2). Our main achievement is to make the constants in our bounds depend on the $C^{0,1}$ norm of a_{ij} and the L^∞ norm of b_i, this being the major difficulty in our proofs. As a consequence, we will obtain a global estimate in $C^{1,1}$ domains. This result is new even when L is the Laplacian, as [24] required a C^3 regularity assumption on the domain. For this, starting from a curved boundary, we flatten it out locally by a change of variables. In the new coordinates, our solution is still a stable solution to an equation of the form (2.1.1), where the new operator L now involves the derivatives of the flattening map. More precisely, the new coefficients a_{ij} depend on the differential of this map, while b_i additionally depend on its Hessian. It follows that the $C^{0,1}$ and L^∞ regularity of the coefficients corresponds to a $C^{1,1}$ domain. In particular, it will suffice to prove a priori estimates in half-balls with the stated dependence on the coefficients.

Furthermore, when $n \leq 9$, our energy bound (as well as the auxiliary Hessian estimates in Theorem 2.1.4 below) will be crucial to establish Hölder estimates up to the boundary in $C^{1,1}$ domains. We tackle this issue in the next chapter, where we extend the optimal C^α bounds of [24] to equations with coefficients. The previous work [24] relied on delicate contradiction-compactness arguments which do not allow to quantify the constants in the estimates. Here, thanks to a new device of Cabré [19] for the Laplacian in flat domains we will be able to give a direct, quantitative proof of all our estimates in [50].

The study of the regularity of stable solutions to (2.1.1) was originally motivated by problems in combustion theory. In that setting, the interest lies in positive, nondecreasing, convex, and superlinear nonlinearities f accounting for the reaction of a combustible mixture. It is also natural to consider a multiple λf of the nonlinearity, where $\lambda > 0$ is a nondimensional parameter measuring the relative strength of the reaction with respect to the processes modeled by L. Applying the implicit function theorem at $\lambda = 0$ and by the properties of f, one obtains a branch of positive classical stable solutions $\{u_\lambda\}_{0<\lambda<\lambda^\star}$ of $-Lu_\lambda = \lambda f(u_\lambda)$ in Ω, $u_\lambda = 0$ on $\partial\Omega$, where $0 < \lambda^\star < \infty$ is the maximal threshold for the existence of classical solutions to this problem. Moreover, by maximum principle, $\lambda \mapsto u_\lambda$ is increasing in $(0, \lambda^\star)$ and converges in L^1 to a weak (distributional) solution $u^\star$, the so called *extremal solution*; see, for instance [11, 44, 48].

By construction, the extremal solution $u^\star$ is a priori only in L^1 and can be unbounded. In [14], Brezis and Vázquez gave a characterization of singular (unbounded) extremal solutions in the energy space $W_0^{1,2}(\Omega)$ when L is the Laplacian. Their result led them to ask whether extremal solutions are necessarily in this space; see [14, Problem 1]. This question has been addressed in various works, always considering the model operator $L = \Delta$. The first result in this direction was obtained by Nedev [78], who showed the validity of the claim for $n \leq 5$. Later, assuming Ω to be convex (or, more generally, "bean shaped"), he was able to extend this result to all dimensions in an unpublished preprint [79] (which is recalled and proven again in [28]). Then, Cabré and Capella studied radial stable solutions in $\Omega = B_1$, showing that $u^\star \in W^{3,2}(B_1)$ in this case. After that, Cabré and Ros-Oton [27] proved the claim for $n \leq 6$ in domains of double revolution, and Villegas [95] obtained the same result in general smooth domains. Recently, Cabré, Figalli, Ros-Oton, and Serra [24] settled the conjecture, showing that $u^\star \in W_0^{1,2+\gamma}(\Omega)$ in all dimensions, where $\gamma > 0$ depends only on n, and Ω is of C^3 class. For this, as mentioned above, they proved a universal energy estimate for smooth stable solutions. Then, they applied it to the functions $\{u_\lambda\}_{0<\lambda<\lambda^\star}$

and passed to the limit as $\lambda \to \lambda^\star$.

For further regularity properties of $u^\star$, the dimension of the space plays a critical role. Notice that, by the linear theory, the smoothness of $u^\star$ follows from its boundedness. When $n \geq 10$, explicit unbounded extremal solutions had been known for a long time, while no such examples were found in lower dimensions. In [11, Open problem 1], Brezis asked whether the extremal solution was always bounded in the latter case. His question prompted a series of works trying to establish L^∞ bounds for classical stable solutions in the range $n \leq 9$. Recently, in the breakthrough paper [24] mentioned above, the question was answered positively for the Laplacian in C^3 domains. For more information on that problem, see the references in [24, 50] or, for instance, in Chapters 1 and 3 of this thesis.

2.1.1 Main results

We assume that the coefficient matrix $A(x) = (a_{ij}(x))$ is uniformly elliptic in Ω, that is, there are positive constants c_0, C_0 such that

$$c_0|p|^2 \leq a_{ij}(x)p_ip_j \leq C_0|p|^2 \quad \text{for all } p \in \mathbb{R}^n. \tag{2.1.3}$$

Our global theorem in $C^{1,1}$ domains requires the assumption

$$a_{ij} \in C^{0,1}(\overline{\Omega}), \quad b_i \in L^\infty(\Omega) \cap C^0(\Omega), \tag{2.1.4}$$

For our local results in half-balls, we further need the auxiliary condition

$$b_i \in C^0(\overline{\Omega}). \tag{2.1.5}$$

We will be able to remove (2.1.5) by an approximation argument, as explained in Remark 2.1.6.

Since we always assume $a_{ij} \in C^{0,1}(\overline{\Omega}) = W^{1,\infty}(\Omega)$, we can write L in divergence form

$$Lu = \operatorname{div}(A(x)\nabla u) + \widehat{b}(x) \cdot \nabla u, \tag{2.1.6}$$

where $\widehat{b}(x) = (\widehat{b}_i(x))$ is the vector field given by

$$\widehat{b}_i(x) = b_i(x) - \partial_k a_{ki}(x). \tag{2.1.7}$$

Notice that $\widehat{b}_i$ is in $L^\infty(\Omega)$ by assumption (2.1.4).

Having specified the regularity of the coefficients, we can give a more precise definition of stable solution. Assuming (2.1.3) and (2.1.4), we consider the class of *strong solutions* to (2.1.1), that is, functions $u \in C^0(\overline{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ such that $-Lu = f(u)$ a.e. in Ω and $u = 0$ on $\partial\Omega$. As commented above, a strong solution u of (2.1.1) is *stable* if the principal eigenvalue of the linearized equation at u is nonnegative. Equivalently (see [9]), the solution u is stable if there exists a function $\varphi \in W^{2,n}_{\mathrm{loc}}(\Omega)$ such that

$$\begin{cases} J_u\varphi \leq 0 & \text{a.e. in } \Omega, \\ \varphi > 0 & \text{in } \Omega, \end{cases} \tag{2.1.8}$$

where, recall, $J_u = L + f'(u)$ denotes the Jacobi operator (the linearization) at u. We would like to point out that the notion of stable solution refers only to the equation satisfied by u and not to its boundary value.

Our energy estimate in $C^{1,1}$ domains will apply to strong stable solutions as above. In a sense, these functions are the natural replacement of the classical solutions for the Laplacian in C^3 domains considered in [24]. Notice that, since $f(u) \in L^\infty(\Omega)$, by L^p estimates in $C^{1,1}$ domains (see [63, Theorem 9.13]), strong solutions belong to $W^{2,p}(\Omega)$ for all $p < \infty$. For further regularity properties, more assumptions on the coefficients and the domain are needed. In fact, our a priori estimates in half-balls below require our solutions to have third weak derivatives, but we will be able to remove this assumption by an approximation argument; see Remark 2.1.6.

We now state the main result of this chapter, an energy estimate up to the boundary in flat domains. For $\rho > 0$, we denote the half-ball of radius ρ centered at 0 by

$$B_\rho^+ := \{x_n > 0\} \cap B_\rho,$$

where $B_\rho = \{|x| < \rho\} \subset \mathbb{R}^n$ is the full-ball. We also write

$$\partial^0 B_\rho^+ = \{x_n = 0\} \cap \partial B_\rho^+.$$

for the lower boundary of B_ρ^+. In the results below, $C = C(\dots)$ denotes a constant C depending only on the quantities appearing inside the parentheses. We have the following:

Theorem 2.1.1. *Let L satisfy conditions* (2.1.3), (2.1.4), *and* (2.1.5) *in $\Omega = B_1^+ \subset \mathbb{R}^n$. Assume that $f \in C^1(\mathbb{R})$ is nonnegative and nondecreasing.*

Let $u \in W^{3,p}(B_1^+)$, for some $p > n$, be a nonnegative stable solution to $-Lu = f(u)$ in B_1^+, with $u = 0$ on $\partial^0 B_1^+$.

Then

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/2}^+)} \leq C\|u\|_{L^1(B_1^+)},$$

where $\gamma = \gamma(n) > 0$ and $C = C(n, c_0, C_0, \|\nabla a_{ij}\|_{L^\infty(B_1^+)}, \|b_i\|_{L^\infty(B_1^+)})$.

Remark 2.1.2. Note that we are further assuming $u \in W^{3,p}(B_1^+)$ with $p > n$. In particular, by Sobolev embedding, u is in $C^2(\overline{B_1^+})$ and the solution is classical.[2] We need third weak derivatives in order to have a cancellation which removes the nonlinearity in the stability condition. This step is crucial for our bounds to be independent of f.

Remark 2.1.3. The continuity $b_i \in C^0(\overline{B_1^+})$ up to the boundary (assumption (2.1.5) above) will allow us to control these coefficients on certain surface integrals over $\partial^0 B_1^+$ arising in the proof. Assuming only $b_i \in L^\infty(B_1^+)$ does not suffice for such estimates on surfaces.

To prove Theorem 2.1.1, the stability condition (2.1.8) will come into play through a useful integral inequality that has already appeared in the previous chapter and was introduced in our work [52]. Recall that since the coefficient matrix $A(x) = (a_{ij}(x))$ is positive definite, it gives rise to a norm

$$|p|_{A(x)} := (a_{ij}(x)p_i p_j)^{1/2} \quad \text{for } p \in \mathbb{R}^n.$$

[2]For the embedding in half-balls, just apply the usual Sobolev embedding in the full ball to a third order reflection of u, for instance, letting

$$u(x', x_n) = -10u(x', -x_n) + 160u(x', -\tfrac{x_n}{2}) - 405u(x', -\tfrac{x_n}{3}) + 256u(x', -\tfrac{x_n}{4})$$

for $x_n < 0$ and $x' \in \mathbb{R}^{n-1}$, which is in $W^{3,p}(B_1) \subset C^2(\overline{B_1})$.

In [52], we showed that if u is stable, then

$$\int_\Omega f'(u)\xi^2\,\mathrm{d}x \le \int_\Omega \left|\nabla\xi - \tfrac{1}{2}\xi A^{-1}(x)\widehat{b}(x)\right|^2_{A(x)}\mathrm{d}x \quad \text{for all } \xi\in C^\infty_c(\Omega), \tag{2.1.9}$$

where $\widehat{b}(x)$ is the vector field introduced in (2.1.7) above. As shown in Chapter 1 above, (2.1.9) follows from the pointwise inequality in (2.1.8) multiplying by ξ^2/φ, integrating by parts, and completing squares. We often refer to (2.1.9) as the "integral stability inequality" to distinguish it from the pointwise condition (2.1.8) above. Moreover, we would like to point out that the inequality (2.1.9) is not equivalent to our stability condition (2.1.8) in general; see [52] or Chapter 1 above.

A fundamental ingredient in the proof of Theorem 2.1.1 will be to control the Hessian of a stable solution in half-balls. The following boundary Hessian estimates can be interpreted as a generalization of a geometric stability condition due to Sternberg and Zumbrun [91]. As above, throughout this chapter, a constant depending only on n, c_0, and C_0 will be called *universal*.

Theorem 2.1.4. *Let $u\in W^{3,p}(B_1^+)$, for some $p>n$, be a nonnegative stable solution of $-Lu=f(u)$ in B_1^+, with $u=0$ on $\partial^0 B_1^+$. Assume that $f\in C^1(\mathbb{R})$ is nonnegative. Assume that L satisfies conditions* (2.1.3), (2.1.4), *and* (2.1.5) *in $\Omega=B_1^+$, and that*

$$\|DA\|_{L^\infty(B_1^+)} + \|b\|_{L^\infty(B_1^+)} \le \varepsilon$$

for some $\varepsilon>0$.

Then

$$\begin{aligned}\int_{B_1^+}\mathcal{A}^2\eta^2\,\mathrm{d}x \le{}& C\int_{B_1^+}|\nabla u|^2\left(|\nabla\eta|^2+|D^2(\eta^2)|+\varepsilon|\nabla(\eta^2)|+\varepsilon^2\eta^2\right)\mathrm{d}x\\ &+C\int_{B_1^+}|D^2u||\nabla u|\left(|\nabla(\eta^2)|+\varepsilon\eta^2\right)\mathrm{d}x\\ &+C\int_{\partial^0B_1^+}|\nabla u|^2(|\nabla(\eta^2)|+\varepsilon\eta^2)\,\mathrm{d}\mathcal{H}^{n-1}\end{aligned} \tag{2.1.10}$$

for all $\eta\in C^\infty_c(B_1)$, where C is a universal constant and

$$\mathcal{A} := \begin{cases} \left(\mathrm{tr}(A(x)D^2uA(0)D^2u) - |\nabla u|^{-2}_{A(0)}|D^2uA(0)\nabla u|^2_{A(x)}\right)^{1/2} & \text{if } \nabla u\neq 0\\ 0 & \text{if } \nabla u = 0.\end{cases} \tag{2.1.11}$$

Assume moreover that f is nondecreasing and $\varepsilon\le\varepsilon_0$. Then

$$\|\nabla u\|_{L^2(\partial^0 B^+_{2/3})} \le C\|\nabla u\|_{L^2(B_1^+)}, \tag{2.1.12}$$

$$\||\nabla u|D^2u\|_{L^1(B^+_{4/7})} \le C\|\nabla u\|^2_{L^2(B_1^+)}, \tag{2.1.13}$$

$$\|\mathcal{A}\|_{L^2(B^+_{1/2})} \le C\|\nabla u\|_{L^2(B_1^+)}, \tag{2.1.14}$$

and

$$\|D^2u\|_{L^1(B^+_{4/7})} \le C\|\nabla u\|_{L^2(B_1^+)}, \tag{2.1.15}$$

where $\varepsilon_0>0$ and C are universal constants.

To prove the first bound (2.1.10) in Theorem 2.1.4, we will exploit the integral stability inequality (2.1.9) by choosing appropriate test functions. Letting $\xi = \mathbf{c}\eta$ in (2.1.9) with $\Omega = B_1^+$, where $\mathbf{c}$, η are smooth functions satisfying $\mathbf{c} = 0$ on $\partial^0 B_1^+$ and $\operatorname{supp}\eta \subset B_1$, if we integrate by parts, then (2.1.9) becomes

$$\int_{B_1^+} \mathbf{c} J_u \mathbf{c}\, \eta^2 \,\mathrm{d}x \leq \int_{B_1^+} \mathbf{c}^2 \left|\nabla\eta - \tfrac{1}{2}\eta A^{-1}(x) b(x)\right|^2_{A(x)} \mathrm{d}x. \tag{2.1.16}$$

In order to obtain universal estimates, the crucial point will be to choose $\mathbf{c}$ in such a way that the Jacobi operator $J_u\mathbf{c}$ in the left-hand side of (2.1.16) becomes independent of the nonlinearity. Thus, in the proof of (2.1.10), our choice will be a smooth approximation of

$$\mathbf{c}(x) = |\nabla u(x)|_{A(0)} - \mathbf{N}\cdot\nabla u(x)$$

for an appropriate constant vector field $\mathbf{N}\colon \mathbb{R}^n_+ \to \mathbb{R}^n$ (given by (2.2.3) in Section 2.2 below). Here, we need $f \geq 0$ to make sure that such an approximation of $\mathbf{c}$ vanishes on $\partial^0 B_1^+$, but otherwise is a technical assumption in this step.

Under a smallness assumption on the coefficients ($\varepsilon \leq \varepsilon_0$), the function $\mathcal{A}$ in (2.1.11) controls part of the Hessian of u (as explained in [52] or in Section 2.2 below). We can further bound the full Hessian by assuming that the equation has a sign $-Lu = f(u) \geq 0$. For the the final form of the Hessian estimates in (2.1.13), (2.1.14), and (2.1.15), we need to control the third term in the right-hand side of (2.1.10), which is a surface integral and arises at every integration by parts. To control such an integral requires both the monotonicity of f and the stability of u, while the previous works [20, 24] only needed the condition on f. The reason for this is an additional Hessian error which does not appear for the Laplacian on C^3 domains when trying to control the boundary integral.

Once Theorem 2.1.4 is available, our main result, Theorem 2.1.1, will follow directly by the ideas of [19, 24] combined with a scaling and covering argument.

To conclude this section, we state our energy estimate in general domains of $C^{1,1}$ class. Approximating (2.1.1) by stable solutions to smoother problems (as explained next in Remark 2.1.6), flattening the boundary, the result will follow easily from Theorem 2.1.1 and by the interior estimates obtained in Chapter 1. This argument requires the convexity of f to ensure that the approximating sequence of stable solutions converges to the original one. The same procedure can be used to obtain Hölder estimates up to the boundary in $C^{1,1}$ domains, which has been carried out in our work [50] and we describe in detail in Chapter 3 below. Since the ideas are very similar, we defer the complete proof of Theorem 2.1.5 below to Chapter 3, where we implement the approximation and flattening argument in full detail. Here, we just give indications in Remark 2.1.6, after the theorem.

Theorem 2.1.5 (Theorem 3.1.1)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{1,1}$ and let L satisfy conditions* (2.1.3) *and* (2.1.4) *in Ω. Assume that $f \in C^1(\mathbb{R})$ is nonnegative, nondecreasing, and convex.*

Let $u \in C^0(\overline{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ be a nonnegative stable solution of $-Lu = f(u)$ in Ω, with $u = 0$ on $\partial\Omega$.

Then

$$\|\nabla u\|_{L^{2+\gamma}(\Omega)} \leq C\|u\|_{L^1(\Omega)},$$

where $\gamma = \gamma(n) > 0$ and $C = C(\Omega, n, c_0, C_0, \|\nabla a_{ij}\|_{L^\infty(\Omega)}, \|b_i\|_{L^\infty(\Omega)})$.

Remark 2.1.6. As mentioned above, our energy estimate in $C^{1,1}$ domains will follow from Theorem 2.1.1 but, unlike this result, it does not require third derivatives of the solution or assumption (2.1.5) (the continuity of b_i up to the boundary). To achieve this, we consider an exhaustion of Ω by smooth sets Ω_k. Using u as a barrier, by monotone iteration, we construct strong stable solutions u_k to a semilinear equation $-L_k u_k = f_k(u_k)$ in Ω_k with smoother coefficients.

Flattening the boundary $\partial\Omega_k$, we obtain solutions in the half-ball, where we would like to apply Theorem 2.1.1. For this, we need to ensure the existence of third weak derivatives in L^p for these solutions, which is guaranteed if the new coefficients $(b_i^k)_k$ are sufficiently regular.[3] The interior continuity of b_i (assumption (2.1.4) above) will make sure that $b_i^k \to b_i$ locally uniformly in Ω, which is essential to show that u is a barrier.

Finally, we need the convexity of f for u_k to converge to the original solution u and not to some other limit. The deeper reason behind this is that stable solutions with convex nonlinearities are unique; see [48, 50] or Appendix F in Chapter 3 below. For C^3 domains and smooth coefficients, we do not need the approximation procedure and we could give the analogue of Theorem 2.1.5 without the convexity assumption on f.

2.1.2 Outline of the chapter

Section 2.2 is devoted to the proof of Theorem 2.1.4 containing the Hessian estimates up to the boundary. In Section 2.3 we prove Theorem 2.1.1, the energy estimate in half-balls.

2.2 Boundary Hessian estimates

Recall the function $\mathcal{A}\colon \overline{B}_1 \to \mathbb{R}$ given by (2.1.11) in the statement of Theorem 2.1.4 (and introduced in (1.3.1) in the previous chapter). It can also be written as

$$\mathcal{A} = \Big(\|A^{1/2}(x) D^2 u A^{1/2}(0)\|_{\mathrm{HS}}^2 - |A^{1/2}(x) D^2 u A^{1/2}(0)\mathbf{n}(x)|^2 \Big)^{1/2} \quad \text{in } \{\nabla u \neq 0\}, \qquad (2.2.1)$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Euclidean Hilbert-Schmidt norm for matrices and $\mathbf{n}(x)$ is the unit vector field $\mathbf{n}\colon (B_1^+ \cup \partial^0 B_1^+) \cap \{\nabla u \neq 0\} \to \mathbb{R}$ given by

$$\mathbf{n}(x) := |\nabla u|_{A(0)}^{-1} A^{1/2}(0) \nabla u(x). \qquad (2.2.2)$$

Here we are following the notation for the Hessian estimates in Chapter 1 above.

First we prove the bound (2.1.10) for $\mathcal{A}$ in Theorem 2.1.4. This is an analogue of the Sternberg-Zumbrun geometric estimate up to the boundary. For this, it is convenient to define the constant vector field

$$\mathbf{N} := |e_n|_{A(0)}^{-1} A(0) e_n, \qquad \mathbf{N}_i := (a_{nn}(0))^{-1/2} a_{in}(0). \qquad (2.2.3)$$

Notice that $\mathbf{N}$ has unit norm with respect to the scalar product defined by the inverse matrix $A^{-1}(0)$, i.e., $|\mathbf{N}|_{A^{-1}(0)} = 1$. Moreover, since u is nonnegative and $u = 0$ on $\partial^0 B_1^+$, we have the identity

$$|\nabla u|_{A(0)} = \mathbf{N} \cdot \nabla u \quad \text{on } \partial^0 B_1^+. \qquad (2.2.4)$$

[3] For instance, suppose that $a_{ij} \in C^{0,1}(\overline{B_1^+})$ and $b_i \in W^{1,p}(B_1^+)$ for some $p > n$, and let $u \in W^{2,p}(B_1^+)$ be a strong solution to $-Lu = f(u)$ in B_1^+, $u = 0$ on $\partial^0 B_1^+$. Since $f(u) \in L^\infty(B_1^+)$, by Calderón-Zygmund estimates (see [63, Theorem 9.13]) we have $u \in W^{2,q}_{\mathrm{loc}}(B_1^+ \cup \partial^0 B_1^+)$ for all $q < \infty$. Formally taking tangential derivatives, for $k = 1, \dots, n-1$ we obtain $-Lu_k = f'(u)u_k + \partial_k a_{ij}(x) u_{ij} + \partial_k b_i(x) u_i \in L^p_{\mathrm{loc}}(B_1^+ \cup \partial^0 B_1^+)$ and $u_k = 0$ on $\partial^0 B_1^+$, hence, again by L^p estimates, we deduce $u \in W^{3,p}_{\mathrm{loc}}(B_1^+ \cup \partial^0 B_1^+)$. It remains to check that the weak derivative u_{nnn} exists and lies in $L^p_{\mathrm{loc}}(B_1^+ \cup \partial^0 B_1^+)$, but this follows easily from the equation.

The vector field $\mathbf{N}$ will also be useful later when controlling the Dirichlet energy on the boundary.

Proof of (2.1.10) *in Theorem 2.1.4.* We test the stability inequality (2.1.16) with a variant of

$$\mathbf{c} := |\nabla u|_{A(0)} - \mathbf{N} \cdot \nabla u,$$

where $\mathbf{N} = (\mathbf{N}_i)$ is the constant vector field defined in (2.2.3) above. Since $|\nabla u|_{A(0)}$ is not necessarily smooth when $\nabla u = 0$, following [24], we take a convex $C^{1,1}$ regularization of the modulus $|\cdot|_{A(0)}$ instead. For each small $\delta > 0$, we define

$$\phi_\delta(z) := |z|_{A(0)} \mathbb{1}_{\{|z|_{A(0)} > \delta\}} + \left(\frac{\delta}{2} + \frac{|z|^2_{A(0)}}{2\delta} \right) \mathbb{1}_{\{|z|_{A(0)} < \delta\}}. \tag{2.2.5}$$

Given that u is nonnegative and superharmonic (in the sense that $Lu = -f(u) \le 0$), unless $u \equiv 0$ (in which case there is nothing to prove), by the Hopf lemma and uniform ellipticity we have $|\nabla u|_{A(0)} \ge c > 0$ on $\partial^0 B_1^+ \cap \operatorname{supp} \eta$, for some constant c. Hence, for $\delta > 0$ sufficiently small we have

$$\phi_\delta(\nabla u) = |\nabla u|_{A(0)} \quad \text{in a neighborhood of } \partial^0 B_1^+ \cap \operatorname{supp} \eta \text{ inside } \overline{B_1^+}. \tag{2.2.6}$$

Choosing $\delta > 0$ small enough such that (2.2.6) holds, we let

$$\mathbf{c}_\delta := \phi_\delta(\nabla u) - \mathbf{N} \cdot \nabla u.$$

Since $\mathbf{c}_\delta$ vanishes on $\partial^0 B_1^+$, this is a valid test function in the stability inequality (2.1.16).

We can write the Jacobi operator acting on $\mathbf{c}_\delta$ as the sum of three terms

$$\begin{aligned} \mathbf{c}_\delta J_u \mathbf{c}_\delta &= \mathbf{c}_\delta (L\mathbf{c}_\delta + f'(u)\mathbf{c}_\delta) \\ &= \phi_\delta(\nabla u) J_u \left[\phi_\delta(\nabla u)\right] - \mathbf{c}_\delta J_u \left[\mathbf{N} \cdot \nabla u\right] - (\mathbf{N} \cdot \nabla u) J_u \left[\phi_\delta(\nabla u)\right]. \end{aligned}$$

Multiplying this identity by η^2 and integrating in B_1^+ yields the left-hand side of (2.1.16), i.e.,

$$\begin{aligned} \int_{B_1^+} \mathbf{c}_\delta J_u \, \mathbf{c}_\delta \eta^2 \, \mathrm{d}x = \int_{B_1^+} \phi_\delta(\nabla u) J_u \left[\phi_\delta(\nabla u)\right] \eta^2 \, \mathrm{d}x - \int_{B_1^+} \mathbf{c}_\delta J_u \left[\mathbf{N} \cdot \nabla u\right] \eta^2 \, \mathrm{d}x \\ - \int_{B_1^+} (\mathbf{N} \cdot \nabla u) J_u \left[\phi_\delta(\nabla u)\right] \eta^2 \, \mathrm{d}x. \end{aligned} \tag{2.2.7}$$

We now study each of the three terms in (2.2.7) separately.

First term. *We prove that*

$$\begin{aligned} &\int_{B_1^+} \phi_\delta(\nabla u) J_u \left[\phi_\delta(\nabla u)\right] \eta^2 \, \mathrm{d}x \\ &\quad \ge \int_{B_1^+} \mathcal{A}^2 \, \mathbb{1}_{\{|\nabla u|_{A(0)} > \delta\}} \, \eta^2 \, \mathrm{d}x - C\delta \int_{B_1^+} |f'(u)|(|\nabla u| + \delta)\eta^2 \, \mathrm{d}x \\ &\qquad - C\varepsilon \int_{B_1^+} (|\nabla u| + \delta) \left(|D^2 u|\eta^2 + |\nabla u||\nabla(\eta^2)|\right) \mathrm{d}x - C\varepsilon \int_{\partial^0 B_1^+} |\nabla u|^2 \eta^2 \, \mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{2.2.8}$$

Recalling that $J_u = L + f'(u)$, we start by computing $L[\phi_\delta(\nabla u)]$ first. Here, since $u \in W^{3,p}(B_1^+)$ with $p > n$, by Sobolev embedding $u \in C^2(\overline{B_1^+})$ and, moreover, the Hessian $D^2 u$ is differentiable a.e. in B_1^+ (for instance, see [55]). Thus we have

$$\begin{aligned} L\left[\phi_\delta(\nabla u)\right] &= a_{ij}(x)\partial^2_{ij}\left[\phi_\delta(\nabla u)\right] + b_i(x)\partial_i\left[\phi_\delta(\nabla u)\right] \\ &= a_{ij}(x)\partial_{z_k}\phi_\delta(\nabla u)u_{ijk} + a_{ij}(x)\partial^2_{z_k z_l}\phi_\delta(\nabla u)u_{jk}u_{il} + b_i(x)\partial_{z_k}\phi_\delta(\nabla u)u_{ik} \end{aligned} \tag{2.2.9}$$

a.e. in B_1^+. By the convexity of ϕ_δ and its definition (2.2.5), it is easy to check that

$$\phi_\delta(\nabla u)a_{ij}(x)\partial^2_{z_k z_l}\phi_\delta(\nabla u)u_{jk}u_{il} \geq \mathcal{A}^2\,\mathbb{1}_{\{|\nabla u|_{A(0)} > \delta\}}. \tag{2.2.10}$$

Therefore, multiplying (2.2.9) by $\phi_\delta(\nabla u)\eta^2$, using (2.2.10), and integrating we obtain

$$\begin{aligned} &\int_{B_1^+} \phi_\delta(\nabla u)L\left[\phi_\delta(\nabla u)\right]\eta^2\,\mathrm{d}x \\ &\geq \int_{B_1^+} a_{ij}(x)\nabla u_{ij}\cdot\nabla\phi_\delta(\nabla u)\phi_\delta(\nabla u)\eta^2\,\mathrm{d}x + \int_{B_1^+}\mathcal{A}^2\,\mathbb{1}_{\{|\nabla u|_{A(0)} > \delta\}}\,\eta^2\,\mathrm{d}x \\ &\qquad + \int_{B_1^+} b_i(x)\partial_{z_k}\phi_\delta(\nabla u)u_{ik}\phi_\delta(\nabla u)\eta^2\,\mathrm{d}x. \end{aligned} \tag{2.2.11}$$

Next, we treat the zero order term $f'(u)\phi_\delta(\nabla u)$ in the linearization $J_u[\phi_\delta(\nabla u)]$. By direct computation $|\phi_\delta(\nabla u) - \nabla\phi_\delta(\nabla u)\cdot\nabla u| \leq \delta$ and hence

$$\int_{B_1^+} f'(u)\phi_\delta(\nabla u)^2\eta^2\,\mathrm{d}x \geq \int_{B_1^+} f'(u)\nabla u\cdot\nabla\phi_\delta(\nabla u)\phi_\delta(\nabla u)\eta^2\,\mathrm{d}x - \delta\int_{B_1^+}|f'(u)|\phi_\delta(\nabla u)\eta^2\,\mathrm{d}x. \tag{2.2.12}$$

Using the equation, we integrate by parts the first term in the right-hand side of (2.2.12) as

$$\begin{aligned} &\int_{B_1^+} f'(u)\nabla u\cdot\nabla\phi_\delta(\nabla u)\,\phi_\delta(\nabla u)\eta^2\,\mathrm{d}x = \int_{B_1^+}\nabla[f(u)]\cdot\nabla\phi_\delta(\nabla u)\,\phi_\delta(\nabla u)\eta^2\,\mathrm{d}x \\ &= \int_{B_1^+} Lu\,\mathrm{div}\left(\nabla\phi_\delta(\nabla u)\,\phi_\delta(\nabla u)\eta^2\right)\mathrm{d}x - \int_{\partial^0 B_1^+} f(u)\partial_{z_n}\phi_\delta(\nabla u)\,\phi_\delta(\nabla u)\eta^2\,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{2.2.13}$$

Moreover, undoing the integration by parts in

$$\begin{aligned} &\int_{B_1^+} a_{ij}(x)u_{ij}\,\mathrm{div}\left(\nabla\phi_\delta(\nabla u)\,\phi_\delta(\nabla u)\eta^2\right)\mathrm{d}x \\ &\qquad = -\int_{B_1^+}\nabla[a_{ij}(x)u_{ij}]\cdot\nabla\phi_\delta(\nabla u)\,\phi_\delta(\nabla u)\eta^2\,\mathrm{d}x \\ &\qquad\qquad + \int_{\partial^0 B_1^+} a_{ij}(x)u_{ij}\partial_{z_n}\phi_\delta(\nabla u)\,\phi_\delta(\nabla u)\eta^2\,\mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

substituting in (2.2.13) and using that $-Lu = f(u)$ up to $\overline{B_1^+}$ by continuity, we deduce

$$\begin{aligned} &\int_{B_1^+} f'(u)\nabla u\cdot\nabla\phi_\delta(\nabla u)\,\phi_\delta(\nabla u)\eta^2\,\mathrm{d}x \\ &= -\int_{B_1^+}\nabla[a_{ij}(x)u_{ij}]\cdot\nabla\phi_\delta(\nabla u)\,\phi_\delta(\nabla u)\eta^2\,\mathrm{d}x + \int_{B_1^+} b_i(x)u_i\,\mathrm{div}\left(\nabla\phi_\delta(\nabla u)\,\phi_\delta(\nabla u)\eta^2\right)\mathrm{d}x \\ &\qquad + \int_{\partial^0 B_1^+} b_i(x)u_i\partial_{z_n}\phi_\delta(\nabla u)\,\phi_\delta(\nabla u)\eta^2\,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{2.2.14}$$

Finally, combining (2.2.11), (2.2.12), and (2.2.14), we obtain

$$\begin{aligned}
\int_{B_1^+} \phi_\delta(\nabla u) J_u\left[\phi_\delta(\nabla u)\right]\eta^2\,\mathrm{d}x &= \int_{B_1^+} \phi_\delta(\nabla u) L\left[\phi_\delta(\nabla u)\right]\eta^2\,\mathrm{d}x + \int_{B_1^+} f'(u)\phi_\delta(\nabla u)^2\eta^2\,\mathrm{d}x \\
&\geq \int_{B_1^+} \mathcal{A}^2\,\mathbb{1}_{\{|\nabla u|_{A(0)}>\delta\}}\,\eta^2\,\mathrm{d}x - \delta\int_{B_1^+} |f'(u)|\phi_\delta(\nabla u)\eta^2\,\mathrm{d}x \\
&\quad - \int_{B_1^+} \nabla a_{ij}(x)u_{ij}\cdot\nabla\phi_\delta(\nabla u)\,\phi_\delta(\nabla u)\eta^2\,\mathrm{d}x \\
&\quad + \int_{B_1^+} b_i(x)\left\{u_i\operatorname{div}\left(\nabla\phi_\delta(\nabla u)\,\phi_\delta(\nabla u)\eta^2\right) + \partial_{z_k}\phi_\delta(\nabla u)u_{ik}\phi_\delta(\nabla u)\eta^2\right\}\mathrm{d}x \\
&\quad + \int_{\partial^0 B_1^+} b_i(x)u_i|e_n|_{A(0)}|\nabla u|_{A(0)}\eta^2\,\mathrm{d}\mathcal{H}^{n-1},
\end{aligned} \tag{2.2.15}$$

where in the boundary term we have used (2.2.6) and (2.2.4) to write

$$\partial_{z_n}\phi_\delta(\nabla u)\phi_\delta(\nabla u) = e_n\cdot A(0)\nabla u = |e_n|_{A(0)}|\nabla u|_{A(0)} \quad \text{on } \partial^0 B_1^+.$$

The claim now follows from (2.2.15) by applying the uniform ellipticity, the coefficient bounds $\|\nabla a_{ij}\|_{L^\infty} + \|b_i\|_{C^0} \leq \varepsilon$, and

$$\phi_\delta(\nabla u) \leq C\left(|\nabla u| + \delta\right), \tag{2.2.16}$$

$$|\nabla\phi_\delta(\nabla u)| + \phi_\delta(\nabla u)|D^2\phi_\delta(\nabla u)| \leq C, \tag{2.2.17}$$

where C are universal constants (and hence independent of δ).

Second term. *We prove that*

$$\begin{aligned}
&\left|\int_{B_1^+} \mathbf{c}_\delta J_u\left[\mathbf{N}\cdot\nabla u\right]\eta^2\,\mathrm{d}x\right| \\
&\qquad\qquad \leq C\varepsilon\int_{B_1^+}\left(|\nabla u| + \delta\right)\left(|D^2u|\eta^2 + |\nabla u||\nabla(\eta^2)|\right)\mathrm{d}x.
\end{aligned} \tag{2.2.18}$$

Computing, we have

$$L[\mathbf{N}\cdot\nabla u] = \mathbf{N}_k a_{ij}(x)u_{ijk} + \mathbf{N}_k b_i(x)u_{ik} \quad \text{a.e. in } B_1^+. \tag{2.2.19}$$

Since $\mathbf{c}_\delta = 0$ on $\partial^0 B_1^+$, integrating by parts and using the equation

$$\begin{aligned}
\int_{B_1^+} f'(u)\left(\mathbf{N}\cdot\nabla u\right)\mathbf{c}_\delta\eta^2\,\mathrm{d}x &= \int_{B_1^+} \mathbf{N}\cdot\nabla[f(u)]\,\mathbf{c}_\delta\eta^2\,\mathrm{d}x = \int_{B_1^+}(Lu)\,\mathbf{N}\cdot\nabla\left(\mathbf{c}_\delta\eta^2\right)\mathrm{d}x \\
&= -\int_{B_1^+} \mathbf{N}\cdot\nabla[a_{ij}(x)u_{ij}]\,\mathbf{c}_\delta\eta^2\,\mathrm{d}x + \int_{B_1^+} b_i(x)u_i\,\mathbf{N}\cdot\nabla\left(\mathbf{c}_\delta\eta^2\right)\mathrm{d}x,
\end{aligned} \tag{2.2.20}$$

where in the last line we have integrated by parts again. Combining (2.2.19) and (2.2.20)

$$\begin{aligned}
&\int_{B_1^+} \mathbf{c}_\delta J_u\left[\mathbf{N}\cdot\nabla u\right]\eta^2\,\mathrm{d}x \\
&= -\int_{B_1^+} \mathbf{N}\cdot\nabla a_{ij}(x)u_{ij}\,\mathbf{c}_\delta\eta^2\,\mathrm{d}x + \int_{B_1^+} \mathbf{N}_k b_i(x)u_{ik}\,\mathbf{c}_\delta\eta^2 + \int_{B_1^+} b_i(x)u_i\,\mathbf{N}\cdot\nabla\left(\mathbf{c}_\delta\eta^2\right)\mathrm{d}x.
\end{aligned} \tag{2.2.21}$$

The claim follows from (2.2.21) by applying (2.2.16), (2.2.17), and the coefficient bounds.

Third term. *We prove that*

$$
\begin{aligned}
&\left|\int_{B_1^+} (\mathbf{N}\cdot\nabla u) J_u\left[\phi_\delta(\nabla u)\right]\eta^2\,\mathrm{d}x\right| \\
&\le C\int_{B_1^+}(|\nabla u|+\delta)\big(|D^2u|\{|\nabla(\eta^2)|+\varepsilon\eta^2\}+\varepsilon|\nabla u||\nabla(\eta^2)|+|\nabla u||D^2(\eta^2)|\big)\,\mathrm{d}x \\
&\qquad + C\int_{\partial^0 B_1^+}|\nabla u|^2\big(|\nabla(\eta^2)|+\varepsilon\eta^2\big)\,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}
\tag{2.2.22}
$$

By definition, we have

$$
(\mathbf{N}\cdot\nabla u)J_u\left[\phi_\delta(\nabla u)\right] = (\mathbf{N}\cdot\nabla u)L[\phi_\delta(\nabla u)] + \phi_\delta(\nabla u)f'(u)(\mathbf{N}\cdot\nabla u). \tag{2.2.23}
$$

The idea is to integrate the first term in (2.2.23), $\int_{B_1^+}(\mathbf{N}\cdot\nabla u)L[\phi_\delta(\nabla u)]\,\eta^2\,\mathrm{d}x$, by parts to get the linearized equation acting on the directional derivative $\mathbf{N}\cdot\nabla u$ instead of on the modulus $\phi_\delta(\nabla u)$. It will then be easy to bound the remaining terms as in Step 2 above.

We write the operator in divergence form $Lu = \operatorname{div}(A(x)\nabla u) + \widehat{b}(x)\cdot\nabla u$ as in (2.1.6). Integrating by parts twice in $\int_{B_1^+}(\mathbf{N}\cdot\nabla u)\operatorname{div}\big(A(x)\nabla\left[\phi_\delta(\nabla u)\right]\big)\eta^2\,\mathrm{d}x$, we have

$$
\begin{aligned}
&\int_{B_1^+}(\mathbf{N}\cdot\nabla u)\operatorname{div}\Big(A(x)\nabla\left[\phi_\delta(\nabla u)\right]\Big)\eta^2\,\mathrm{d}x \\
&\quad = \int_{B_1^+}\phi_\delta(\nabla u)\operatorname{div}\Big(A(x)\nabla(\mathbf{N}\cdot\nabla u)\Big)\eta^2\,\mathrm{d}x \\
&\qquad + \int_{B_1^+}\phi_\delta(\nabla u)\Big(2A(x)\nabla(\mathbf{N}\cdot\nabla u)\cdot\nabla(\eta^2) + (\mathbf{N}\cdot\nabla u)\operatorname{div}\big\{A(x)\nabla(\eta^2)\big\}\Big)\,\mathrm{d}x \\
&\qquad + \int_{\partial^0 B_1^+}\Big(\phi_\delta(\nabla u)A(x)\nabla\big\{(\mathbf{N}\cdot\nabla u)\eta^2\big\}\cdot e_n - (\mathbf{N}\cdot\nabla u)A(x)\nabla\left[\phi_\delta(\nabla u)\right]\cdot e_n\,\eta^2\Big)\,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}
\tag{2.2.24}
$$

Since u is nonnegative and $u=0$ on $\partial^0 B_1^+$, we have $\nabla u = |\nabla u|_{A(0)}|e_n|_{A(0)}^{-1}e_n$ and hence, using (2.2.4) and (2.2.6), the boundary integrand in (2.2.24) can be written simply as

$$
\phi_\delta(\nabla u)A(x)\nabla\big\{(\mathbf{N}\cdot\nabla u)\eta^2\big\}\cdot e_n - (\mathbf{N}\cdot\nabla u)A(x)\nabla\left[\phi_\delta(\nabla u)\right]\cdot e_n\,\eta^2 = |\nabla u|_{A(0)}^2A(x)\nabla(\eta^2)\cdot e_n. \tag{2.2.25}
$$

Combining (2.2.24) and (2.2.25), we deduce

$$
\begin{aligned}
&\int_{B_1^+} (\mathbf{N}\cdot\nabla u) L[\phi_\delta(\nabla u)]\,\eta^2\,\mathrm{d}x \\
&= \int_{B_1^+} (\mathbf{N}\cdot\nabla u)\mathrm{div}\Big(A(x)\nabla\left[\phi_\delta(\nabla u)\right]\Big)\,\eta^2\,\mathrm{d}x + \int_{B_1^+} (\mathbf{N}\cdot\nabla u)\big(\widehat{b}(x)\cdot\nabla\left[\phi_\delta(\nabla u)\right]\big)\,\eta^2\,\mathrm{d}x \\
&= \int_{B_1^+} \phi_\delta(\nabla u) L[\mathbf{N}\cdot\nabla u]\,\eta^2\,\mathrm{d}x \\
&\quad + \int_{B_1^+} \phi_\delta(\nabla u)\Big(2A(x)\nabla(\mathbf{N}\cdot\nabla u)\cdot\nabla(\eta^2) + (\mathbf{N}\cdot\nabla u)\,\mathrm{div}\big\{A(x)\nabla(\eta^2)\big\}\Big)\,\mathrm{d}x \\
&\qquad + \int_{B_1^+} \Big(-\phi_\delta(\nabla u)\,\widehat{b}(x)\cdot\nabla(\mathbf{N}\cdot\nabla u) + (\mathbf{N}\cdot\nabla u)\,\widehat{b}(x)\cdot\nabla\left[\phi_\delta(\nabla u)\right]\Big)\eta^2\,\mathrm{d}x \\
&\qquad\quad + \int_{\partial^0 B_1^+} |\nabla u|^2_{A(0)} A(x)\nabla(\eta^2)\cdot e_n\,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}
\tag{2.2.26}
$$

We now treat the second term in (2.2.23). Integrating by parts twice as in the proof of Step 2 (this time including boundary terms) and using the equation, it follows that

$$
\begin{aligned}
&\int_{B_1^+} f'(u)\,(\mathbf{N}\cdot\nabla u)\;\phi_\delta(\nabla u)\eta^2\,\mathrm{d}x = \int_{B_1^+} \mathbf{N}\cdot\nabla[f(u)]\,\phi_\delta(\nabla u)\eta^2\,\mathrm{d}x \\
&\quad = \int_{B_1^+} (Lu)\,\mathbf{N}\cdot\nabla\left(\phi_\delta(\nabla u)\eta^2\right)\mathrm{d}x - \int_{\partial^0 B_1^+} f(u)|e_n|_{A(0)}\,\phi_\delta(\nabla u)\eta^2\,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad = -\int_{B_1^+} \mathbf{N}\cdot\nabla[a_{ij}(x)u_{ij}]\,\phi_\delta(\nabla u)\eta^2\,\mathrm{d}x + \int_{B_1^+} b_i(x)u_i\,\mathbf{N}\cdot\nabla\left(\phi_\delta(\nabla u)\eta^2\right)\mathrm{d}x \\
&\qquad\quad + \int_{\partial^0 B_1^+} b_i(x)u_i|e_n|_{A(0)}\,\phi_\delta(\nabla u)\eta^2\,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}
\tag{2.2.27}
$$

Finally, summing (2.2.26) and (2.2.27), we obtain

$$
\begin{aligned}
&\int_{B_1^+} (\mathbf{N}\cdot\nabla u) J_u[\phi_\delta(\nabla u)]\,\eta^2\,\mathrm{d}x \\
&= \int_{B_1^+} \Big(L[\mathbf{N}\cdot\nabla u] - \mathbf{N}\cdot\nabla[a_{ij}(x)u_{ij}]\Big)\phi_\delta(\nabla u)\eta^2\,\mathrm{d}x \\
&\quad + \int_{B_1^+} \phi_\delta(\nabla u)\Big(2A(x)\nabla(\mathbf{N}\cdot\nabla u)\cdot\nabla(\eta^2) + (\mathbf{N}\cdot\nabla u)\,\mathrm{div}\big\{A(x)\nabla(\eta^2)\big\}\Big)\,\mathrm{d}x \\
&\qquad + \int_{B_1^+} \widehat{b}(x)\cdot\Big((\mathbf{N}\cdot\nabla u)\,\nabla\left[\phi_\delta(\nabla u)\right] - \phi_\delta(\nabla u)\,\nabla(\mathbf{N}\cdot\nabla u)\Big)\eta^2\,\mathrm{d}x \\
&\qquad\quad + \int_{B_1^+} b_i(x)u_i\,\mathbf{N}\cdot\nabla\left(\phi_\delta(\nabla u)\eta^2\right)\mathrm{d}x \\
&\qquad\qquad + \int_{\partial^0 B_1^+} \Big(|\nabla u|^2_{A(0)} A(x)\nabla(\eta^2)\cdot e_n + b_i(x)u_i|e_n|_{A(0)}\,\phi_\delta(\nabla u)\eta^2\Big)\,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}
\tag{2.2.28}
$$

Noticing that $L[\mathbf{N}\cdot\nabla u] - \mathbf{N}\cdot\nabla[a_{ij}(x)u_{ij}] = -\mathbf{N}\cdot\nabla a_{ij}(x)u_{ij} + \mathbf{N}_k b_i(x)u_{ik}$, every term in the right-hand side of (2.2.28) can be bounded as claimed in (2.2.22). For this, apply the uniform ellipticity, the coefficient bounds $\|\nabla a_{ij}\|_{L^\infty} + \|b_i\|_{C^0} + \|\widehat{b}_i\|_{L^\infty} \le 2\varepsilon$, and the estimates (2.2.16) and (2.2.17).

Conclusion. Applying the three estimates (2.2.8), (2.2.18), and (2.2.22) in (2.2.7) yields the lower bound

$$\begin{aligned}
&\int_{B_1^+} \mathbf{c}_\delta J_u \mathbf{c}_\delta\, \eta^2\,\mathrm{d}x \\
&\quad\ge \int_{B_1^+\cap\{|\nabla u|_{A(0)}>\delta\}} \mathcal{A}^2\,\eta^2\,\mathrm{d}x - C\delta \int_{B_1^+} |f'(u)|(|\nabla u|+\delta)\eta^2\,\mathrm{d}x \\
&\qquad - C\int_{B_1^+} (|\nabla u|+\delta)\left(|D^2u|\{|\nabla(\eta^2)|+\varepsilon\eta^2\} + \varepsilon|\nabla u||\nabla(\eta^2)| + |\nabla u||D^2(\eta^2)|\right)\mathrm{d}x \\
&\qquad - C\int_{\partial^0 B_1^+} |\nabla u|^2(|\nabla(\eta^2)|+\varepsilon\eta^2)\,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned} \tag{2.2.29}$$

By the integral stability inequality (2.1.16) with $\mathbf{c} = \mathbf{c}_\delta$, we also have the upper bound

$$\begin{aligned}
\int_{B_1^+} \mathbf{c}_\delta J_u \mathbf{c}_\delta\, \eta^2\,\mathrm{d}x &\le \int_{B_1^+} \phi_\delta(\nabla u)^2 |\nabla\eta - \tfrac{1}{2}\eta A^{-1}(x)\widehat{b}(x)|^2_{A(x)}\,\mathrm{d}x \\
&\le C\int_{B_1^+} (|\nabla u|+\delta)^2\left(|\nabla\eta|^2+\varepsilon^2\eta^2\right)\mathrm{d}x.
\end{aligned} \tag{2.2.30}$$

Hence, combining (2.2.29) and (2.2.30) and taking the limit as $\delta\to 0$, we deduce the claim $\square$

In order to prove the remaining estimates in Theorem 2.1.4, we need to control the right-hand side of (2.1.10). For this, next we prove two basic Hessian estimates for (generalized) superharmonic functions. We essentially follow the proof of Theorem 1.1.2 above, but now including boundary terms.

Lemma 2.2.1. *Let $u \in C^2(\overline{B_1^+})$ be superharmonic in the sense that $Lu \le 0$ in B_1^+, where L satisfies conditions* (2.1.3) *and* (2.1.4) *in $\Omega = B_1^+$. Assume that*

$$\|DA\|_{L^\infty(B_1^+)} + \|b\|_{L^\infty(B_1^+)} \le \varepsilon$$

for some $\varepsilon > 0$.

Then, there exists a universal $\varepsilon_0 > 0$ with the following property: if $\varepsilon \le \varepsilon_0$, then, for all $\zeta \in C^{0,1}_c(B_1)$ with $\zeta \ge 0$, we have

$$\int_{B_1^+} |D^2u|\,\zeta\,\mathrm{d}x \le C\int_{B_1^+} |\nabla u|\,\left(|\nabla\zeta|+\varepsilon\zeta\right)\mathrm{d}x + C\int_{B_1^+} \mathcal{A}\,\zeta\,\mathrm{d}x + C\int_{\partial^0 B_1^+} |\nabla u|\,\zeta\,\mathrm{d}x \tag{2.2.31}$$

and

$$\begin{aligned}
\int_{B_1^+} |D^2u||\nabla u|\,\zeta\,\mathrm{d}x \le C\int_{B_1^+} |\nabla u|^2\,\left(|\nabla\zeta|+\varepsilon\zeta\right)\mathrm{d}x + C\int_{B_1^+} \mathcal{A}|\nabla u|\,\zeta\,\mathrm{d}x \\
+ C\int_{\partial^0 B_1^+} |\nabla u|^2\zeta\,\mathrm{d}\mathcal{H}^{n-1},
\end{aligned} \tag{2.2.32}$$

where C is a universal constant.

Proof. Consider the auxiliary function

$$\mathcal{A}_0 := \begin{cases} \left(\|A^{1/2}(0)D^2uA^{1/2}(0)\|_{\mathrm{HS}}^2 - |A^{1/2}(0)D^2uA^{1/2}(0)\mathbf{n}(x)|^2\right)^{1/2} & \text{if } \nabla u \neq 0 \\ 0 & \text{if } \nabla u = 0, \end{cases}$$

where the vector field $\mathbf{n}(x)$ has been introduced in (2.2.2) in the definition of $\mathcal{A}$ in (2.2.1). Using that $\|DA\|_{L^\infty(B_1^+)} \le \varepsilon$, it is easy to show (see [52] or Step 2 in the proof of Theorem 1.1.2 above) that

$$|\mathcal{A}^2 - \mathcal{A}_0^2| \le C\varepsilon|x|\mathcal{A}_0^2 \quad \text{in } B_1^+, \tag{2.2.33}$$

where C always denotes a universal constant. In particular, the functions $\mathcal{A}$ and $\mathcal{A}_0$ are comparable for ε small. Using that $Lu \le 0$, following [52], it is not hard to show that

$$|D^2u| \le -C\mathrm{tr}\big(A(0)D^2u\big) + C\mathcal{A}_0 + C\varepsilon|x||D^2u| + C\varepsilon|\nabla u| \quad \text{a.e. in } B_1^+. \tag{2.2.34}$$

First we prove the Hessian bound (2.2.31). Multiplying (2.2.34) by ζ and integrating in B_1^+

$$\begin{aligned}\int_{B_1^+} |D^2u|\zeta \,\mathrm{d}x \le -C\int_{B_1^+} \mathrm{tr}\big(A(0)D^2u\big)\,\zeta\,\mathrm{d}x + C\int_{B_1^+} \mathcal{A}_0\,\zeta\,\mathrm{d}x \\ + C\varepsilon\int_{B_1^+} |x||D^2u|\zeta\,\mathrm{d}x + C\varepsilon\int_{B_1^+} |\nabla u|\,\zeta\,\mathrm{d}x.\end{aligned} \tag{2.2.35}$$

Integrating by parts, we have

$$-\int_{B_1^+} \mathrm{tr}\big(A(0)D^2u\big)\,\zeta\,\mathrm{d}x = \int_{B_1^+} A(0)\nabla u\cdot\nabla\zeta\,\mathrm{d}x - \int_{\partial^0 B_1^+} A(0)\nabla u\cdot e_n\,\zeta\,\mathrm{d}\mathcal{H}^{n-1},$$

and substituting in (2.2.35), by uniform ellipticity,

$$\begin{aligned}\int_{B_1^+} |D^2u|\zeta\,\mathrm{d}x \le C\int_{B_1^+} |\nabla u|\,|\nabla\zeta|\,\mathrm{d}x + C\int_{B_1^+} \mathcal{A}_0\,\zeta\,\mathrm{d}x + C\int_{\partial^0 B_1^+} |\nabla u|\,\zeta\,\mathrm{d}x \\ + C\varepsilon\int_{B_1^+} |x||D^2u|\zeta\,\mathrm{d}x + C\varepsilon\int_{B_1^+} |\nabla u|\,\zeta\,\mathrm{d}x.\end{aligned} \tag{2.2.36}$$

Choosing $\varepsilon_0 > 0$ universal sufficiently small, we can absorb the Hessian term in the right-hand side of (2.2.36), and by (2.2.33) (taking ε_0 smaller) we deduce the first claim.

For the second estimate (2.2.32), multiplying (2.2.34) by $|\nabla u|_{A(0)}\zeta$ and integrating in B_1^+

$$\begin{aligned}\int_{B_1^+} |D^2u||\nabla u|_{A(0)}\zeta\,\mathrm{d}x \le -C\int_{B_1^+} |\nabla u|_{A(0)}\mathrm{tr}\big(A(0)D^2u\big)\,\zeta\,\mathrm{d}x + C\int_{B_1^+} \mathcal{A}_0|\nabla u|_{A(0)}\,\zeta\,\mathrm{d}x \\ + C\varepsilon\int_{B_1^+} |x||D^2u||\nabla u|_{A(0)}\zeta\,\mathrm{d}x + C\varepsilon\int_{B_1^+} |\nabla u|^2\,\zeta\,\mathrm{d}x.\end{aligned} \tag{2.2.37}$$

The first integrand in the right-hand side of (2.2.37) can be bounded by

$$-|\nabla u|_{A(0)}\mathrm{tr}\big(A(0)D^2u\big) \le -\frac{1}{2}\mathrm{div}\big(|\nabla u|_{A(0)}A(0)\nabla u\big) + C\mathcal{A}_0|\nabla u|_{A(0)} \quad \text{a.e. in } B_1^+. \tag{2.2.38}$$

Substituting (2.2.38) in (2.2.37) leads to

$$\begin{aligned}\int_{B_1^+} |D^2u||\nabla u|_{A(0)}\zeta\,\mathrm{d}x \le -C\int_{B_1^+} \operatorname{div}\big(|\nabla u|_{A(0)}A(0)\nabla u\big)\,\zeta\,\mathrm{d}x + C\int_{B_1^+} \mathcal{A}_0|\nabla u|_{A(0)}\,\zeta\,\mathrm{d}x \\ + C\varepsilon\int_{B_1^+} |x||D^2u||\nabla u|_{A(0)}\,\zeta\,\mathrm{d}x + C\varepsilon\int_{B_1^+} |\nabla u|^2\,\zeta\,\mathrm{d}x,\end{aligned}$$

and integrating by parts the divergence term, we obtain the inequality

$$\begin{aligned}\int_{B_1^+} |D^2u||\nabla u|_{A(0)}\zeta\,\mathrm{d}x \le C\int_{B_1^+} |\nabla u|^2\,\left(|\nabla\zeta| + \varepsilon\zeta\right)\mathrm{d}x + C\int_{B_1^+} \mathcal{A}_0|\nabla u|_{A(0)}\,\zeta\,\mathrm{d}x \\ + C\varepsilon\int_{B_1^+} |x||D^2u||\nabla u|_{A(0)}\,\zeta\,\mathrm{d}x + C\int_{\partial^0 B_1^+} |\nabla u|^2\zeta\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned} \tag{2.2.39}$$

Once again, choosing $\varepsilon_0 > 0$ universal small, we can absorb the "Hessian times the gradient" error in (2.2.39) into the left-hand side, and by (2.2.33) we deduce the second claim. □

Thanks to Lemma 2.2.1, we can get rid of the Hessian terms appearing in the right-hand side of the first inequality (2.1.10) in Theorem 2.1.4:

Lemma 2.2.2. *Let $u \in W^{3,p}(B_1^+)$, for some $p > n$, be a nonnegative stable solution of $-Lu = f(u)$ in B_1^+, with $u = 0$ on $\partial^0 B_1^+$. Assume that $f \in C^1(\mathbb{R})$ is nonnegative. Assume that L satisfies conditions* (2.1.3), (2.1.4), *and* (2.1.5) *in $\Omega = B_1^+$, and that*

$$\|DA\|_{L^\infty(B_1^+)} + \|b\|_{L^\infty(B_1^+)} \le \varepsilon$$

for some $\varepsilon > 0$.

If $\varepsilon \le \varepsilon_0$, then

$$\int_{B_{8/9}^+} \mathcal{A}^2\,\mathrm{d}x \le C\int_{B_1^+} |\nabla u|^2\,\mathrm{d}x + C\int_{\partial^0 B_1^+} |\nabla u|^2\,\mathrm{d}\mathcal{H}^{n-1},$$

where $\varepsilon_0 > 0$ and C are universal constants.

Proof. Let $\varepsilon_0 > 0$ be the universal constant in the conclusion of Lemma 2.2.1. Applying (2.2.32) in Lemma 2.2.1 with $\zeta = |\nabla(\eta^2)| + \varepsilon\eta^2 \in C_c^{0,1}(B_1)$ yields

$$\begin{aligned}&\int_{B_1^+} |D^2u||\nabla u|(|\nabla(\eta^2)| + \varepsilon\eta^2)\,\mathrm{d}x \\ &\le C\int_{B_1^+} |\nabla u|^2\,\left(|D^2(\eta^2)| + \varepsilon|\nabla(\eta^2)| + \varepsilon^2\eta^2\right)\mathrm{d}x + C\int_{B_1^+} \mathcal{A}|\nabla u|\,(|\nabla(\eta^2)| + \varepsilon\eta^2)\,\mathrm{d}x \\ &\qquad + C\int_{\partial^0 B_1^+} |\nabla u|^2(|\nabla(\eta^2)| + \varepsilon\eta^2)\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned} \tag{2.2.40}$$

Since $|\nabla(\eta^2)| + \varepsilon\eta^2 = |\eta|\,(2|\nabla\eta| + \varepsilon|\eta|)$, by Cauchy-Schwarz, the second term in (2.2.40) can be bounded by

$$\int_{B_1^+} \mathcal{A}|\nabla u|\,(|\nabla(\eta^2)| + \varepsilon\eta^2)\,\mathrm{d}x \le C\left(\int_{B_1^+} \mathcal{A}^2\eta^2\,\mathrm{d}x\right)^{1/2}\left(\int_{B_1^+} |\nabla u|^2\,\left(|\nabla\eta|^2 + \varepsilon^2\eta^2\right)\mathrm{d}x\right)^{1/2}. \tag{2.2.41}$$

Hence, applying (2.2.40) and (2.2.41) to the Hessian errors in the right-hand side of (2.1.10) in Theorem 2.1.4, we obtain

$$
\begin{aligned}
\int_{B_1^+} \mathcal{A}^2\eta^2 \,\mathrm{d}x \leq C \left(\int_{B_1^+} \mathcal{A}^2\eta^2 \,\mathrm{d}x\right)^{1/2} &\left(\int_{B_1^+} |\nabla u|^2 \left(|\nabla\eta|^2 + \varepsilon^2\eta^2\right) \mathrm{d}x\right)^{1/2} \\
+ C \int_{B_1^+} &|\nabla u|^2 \left(|\nabla\eta|^2 + |D^2(\eta^2)| + \varepsilon|\nabla(\eta^2)| + \varepsilon^2\eta^2\right) \mathrm{d}x \\
&+ C \int_{\partial^0 B_1^+} |\nabla u|^2 (|\nabla(\eta^2)| + \varepsilon\eta^2) \,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}
\tag{2.2.42}
$$

Therefore, by Young's inequality, we can absorb the $\int_{B_1^+} \mathcal{A}^2\,\eta^2\,\mathrm{d}x$ term in (2.2.42) into the left-hand side. Choosing $\eta \in C_c^\infty(B_1)$ with $0 \leq \eta \leq 1$ in B_1 and $\eta = 1$ in $B_{8/9}$, by the universal bound $\varepsilon \leq \varepsilon_0$, we deduce the claim. □

Thanks to the preliminary lemmas above, we are now in position to conclude the proof of Theorem 2.1.4:

Proof of the boundary estimates (2.1.12), (2.1.13), (2.1.14), *and* (2.1.15) *in Theorem 2.1.4.* Once we obtain the boundary gradient estimate 2.1.12, the remaining inequalities (2.1.13), (2.1.14), and (2.1.15) will follow easily from Lemmas 2.2.1 and 2.2.2.

To control the gradient on the boundary, we proceed in two steps. First we employ the Pohozaev trick to bound the L^2 norm of ∇u on the lower boundary by the Dirichlet energy up to Hessian errors. Secondly, we use Lemmas 2.2.1 and 2.2.2 to control these Hessian errors and apply Simon's lemma (recalled in Appendix D).

Step 1. *We prove that*

$$
\|\nabla u\|^2_{L^2(\partial^0 B^+_{2/3})} \leq C(1+\varepsilon)\|\nabla u\|^2_{L^2(B^+_{7/9})} + C\varepsilon \||D^2u|\,|\nabla u|\|_{L^1(B^+_{7/9})},
$$

where C is a universal constant.

Let $\eta \in C_c^\infty(B_{7/9})$. Integrating by parts, by the properties of u and the vector field $\mathbf{N}$ defined in (2.2.3), it is easy to check that

$$
\begin{aligned}
&|e_n|_{A(0)} \int_{\partial^0 B_1^+} |\nabla u|^2_{A(0)}\eta^2 \,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad = \int_{B_1^+} \operatorname{div}\left(|\nabla u|^2_{A(0)}\mathbf{N} - 2(\mathbf{N}\cdot\nabla u)A(0)\nabla u\right)\eta^2 \,\mathrm{d}x \\
&\qquad + \int_{B_1^+} \left(|\nabla u|^2_{A(0)}\mathbf{N} - 2(\mathbf{N}\cdot\nabla u)A(0)\nabla u\right)\cdot\nabla(\eta^2)\,\mathrm{d}x.
\end{aligned}
\tag{2.2.43}
$$

The divergence term in (2.2.43) can be written as

$$
\begin{aligned}
&\operatorname{div}\left(|\nabla u|^2_{A(0)}\mathbf{N} - 2(\mathbf{N}\cdot\nabla u)A(0)\nabla u\right) = -2(\mathbf{N}\cdot\nabla u)\operatorname{tr}(A(0)D^2u) \\
&\quad = -2(\mathbf{N}\cdot\nabla u)Lu + 2(\mathbf{N}\cdot\nabla u)(b(x)\cdot\nabla u) + 2(\mathbf{N}\cdot\nabla u)\operatorname{tr}\left(\{A(x)-A(0)\}D^2u\right) \\
&\quad \leq -2(\mathbf{N}\cdot\nabla u)Lu + C\varepsilon|\nabla u|^2 + C\varepsilon|x||D^2u||\nabla u|,
\end{aligned}
$$

where in the last line we have used the bounds $\|b\|_{L^\infty(B_1^+)} \le \varepsilon$ and $|A(x) - A(0)| \le \varepsilon|x|$ for $x \in B_1^+$. It follows that

$$
\begin{aligned}
&|e_n|_{A(0)} \int_{\partial^0 B_1^+} |\nabla u|^2_{A(0)} \eta^2 \, \mathrm{d}\mathcal{H}^{n-1} \\
&\qquad \le -2 \int_{B_1^+} (\mathbf{N} \cdot \nabla u) Lu \, \eta^2 \, \mathrm{d}x + C \int_{B_1^+} |\nabla u|^2 \big(|\nabla(\eta^2)| + \varepsilon\eta^2\big) \, \mathrm{d}x \\
&\qquad\qquad + C\varepsilon \int_{B_1^+} |x||D^2 u||\nabla u|\eta^2 \, \mathrm{d}x
\end{aligned} \tag{2.2.44}
$$

and, thus, it remains to control the term $-2\int_{B_1^+} (\mathbf{N} \cdot \nabla u) Lu \, \eta^2 \, \mathrm{d}x$ in (2.2.44).

Since $-Lu = f(u)$ in B_1^+, the primitive $F(t) := \int_0^t f(s)\,\mathrm{d}s$ of f satisfies

$$
\mathbf{N} \cdot \nabla[F(u)] = (\mathbf{N} \cdot \nabla u) f(u) = -(\mathbf{N} \cdot \nabla u) Lu,
$$

and the first term on the right hand side of (2.2.44) can be integrated by parts as

$$
-\int_{B_1^+} (\mathbf{N} \cdot \nabla u) Lu \, \eta^2 \, \mathrm{d}x = \int_{B_1^+} \mathbf{N} \cdot \nabla[F(u)]\eta^2 \, \mathrm{d}x = -\int_{B_1^+} F(u)\big(\mathbf{N} \cdot \nabla(\eta^2)\big) \, \mathrm{d}x. \tag{2.2.45}
$$

By the monotonicity of f, since u and f are nonnegative, we have $|F(u)| \le uf(u) = -uLu$. Hence, writing L in divergence form $Lu = \operatorname{div}(A(x)\nabla u) + \widehat{b}(x) \cdot \nabla u$ as in (2.1.6), by the coefficient bound $\|\widehat{b}\|_{L^\infty(B_1^+)} \le C\varepsilon$ we deduce

$$
|F(u)| \le -u \operatorname{div}(A(x)\nabla u) + C\varepsilon u\,|\nabla u|. \tag{2.2.46}
$$

Using (2.2.46), we estimate the right-hand side of (2.2.45) by

$$
\Big| -\int_{B_1^+} F(u)\big(\mathbf{N}\cdot\nabla(\eta^2)\big) \, \mathrm{d}x \Big| \le -C \int_{B_1^+} u \operatorname{div}\big(A(x)\nabla u\big)|\nabla(\eta^2)| \, \mathrm{d}x + C\varepsilon \int_{B_1^+} u\,|\nabla u||\nabla(\eta^2)| \, \mathrm{d}x, \tag{2.2.47}
$$

and since $|\nabla(\eta^2)|$ is Lipschitz, the divergence term in (2.2.47) can be integrated by parts as

$$
-\int_{B_1^+} u \operatorname{div}\big(A(x)\nabla u\big)|\nabla(\eta^2)| \, \mathrm{d}x = \int_{B_1^+} |\nabla u|^2_{A(x)}|\nabla(\eta^2)| \, \mathrm{d}x + \int_{B_1^+} u\, A(x)\nabla u \cdot \nabla|\nabla(\eta^2)| \, \mathrm{d}x. \tag{2.2.48}
$$

Therefore, combining (2.2.45), (2.2.47), and (2.2.48), we deduce

$$
-\int_{B_1^+} (\mathbf{N} \cdot \nabla u) Lu \, \eta^2 \, \mathrm{d}x \le \int_{B_1^+} |\nabla u|^2_{A(x)}|\nabla(\eta^2)| \, \mathrm{d}x + C \int_{B_1^+} u|\nabla u| \left(|D^2(\eta^2)| + \varepsilon|\nabla(\eta^2)|\right) \mathrm{d}x. \tag{2.2.49}
$$

Moreover, we can bound the last term in (2.2.49) by Cauchy-Schwarz and the Poincaré inequality (valid since $u = 0$ on $\partial^0 B_1^+$) as

$$
\begin{aligned}
&\int_{B_1^+} u|\nabla u| \left(|D^2(\eta^2)| + \varepsilon|\nabla(\eta^2)|\right) \mathrm{d}x \\
&\le C \left(\int_{B_{7/9}^+} |\nabla u|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_{B_1^+} |\nabla u|^2 \left(|D^2(\eta^2)| + \varepsilon|\nabla(\eta^2)|\right)^2 \mathrm{d}x \right)^{1/2}.
\end{aligned} \tag{2.2.50}
$$

Applying the bounds (2.2.49) and (2.2.50) in (2.2.44), by uniform ellipticity, we obtain

$$\begin{aligned}\int_{\partial^0 B_1^+} &|\nabla u|^2\eta^2 \,\mathrm{d}\mathcal{H}^{n-1} \\ &\leq C\left(\int_{B_{7/9}^+}|\nabla u|^2\,\mathrm{d}x\right)^{1/2}\left(\int_{B_1^+}|\nabla u|^2\left(|D^2(\eta^2)|+\varepsilon|\nabla(\eta^2)|\right)^2\mathrm{d}x\right)^{1/2} \\ &\qquad + C\int_{B_1^+}|\nabla u|^2\big(|\nabla(\eta^2)|+\varepsilon\eta^2\big)\,\mathrm{d}x + C\varepsilon\int_{B_1^+}|x||D^2u||\nabla u|\eta^2\,\mathrm{d}x.\end{aligned} \tag{2.2.51}$$

Finally, choosing $\eta \in C_c^\infty(B_{7/9})$ in (2.2.51) satisfying $\eta = 1$ in $B_{2/3}$ and $0 \leq \eta \leq 1$ in $B_{7/9}$, we deduce

$$\int_{\partial^0 B_{2/3}^+}|\nabla u|^2\,\mathrm{d}\mathcal{H}^{n-1} \leq C\,(1+\varepsilon)\int_{B_{7/9}^+}|\nabla u|^2\,\mathrm{d}x + C\varepsilon\int_{B_{7/9}^+}|x||D^2u||\nabla u|\,\mathrm{d}x,$$

which yields the claim.

Step 2. *Conclusion.*

Let $\varepsilon_0 > 0$ be the universal constant in the conclusion of Lemma 2.2.1. Applying this result with a cut-off $\zeta \in C_c^1(B_{8/9})$ such that $0 \leq \zeta \leq 1$ and $\zeta = 1$ in $B_{7/9}$, if $\varepsilon \leq \varepsilon_0$, then

$$\||D^2u|\,|\nabla u|\|_{L^1(B_{7/9}^+)} \leq C\|\nabla u\|^2_{L^2(B_{8/9}^+)} + C\|\nabla u\|^2_{L^2(\partial^0 B_{8/9}^+)} + C\|\mathcal{A}\,|\nabla u|\|_{L^1(B_{8/9}^+)}. \tag{2.2.52}$$

Hence, applying Cauchy-Schwarz in (2.2.52) and by Lemma 2.2.2, we deduce

$$\||D^2u|\,|\nabla u|\|_{L^1(B_{7/9}^+)} \leq C\|\nabla u\|^2_{L^2(B_1^+)} + C\|\nabla u\|^2_{L^2(\partial^0 B_1^+)}. \tag{2.2.53}$$

Let $\delta > 0$. Using (2.2.53) in Step 1 above, letting $\varepsilon_\delta := \min\{\varepsilon_0, \delta/C\}$, we obtain

$$\|\nabla u\|^2_{L^2(\partial^0 B_{2/3}^+)} \leq \delta\|\nabla u\|^2_{L^2(\partial^0 B_1^+)} + C\|\nabla u\|^2_{L^2(B_1^+)} \qquad \text{for } \varepsilon \leq \varepsilon_\delta. \tag{2.2.54}$$

Hence, by translation and rescaling of (2.2.54), for all $y \in \partial^0 B_1^+$ and $\rho > 0$ such that $B_\rho^+(y) \subset B_1^+$, we have

$$\begin{aligned}\rho\int_{\partial^0 B_{2\rho/3}^+(y)}|\nabla u|^2\,\mathrm{d}\mathcal{H}^{n-1} &\leq \delta\rho\int_{\partial^0 B_\rho^+(y)}|\nabla u|^2\,\mathrm{d}\mathcal{H}^{n-1} + C\int_{B_\rho^+(y)}|\nabla u|^2\,\mathrm{d}x \\ &\leq \delta\rho\int_{\partial^0 B_\rho^+(y)}|\nabla u|^2\,\mathrm{d}\mathcal{H}^{n-1} + C\int_{B_1^+}|\nabla u|^2\,\mathrm{d}x \qquad \text{for } \varepsilon \leq \varepsilon_\delta.\end{aligned} \tag{2.2.55}$$

Since $y \in \partial^0 B_1^+$, we have $y = (y', 0)$ for some $y' \in \mathbb{R}^{n-1}$, and the lower boundary $\partial^0 B_\rho^+(y)$ is simply the $(n-1)$-dimensional ball $B'_\rho(y') := \{x \in \mathbb{R}^{n-1} \colon |x - y'| < \rho\} \subset \mathbb{R}^{n-1} = \partial^0\mathbb{R}^n$. By (2.2.55), we can apply the Simon lemma to the subadditive quantity

$$B' \mapsto \int_{B'}|\nabla u|^2\,\mathrm{d}\mathcal{H}^{n-1}$$

on balls $B' \subset B'_1 \subset \mathbb{R}^{n-1} = \partial^0\mathbb{R}^n$ to deduce the bound

$$\int_{\partial^0 B_{2/3}^+}|\nabla u|^2\,\mathrm{d}\mathcal{H}^{n-1} \leq C\int_{B_1^+}|\nabla u|^2\,\mathrm{d}x \quad \text{for } \varepsilon \leq \varepsilon_\delta, \tag{2.2.56}$$

for some universal $\delta > 0$. In particular, we may take ε_0 universal equal to ε_δ and this concludes the proof of (2.1.12).

Finally, to deduce the remaining Hessian estimates we proceed as in the proof of (2.2.53). To prove (2.1.13), we apply (2.2.32) from Lemma 2.2.1 with a cut-off function $\zeta \in C^1_c(B_{16/27})$ such that $0 \le \zeta \le 1$ and $\zeta = 1$ in $B_{4/7=16/28} \subset B_{16/27}$, and by Cauchy-Schwarz

$$\begin{aligned} \||D^2u|\,|\nabla u|\|_{L^1(B^+_{4/7})} &\le C\|\nabla u\|^2_{L^2(B^+_{16/27})} + C\|\nabla u\|^2_{L^2(\partial^0 B^+_{16/27})} + C\|\mathcal{A}\|^2_{L^2(B^+_{16/27})} \\ &\le C\|\nabla u\|^2_{L^2(B^+_{2/3})} + C\|\nabla u\|^2_{L^2(\partial^0 B^+_{2/3})}, \end{aligned} \tag{2.2.57}$$

where in the last line we have used Lemma 2.2.2 applied to the rescaled function $u(\frac{2}{3}\cdot)$. Applying (2.2.56) to (2.2.57) now leads to (2.1.13).

Now, the bound (2.1.14) is easily obtained combining Lemma 2.2.2 with the boundary estimate (2.1.12). The final estimate (2.1.15) follows from Lemma 2.2.1 and the above. $\square$

2.3 Boundary $W^{1,2+\gamma}$ estimate

First we control the Dirichlet energy by the L^1 norm of the solution under a smallness condition on the coefficients. This follows from Theorem 2.1.4 and the interpolation inequalities of Cabré in [19] (recalled in Appendix C below).

Lemma 2.3.1. *Let $u \in W^{3,p}(B_1^+)$, for some $p > n$, be a nonnegative stable solution of $-Lu = f(u)$ in B_1^+, with $u = 0$ on $\partial^0 B_1^+$. Assume that $f \in C^1(\mathbb{R})$ is nonnegative and nondecreasing. Assume that L satisfies conditions* (2.1.3), (2.1.4), *and* (2.1.5) *in $\Omega = B_1^+$, and*

$$\|DA\|_{L^\infty(B_1^+)} + \|b\|_{L^\infty(B_1^+)} \le \varepsilon$$

for some $\varepsilon > 0$.

If $\varepsilon \le \varepsilon_0$, then

$$\|\nabla u\|_{L^2(B^+_{1/2})} \le C\|u\|_{L^1(B_1^+)},$$

where $\varepsilon_0 > 0$ and C are universal constants.

Proof. We cover $B^+_{1/2}$ (except for a set of measure zero) with a family of disjoint open cubes $Q_j \subset \mathbb{R}^n_+$ of the same side-length and small enough so that $Q_j \subset B^+_{4/7}$. The side-length and the number of cubes depend only on n. Combining the interpolation inequalities of Proposition C.1 (with $p = 2$) and Proposition C.3, rescaled from the unit cube to Q_j, with $\tilde{\delta} = \delta^{3/2}$ for a given $\delta \in (0,1)$, we have

$$\int_{Q_j} |\nabla u|^2 dx \le C\delta \int_{Q_j} |D^2u||\nabla u|\,dx + C\delta \int_{Q_j} |\nabla u|^2 dx + C\delta^{-2-\frac{3n}{2}} \left(\int_{Q_j} |u|\,dx\right)^2.$$

Since $Q_j \subset B^+_{4/7}$, applying (2.1.13) from Theorem 2.1.4, for $\varepsilon \le \varepsilon_0$ we deduce

$$\int_{Q_j} |\nabla u|^2 dx \le C\delta \int_{B_1^+} |\nabla u|^2 dx + C\delta^{-2-\frac{3n}{2}} \left(\int_{B_1^+} |u|\,dx\right)^2.$$

Adding up these inequalities, we obtain

$$\|\nabla u\|^2_{L^2(B^+_{1/2})} \le C\delta\|\nabla u\|^2_{L^2(B_1^+)} + C\delta^{-2-\frac{3n}{2}}\|u\|^2_{L^1(B_1^+)} \quad \text{for } \delta \in (0,1) \text{ and } \varepsilon \le \varepsilon_0. \tag{2.3.1}$$

For $B_\rho^+(y) \subset B_1^+$ with $y \in \partial^0 B_1^+$, the function $u^{y,\rho} := u(y+\rho\,\cdot)$ is a stable solution to a semilinear equation with coefficients $A^{y,\rho} = A(y+\rho\,\cdot)$ and $b^{y,\rho} = \rho\, b(y+\rho\,\cdot)$. In particular, since $\rho \le 1$, for $\varepsilon \le \varepsilon_0$ we have

$$\|DA^{y,\rho}\|_{L^\infty(B_1^+)} + \|b^{y,\rho}\|_{L^\infty(B_1^+)} \le \rho\varepsilon \le \varepsilon_0,$$

and we may apply (2.3.1) to $u^{y,\rho}$, which yields

$$\rho^{n+2}\int_{B^+_{\rho/2}(y)} |\nabla u|^2\,dx \le C\delta\rho^{n+2}\int_{B^+_\rho(y)} |\nabla u|^2\,dx + C\delta^{-2-\frac{3n}{2}}\left(\int_{B^+_\rho(y)} |u|\,dx\right)^2,$$

hence

$$\rho^{n+2}\int_{B^+_{\rho/2}(y)} |\nabla u|^2\,dx \le C\delta\rho^{n+2}\int_{B^+_\rho(y)} |\nabla u|^2\,dx + C\delta^{-2-\frac{3n}{2}}\|u\|^2_{L^1(B_1^+)} \qquad \text{for all } B_\rho^+(y) \subset B_1^+ \text{ with } y \in \partial^0 B_1^+ \text{ and } \delta \in (0,1). \tag{2.3.2}$$

To deduce the desired bound, we must combine (2.3.2) with the following interior estimates derived in [52, Proposition 1.3]:

$$\rho^{n+2}\int_{B_{\rho/2}(y)} |\nabla u|^2\,\mathrm{d}x \le C\|u\|^2_{L^1(B_1^+)} \quad \text{ for all } B_\rho(y) \subset B_1^+. \tag{2.3.3}$$

We now claim that for all balls $B_\rho(y) \subset B_1$ (not necessarily contained in B_1^+) and every $\delta \in (0,1)$, we have

$$\rho^{n+2}\int_{\partial\mathbb{R}^n_+\cap B_{\rho/2}(y)} |\nabla u|^2\,dx \le C\delta\rho^{n+2}\int_{\partial\mathbb{R}^n_+\cap B_\rho(y)} |\nabla u|^2\,dx + C\delta^{-2-\frac{3n}{2}}\|u\|^2_{L^1(B_1^+)}. \tag{2.3.4}$$

This is achieved by a simple covering argument. The key observation is that $\mathbb{R}^n_+ \cap B_{\rho/2}(y)$ can be covered by a dimensional number of balls $\{B_{\rho/16}(y_i)\}_i$ and $\{B_{3\rho/16}(z_j)\}_j$, where y_i are such that $B_{\rho/8}(y_i) \subset \mathbb{R}^n_+ \cap B_\rho(y) \subset B_1^+$ are interior balls, while $z_j \in \partial\mathbb{R}^n_+$ satisfy $B^+_{3\rho/8}(z_j) \subset \mathbb{R}^n_+ \cap B_\rho(y) \subset B_1^+$. Applying (2.3.3) to the interior balls and (2.3.2) to the boundary balls, it is not hard to deduce (2.3.4). For more details, we refer the reader to the proof of Lemma 8.2 in [20].

By (2.3.4), applying Simon's lemma to the subadditive quantity $B \mapsto \|\nabla u\|^2_{L^2(\mathbb{R}^n_+\cap B)}$ now yields the claim. $\square$

Following ideas from [24], the higher integrability estimate in Theorem 2.1.1 will now be a direct consequence of the Hessian estimates in Theorem 2.1.4 and of Lemma 2.3.1.

Proof of Theorem 2.1.1. There are three steps in our proof. First, by the divergence theorem and Theorem 2.1.4, we control the surface integral of $|\nabla u|^2$ on every level set of u by the Dirichlet energy. Secondly, using coarea formula, Hölder, and Sobolev inequality, we will bound the $L^{2+\gamma}$ norm of the gradient by the L^2 norm. Finally, Lemma 2.3.1 will yield the final estimate in terms of the L^1 norm of the solution. All these bounds are shown under a smallness condition on the coefficients which is removed in the last step.

Step 1: *We prove that, if $\varepsilon \le \varepsilon_0$, then for a.e. $t \in \mathbb{R}$ we have*

$$\int_{\{u=t\}\cap B_{1/2}} |\nabla u|^2\,\mathrm{d}\mathcal{H}^{n-1} \le C\|\nabla u\|^2_{L^2(B_1)},$$

where $\varepsilon_0 > 0$ and C are universal.

Since $\left|\operatorname{div}\big(|\nabla u|\nabla u\big)\right| \le C|D^2u||\nabla u|$, by (2.1.13) in Theorem 2.1.4, for $\varepsilon \le \varepsilon_0$ we have

$$\left\|\operatorname{div}\big(|\nabla u|\nabla u\big)\right\|_{L^1(B^+_{4/7})} \le C\|\nabla u\|^2_{L^2(B_1^+)}. \tag{2.3.5}$$

Consider a cut-off function $\eta \in C_c^\infty(B_{4/7})$ with $\eta = 1$ in $B_{1/2}$ and $0 \le \eta \le 1$. By the divergence theorem, for a.e. $t \in \mathbb{R}$ we have

$$\begin{aligned}
&\int_{\{u=t\}\cap B^+_{1/2}} |\nabla u|^2 \,\mathrm{d}\mathcal{H}^{n-1}\\
&\quad\le \int_{\{u=t\}\cap B_1^+\cap\{\nabla u\neq 0\}} |\nabla u|^2\eta^2 \,\mathrm{d}\mathcal{H}^{n-1}\\
&\quad= -\int_{\{u>t\}\cap B_1^+\cap\{\nabla u\neq 0\}} \operatorname{div}\big(|\nabla u|\nabla u\,\eta^2\big)\,\mathrm{d}x - \int_{\{u>t\}\cap\partial^0 B_1^+\cap\{\nabla u\neq 0\}} |\nabla u|^2\,\eta^2\,\mathrm{d}x\\
&\quad\le \int_{B^+_{4/7}} |\nabla u|^2|\nabla(\eta^2)|\,\mathrm{d}x + \int_{B^+_{4/7}} \left|\operatorname{div}\big(|\nabla u|\nabla u\big)\right|\eta^2\,\mathrm{d}x
\end{aligned}$$

and (2.3.5) now yields the claim

Step 2: *We prove that, if $\varepsilon \le \varepsilon_0$, then*

$$\|\nabla u\|_{L^{2+\gamma}(B^+_{1/2})} \le C\|\nabla u\|_{L^2(B_1^+)},$$

where $\gamma > 0$ is dimensional and $\varepsilon_0 > 0$ and C are universal constants.

Multiplying by a constant, we may assume that $\|\nabla u\|_{L^2(B_1^+)} = 1$.

Letting $h(t) = \max\{1,t\}$, by the Sobolev embedding for functions vanishing on $\partial^0 B_1^+$,

$$\begin{aligned}
&\int_{\mathbb{R}^+} \mathrm{d}t \int_{\{u=t\}\cap B_1^+\cap\{|\nabla u|\neq 0\}} \mathrm{d}\mathcal{H}^{n-1} h(t)^p |\nabla u|^{-1}\\
&\qquad\qquad \le |B_1^+\cap\{u<1\}| + \int_{B_1^+} u^p\,\mathrm{d}x \le C
\end{aligned} \tag{2.3.6}$$

for some $p > 2$. Choosing dimensional constants $q > 1$ and $\theta \in (0, 1/3)$ such that $p/q = (1-\theta)/\theta$, we obtain

$$\begin{aligned}
\int_{B^+_{1/2}} |\nabla u|^{3-3\theta}\,\mathrm{d}x &= \int_{\mathbb{R}^+}\mathrm{d}t \int_{\{u=t\}\cap B^+_{1/2}\cap\{|\nabla u|\neq 0\}} \mathrm{d}\mathcal{H}^{n-1} h(t)^{p\theta - q(1-\theta)}|\nabla u|^{-\theta+2(1-\theta)}\\
&\le \left(\int_{\mathbb{R}^+}\mathrm{d}t \int_{\{u=t\}\cap B_1^+\cap\{|\nabla u|\neq 0\}} \mathrm{d}\mathcal{H}^{n-1} h(t)^p|\nabla u|^{-1}\right)^{\theta}\\
&\qquad \cdot \left(\int_{\mathbb{R}^+} h(t)^{-q}\,\mathrm{d}t \int_{\{u=t\}\cap B^+_{1/2}} \mathrm{d}\mathcal{H}^{n-1}|\nabla u|^2\right)^{1-\theta}.
\end{aligned}$$

By Step 1 and (2.3.6), it follows that

$$\int_{B^+_{1/2}} |\nabla u|^{3-3\theta}\,\mathrm{d}x \le C,$$

which was the claim.

Step 3: *Conclusion.*

Combining Step 2 (rescaled) and Lemma 2.3.1, we deduce that our class of stable solutions satifies

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/4}^+)} \le C\|u\|_{L^1(B_1^+)} \quad \text{for } \varepsilon \le \varepsilon_0, \tag{2.3.7}$$

where $\gamma > 0$ is dimensional and $\varepsilon_0 > 0$ and C are universal.

To conclude, we apply a simple covering argument. Let $\delta \in (0,1)$ be sufficiently small such that

$$\delta\left(\|DA\|_{L^\infty(B_1^+)} + \|b\|_{L^\infty(B_1^+)}\right) \le \varepsilon_0. \tag{2.3.8}$$

First, we cover the lower boundary $\partial^0 B_{1/2}^+$ by a finite number of balls $B_{\delta/4}(y_i)$ with $y_i \in \partial^0 B_1^+$, taking $\delta > 0$ smaller if necessary so that $B_\delta(y_i) \subset B_1$. Next, we cover $\overline{B_{1/2}^+} \setminus \left(\cup_i B_{\delta/4}(y_i)\right)$ by balls $B_{\widetilde{\delta}/2}(z_i)$ with a smaller radius $\widetilde{\delta} > 0$ such that $B_{\widetilde{\delta}}(z_i) \subset B_1^+$. Thus we obtain a covering of $B_{1/2}^+$ by half-balls $\{B_{\delta/4}^+(y_i)\}_i$ (centered at the boundary) and interior balls $\{B_{\widetilde{\delta}/2}(z_i)\}_i$, satisfying $B_\delta^+(y_i) \subset B_1^+$ and $B_{\widetilde{\delta}}(z_i) \subset B_1^+$, respectively. Notice that, by (2.3.8), the radii δ and $\widetilde{\delta}$ as well as the number of balls depend only on n, ε_0, $\|DA\|_{L^\infty(B_1^+)}$, and $\|b\|_{L^\infty(B_1^+)}$.

Thanks to (2.3.8), the function $u(y_i + \delta\cdot)$ vanishing on $\partial^0 B_1^+$ is a stable solution of a semilinear equation in B_1^+, with coefficients $A^{y_i,\delta} = A(y_i + \delta\cdot)$ and $b^{y_i,\delta} = \delta\, b(y_i + \delta\cdot)$ such that $\|DA^{y_i,\delta}\|_{L^\infty} + \|b^{y_i,\delta}\|_{L^\infty} \le \varepsilon_0$. From (2.3.7) now we deduce

$$\|\nabla u\|_{L^{2+\gamma}(B_{\delta/4}^+(y_i))} \le C_\delta \|u\|_{L^1(B_\delta^+(y_i))}, \tag{2.3.9}$$

where C_δ depends only on n, c_0, C_0, and δ. For the interior balls $B_{\widetilde{\delta}/4}(z_i)$, we need the following interior energy estimates from Theorem 1.1.1 above (rescaled):

$$\|\nabla u\|_{L^{2+\gamma}(B_{\widetilde{\delta}/2}(z_i))} \le C_{\widetilde{\delta}} \|u\|_{L^1(B_{\widetilde{\delta}}(z_i))}, \tag{2.3.10}$$

where $C_{\widetilde{\delta}}$ depends only on n, c_0, C_0, and $\widetilde{\delta}$.

By (2.3.9) and (2.3.10), we finally obtain

$$\begin{aligned}
\|\nabla u\|_{L^{2+\gamma}(B_{1/2}^+)} &\le \sum_i \|\nabla u\|_{L^{2+\gamma}(B_{\delta/4}^+(y_j))} + \sum_i \|\nabla u\|_{L^{2+\gamma}(B_{\widetilde{\delta}/2}(z_i))} \\
&\le C_\delta \sum_i \|u\|_{L^1(B_\delta^+(y_i))} + C_{\widetilde{\delta}} \sum_i \|u\|_{L^1(B_{\widetilde{\delta}}(z_i))} \\
&\le C\|u\|_{L^1(B_1^+)},
\end{aligned}$$

where the last constant depends only on n, c_0, C_0, $\|DA\|_{L^\infty(B_1^+)}$, and $\|b\|_{L^\infty(B_1^+)}$. This concludes the proof of the theorem. □

Remark 2.3.2. It is also possible to deduce a higher integrability of the gradient from Lemma 2.3.1 directly by applying Gehring's lemma [59]. However, by that method, the integrability exponent in Theorem 2.1.1 would no longer be dimensional (i.e., depending only on n), but would additionally depend on the ellipticity constants.[4] Thus, the techniques

[4]Indeed, combining Lemma 2.3.1 with the analogous interior estimates in [52, Proposition 1.3], by Poincaré's inequality and a scaling and covering argument, it is not hard to show that the (say) even reflection of ∇u with respect to $\{x_n = 0\}$ satisfies $\left(R^{-n}\int_{B_R(x)} |\nabla u|^2\right)^{1/2} \le C_1 R^{-n}\int_{B_{2R}(x)} |\nabla u|$ for any ball $B_{2R}(x) \subset B_1$, where $C_1 = C_1(n, c_0, C_0)$ is a universal constant. Applying Gehring's lemma (for instance, by Theorem 6.38 in [62]) we now obtain an estimate $\|\nabla u\|_{L^p(B_{1/2}^+)} \le C\|\nabla u\|_{L^2(B_1^+)}$ for some $p = p(n, C_1) > 2$ and $C = C(n, C_1)$.

in [24] give a more precise control of the integrability exponent than Gehring's lemma. For instance, following the proof above, it is easy to see that one can take any $\gamma(n) < \frac{4}{3n-2}$.

We conclude this section by stating a corollary of the higher integrability and Hessian estimates that will be useful in the next chapter. It consists of two simple estimates on annuli that can be proven by a standard covering argument, combining Theorem 2.1.1 (respectively Theorem 2.1.4 and Lemma 2.3.1) with the analogous interior estimates Theorem 1.1.1 (respectively in Proposition 1.1.3 and Remark 1.3.4) obtained in Chapter 1.

Corollary 2.3.3. *Let $u \in W^{3,p}(B_1^+)$, for some $p > n$, be a nonnegative stable solution of $-Lu = f(u)$ in B_1^+, with $u = 0$ on $\partial^0 B_1^+$. Assume that $f \in C^1(\mathbb{R})$ is nonnegative and nondecreasing. Assume that L satisfies conditions* (2.1.3), (2.1.4), *and* (2.1.5) *in $\Omega = B_1^+$, and*

$$\|DA\|_{L^\infty(B_1^+)} + \|b\|_{L^\infty(B_1^+)} \leq \varepsilon$$

for some $\varepsilon > 0$. Let $0 < \rho_1 < \rho_2 < \rho_3 < \rho_4 \leq 1$.

Then

$$\|\nabla u\|_{L^{2+\gamma}(A^+_{\rho_2,\rho_3})} \leq C_{\varepsilon,\rho_i} \|u\|_{L^1(A^+_{\rho_1,\rho_4})}$$

and

$$\|D^2 u\|_{L^1(A^+_{\rho_2,\rho_3})} \leq C_{\varepsilon,\rho_i} \|u\|_{L^1(A^+_{\rho_1,\rho_4})},$$

where C_{ε,ρ_i} is a constant depending only on n, c_0, C_0, ε, ρ_1, ρ_2, ρ_3, and ρ_4.

Chapter 3

Boundary regularity in $C^{1,1}$ domains

This chapter establishes the boundary Hölder continuity of stable solutions to semilinear elliptic problems in the optimal range of dimensions $n \leq 9$, for $C^{1,1}$ domains. We consider equations $-Lu = f(u)$ in a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^n$, with $u = 0$ on $\partial\Omega$, where L is a linear elliptic operator with variable coefficients and $f \in C^1$ is nonnegative, nondecreasing, and convex. The stability of u amounts to the nonnegativity of the principal eigenvalue of the linearized equation $-L - f'(u)$. Our result is new even for the Laplacian, for which [24] proved the Hölder continuity in C^3 domains.

3.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f\colon \mathbb{R} \to \mathbb{R}$ a C^1 function. In this chapter, we consider stable solutions $u\colon \overline{\Omega} \to \mathbb{R}$ to the semilinear problem

$$\begin{cases} -Lu = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.1}$$

where L is a second order linear elliptic differential operator of the form

$$Lu = a_{ij}(x)u_{ij} + b_i(x)u_i, \quad a_{ij}(x) = a_{ji}(x). \tag{3.1.2}$$

We assume that the coefficient matrix $A(x) = (a_{ij}(x))$ is uniformly elliptic in Ω, i.e., there are positive constants c_0, C_0 such that

$$c_0|p|^2 \leq a_{ij}(x)p_ip_j \leq C_0|p|^2 \quad \text{for all } p \in \mathbb{R}^n. \tag{3.1.3}$$

This last condition is denoted by $c_0 \leq A(x) \leq C_0$. In addition, we will always assume that

$$a_{ij} \in C^{0,1}(\overline{\Omega}), \quad b_i \in L^\infty(\Omega) \cap C^0(\Omega). \tag{3.1.4}$$

For some auxiliary results, we will further need that

$$b_i \in C^0(\overline{\Omega}). \tag{3.1.5}$$

A strong solution $u \in C^0(\overline{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ of (3.1.1) is *stable* if the principal eigenvalue of the linearized equation at u is nonnegative. Equivalently, the solution u is stable if there is a function $\varphi \in W^{2,n}_{\mathrm{loc}}(\Omega)$ satisfying

$$\begin{cases} J_u\varphi \leq 0 & \text{a.e. in } \Omega, \\ \varphi > 0 & \text{in } \Omega, \end{cases} \tag{3.1.6}$$

where $J_u := L + f'(u)$ denotes the Jacobi operator (the linearization) at u. For variational problems, the stability condition amounts to the nonnegativity of the second variation. In particular, the class of stable solutions contains (local and global) minimizers. Instead, here we are interested in non-variational equations as above. For fundamental properties of the principal eigenvalue of linear non self-adjoint operators such as J_u, we refer to the classic work of Berestycki, Nirenberg, and Varadhan [9].

The aim of this chapter is to investigate the regularity up to the boundary of stable solutions to (3.1.1). Here, the question reduces to showing that solutions are bounded, since the linear theory then allows to prove further smoothness properties. Our present work extends the boundary regularity results of Cabré, Figalli, Ros-Oton, and Serra [24] and of Cabré [19] for the Laplacian to the above operators with variable coefficients. When $n \geq 10$, examples of singular stable solutions have been known for a long time. In the breakthrough article [24], the authors solved the long-standing conjecture: if $n \leq 9$, then all stable solutions are bounded (when $L = \Delta$). Their proof was quite delicate and relied on a contradiction-compactness argument which did not allow to quantify the constants in the estimates. An alternative quantitative proof has been recently found in [19], although it only applies to the Laplacian in flat domains. Generalizing and combining ideas form these two works, we will give, for the first time (even for the Laplacian), a quantitative proof valid in curved domains.

For $L = \Delta$, a key assumption needed in [24] was for the domain to be of class C^3. As we explain next, our analysis will allow us to weaken this condition to a $C^{1,1}$ regularity assumption. For this, starting from a curved boundary, we flatten it out (locally) by a change of variables. This transformation does not alter the semilinear nature of the equation but modifies the coefficients, which now involve first and second order derivatives of the flattening map. Namely, the new coefficients a_{ij} include first derivatives of this map, while the b_i contain second derivatives. Now, the crucial point is to obtain universal a priori estimates independent of the nonlinearity (in the spirit of [24]) and having a specific dependence on the coefficients. Our bounds will depend only on the ellipticity constants in (3.1.3) and on the norms $\|\nabla a_{ij}\|_{L^\infty}$ and $\|b_i\|_{L^\infty}$ of the coefficients, corresponding to the flattening map of a $C^{1,1}$ domain. As mentioned above, thanks to a device of [19] for flat domains, our estimates in the new coordinates will all be quantitative. Combined with the interior bounds that we established in [52], they will yield a global estimate.

We believe that our ideas can also be applied to study the boundary regularity of stable solutions for other equations. Our method provides a robust, direct way of proving quantitative estimates up to the boundary. In particular, when L is the p-Laplacian, we could extend the optimal interior bounds of Cabré, Miraglio, and Sanchón in [25] up to the boundary. By contrast, the previous work [24] relies on an intricate blow-up and Liouville theorem argument. The authors of [24] need this in order to apply a result of theirs only available on a flat boundary, which they could only prove by contradiction-compactness. This critical step does not allow them to quantify the constants in their inequalities.

Variational problems have a long history of regularity results for stable solutions, starting with the pioneering work of Crandall and Rabinowitz [44] in the seventies. For exponential and power nonlinearities, they showed that stable solutions are bounded in smooth domains when $n \leq 9$ (see also Joseph and Lundgren [64] for an exhaustive analysis of the radial case). Their result is optimal, since the logarithm $u(x) = \log(1/|x|^2) \in W_0^{1,2}(\Omega)$ solves (3.1.1) (in the weak sense) with $\Omega = B_1$, $L = \Delta$, and $f(u) = 2(n-2)e^u$, and is stable for $n \geq 10$. This last fact follows immediately from Hardy's inequality. Surprisingly, [44] appears to be the only variational paper where variable coefficients have been considered.

Namely, the a priori estimates in [44] apply to self-adjoint operators in divergence form, with merely bounded coefficients. However, the methods used cannot be extended to treat more general nonlinearities.

The motivation for considering exponential nonlinearities in [44] came from problems in combustion theory, namely, from the so called explosion or Gelfand problem [60] (recalled in Subsection 3.1.2 below). In the nineties, Brezis [11] asked whether the optimal dimension could be the same for more general nonlinearities. He was interested in a natural class of nonlinearities for which the Gelfand problem admits stable solutions, namely: nonnegative, nondecreasing, convex, and superlinear ones. This question motivated a series of works trying to establish global a priori estimates for stable solutions to (3.1.1) in the model case, i.e., when $L = \Delta$. First, Nedev [78] proved their boundedness for $n \leq 3$. Then, Cabré and Capella [21] reached the optimal dimension $n \leq 9$ in the radial case. Later, Cabré [16] and Villegas [95] showed the boundedness when $n \leq 4$. Afterwards, Cabré and Ros-Oton [27] proved the boundedness for $n \leq 7$ when Ω is a domain of double revolution. Finally, Cabré, Figalli, Ros-Oton, and Serra [24] solved the conjecture in C^3 domains.

Concerning non-variational problems, there is only one paper, to the best of our knowledge, studying the regularity of stable solutions in our setting. In [40], Cowan and Ghoussoub consider operators of the form (3.1.2), assuming that $a_{ij} = \delta_{ij}$ and $b_i \in C^\infty(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. They showed that stable solutions are bounded when $n \leq 9$ for the exponential nonlinearity. In particular, adding an advection term does not modify the optimal dimension in this case. This is in accordance with our present work, where our mild smoothness assumption on the coefficients guarantees the invariance of the optimal dimension for general nonlinearities. It is worth noting that the interest of the authors in [40] was in singular nonlinearities appearing in the modeling of MEMS devices, namely, nonlinearities $f = f(u)$ defined for $u \in [0, 1)$ which blow up at $u = 1$.

Finally, we would also like to mention the recent work of Costa, de Souza, and Montenegro [38] for a more general non-variational setting. In that paper, the authors consider the Gelfand problem for systems of equations including operators of the form (3.1.2). While they are mostly concerned with the existence of stable solutions to this problem, they also address the question of regularity, but only for the Laplacian. More precisely, they study stable solutions $u \colon \mathbb{R}^n \to \mathbb{R}^m$ (with $m \geq 2$) of $-\Delta u = F(u)$, where $F \colon \mathbb{R}^m \to \mathbb{R}^m$ satisfies natural assumptions analogous to ours. In the radial case, they are able to show that they are bounded for $n \leq 9$, which is the optimal dimension for scalar equations. For convex $C^{1,1}$ domains, they show their boundedness for $n \leq 3$ by adapting the interior estimates of Cabré [16]. We believe that our techniques can also be used to reach the optimal dimension for systems of equations in any $C^{1,1}$ domain.

3.1.1 Main results

Our main result provides two types of a priori estimates for strong stable solutions on domains of class $C^{1,1}$. The first one is an energy estimate valid in all dimensions. It has been announced in the previous chapter (Theorem 2.1.5 above), where we have proved it in flat domains (see Theorem 2.1.5 above or Theorem 3.1.4 below). Here, we will complete the proof, which involves a covering and approximation procedure. The second estimate is a bound of the Hölder norm in the optimal range of dimensions $n \leq 9$. As usual, here and throughout the chapter, when we write $C = C(\dots)$ for a constant C we mean that C depends only on the quantities appearing inside the parentheses.

Theorem 3.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{1,1}$ and let L satisfy conditions* (3.1.3) *and* (3.1.4) *in Ω. Assume that $f \in C^1(\mathbb{R})$ is nonnegative, nondecreasing, and convex.*

Let $u \in C^0(\overline{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ be a nonnegative stable solution of $-Lu = f(u)$ in Ω, with $u = 0$ on $\partial\Omega$.

Then

$$\|\nabla u\|_{L^{2+\gamma}(\Omega)} \leq C\|u\|_{L^1(\Omega)}, \tag{3.1.7}$$

where $\gamma = \gamma(n) > 0$ and $C = C(\Omega, n, c_0, C_0, \|\nabla a_{ij}\|_{L^\infty(\Omega)}, \|b_i\|_{L^\infty(\Omega)})$. In addition,

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C\|u\|_{L^1(\Omega)} \qquad \text{if } n \leq 9, \tag{3.1.8}$$

where $\alpha = \alpha(n, c_0, C_0) > 0$ and $C = C(\Omega, n, c_0, C_0, \|\nabla a_{ij}\|_{L^\infty(\Omega)}, \|b_i\|_{L^\infty(\Omega)})$.

The proof of Theorem 3.1.1 relies on analogous boundary estimates in half-balls, given next, as well as in the interior bounds from Chapter 1. As before, below, we always write $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ and, for each $\rho > 0$, we let

$$B^+_\rho := \mathbb{R}^n_+ \cap B_\rho.$$

Moreover, for any open set $\Omega \subset \mathbb{R}^n_+$, we denote its lower and upper boundaries by

$$\partial^0\Omega = \{x_n = 0\} \cap \partial\Omega, \quad \partial^+\Omega = \mathbb{R}^n_+ \cap \partial\Omega.$$

The Hölder estimate (3.1.8) will be a consequence of the following:

Theorem 3.1.2. *Let L satisfy conditions* (3.1.3), (3.1.4), *and* (3.1.5) *in $\Omega = B^+_1$. Assume that $f \in C^1(\mathbb{R})$ is nonnegative, nondecreasing, and convex.*

Let $u \in W^{3,p}(B^+_1)$, for some $p > n$, be a nonnegative stable solution to $-Lu = f(u)$ in B^+_1, with $u = 0$ on $\partial^0 B^+_1$.

Then

$$\|u\|_{C^\alpha(\overline{B^+_{1/2}})} \leq C\|u\|_{L^1(B^+_1)} \qquad \text{if } n \leq 9, \tag{3.1.9}$$

where $\alpha = \alpha(n, c_0, C_0) > 0$ and $C = C(n, c_0, C_0, \|\nabla a_{ij}\|_{L^\infty(B^+_1)}, \|b_i\|_{L^\infty(B^+_1)})$.

Remark 3.1.3. In contrast to Theorem 3.1.1 above, here we require additional hypotheses on the solution and the coefficients. Namely, we need third weak derivatives of u to remove the nonlinearity from the stability condition, making our bounds independent of f. We further need the continuity of b_i up to the boundary, assumption (3.1.5), to control certain surface integrals over $\partial^0 B^+_1$ appearing in the proof.

To prove the estimate in $C^{1,1}$ domains from the one in half-balls, we carry out an approximation and flattening procedure. We consider an exhaustion by smooth domains Ω_k and, in each of them, we construct a stable solution u_k to a semilinear equation with more regular coefficients. The smoothness of the data will guarantee that u_k is in $W^{3,p}(\Omega_k)$ and hence, flattening the boundary $\partial\Omega_k$, we may apply Theorem 3.1.2. Thanks to the $C^{1,1}$ regularity assumption, the constants in the bounds for u_k in half-balls will be independent of k. By convexity of f, the functions u_k converge to the original solution and taking limits we deduce the theorem.

As mentioned above, the energy estimate (3.1.7) in $C^{1,1}$ domains uses the analogue result in half-balls. In the following result from Chapter 2 (see Theorem 2.1.1), we obtained such a bound via Hessian estimates for stable solutions in the spirit of Sternberg and Zumbrun [91]. To prove Theorem 3.1.2, we will need both the energy and Hessian estimates.

Theorem 3.1.4 (Theorem 2.1.4)**.** *Let L satisfy conditions* (3.1.3), (3.1.4), *and* (3.1.5) *in* $\Omega = B_1^+$. *Assume that $f \in C^1(\mathbb{R})$ is nonnegative and nondecreasing.*

Let $u \in W^{3,p}(B_1^+)$, for some $p > n$, be a nonnegative stable solution to $-Lu = f(u)$ in B_1^+, with $u = 0$ on $\partial^0 B_1^+$.

Then

$$\| |\nabla u|\, D^2 u\|_{L^1(B_{1/2}^+)} \le C \|\nabla u\|^2_{L^2(B_1^+)}, \tag{3.1.10}$$

and

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/2}^+)} \le C \|u\|_{L^1(B_1^+)}, \tag{3.1.11}$$

where $\gamma = \gamma(n) > 0$ and $C = C(n, c_0, C_0, \|\nabla a_{ij}\|_{L^\infty(B_1^+)}, \|b_i\|_{L^\infty(B_1^+)})$.

3.1.2 Application: Regularity of the extremal solution in $C^{1,1}$ domains.

Let $f \colon [0, +\infty) \to \mathbb{R}$ satisfy $f(0) > 0$ and be nondecreasing, convex, and superlinear at $+\infty$, meaning that

$$\lim_{u \to +\infty} \frac{f(u)}{u} = +\infty.$$

Given a constant $\lambda > 0$, we consider the problem

$$\begin{cases} -Lu = \lambda f(u) & \text{in } \Omega \\ \quad u > 0 & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.12}$$

where $\Omega \subset \mathbb{R}^n$ is a $C^{1,1}$ bounded domain, and L satisfies conditions (3.1.3) and (3.1.4) in Ω.

The boundary value problem (3.1.12) is the *Gelfand problem* mentioned above. It was first presented by Barenblatt in a volume edited by Gelfand [60]. Originally, (3.1.12) was introduced to study ignition and explosion phenomena in the theory of thermal combustion. In that framework, u can be understood as the temperature of a combustible mixture, while λ measures the relative strength of the reaction $f(u)$ with respect to the diffusion-advection processes modeled by L. When λ is large, solutions are not expected to exist, which is interpreted as the occurrence of an explosion.

Stable solutions play a prominent role in the Gelfand problem, as evidenced by the next proposition below. For an account of the history and references for (3.1.12), we refer the reader to the monograph of Dupaigne [48]. Here, instead, we only recall a basic, well-known result concerning the existence of solutions to (3.1.12).

Note that by the identification $a_{ij} \in C^{0,1}(\overline{\Omega}) = W^{1,\infty}(\Omega)$ we can always write our operator L in divergence form

$$Lu = \operatorname{div}(A(x)\nabla u) + \widehat{b}(x) \cdot \nabla u, \tag{3.1.13}$$

where $\widehat{b}(x) = (\widehat{b}_i(x))$ is the vector field given by

$$\widehat{b}_i(x) = b_i(x) - \partial_k a_{ki}(x). \tag{3.1.14}$$

In particular, since we always assume that $\partial_k a_{ij} \in L^\infty(\Omega)$ and $b_i \in L^\infty(\Omega)$, we also have $\widehat{b}_i \in L^\infty(\Omega)$. The following result for non-variational problems has appeared in a slightly different form in [8, 38]. For the classical variational version, see, for instance, [11, 48].

Proposition 3.1.5. *There exists a constant $\lambda^\star \in (0, +\infty)$ such that:*

(i) *For each $\lambda \in (0, \lambda^\star)$ there is a unique strong stable solution $u_\lambda \in C^0(\overline{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ of* (3.1.12)*. Moreover, we have $u_\lambda < u_{\lambda'}$ in Ω for $\lambda < \lambda'$.*

(ii) *For $\lambda > \lambda^\star$ there is no strong solution.*

Assume moreover that

$$\operatorname{div} \widehat{b} \in L^\infty(\Omega), \tag{3.1.15}$$

so that the adjoint operator

$$L^T \zeta = \operatorname{div}(A(x)\nabla\zeta) - \widehat{b}(x) \cdot \nabla\zeta - \operatorname{div} \widehat{b}(x)\, \zeta$$

is well defined for $\zeta \in W^{2,n}_{\mathrm{loc}}(\Omega)$ and has bounded coefficients. Then:

(iii) *For $\lambda = \lambda^\star$ there exists a unique L^1-weak solution $u^\star$, in the following sense: $u^\star \in L^1(\Omega)$, $f(u^\star)\mathrm{dist}(\cdot, \partial\Omega) \in L^1(\Omega)$, and*

$$-\int_\Omega u^\star L^T\zeta \,\mathrm{d}x = \lambda^\star \int_\Omega f(u^\star)\zeta \,\mathrm{d}x \quad \textit{for all } \zeta \in W^{2,n}(\Omega) \textit{ with } L^T\zeta \in L^\infty(\Omega) \textit{ and } \zeta|_{\partial\Omega} = 0.$$

The solution $u^\star$ is called the extremal solution *of* (3.1.12) *and satisfies $u_\lambda \uparrow u^\star$ as $\lambda \uparrow \lambda^\star$.*

Remark 3.1.6. The uniqueness of $u^\star$ is due to Martel [73]. Although he proved it in the model case $L = \Delta$, the same ideas extend to the operators considered in this chapter.

Remark 3.1.7. The additional regularity (3.1.15) of the drift $\widehat{b}_i$ is needed in (iii) to guarantee that $u^\star \in L^1(\Omega)$ and $f(u^\star)\mathrm{dist}(\cdot, \partial\Omega) \in L^1(\Omega)$. For this, testing the equation with the principal eigenfunction ϕ of L^T, by superlinearity of f, it is easy to show that

$$\int_\Omega f(u_\lambda)\phi \,\mathrm{d}x \le C \quad \text{for } \lambda \in (0, \lambda^\star),$$

where C does not depend on λ. By regularity $\phi \in W^{2,p}(\Omega)$ for all $p < \infty$, hence $\phi \in C^1(\overline{\Omega})$, and by maximum principle $\phi \ge c\,\mathrm{dist}(\cdot, \partial\Omega)$, whence $f(u^\star)\mathrm{dist}(\cdot, \partial\Omega) \in L^1(\Omega)$. Now, testing the equation with the unique solution to $-L^T\vartheta = 1$ in Ω, $\vartheta = 0$ on $\partial\Omega$, we also have

$$\int_\Omega u_\lambda \,\mathrm{d}x = \lambda \int_\Omega f(u_\lambda)\vartheta \,\mathrm{d}x \quad \text{for } \lambda \in (0, \lambda^\star).$$

Again, by regularity $\vartheta \le C\phi$, and using the inequality above, we conclude that $u^\star \in L^1(\Omega)$.

Since, a priori, the extremal solution $u^\star$ is only in $L^1(\Omega)$, it is natural to investigate further regularity properties. In this direction, Brezis and Vázquez [14, Problem 1] asked whether the extremal solution for the model operator $L = \Delta$ was always in $W^{1,2}_0(\Omega)$, this being the natural energy space of the variational problem. Similarly, as explained above, Brezis [11, Open problem 1] asked if the extremal solution $u^\star$ was always bounded for $n \le 9$. Both of these questions were answered positively by Cabré, Figalli, Ros-Oton, and Serra [24] for the Laplacian in C^3 domains. For this, they applied their a priori estimates to the classical stable solutions $\{u_\lambda\}_{\lambda<\lambda^\star}$ and, using that they are bounded in $L^1(\Omega)$, they passed to the limit as $\lambda \uparrow \lambda^\star$. By this same procedure, our main theorem, Theorem 3.1.1, extends their result to operators with coefficients in $C^{1,1}$ domains:

Corollary 3.1.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{1,1}$ and let L satisfy conditions* (3.1.3), (3.1.4), *and* (3.1.15) *in* Ω. *Assume that $f \in C^1(\mathbb{R})$ is positive, nondecreasing, convex, and superlinear at $+\infty$.*

Then the extremal solution $u^\star$ to (3.1.12) *is in $W_0^{1,2+\gamma}(\Omega)$ for some $\gamma = \gamma(n) > 0$. Moreover, if $n \le 9$, then $u^\star$ is bounded (and hence is a strong solution in $W^{2,p}(\Omega)$ for all $p < \infty$).*

3.1.3 Structure of the proof

By approximation, it will suffice to prove Theorem 3.1.1 in smooth domains. Flattening the boundary, we further reduce the problem to half-balls and, hence, the core of the proof is to show the Hölder estimate in Theorem 3.1.2. Moreover, by a scaling and covering argument, we may assume that the operator L is close to the Laplacian, i.e., the coefficients satisfy $A(0) = I$ and $\|DA\|_{L^\infty} + \|b\|_{L^\infty} \le \varepsilon$, with ε small.

As in the previous chapters, to obtain a priori estimates, we will use the stability of the solution via a more convenient integral inequality. Following the notation above, we denote the norm induced by the positive definite matrix $A(x) = (a_{ij}(x))$ by

$$|p|_{A(x)} := (a_{ij}(x)p_ip_j)^{1/2} \qquad \text{for } p \in \mathbb{R}^n.$$

In Chapter 1, we showed that if u is stable, then

$$\int_\Omega f'(u)\xi^2\,\mathrm{d}x \le \int_\Omega \left|\nabla\xi - \tfrac{1}{2}\xi A^{-1}(x)\widehat{b}(x)\right|^2_{A(x)} \mathrm{d}x \qquad \text{for all } \xi \in C_c^\infty(\Omega), \tag{3.1.16}$$

where the vector field $\widehat{b}(x) = (\widehat{b}_i(x))$ is given by (3.1.14) above (and introduced in (2.1.7)).

Using the integral stability inequality (3.1.16) and thanks to the energy and Hessian estimates in Theorem 3.1.4, we will be able to prove two key auxiliary results: Propositions 3.1.9 and 3.1.10 below, which we comment on next. Combined, they will yield the Hölder estimate (3.1.9) in Theorem 3.1.2.

The first proposition provides a weighted L^2 estimate for the radial derivative

$$u_r = \frac{x}{|x|}\cdot\nabla u \quad (r = |x|)$$

in a half-ball by the L^2 norm of the full gradient in a half-annulus, under a smallness condition on the coefficients. This bound requires $n \le 9$ and will be essential in the proof of the Hölder regularity of stable solutions. As above, here and throughout this chapter, a constant depending only on n, c_0, and C_0 will be called *universal*.

Proposition 3.1.9. *Let $u \in W^{3,p}(B_1^+)$, for some $p > n$, be a nonnegative stable solution of $-Lu = f(u)$ in B_1^+, with $u = 0$ on $\partial^0 B_1^+$. Assume that $f \in C^1(\mathbb{R})$ is nonnegative and nondecreasing. Assume that L satisfies conditions* (3.1.3), (3.1.4), *and* (3.1.5) *in $\Omega = B_1^+$, and*

$$\|DA\|_{L^\infty(B_1^+)} + \|b\|_{L^\infty(B_1^+)} \le \varepsilon$$

for some $\varepsilon > 0$.

If $3 \le n \le 9$ and $\varepsilon \le \varepsilon_0$, then

$$\int_{B_\rho^+} r^{2-n}u_r^2\,\mathrm{d}x \le C\int_{B_{2\rho}^+\setminus B_\rho^+} r^{2-n}|\nabla u|^2\,\mathrm{d}x + C\varepsilon\int_{B_{4\rho}^+} r^{3-n}|\nabla u|^2\,\mathrm{d}x$$

for all $\rho \le 1/4$, where $\varepsilon_0 > 0$ and C are universal constants.

Although Proposition 3.1.9 requires $n \geq 3$, adding superfluous variables, we may be able to use it to prove the C^α estimate when $n \leq 2$ as well.

To prove Proposition 3.1.9, we use the integral stability inequality (3.1.16) with appropriate test functions. Letting $\xi = \mathbf{c}\eta$ in (3.1.16) with $\Omega = B_1^+$, where $\mathbf{c}$, η are smooth functions satisfying $\mathbf{c} = 0$ on $\partial^0 B_1^+$ and $\operatorname{supp} \eta \subset B_1$, if we integrate by parts, then (3.1.16) becomes

$$\int_{B_1^+} \mathbf{c} J_u \mathbf{c}\, \eta^2 \, \mathrm{d}x \leq \int_{B_1^+} \mathbf{c}^2 \left| \nabla \eta - \tfrac{1}{2} \eta A^{-1}(x) b(x) \right|_{A(x)}^2 \mathrm{d}x. \tag{3.1.17}$$

By approximation, we may choose

$$\mathbf{c}(x) = x \cdot \nabla u(x) = r u_r \quad \text{ and } \quad \eta(x) = |x|_{A^{-1}(0)}^{\frac{2-n}{2}} \zeta(x),$$

where $\zeta \in C_c^\infty(B_1)$ is a cut-off. A test function of this type appeared for the first time in the work of Cabré and Capella [21] for the Laplacian in the radial case. A similar choice was used in [24] to establish the boundedness in C^3 domains. Our function is a linear transformation of the latter one.

This choice of test function will lead to the desired inequality, but also produces weighted Hessian errors that we must control. For this, we invoke Theorem 3.1.4, which is why we need the continuity of the coefficient $b_i \in C^0(\overline{B_1^+})$ and the assumptions $f \geq 0$ and $f' \geq 0$ on the nonlinearity. By contrast, for the Laplacian in C^3 domains, no such errors arise, which is why the previous works [19, 24] did not need any assumptions on f at this step.

In the second proposition, we control the L^1 norm of our stable solution by the L^1 norm of its radial derivative. This estimate is an extension of a device in [19] for the Laplacian in half-balls, and it is also the key step which makes our proofs quantitative. In addition to a smallness condition on the coefficients, this is the only place where we need the assumption $A(0) = I$ and where we use the convexity of f crucially (aside from the approximation argument for $C^{1,1}$ domains). It is worth mentioning that such a tool is also available for interior estimates, where the proof is entirely different and only requires f to be nonnegative; see [19, 52].

Proposition 3.1.10. *Let $u \in W^{3,p}(B_1^+)$, for some $p > n$, be a nonnegative stable solution of $-Lu = f(u)$ in B_1^+, with $u = 0$ on $\partial^0 B_1^+$. Assume that $f \in C^1(\mathbb{R})$ is nonnegative, nondecreasing, and convex. Assume that L satisfies conditions* (3.1.3), (3.1.4), *and* (3.1.5) *in $\Omega = B_1^+$, and that*

$$A(0) = I \quad \textit{and} \quad \|DA\|_{L^\infty(B_1^+)} + \|b\|_{L^\infty(B_1^+)} \leq \varepsilon$$

for some $\varepsilon > 0$.

If $\varepsilon \leq \varepsilon_0$, then

$$\|u\|_{L^1(B_1^+ \setminus B_{1/2}^+)} \leq C \|u_r\|_{L^1(B_1^+ \setminus B_{1/2}^+)},$$

where $\varepsilon_0 > 0$ and C are universal constants.

The proof of the Hölder estimate (3.1.9) in Theorem 3.1.2 requires the previous results. Combining the energy estimate (3.1.11) with Proposition 3.1.10 on dyadic annuli and rescaling, it is easy to show that the weighted Dirichlet energy in a ball is controlled by the weighted L^2 norm of the radial derivative in a larger ball. Applying Proposition 3.1.9 and by hole-filling, we now deduce a decay of the former quantity, leading to a C^α bound.

3.1.4 Outline of the chapter

In Section 3.2, we prove Proposition 3.1.9, the weighted inequality for the radial derivative. Section 3.3 focuses on the proof of Proposition 3.1.10, controlling the solution by its radial derivative. In Section 3.4, we obtain the Hölder estimates in Theorem 3.1.2. Finally, in Section 3.5 we proof our main result, Theorem 3.1.1.

3.2 The boundary weighted L^2 estimate for radial derivatives

Here we obtain the weighted estimates in half-balls for the radial derivative leading to Proposition 3.1.9. We will test the stability inequality (3.1.17) with the functions

$$\mathbf{c} := x \cdot \nabla u \quad \text{and} \quad \eta = r^{\frac{2-n}{2}}\zeta,$$

where $\zeta \in C_c^\infty(B_1)$ is a cut-off. This is a valid test function, since $x \cdot \nabla u = 0$ on $\partial^0 B_1^+$. Thus, the proof is essentially the same as in the interior case; see Chapter 1 or [52].

First we test with a generic cut-off η, not necessarily a power function:

Lemma 3.2.1. *Let $u \in W^{3,p}(B_1^+)$, for some $p > n$, be a stable solution of $-Lu = f(u)$ in B_1^+, with $u = 0$ on $\partial^0 B_1^+$, for some function $f \in C^1(\mathbb{R})$. Assume that L satisfies conditions* (3.1.3) *and* (3.1.4) *in $\Omega = B_1^+$, and that*

$$\|DA\|_{L^\infty(B_1^+)} + \|b\|_{L^\infty(B_1^+)} \le \varepsilon$$

for some $\varepsilon > 0$.

Then

$$\begin{aligned}
&\int_{B_1^+} |\nabla u|^2_{A(x)}\Big((n-2)\eta^2 + x\cdot\nabla(\eta^2)\Big)\,\mathrm{d}x \\
&+ \int_{B_1^+} \Big(-2(x\cdot\nabla u)A(x)\nabla u\cdot\nabla(\eta^2) - |x\cdot\nabla u|^2|\nabla\eta|^2_{A(x)}\Big)\,\mathrm{d}x \\
&\qquad \le C\varepsilon\int_{B_1^+}|D^2u||\nabla u||x|^2\eta^2\,\mathrm{d}x + C\varepsilon\int_{B_1^+}|\nabla u|^2\Big(|x|^2|\nabla(\eta^2)| + (|x|+\varepsilon|x|^2)\eta^2\Big)\,\mathrm{d}x,
\end{aligned}$$

for all $\eta \in C_c^\infty(B_1)$, where C is universal.

Proof. We will test the stability inequality (3.1.17) with $\mathbf{c} = x\cdot\nabla u$. First, we compute the left-hand side of (3.1.17), i.e., $\int_{B_1^+}(x\cdot\nabla u)J_u[x\cdot\nabla u]\,\eta^2\,\mathrm{d}x$. Computing, we have

$$L[x\cdot\nabla u] = x_k a_{ij}(x)u_{ijk} + 2a_{ij}(x)u_{ij} + x_k b_i(x)u_{ik} + b_i(x)u_i \quad \text{for a.e. } x \in B_1^+. \tag{3.2.1}$$

For the zero order term, integrating by parts and using the equation, we have

$$\begin{aligned}
&\int_{B_1^+} f'(u)(x\cdot\nabla u)^2\,\eta^2\,\mathrm{d}x = \int_{B_1^+}\nabla[f(u)]\cdot(x\cdot\nabla u)x\,\eta^2\,\mathrm{d}x = \int_{B_1^+} Lu\,\mathrm{div}\,\big\{(x\cdot\nabla u)x\,\eta^2\big\}\,\mathrm{d}x \\
&= -\int_{B_1^+} x\cdot\nabla[a_{ij}(x)u_{ij}]\,(x\cdot\nabla u)\,\eta^2\,\mathrm{d}x + \int_{B_1^+} b_i(x)u_i\,\mathrm{div}\,\big\{(x\cdot\nabla u)x\,\eta^2\big\}\,\mathrm{d}x,
\end{aligned} \tag{3.2.2}$$

where in the last line we have integrated by parts again. Combining (3.2.1) and (3.2.2), it follows that

$$
\begin{aligned}
&\int_{B_1^+}(x\cdot\nabla u)J_u[x\cdot\nabla u]\,\eta^2\,\mathrm{d}x = \int_{B_1^+}(x\cdot\nabla u)L[x\cdot\nabla u]\,\eta^2\,\mathrm{d}x + \int_{B_1^+} f'(u)(x\cdot\nabla u)^2\,\eta^2\,\mathrm{d}x\\
&= \int_{B_1^+} 2(x\cdot\nabla u)a_{ij}(x)u_{ij}\eta^2\,\mathrm{d}x - \int_{B_1^+} x\cdot\nabla a_{ij}(x)\,u_{ij}\,(x\cdot\nabla u)\,\eta^2\,\mathrm{d}x\\
&\quad + \int_{B_1^+} x_k b_i(x)u_{ik}(x\cdot\nabla u)\eta^2\,\mathrm{d}x + \int_{B_1^+} b_i(x)u_i\Big((x\cdot\nabla u)\eta^2 + \operatorname{div}\big\{(x\cdot\nabla u)x\,\eta^2\big\}\Big)\,\mathrm{d}x.
\end{aligned}
\tag{3.2.3}
$$

Notice that the first integrand in the right-hand side of (3.2.3) can be written as

$$
\begin{aligned}
2(x\cdot\nabla u)a_{ij}(x)u_{ij} = \operatorname{div}\big(2(x\cdot\nabla u)A(x)\nabla u - |\nabla u|^2_{A(x)}x\big) + (n-2)|\nabla u|^2_{A(x)}\\
- 2(x\cdot\nabla u)\partial_i a_{ij}(x)u_j + x\cdot\nabla a_{ij}(x)u_iu_j.
\end{aligned}
\tag{3.2.4}
$$

Hence, substituting (3.2.4) in (3.2.3) we deduce

$$
\begin{aligned}
&\int_{B_1^+}(x\cdot\nabla u)J_u[x\cdot\nabla u]\,\eta^2\,\mathrm{d}x\\
&= \int_{B_1^+}\operatorname{div}\big(2(x\cdot\nabla u)A(x)\nabla u - |\nabla u|^2_{A(x)}x\big)\,\eta^2\,\mathrm{d}x + \int_{B_1^+}(n-2)|\nabla u|^2_{A(x)}\eta^2\,\mathrm{d}x\\
&\quad + \int_{B_1^+}\Big(-2\partial_i a_{ij}(x)u_j\,(x\cdot\nabla u) + x\cdot\nabla a_{ij}(x)\,\big\{u_iu_j - u_{ij}\,(x\cdot\nabla u)\big\}\Big)\eta^2\,\mathrm{d}x\\
&\quad + \int_{B_1^+} b_i(x)x_ku_{ik}(x\cdot\nabla u)\eta^2\,\mathrm{d}x + \int_{B_1^+} b_i(x)u_i\Big((x\cdot\nabla u)\eta^2 + \operatorname{div}\big\{(x\cdot\nabla u)x\,\eta^2\big\}\Big)\,\mathrm{d}x.
\end{aligned}
\tag{3.2.5}
$$

Thus, integrating by parts the divergence term in (3.2.5) and applying the coefficient estimates $\|\nabla a_{ij}\|_{L^\infty} + \|b_i\|_{L^\infty} \le \varepsilon$, we obtain the lower bound

$$
\begin{aligned}
&\int_{B_1^+}(x\cdot\nabla u)J_u[x\cdot\nabla u]\,\eta^2\,\mathrm{d}x\\
&\ge -\int_{B_1^+} 2(x\cdot\nabla u)A(x)\nabla u\cdot\nabla(\eta^2) + \int_{B_1^+}|\nabla u|^2_{A(x)}\Big((n-2)\eta^2 + x\cdot\nabla(\eta^2)\Big)\,\mathrm{d}x\\
&\quad - C\varepsilon\int_{B_1^+}\Big(|D^2u||\nabla u||x|^2\eta^2 + |\nabla u|^2|x|\eta^2 + |\nabla u|^2|x|^2|\nabla(\eta^2)|\Big)\,\mathrm{d}x.
\end{aligned}
\tag{3.2.6}
$$

On the other hand, testing the integral stability inequality (3.1.17) with $\mathbf{c} = x\cdot\nabla u$, we deduce the upper bound

$$
\begin{aligned}
&\int_{B_1^+}(x\cdot\nabla u)J_u[x\cdot\nabla u]\,\eta^2\,\mathrm{d}x\\
&\qquad\le \int_{B_1^+}|x\cdot\nabla u|^2\left|\nabla\eta - \tfrac{1}{2}\eta A^{-1}(x)\widehat{b}(x)\right|^2_{A(x)}\mathrm{d}x\\
&\qquad\le \int_{B_1^+}|x\cdot\nabla u|^2|\nabla\eta|^2_{A(x)}\,\mathrm{d}x + C\varepsilon\int_{B_1}|\nabla u|^2|x|^2\big(|\nabla(\eta^2)| + \varepsilon\eta^2\big)\,\mathrm{d}x.
\end{aligned}
\tag{3.2.7}
$$

Combining (3.2.6) and (3.2.7) and rearranging terms yields the claim. □

Recall our notation for the radial derivative

$$r = |x|, \qquad u_r = \frac{x}{|x|} \cdot \nabla u.$$

Given $\rho \in (0, 1/2]$, we consider a cut-off $\zeta \in C_c^\infty(B_{2\rho})$ with $0 \le \zeta \le 1$, $\zeta = 1$ in B_ρ, and $|\nabla \zeta| \le C/\rho$ in $\operatorname{supp}|\nabla\zeta| \subset \overline{B_{2\rho}} \setminus B_\rho$. For $a \ge 0$, by approximation, we may take the singular test function $\eta = r^{-a/2}\zeta$ in Lemma 3.2.1 (see Lemma 1.5.4 above), which yields:

Lemma 3.2.2. *Let $u \in W^{3,p}(B_1^+)$, for some $p > n$, be a stable solution of $-Lu = f(u)$ in B_1^+, with $u = 0$ on $\partial^0 B_1^+$, for some function $f \in C^1(\mathbb{R})$. Assume that L satisfies conditions* (3.1.3) *and* (3.1.4) *in $\Omega = B_1^+$, and that*

$$\|DA\|_{L^\infty(B_1^+)} + \|b\|_{L^\infty(B_1^+)} \le \varepsilon$$

for some $\varepsilon > 0$.

If $0 \le a \le \min\{10, n\} - 2$, then

$$\begin{aligned}(n-2-a)\int_{B_\rho^+} r^{-a}|\nabla u|^2\,\mathrm{d}x + \frac{a(8-a)}{4}\int_{B_\rho^+} r^{-a}u_r^2\,\mathrm{d}x \\ \le C\int_{B_{2\rho}^+\setminus B_\rho^+} r^{-a}|\nabla u|^2\,\mathrm{d}x + C\varepsilon\int_{B_{2\rho}^+} r^{2-a}|D^2u||\nabla u|\,\mathrm{d}x \\ + C\varepsilon\int_{B_{2\rho}^+}\left(r^{1-a} + \varepsilon r^{2-a}\right)|\nabla u|^2\,\mathrm{d}x.\end{aligned}$$

for all $\rho \le 1/2$, where C is a universal constant.

The proof of this lemma is the same as the one for interior estimates in Lemma 1.5.4, hence we omit it. Lemma 3.2.2 now allows us to prove Proposition 3.1.9:

Proof of Proposition 3.1.9. Since $3 \le n \le 9$, we have that $\min\{10, n\} - 2 = n - 2$ and we may choose the exponent $a = n - 2$ in Lemma 3.2.2, leading us to the inequality

$$\begin{aligned}\frac{(n-2)(10-n)}{4}\int_{B_\rho^+} r^{2-n}u_r^2\,\mathrm{d}x \\ \le C\int_{B_{2\rho}^+\setminus B_\rho^+} r^{2-n}|\nabla u|^2\,\mathrm{d}x + C\varepsilon\int_{B_{2\rho}^+} r^{4-n}|D^2u||\nabla u|\,\mathrm{d}x \\ + C\varepsilon\int_{B_{2\rho}^+}\left(r^{3-n} + \varepsilon r^{4-n}\right)|\nabla u|^2\,\mathrm{d}x\end{aligned} \tag{3.2.8}$$

for $\rho \le 1/2$.

It remains to control the weighted Hessian error in (3.2.8). For this, combining the boundary "Hessian times the gradient" estimate (3.1.10) in Theorem 3.1.4 with the analogous interior estimates in Theorem 1.1.2 above, by a simple scaling and covering argument we have

$$\int_{B_{\delta/2}^+\setminus B_{\delta/4}^+} r^{4-n}|D^2u||\nabla u|\,\mathrm{d}x \le C\int_{B_\delta^+\setminus B_{\delta/8}^+} r^{3-n}|\nabla u|^2\,\mathrm{d}x \quad \text{for all } \delta \in (0,1) \text{ and } \varepsilon \le \varepsilon_0, \tag{3.2.9}$$

where $\varepsilon_0 > 0$ and C are universal. Letting $\delta = 2^{3-k}\rho$ in (3.2.9) and summing in $k \in \mathbb{N}$, we obtain

$$\int_{B_{2\rho}^+} r^{4-n}|D^2u||\nabla u|\,\mathrm{d}x \le C\int_{B_{4\rho}^+} r^{3-n}|\nabla u|^2\,\mathrm{d}x \quad \text{for all } \rho \le 1/4 \text{ and } \varepsilon \le \varepsilon_0. \tag{3.2.10}$$

Applying (3.2.10) in (3.2.8), using that $(10-n)(n-2) > 0$, we deduce the claim. $\square$

3.3 In half-annuli the radial derivative controls the function in L^1

Here we take advantage of the homogeneity of the equation to control the L^1 norm of a solution by the L^1 norm of its radial derivative. This is an extension of a device due to Cabré [19] which provided quantitative proofs of the regularity of stable solutions for the Laplacian in flat domains. Our proofs remain quantitative thanks to this idea.

Let $\tau \geq 1$ be a parameter close to 1. Given any function $v\colon B_1^+ \to \mathbb{R}$, we denote its L^∞ rescaling by $v^\tau := v(\tau\cdot)$. Consider the elliptic operator $Lv = a_{ij}(x)v_{ij} + b_i(x)v_i$, with coefficients $a_{ij} \in C^{0,1}(\overline{B_1^+})$ and $b_i \in C^0(\overline{B_1^+})$. For each τ, we define L^τ to be the operator given by the rescaling

$$L^\tau v := \tau^{-2}a_{ij}^\tau(x)v_{ij} + \tau^{-1}b_i^\tau(x)v_i. \tag{3.3.1}$$

Our principal motivation for considering L^τ is the invariance property $(Lv)^\tau = L^\tau v^\tau$. In particular, given $u \in W^{3,p}(B_1^+)$ a solution to $-Lu = f(u)$ in B_1^+, we have

$$-L^\tau u^\tau = f(u^\tau) \quad \text{in } B_{1/\tau}^+. \tag{3.3.2}$$

Notice that if $1 \leq \tau < 1+\delta$, then this last equation is satisfied in $B_{1/(1+\delta)}^+ \subset B_{1/\tau}^+$. To prove Proposition 3.1.10, we will take a derivative of (3.3.2) with respect to τ.

Before giving the proof of the proposition, it is convenient to recall the following simple corollary of the Hessian and higher integrability estimates proven in Chapter 2. Here and throughout this section, we use the notation for half-annuli from [19], namely, for $\rho_2 > \rho_1 > 0$, we let

$$A_{\rho_1,\rho_2}^+ := B_{\rho_2}^+ \setminus \overline{B_{\rho_1}^+} = \{x \in \mathbb{R}^n \colon x_n > 0, \rho_1 < |x| < \rho_2\}.$$

Corollary 3.3.1 (Corollary 2.3.3 in Chapter 2). *Let $u \in W^{3,p}(B_1^+)$, for some $p > n$, be a nonnegative stable solution of $-Lu = f(u)$ in B_1^+, with $u = 0$ on $\partial^0 B_1^+$. Assume that $f \in C^1(\mathbb{R})$ is nonnegative and nondecreasing. Assume that L satisfies conditions* (3.1.3)*,* (3.1.4)*, and* (3.1.5) *in $\Omega = B_1^+$, and*

$$\|DA\|_{L^\infty(B_1^+)} + \|b\|_{L^\infty(B_1^+)} \leq \varepsilon$$

for some $\varepsilon > 0$. Let $0 < \rho_1 < \rho_2 < \rho_3 < \rho_4 \leq 1$.

Then

$$\|\nabla u\|_{L^{2+\gamma}(A_{\rho_2,\rho_3}^+)} \leq C_{\varepsilon,\rho_i}\|u\|_{L^1(A_{\rho_1,\rho_4}^+)}$$

and

$$\|D^2 u\|_{L^1(A_{\rho_2,\rho_3}^+)} \leq C_{\varepsilon,\rho_i}\|u\|_{L^1(A_{\rho_1,\rho_4}^+)},$$

where C_{ε,ρ_i} is a constant depending only on n, c_0, C_0, ε, ρ_1, ρ_2, ρ_3, and ρ_4.

Proof of Proposition 3.1.10. Considering the rescaled function $u(\frac{\cdot}{6})$, we may assume that we have a stable solution in B_6^+.

Let $\zeta \in C_c^\infty(A_{4,5})$ be a nonnegative cut-off function with $\zeta = 1$ in $A_{4.1,4.9}$. We consider the function $\xi := x_n\zeta$, which satisfies

$$\xi \geq 0 \text{ in } A_{4,5}^+, \quad \xi = 0 \text{ on } \partial^0 A_{4,5}^+, \quad \xi = \xi_\nu = 0 \text{ on } \partial^+ A_{4,5}^+, \text{ and } \quad \xi = x_n \text{ in } A_{4.1,4.9}^+.$$

Multiplying (3.3.2) (rescaled) by ξ for each $1 \leq \tau \leq 1.1$ and integrating in $A_{4,5}^+$, we have

$$\int_{A_{4,5}^+} (L^\tau u^\tau)\xi \,\mathrm{d}x = -\int_{A_{4,5}^+} f(u^\tau)\,\mathrm{d}x \quad \text{for all } \tau \in [1, 1.1]. \tag{3.3.3}$$

Differentiating (3.3.3) with respect to τ and integrating, we also have

$$\int_1^{1.1} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int_{A^+_{4,5}} (L^\tau u^\tau)\xi \,\mathrm{d}x\right) \mathrm{d}\tau = \int_{A^+_{4,5}} \big(f(u) - f(u^{1.1})\big)\,\xi \,\mathrm{d}x. \tag{3.3.4}$$

Our claim will be a consequence of this last identity. For this, we first establish lower bounds for the left-hand side of (3.3.4) by using that $L^\tau u^\tau \le 0$. Later, with the help of the stability inequality and the convexity of f, we obtain upper bounds of the right-hand side. Finally, we will control the remaining Hessian errors by applying Corollary 3.3.1.

Step 1. *We prove that*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{A^+_{4,5}} (L^\tau u^\tau)\xi \,\mathrm{d}x \ge c\|u\|_{L^1(A^+_{4.7,4.8})} &- C\|u_r\|_{L^1(A^+_{4,5.5})} \\ &- C\varepsilon\|D^2u\|_{L^1(A^+_{4,5.5})} - C\varepsilon\|\nabla u\|_{L^1(A^+_{4,5.5})}\end{aligned}$$

for all $\tau \in [1, 1.1]$, *where* c *and* C *are universal constants.*

By definition (3.3.1), we have

$$\int_{A^+_{4,5}} (L^\tau u^\tau)\xi \,\mathrm{d}x = \int_{A^+_{4,5}} \tau^{-2} a^\tau_{ij}(x) u^\tau_{ij}\,\xi dx + \int_{A^+_{4,5}} \tau^{-1} b^\tau_i(x) u^\tau_i\,\xi \,\mathrm{d}x \tag{3.3.5}$$

for all $\tau \in [1, 1.1]$.

On the one hand, since $\frac{du^\tau}{\mathrm{d}\tau} = \tau^{-1}x\cdot\nabla u^\tau$, differentiating under the integral sign

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\int_{A^+_{4,5}} \tau^{-2} a^\tau_{ij}(x) u^\tau_{ij}\,\xi dx\right\} \\
&= \int_{A^+_{4,5}} \tau^{-3} a^\tau_{ij}(x)[x\cdot\nabla u^\tau]_{ij}\,\xi dx - 2\int_{A^+_{4,5}} \tau^{-3} a^\tau_{ij}(x) u^\tau_{ij}\,\xi dx + \int_{A^+_{4,5}} \tau^{-3} x\cdot\nabla a^\tau_{ij}(x) u^\tau_{ij}\,\xi dx \\
&= \int_{A^+_{4,5}} \tau^{-1} L^\tau[x\cdot\nabla u^\tau]\,\xi dx - 2\int_{A^+_{4,5}} \tau^{-1}(L^\tau u^\tau)\,\xi dx - \int_{A^+_{4,5}} \tau^{-2} b^\tau_i(x)[x\cdot\nabla u^\tau]_i\,\xi dx \\
&\qquad + \int_{A^+_{4,5}} \tau^{-3} x\cdot\nabla a^\tau_{ij}(x) u^\tau_{ij}\,\xi dx + 2\int_{A^+_{4,5}} \tau^{-2} b^\tau_i(x) u^\tau_i\,\xi dx.
\end{aligned} \tag{3.3.6}$$

On the other hand, since $\operatorname{supp}\xi^{1/\tau} \subset A^+_{4\tau,5\tau} \subset A^+_{4,5.5}$, by a change of variables

$$\int_{A^+_{4,5}} \tau^{-1} b^\tau_i(x) u^\tau_i \xi \,\mathrm{d}x = \int_{A^+_{4,5}} b^\tau_i(x)(u_i)^\tau \xi \,\mathrm{d}x = \int_{A^+_{4,5.5}} \tau^{-n} b_i(x) u_i \xi^{1/\tau} \,\mathrm{d}x$$

and taking a derivative

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\int_{A^+_{4,5}} \tau^{-1} b^\tau_i(x) u^\tau_i \xi \,\mathrm{d}x\right\} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\int_{A^+_{4,5.5}} \tau^{-n} b_i(x) u_i \xi^{1/\tau} \,\mathrm{d}x\right\} \\
&= -n\int_{A^+_{4,5.5}} \tau^{-(n+1)} b_i(x) u_i \xi^{1/\tau} \,\mathrm{d}x - \int_{A^+_{4,5.5}} \tau^{-(n+2)} b_i(x) u_i\, x\cdot\nabla\xi^{1/\tau} \,\mathrm{d}x \\
&= -n\int_{A^+_{4,5}} \tau^{-2} b^\tau_i(x) u^\tau_i \xi \,\mathrm{d}x - \int_{A^+_{4,5}} \tau^{-3} b^\tau_i(x) u^\tau_i\, x\cdot\nabla\xi \,\mathrm{d}x,
\end{aligned} \tag{3.3.7}$$

where in the last line we have undone the change of variables. Thus, combining (3.3.6) and (3.3.7), by (3.3.5) we deduce

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\int_{A^+_{4,5}}(L^\tau u^\tau)\xi\,\mathrm{d}x\right\}\\
&=\int_{A^+_{4,5}}\tau^{-1}L^\tau[x\cdot\nabla u^\tau]\,\xi dx-2\int_{A^+_{4,5}}\tau^{-1}(L^\tau u^\tau)\,\xi dx\\
&\qquad\qquad+\int_{A^+_{4,5}}\tau^{-3}x\cdot\nabla a^\tau_{ij}(x)u^\tau_{ij}\,\xi dx-\int_{A^+_{4,5}}\tau^{-2}b^\tau_i(x)[x\cdot\nabla u^\tau]_i\,\xi dx\\
&\qquad\qquad+(2-n)\int_{A^+_{4,5}}\tau^{-2}b^\tau_i(x)u^\tau_i\,\xi dx-\int_{A^+_{4,5}}\tau^{-3}b^\tau_i(x)u^\tau_i\,x\cdot\nabla\xi\,\mathrm{d}x,
\end{aligned}$$

and hence, by the bounds $1\le\tau\le 1.1$ and $\|\nabla a_{ij}\|_{L^\infty}+\|b_i\|_{L^\infty}\le\varepsilon$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\int_{A^+_{4,5}}(L^\tau u^\tau)\xi\,\mathrm{d}x\right\}\ge\int_{A^+_{4,5}}\tau^{-1}L^\tau[x\cdot\nabla u^\tau]\,\xi dx-2\int_{A^+_{4,5}}\tau^{-1}(L^\tau u^\tau)\,\xi dx\\
-C\varepsilon\|D^2u\|_{L^1(A^+_{4,5.5})}-C\varepsilon\|\nabla u\|_{L^1(A^+_{4,5.5})}.
\end{aligned}\tag{3.3.8}$$

Next, we bound the two terms in the right-hand side of (3.3.8) by below.

For the first term, we write $L^\tau v=\operatorname{div}(\tau^{-2}A^\tau(x)v)+\tau^{-1}\widehat{b}^\tau(x)\cdot\nabla v$ in divergence form as in (3.1.13) and integrate by parts. Recalling that ξ and ξ_ν vanish on $\partial^+A^+_{4,5}$, and ξ and $x\cdot\nabla u^\tau$ vanish on $\partial^0A^+_{4,5}$, integrating by parts twice we have

$$\begin{aligned}
&\int_{A^+_{4,5}}\tau^{-1}L^\tau[x\cdot\nabla u^\tau]\,\xi\,\mathrm{d}x\\
&\quad=\int_{A^+_{4,5}}\tau^{-3}(x\cdot\nabla u^\tau)\operatorname{div}\big(A^\tau(x)\nabla\xi\big)\,\mathrm{d}x+\int_{A^+_{4,5}}\tau^{-2}\widehat{b}^\tau(x)\cdot\nabla[x\cdot\nabla u^\tau]\,\xi\,\mathrm{d}x\\
&\quad\ge -C\|u_r\|_{L^1(A^+_{4,5.5})}-C\varepsilon\|D^2u\|_{L^1(A^+_{4,5.5})}-C\varepsilon\|\nabla u\|_{L^1(A^+_{4,5.5})},
\end{aligned}\tag{3.3.9}$$

where in the last line we have used that $1\le\tau\le 1.1$, as well as the uniform ellipticity and the bounds $\|\nabla a_{ij}\|_{L^\infty}+\|b_i\|_{L^\infty}\le\varepsilon$.

The lower bounds for the second term in (3.3.8) are the most delicate. Given $\rho_1\in(4.1,4.2)$ and $\rho_2\in(4.8,4.9)$, we consider the solution φ of the mixed boundary value problem

$$\begin{cases}-\Delta\varphi=1 & \text{in } A^+_{\rho_1,\rho_2}\\ \varphi=0 & \text{on } \partial^0A^+_{\rho_1,\rho_2}\\ \varphi_\nu=0 & \text{on } \partial^+A^+_{\rho_1,\rho_2}.\end{cases}$$

Notice that $\varphi\ge 0$ in $A^+_{\rho_1,\rho_2}$ by the maximum principle. Moreover, we have the a priori bounds $|\varphi|+|\nabla\varphi|\le C$ in $A^+_{\rho_1,\rho_2}$, where $C=C(n)$ is a dimensional constant, and hence

$$\xi\ge c\varphi\quad\text{in } A^+_{\rho_1,\rho_2},\tag{3.3.10}$$

for some small dimensional $c=c(n)>0$; for the details see [19, Appendix B].

Using (3.3.10) and by the nonnegativity of $-L^\tau u^\tau=f(u^\tau)\ge 0$, we have

$$-\int_{A^+_{4,5}}\tau^{-1}(L^\tau u^\tau)\,\xi\,\mathrm{d}x\ge c\int_{A^+_{\rho_1,\rho_2}}\tau^{-1}(-L^\tau u^\tau)\,\varphi\,\mathrm{d}x.\tag{3.3.11}$$

Since $A(0) = I$, we have $|A^\tau(x) - I| \le \varepsilon\tau|x|$ and writing

$$L^\tau u^\tau = \tau^{-2}\Delta u^\tau + \tau^{-2}\mathrm{tr}\big((A^\tau(x) - I)D^2u^\tau\big) + \tau^{-1}b^\tau(x)\cdot\nabla u^\tau,$$

by the bounds for φ, τ, and the coefficients, the right-hand side of (3.3.11) can be further bounded by below as

$$\begin{aligned}
&\int_{A^+_{\rho_1,\rho_2}} \tau^{-1}(-L^\tau u^\tau)\,\varphi\,\mathrm{d}x \\
&\qquad \ge \tau^{-3}\int_{A^+_{\rho_1,\rho_2}} (-\Delta u^\tau)\,\varphi\,\mathrm{d}x - C\varepsilon\int_{A^+_{\rho_1,\rho_2}} \tau^{-1}\left(|D^2u^\tau||x| + |\nabla u^\tau|\right)\mathrm{d}x \\
&\qquad \ge c\int_{A^+_{\rho_1,\rho_2}} (-\Delta u^\tau)\,\varphi\,\mathrm{d}x - C\varepsilon\|D^2u\|_{L^1(A^+_{4,5.5})} - C\varepsilon\|\nabla u\|_{L^1(A^+_{4,5.5})}.
\end{aligned} \tag{3.3.12}$$

Following [19], we integrate by parts the Laplacian in (3.3.12) as

$$\begin{aligned}
\int_{A^+_{\rho_1,\rho_2}} (-\Delta u^\tau)\,\varphi\,\mathrm{d}x &= \int_{A^+_{\rho_1,\rho_2}} u^\tau\,\mathrm{d}x - \int_{\partial^+A^+_{\rho_1,\rho_2}} (u^\tau)_\nu\,\mathrm{d}\mathcal{H}^{n-1} \\
&\ge c\|u\|_{L^1(A^+_{\tau\rho_1,\tau\rho_2})} - \int_{\partial^+B^+_{\rho_1}} |(u^\tau)_r|\,\mathrm{d}\mathcal{H}^{n-1} - \int_{\partial^+B^+_{\rho_2}} |(u^\tau)_r|\,\mathrm{d}\mathcal{H}^{n-1} \\
&\ge c\|u\|_{L^1(A^+_{4.7,4.8})} - C\int_{\partial^+B^+_{\tau\rho_1}} |u_r|\,\mathrm{d}\mathcal{H}^{n-1} - C\int_{\partial^+B^+_{\tau\rho_2}} |u_r|\,\mathrm{d}\mathcal{H}^{n-1},
\end{aligned} \tag{3.3.13}$$

where in the last line we have used that $\tau\rho_1 \le 1.1\cdot 4.2 \le 4.7$ and $\tau\rho_2 \ge 4.8$. Now, combining (3.3.11), (3.3.12), and (3.3.13), we deduce

$$\begin{aligned}
-2\int_{A^+_{4,5}} \tau^{-1}(L^\tau u^\tau)\,\xi\,\mathrm{d}x \ge c\|u\|_{L^1(A^+_{4.7,4.8})} &- C\int_{\partial^+B^+_{\tau\rho_1}} |u_r|\,\mathrm{d}\mathcal{H}^{n-1} - C\int_{\partial^+B^+_{\tau\rho_2}} |u_r|\,\mathrm{d}\mathcal{H}^{n-1} \\
&- C\varepsilon\|D^2u\|_{L^1(A^+_{4,5.5})} - C\varepsilon\|\nabla u\|_{L^1(A^+_{4,5.5})}.
\end{aligned} \tag{3.3.14}$$

Integrating (3.3.14) in $\rho_1 \in (4.1, 4.2)$ and $\rho_2 \in (4.8, 4.9)$, using that $\tau\rho_1 \ge 4.1 \ge 4$ and $\tau\rho_2 \le 1.1\cdot 4.9 \le 5.5$, we finally obtain

$$\begin{aligned}
-2\int_{A^+_{4,5}} \tau^{-1}(L^\tau u^\tau)\,\xi\,\mathrm{d}x &\ge c\|u\|_{L^1(A^+_{4.7,4.8})} - C\|u_r\|_{L^1(A^+_{4,5.5})} \\
&\qquad - C\varepsilon\|D^2u\|_{L^1(A^+_{4,5.5})} - C\varepsilon\|\nabla u\|_{L^1(A^+_{4,5.5})}.
\end{aligned} \tag{3.3.15}$$

Applying (3.3.9) and (3.3.15) in (3.3.8) now yields the claim.

Step 2. *We prove that for every $\delta \in (0,1)$ and $\varepsilon_0 > 0$, we have*

$$\begin{aligned}
&\int_{A^+_{4,5}} \big(f(u) - f(u^{1.1})\big)\,\xi\,\mathrm{d}x \\
&\qquad \le C_{\varepsilon_0}\left(\delta\|u\|_{L^1(A^+_{3,6})} + \delta\|D^2u\|_{L^1(A^+_{3.8,5.8})} + \delta^{-1-2\frac{2+\gamma}{\gamma}}\|u_r\|_{L^1(A^+_{3,6})}\right)
\end{aligned}$$

for all $\varepsilon \le \varepsilon_0$, where $\gamma = \gamma(n) > 0$ and C_{ε_0} depends only on n, c_0, C_0, and ε_0.

Let $\phi \in C_c^\infty(A_{3.9,5.1})$ be a nonnegative test function with $\phi = 1$ in $A_{4,5}$. Since $\xi = 0$ on $\partial A^+_{4,5}$ and $u - u^{1.1} = 0$ on $\partial^0 A^+_{3.9,5.1}$, the functions ξ and $(u - u^{1.1})\phi$ are valid test functions in the integral stability inequality (3.1.16) with $\Omega = B_1^+$.

Since f is nondecreasing, we have $f' \geq 0$. By convexity $f(u) - f(u^{1.1}) \leq f'(u)(u - u^{1.1})$, hence, multiplying by ξ, integrating, and using the stability inequality (3.1.16) twice, we obtain

$$\begin{aligned}
&\int_{A^+_{4,5}} \big(f(u) - f(u^{1.1})\big)\xi \,\mathrm{d}x \leq \int_{A^+_{4,5}} f'(u)(u - u^{1.1})\xi \,\mathrm{d}x \\
&\leq \left(\int_{A^+_{4,5}} f'(u)\xi^2 \,\mathrm{d}x\right)^{1/2} \left(\int_{A^+_{3.9,5.1}} f'(u)\left((u - u^{1.1})\phi\right)^2 \mathrm{d}x\right)^{1/2} \\
&\leq \left(\int_{A^+_{4,5}} |\nabla\xi - \tfrac{1}{2}\xi A^{-1}(x)\widehat{b}(x)|^2_{A(x)} \,\mathrm{d}x\right)^{1/2} \\
&\quad \cdot \left(\int_{A^+_{3.9,5.1}} \left|\nabla\{(u - u^{1.1})\phi\} - \tfrac{1}{2}(u - u^{1.1})\phi A^{-1}(x)\widehat{b}(x)\right|^2_{A(x)} \mathrm{d}x\right)^{1/2}.
\end{aligned} \tag{3.3.16}$$

Using that $\|\xi\|_{C^1} + \|\phi\|_{C^1} \leq C$ and the coefficient bounds, from (3.3.16) it follows that

$$\begin{aligned}
&\int_{A^+_{4,5}} \big(f(u) - f(u^{1.1})\big)\xi \,\mathrm{d}x \\
&\quad \leq C(1+\varepsilon)\|\nabla(u - u^{1.1})\|_{L^2(A^+_{3.9,5.1})} + C(1+\varepsilon)^2\|u - u^{1.1}\|_{L^2(A^+_{3.9,5.1})} \\
&\quad \leq C_{\varepsilon_0}\|\nabla(u - u^{1.1})\|_{L^2(A^+_{3.9,5.1})},
\end{aligned} \tag{3.3.17}$$

where in the last line we have applied the Poincaré inequality for functions vanishing on $\partial^0 A^+_{3.9,5.1}$.

It remains to control the norm $\|\nabla(u - u^{1.1})\|_{L^2(A^+_{3.9,5.1})}$ in (3.3.17). First we interpolate between L^1 and $L^{2+\gamma}$. Letting $q = \frac{2(1+\gamma)}{2+\gamma}$, we have

$$\|\nabla(u - u^{1.1})\|_{L^2(A^+_{3.9,5.1})} \leq C\|\nabla u\|^{1/q}_{L^{2+\gamma}(A_{3.9,5.1\cdot 1.1})}\|\nabla(u - u^{1.1})\|^{1/q'}_{L^1(A^+_{3.9,5.1})}. \tag{3.3.18}$$

From (3.3.18), by Corollary 3.3.1 we deduce

$$\begin{aligned}
\|\nabla(u - u^{1.1})\|_{L^2(A^+_{3.9,5.1})} &\leq C_{\varepsilon_0}\|u\|^{1/q}_{L^1(A^+_{3,6})}\|\nabla(u - u^{1.1})\|^{1/q'}_{L^1(A^+_{3.9,5.1})} \\
&\leq \delta\|u\|_{L^1(A^+_{3,6})} + C_{\varepsilon_0}\delta^{-q'/q}\|\nabla(u - u^{1.1})\|_{L^1(A^+_{3.9,5.1})}.
\end{aligned} \tag{3.3.19}$$

By Corollary C.2 (with $\widetilde{\delta} = \delta^{1+q'/q}$), we also have

$$\begin{aligned}
&\|\nabla(u - u^{1.1})\|_{L^1(A^+_{3.9,5.1})} \\
&\quad \leq C\delta^{1+\frac{q'}{q}}\|D^2(u - u^{1.1})\|_{L^1(A^+_{3.8,5.2})} + C\delta^{-1-\frac{q'}{q}}\|u - u^{1.1}\|_{L^1(A^+_{3.8,5.2})}.
\end{aligned} \tag{3.3.20}$$

Hence, applying (3.3.20) in (3.3.19), we obtain the following estimate for the Dirichlet energy

$$\begin{aligned}
&\|\nabla(u - u^{1.1})\|_{L^2(A^+_{3.9,5.1})} \\
&\leq \delta\|u\|_{L^1(A^+_{3,6})} + C_{\varepsilon_0}\delta\|D^2 u\|_{L^1(A^+_{3.8,5.8})} + C_{\varepsilon_0}\delta^{-1-2\frac{q'}{q}}\|u - u^{1.1}\|_{L^1(A^+_{3.8,5.2})}.
\end{aligned} \tag{3.3.21}$$

Finally, since $\frac{\mathrm{d}}{\mathrm{d}\tau}u^\tau(x) = ru_r(\tau x)$, we have $u(x) - u^{1.1}(x) = -r\int_1^{1.1} u_r(\tau x)\,\mathrm{d}\tau$ and hence

$$\|u - u^{1.1}\|_{L^1(A^+_{3.8,5.2})} \le C\|u_r\|_{L^1(A^+_{3.8,5.2\cdot 1.1})} \le C\|u_r\|_{L^1(A^+_{3,6})}. \tag{3.3.22}$$

Using (3.3.22) in (3.3.21), and by (3.3.17) we deduce the claim.

Step 3. *Conclusion.*

Combining Steps 1 and 2 in (3.3.4), for $\delta \in (0,1)$ and (say) $\varepsilon \le 1$, we have

$$\begin{aligned} \|u\|_{L^1(A^+_{4.7,4.8})} \le C\delta\|u\|_{L^1(A^+_{3,6})} + C\,(\varepsilon+\delta)\,\|D^2u\|_{L^1(A^+_{3.8,5.8})} + C\varepsilon\|\nabla u\|_{L^1(A^+_{4,5.5})} \\ + C\delta^{-1-2\frac{2+\gamma}{\gamma}}\|u_r\|_{L^1(A^+_{3,6})}. \end{aligned} \tag{3.3.23}$$

Thanks to Corollary 3.3.1 we can control the Hessian and gradient errors in (3.3.23) by the L^1 norm of the function. Namely, we have $\|D^2u\|_{L^1(A^+_{3.8,5.8})} \le C\|u\|_{L^1(A^+_{3,6})}$ and by Hölder's inequality $\|\nabla u\|_{L^1(A^+_{4,5.5})} \le C\|\nabla u\|_{L^{2+\gamma}(A^+_{4,5.5})} \le C\|u\|_{L^1(A^+_{3,6})}$. It follows that

$$\|u\|_{L^1(A^+_{4.7,4.8})} \le C\delta\|u\|_{L^1(A^+_{3,6})} + C\delta^{-1-2\frac{2+\gamma}{\gamma}}\|u_r\|_{L^1(A^+_{3,6})} \quad \text{for } \varepsilon \le \delta. \tag{3.3.24}$$

Now, proceeding as in [19], if we write the function in terms of the radial derivative as $u(s\sigma) = u(t\sigma) - \int_s^t u_r(r\sigma)\,\mathrm{d}r$ for $s \in (3,6)$, $t \in (4.7,4.8)$, and $\sigma \in \mathbb{S}^{n-1}$, then integrating in $\int_3^6 s^{n-1}\,\mathrm{d}s\int_{\partial^+B_1^+}\mathrm{d}\sigma$ it is not hard to show that

$$\|u\|_{L^1(A^+_{3,6})} \le C\left(\|u\|_{L^1(A^+_{4.7,4.8})} + \|u_r\|_{L^1(A^+_{3,6})}\right). \tag{3.3.25}$$

Combining (3.3.25) and (3.3.24), we deduce

$$\|u\|_{L^1(A^+_{3,6})} \le C\delta\|u\|_{L^1(A^+_{3,6})} + C\delta^{-1-2\frac{2+\gamma}{\gamma}}\|u_r\|_{L^1(A^+_{3,6})} \quad \text{for } \varepsilon\delta,$$

and choosing $\delta > 0$ universal small in this last inequality, we can absorb the L^1 norm of u into the left-hand side, concluding the proof. □

3.4 Boundary C^α estimate

We prove the Hölder estimate in half-balls. At this point, the proof amounts to combining Propositions 3.1.9 and 3.1.10 with Theorem 3.1.4, to deduce a decay of the weighted Dirichlet energy and applying a scaling and covering argument.

Proof of the Hölder estimate (3.1.9) *in Theorem 3.1.2.* We may assume that $3 \le n \le 9$. Indeed, when $n = 2$, we recover the estimate by applying Theorem 3.1.2 to the function $\widetilde{u}(x_1,x_2,x_3) := u(x_2,x_3)$, a stable solution to the elliptic equation $c_0\widetilde{u}_{x_1x_1} + L\widetilde{u} = f(\widetilde{u})$ in $B_1^+ \subset \mathbb{R}^3$ where L acts only in the (x_2,x_3) variables. Similarly, when $n = 1$, one considers the function $\widetilde{u}(x_1,x_2,x_3) := u(x_3)$.

Throughout the proof, C denotes a generic universal constant unless stated otherwise. The proof is divided in three steps. In Step 1, we prove the decay of the weighted Dirichlet energy under the assumption $A(0) = I$. Later, in Step 2, we remove this assumption and prove a C^α estimate in universally small balls. Finally, in Step 3, we deduce the theorem by a scaling an covering argument.

Step 1: *Under the assumption that*

$$A(0) = I \quad \text{and} \quad \|DA\|_{L^\infty(B_1^+)} + \|b\|_{L^\infty(B_1^+)} \le \varepsilon,$$

we prove that if $\varepsilon \le \varepsilon_0$, then

$$\int_{B_\rho^+} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \le C\|u\|^2_{L^1(B_1^+)}\rho^{2\alpha} \quad \text{for all } \rho \le 1/16, \tag{3.4.1}$$

where $\varepsilon_0 > 0$, $\alpha > 0$, and C are universal constants.

First we write the weighted Dirichlet integral as an infinite sum on dyadic annuli, applying Corollary 3.3.1 on each annulus. We treat the case $\rho = 1/2$ and recover the result for general ρ by rescaling. This is the same approach used for the interior estimates in Chapter 1.

Let $r_j := 2^{-j}$ for $j \ge 0$. Then

$$\int_{B_{1/2}^+} r^{2-n}|\nabla u|^2 \,\mathrm{d}x = \sum_{j=0}^{\infty} \int_{A^+_{r_{j+2},r_{j+1}}} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \le C\sum_{j=0}^{\infty} r_j^{2-n} \int_{A^+_{r_{j+2},r_{j+1}}} |\nabla u|^2 \,\mathrm{d}x, \tag{3.4.2}$$

and hence, we must control each of the summands in (3.4.2).

Combining Corollary 3.3.1 and Proposition 3.1.10 applied to the functions $u(r_j\cdot)$ and by Hölder inequality, we have

$$r_j^{2-n} \int_{A^+_{r_{j+2},r_{j+1}}} |\nabla u|^2 \,\mathrm{d}x \le C r_j^{2-n} \int_{A^+_{r_{j+3},r_j}} u_r^2 \,\mathrm{d}x \quad \text{for } \varepsilon \le \varepsilon_0. \tag{3.4.3}$$

Therefore, using (3.4.3) in (3.4.2), we obtain

$$\int_{B_{1/2}^+} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \le C\sum_{j=0}^{\infty} r_j^{2-n} \int_{A^+_{r_{j+3},r_j}} u_r^2 \,\mathrm{d}x \le C\int_{B_1^+} r^{2-n}u_r^2 \,\mathrm{d}x \quad \text{for } \varepsilon \le \varepsilon_0. \tag{3.4.4}$$

Applying (3.4.4) to the functions $u(2\rho\cdot)$, we deduce

$$\int_{B_\rho^+} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \le C\int_{B_{2\rho}^+} r^{2-n}u_r^2 \,\mathrm{d}x \quad \text{for all } \rho \le 1/2 \text{ and } \varepsilon \le \varepsilon_0,$$

and, by Proposition 3.1.9 (with 2ρ in place of ρ), it follows that

$$\int_{B_\rho^+} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \le C\int_{B_{4\rho}^+\setminus B_{2\rho}^+} r^{2-n}|\nabla u|^2 \,\mathrm{d}x + C\varepsilon\int_{B_{8\rho}^+} r^{3-n}|\nabla u|^2 \,\mathrm{d}x \tag{3.4.5}$$
$$\text{for all } \rho \le 1/8 \text{ and } \varepsilon \le \varepsilon_0.$$

Now, splitting the last integral into $B_{8\rho}^+ = (B_{8\rho}^+ \setminus B_\rho^+) \cup B_\rho^+$, since $r^{3-n} \le r^{2-n}$ in B_1^+, and using that the bounds $\rho \le 1/8$ and $\varepsilon \le \varepsilon_0$ are universal, from (3.4.5) we deduce

$$\int_{B_\rho^+} r^{2-n}|\nabla u|^2 \,\mathrm{d}x \le C\int_{B_{8\rho}^+\setminus B_\rho^+} r^{2-n}|\nabla u|^2 \,\mathrm{d}x + C\varepsilon\int_{B_\rho^+} r^{2-n}|\nabla u|^2 \,\mathrm{d}x$$
$$\text{for all } \rho \le 1/8 \text{ and } \varepsilon \le \varepsilon_0.$$

Taking $\varepsilon_0 > 0$ universal smaller if necessary, we can absorb the last integral into the left-hand side, which yields

$$\int_{B_\rho^+} r^{2-n}|\nabla u|^2\,\mathrm{d}x \le C\int_{B_{8\rho}^+\setminus B_\rho^+} r^{2-n}|\nabla u|^2\,\mathrm{d}x \quad \text{for all } \rho\le 1/8 \text{ and } \varepsilon\le\varepsilon_0. \tag{3.4.6}$$

Hole-filling (3.4.6), we also have

$$\int_{B_\rho^+} r^{2-n}|\nabla u|^2\,\mathrm{d}x \le \theta\int_{B_{8\rho}^+} r^{2-n}|\nabla u|^2\,\mathrm{d}x \quad \text{for all } \rho\le 1/8 \text{ and } \varepsilon\le\varepsilon_0, \tag{3.4.7}$$

where $\theta = \frac{C}{1+C}\in(0,1)$ is universal. Iterating (3.4.7), for $8^{-(k+1)} < \rho \le 8^{-k}$ we deduce

$$\int_{B_\rho^+} r^{2-n}|\nabla u|^2\,\mathrm{d}x \le \theta^k\int_{B_{8^k\rho}^+} r^{2-n}|\nabla u|^2\,\mathrm{d}x \le \frac{1}{\theta}\rho^{2\alpha}\int_{B_1^+} r^{2-n}|\nabla u|^2\,\mathrm{d}x,$$

with $\alpha = -\frac{1}{2}\log_8\theta > 0$, and hence

$$\int_{B_\rho^+} r^{2-n}|\nabla u|^2\,\mathrm{d}x \le C\rho^{2\alpha}\int_{B_1^+} r^{2-n}|\nabla u|^2\,\mathrm{d}x \quad \text{for all } \rho\le 1/8 \text{ and } \varepsilon\le\varepsilon_0. \tag{3.4.8}$$

Finally, we can estimate the integral in the right-hand side of (3.4.8) by splitting $B_1^+ = (B_1^+\setminus B_{1/8}^+)\cup B_{1/8}^+$ and applying (3.4.6) with $\rho = 1/8$ to bound the integral in the annulus, which results in

$$\int_{B_\rho^+} r^{2-n}|\nabla u|^2\,\mathrm{d}x \le C\rho^{2\alpha}\|\nabla u\|_{L^2(B_1^+)} \quad \text{for all } \rho\le 1/8 \text{ and } \varepsilon\le\varepsilon_0. \tag{3.4.9}$$

Applying the energy estimate (3.1.11) in Theorem 3.1.4 (rescaled) to (3.4.9) now yields the claim.

Step 2: *Assuming*

$$\|DA\|_{L^\infty(B_1^+)} + \|b\|_{L^\infty(B_1^+)} \le \varepsilon,$$

we prove that if $\varepsilon\le\varepsilon_0$, then

$$\|u\|_{C^\alpha(\overline{B_{\rho_0}^+})} \le C\|u\|_{L^1(B_1^+)},$$

where $\varepsilon_0 > 0$, $\alpha > 0$, $\rho_0 > 0$, and C are universal.

Notice that for each $y\in\partial^0 B_1^+$ and each half-ball $B_d(y)\subset B_1^+$, there is a rotation matrix $R = R(y)\in SO(n)$ such that the function $u^{y,d}(x) := u\big(y + \frac{d}{\sqrt{C_0}}A^{1/2}(y)Rx\big)$ satisfies $u = 0$ on $\partial^0 B_1^+$, and is a stable solution to a semilinear equation in B_1^+ with coefficients

$$A^{y,d}(x) := R^T A^{-1/2}(y) A\big(y + \tfrac{d}{\sqrt{C_0}}A^{1/2}(y)Rx\big)A^{-1/2}(y)R,$$

$$b^{y,d}(x) := \tfrac{d}{\sqrt{C_0}}R^T A^{-1/2}(y) b\big(y + \tfrac{d}{\sqrt{C_0}}A^{1/2}(y)Rx\big).$$

The new matrix is uniformly elliptic $\frac{c_0}{C_0}\le A^{y,d}\le\frac{C_0}{c_0}$ with $A^{y,d}(0) = I$, and the coefficients can be bounded by $\|DA^{y,d}\|_{L^\infty(B_1^+)} + \|b^{y,d}\|_{L^\infty(B_1^+)} \le Cd\left(\|DA\|_{L^\infty(B_1^+)} + \|b\|_{L^\infty(B_1^+)}\right)$. Choosing $d > 0$ universal sufficiently small so that $Cd\le 1$, we further have

$$\|DA^{y,d}\|_{L^\infty(B_1^+)} + \|b^{y,d}\|_{L^\infty(B_1^+)} \le \varepsilon \quad \text{for all } y\in\partial^0 B_{1-d}^+.$$

Since $\varepsilon \le \varepsilon_0$ (with $\varepsilon_0 > 0$ as in Step 1), by (3.4.1) it follows that

$$\int_{B_\rho^+} r^{2-n} |\nabla u^{y,d}|^2 \, dx \le C \|u^{y,d}\|^2_{L^1(B_1^+)} \rho^{2\alpha} \quad \text{for } y \in \partial^0 B^+_{1-d} \text{ and } \rho \le 1/8,$$

and using that $r^{2-n} \ge \rho^{2-n}$ in B_ρ^+, we also have

$$\int_{B_\rho^+} |\nabla u^{y,d}|^2 \, dx \le C \|u^{y,d}\|^2_{L^1(B_1^+)} \rho^{2\alpha+n-2} \quad \text{for } y \in \partial^0 B^+_{1-d} \text{ and } \rho \le 1/8. \tag{3.4.10}$$

Let us express (3.4.10) in terms of the original function u. By the change of variables $z = y + \frac{d}{\sqrt{C_0}} A^{1/2}(y) R x$ and by uniform ellipticity, using that $B^+_{\sqrt{c_0}\rho} \subset A^{1/2}(y) R(B_\rho^+)$, the left-hand side of (3.4.10) can be bounded from below by

$$\int_{B_\rho^+} |\nabla u^{y,d}|^2 \, dx \ge c\, d^{2-n} \int_{B^+_{d\sqrt{\frac{c_0}{C_0}}\rho}(y)} |\nabla u|^2 \, dz, \tag{3.4.11}$$

where $c > 0$ is a universal constant. Similarly, we also have $\|u^{y,d}\|_{L^1(B_1^+)} \le C d^{-n} \|u\|_{L^1(B_1^+)}$ and hence, from (3.4.10) and (3.4.11) we deduce

$$\int_{B^+_{d\sqrt{\frac{c_0}{C_0}}\rho}(y)} |\nabla u|^2 \, dz \le C d^{-2} \|u\|^2_{L^1(B_1^+)} \rho^{n-2+2\alpha} \quad \text{for } y \in \partial^0 B^+_{1-d} \text{ and } \rho \le 1/8. \tag{3.4.12}$$

We let $\rho_0 := \frac{d}{16}\sqrt{\frac{c_0}{C_0}}$. Making d smaller if necessary, we may assume that $B^+_{2\rho_0} \subset B^+_{1-d}$. Dividing ρ by $d\sqrt{\frac{c_0}{C_0}}$ in (3.4.12), using that d is universal, and by Cauchy-Schwarz we have

$$\int_{B_\rho^+(y)} |\nabla u| \, dz \le C \|u\|_{L^1(B_1^+)} \rho^{n-1+\alpha} \quad \text{for } y \in \partial^0 B^+_{2\rho_0} \text{ and } \rho \le 2\rho_0. \tag{3.4.13}$$

With (3.4.13) on hand, we are finally ready to prove the boundary Hölder estimate.

Let $x = (x', x_n) \in B^+_{\rho_0} \subset \mathbb{R}^n \times \mathbb{R}_+$. Since $u = 0$ on $\partial^0 B_1^+$, by the Poincaré inequality

$$\|u\|_{L^1(B^+_{2x_n}(x',0))} \le C x_n \|\nabla u\|_{L^1(B^+_{2x_n}(x',0))}. \tag{3.4.14}$$

Applying (3.4.13) with $\rho = 2x_n$ and $y = (x', 0)$, from (3.4.14) we deduce

$$\|u\|_{L^1(B^+_{2x_n}(x',0))} \le C \|u\|_{L^1(B_1^+)} (x_n)^{n+\alpha}. \tag{3.4.15}$$

By the interior Hölder estimates from Theorem 1.1.1 (see Chapter 1) applied in the ball $B_{x_n}(x) \subset B_1^+$, we have

$$\|u\|_{L^\infty(B_{x_n/2}(x))} + (x_n)^\alpha [u]_{C^\alpha(\overline{B}_{x_n/2}(x))} \le C \|u\|_{L^1(B_{x_n}(x))} (x_n)^{-n}, \tag{3.4.16}$$

where $\alpha > 0$ and C are universal constants (since we are assuming a universal bound $\varepsilon \le \varepsilon_0$ on the coefficients). Since $B_{x_n}(x) \subset B^+_{2x_n}(x', 0)$, combining (3.4.16) and (3.4.15)

$$\|u\|_{L^\infty(B_{x_n/2}(x))} + (x_n)^\alpha [u]_{C^\alpha(\overline{B}_{x_n/2}(x))} \le C \|u\|_{L^1(B_1^+)} (x_n)^\alpha \quad \text{for } x \in B^+_{\rho_0}. \tag{3.4.17}$$

In particular, from (3.4.17) it follows that $|u(x)| \le C \|u\|_{L^1(B_1^+)} (x_n)^\alpha$ in $B^+_{\rho_0}$, and we have controlled the L^∞ norm of u in $B^+_{\rho_0}$. To bound the Hölder norm in $B^+_{\rho_0}$, consider

$x, y \in B^+_{\rho_0}$ such that $x \neq y$. Without loss of generality we may assume $y_n \leq x_n$. On the one hand, if $|x - y| \leq x_n/2$, then from (3.4.17) we deduce

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq [u]_{C^\alpha(\overline{B}_{x_n/2}(x))} \leq C\|u\|_{L^1(B_1^+)}.$$

On the other hand, if $|x - y| > x_n/2$, then by the L^∞ estimates in (3.4.17) it follows that

$$\begin{aligned}|u(x) - u(y)| \leq u(x) + u(y) &\leq C\|u\|_{L^1(B_1^+)} ((x_n)^\alpha + (y_n)^\alpha) \leq C\|u\|_{L^1(B_1^+)}(x_n)^\alpha \\ &\leq C\|u\|_{L^1(B_1^+)}|x - y|^\alpha.\end{aligned}$$

Combined, the two inequalities above yield a bound for $[u]_{C^\alpha(B^+_{\rho_0})}$, which was the claim.

Step 3: *Conclusion.*

Arguing as in the proof of the higher integrability estimate in Chapter 2, by a scaling and covering argument and using the interior estimates from Chapter 1, is is not hard to deduce the theorem in its final form from Step 2.

□

3.5 Approximation and proof in $C^{1,1}$ domains

Here we give the complete proof of our main result, Theorem 3.1.1, which establishes a priori estimates in $C^{1,1}$ domains. By an approximation argument (carried out in the proof at the end of this section), it will suffice to prove this result in smooth domains.

First, we comment on the invariance of our class of solutions under general transformations flattening the boundary. This will allow us to reduce the problem in smooth domains to proving estimates in half-balls, which have already been obtained in Theorem 3.1.2 above.

Given a smooth bounded domain $\Omega \subset \mathbb{R}^n$, we can always write it as the superlevel set of a smooth function $\Phi \in C^\infty(\mathbb{R}^n)$, namely,

$$\Omega = \{x \in \mathbb{R}^n \colon \Phi(x) > 0\} = \{\Phi > 0\}.$$

Moreover, Φ can be chosen so that $\nabla\Phi \neq 0$ on $\partial\Omega$; see Appendix E.

Let $x_0 \in \partial\Omega$. Upon rotating the coordinate axes, we may assume that $\nabla\Phi(x_0) = \partial_n\Phi(x_0)e_n$, with $\partial_n\Phi(x_0) = |\nabla\Phi(x_0)| > 0$. Then, writing $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, the map $\Psi(x) := \big((x - x_0)', \frac{\Phi(x)}{\partial_n\Phi(x_0)}\big)$ is a local diffeomorphism around x_0 which flattens out the boundary $\partial\Omega$. More precisely, by a quantitative version of the Inverse Function Theorem, we have the following:

Lemma 3.5.1. *There are numbers $0 < R_1 < R_2$ and $\rho > 0$ depending only on $\|\nabla\Phi\|_{C^{0,1}(\mathbb{R}^n)}$ and $\||\nabla\Phi|^{-1}\|_{L^\infty(\partial\Omega)}$ such that*

$$\Psi(B_{R_1}(x_0) \cap \Omega) \subset B^+_{\rho/2} \subset B^+_\rho \subset \Psi(B_{R_2}(x_0) \cap \Omega). \tag{3.5.1}$$

Proof. By a translation, we may assume that $x_0 = 0 \in \partial\Omega$. Since the map Ψ satisfies $\Psi(0) = 0$ and $D\Psi(0) = I$, choosing $R_2 > 0$ small such that

$$|D\Psi(x) - D\Psi(z)| \leq 1/2 \quad \text{for all } x, z \in B_{R_2}, \tag{3.5.2}$$

by Lemma 1.3 in [67, Chapter XIV] we deduce that for all $y \in B_{R_2/2}$ there is a unique $x \in B_{R_2}$ such that $\Psi(x) = y$. Thus, we obtain the second inclusion

$$B^+_{R_2/2} \subset \Psi(B_{R_2} \cap \Omega).$$

Using that $[D\Psi]_{C^{0,1}(\mathbb{R}^n)} \leq [\nabla\Phi]_{C^{0,1}(\mathbb{R}^n)} \| |\nabla\Phi|^{-1}\|_{L^\infty(\partial\Omega)}$, it is easy to check that condition (3.5.2) is fulfilled if

$$R_2 \leq (4[\nabla\Phi]_{C^{0,1}(\mathbb{R}^n)} \| |\nabla\Phi|^{-1}\|_{L^\infty(\partial\Omega)})^{-1}. \tag{3.5.3}$$

To show the first inclusion in (3.5.1), we proceed as above but considering the inverse map Ψ^{-1} instead. If $R_1 > 0$ is sufficiently small such that

$$|D\Psi^{-1}(\widetilde{x}) - D\Psi^{-1}(\widetilde{z})| \leq 1/2 \quad \text{for all } \widetilde{x}, \widetilde{z} \in B_{2R_1}, \tag{3.5.4}$$

then, again by Lemma 1.3 in [67, Chapter XIV], we deduce

$$\Psi(B_{R_1} \cap \Omega) \subset B^+_{2R_1}. \tag{3.5.5}$$

For (3.5.4) to hold in this case, it suffices to take $R_1 > 0$ sufficiently small such that

$$[D\Psi^{-1}]_{C^{0,1}(B_{2R_1})} 4R_1 \leq 1/2. \tag{3.5.6}$$

It remains to estimate $[D\Psi^{-1}]_{C^{0,1}(B_{2R_1})}$. Let $\widetilde{x} = \Psi(x)$, $\widetilde{y} = \Psi(y) \in B_{2R_1}$. If $R_1 \leq R_2$, with R_2 as in (3.5.3), then $\frac{1}{\partial_n\Phi(x)} \leq 2\| |\nabla\Phi|^{-1}\|_{L^\infty(\partial\Omega)}$ and hence

$$\begin{aligned}
&|D\Psi^{-1}(\widetilde{x}) - D\Psi^{-1}(\widetilde{y})|^2 \\
&\leq 2\frac{|\nabla'\Phi(x) - \nabla'\Phi(y)|^2}{|\partial_n\Phi(x)|^2} + \frac{(2|\nabla'\Phi(y)|^2 + |\partial_n\Phi(0)|^2)\, |\partial_n\Phi(x) - \partial_n\Phi(y)|^2}{|\partial_n\Phi(x)|^2|\partial_n\Phi(y)|^2} \\
&\leq 8(1 + 6\|\nabla\Phi\|^2_{L^\infty(\mathbb{R}^n)} \| |\nabla\Phi|^{-1}\|^2_{L^\infty(\partial\Omega)}) \| |\nabla\Phi|^{-1}\|^2_{L^\infty(\partial\Omega)} [\nabla\Phi]^2_{C^{0,1}(\mathbb{R}^n)} |x-y|^2.
\end{aligned}$$

Moreover, under the same assumption on R_1, we have

$$|x-y| \leq [D\Psi^{-1}]_{L^\infty(B_{2R_1})} |\widetilde{x} - \widetilde{y}| \leq (1 + 8\|\nabla\Phi\|^2_{L^\infty} \| |\nabla\Phi|^{-1}\|^2_{L^\infty(\partial\Omega)})^{1/2} |\widetilde{x} - \widetilde{y}|.$$

Combining the last two bounds, it follows that if $R_1 \leq R_2$ is such that

$$R_1 \leq \left(16\sqrt{2}\big(1 + 8\|\nabla\Phi\|^2_{L^\infty(\mathbb{R}^n)} \| |\nabla\Phi|^{-1}\|^2_{L^\infty(\partial\Omega)}\big) \| |\nabla\Phi|^{-1}\|_{L^\infty(\partial\Omega)} [\nabla\Phi]_{C^{0,1}(\mathbb{R}^n)}\right)^{-1}, \tag{3.5.7}$$

then (3.5.6) holds, and hence (3.5.5) also.

Finally, choosing R_1, $R_2 > 0$ satisfying (3.5.3), (3.5.7), and $8R_1 \leq R_2$, we may take $\rho = 4R_1$, which concludes the proof of (3.5.1). □

Consider now the operator $Lu(x) = a_{ij}(x)u_{ij} + b_i(x)u_i$ introduced in (3.1.2) acting on functions $u \in C^2(B_{R_2}(x_0) \cap \overline{\Omega})$, with R_2 as in Lemma 3.5.1. In the new coordinates $\widetilde{x} = \Psi(x)$, the function $\widetilde{u} = u \circ \Psi^{-1}$ satisfies

$$\widetilde{L}\widetilde{u} := (Lu)(\Psi^{-1}(\widetilde{x})) = \widetilde{a}_{ij}(\widetilde{x})\widetilde{u}_{ij} + \widetilde{b}_i(\widetilde{x})\widetilde{u}_i,$$

where the new coefficients are given by

$$\widetilde{a}_{ij} \circ \Psi(x) = a_{kl}(x)\partial_k\Psi_i(x)\partial_l\Psi_j(x)$$

and

$$\widetilde{b}_i \circ \Psi(x) = b_k(x)\partial_k \Psi_i(x) + a_{jk}(x)\partial^2_{jk}\Psi_i(x)$$

If $0 < c_0 \le A(x) \le C_0$, then, taking $R_2 = R_2(\|\nabla\Phi\|_{C^{0,1}(\mathbb{R}^n)}, \||\nabla\Phi|^{-1}\|_{L^\infty(\partial\Omega)}) > 0$ smaller if necessary (uniformly in the norms of $\nabla\Phi$), the new matrix $\widetilde{A}(\widetilde{x}) = (\widetilde{a}_{ij}(\widetilde{x}))$ is uniformly elliptic with (say)

$$0 < \frac{1}{2}c_0 \le \widetilde{A}(\widetilde{x}) \le \frac{3}{2}C_0.$$[1]

If u is a stable solution of $-Lu = f(u)$ in $B_{R_2}(x_0) \cap \Omega$, then $\widetilde{u}$ is a stable solution of $-\widetilde{L}\widetilde{u} = f(\widetilde{u})$ in $\Psi(B_{R_2}(x_0) \cap \Omega) \subset \mathbb{R}^n_+$. We have already shown that $\widetilde{u}$ solves this equation. To show that it is stable, notice that if $\varphi > 0$ is the function in the definition of stability (3.1.6), then $\widetilde{\varphi} = \varphi \circ \Psi^{-1}$ satisfies the same condition with respect to $\widetilde{L}$.

Finally, notice that the norms of the coefficients $\|D\widetilde{A}\|_{L^\infty}$ and $\|\widetilde{b}\|_{L^\infty}$ involve the norms $\|DA\|_{L^\infty}$, $\|b\|_{L^\infty}$, $\|D\Phi\|_{L^\infty}$, and $\|D^2\Phi\|_{L^\infty}$. This dependence will be crucial to extend our results to $C^{1,1}$ domains, which are given by a $C^{1,1} = W^{2,\infty}$ function Φ.

Now, thanks to all the preliminaries, we can finally upgrade our estimates in half-balls to $C^{1,1}$ domains by an approximation argument:

Proof of Theorem 3.1.1. We proceed in two steps. First, we prove the theorem in smooth domains for solutions in $u \in W^{3,p}(\Omega)$, that is, with weak derivatives integrable *up to the boundary*. Then, we approximate our $C^{1,1}$ domain from the interior by smooth domains and apply the first step on a suitable sequence of stable solutions.

As commented above (see also Appendix E), recall that given a bounded domain $\Omega \subset \mathbb{R}^n$ of class $C^{1,1}$, there is a function $\Phi \in C^{1,1}(\mathbb{R}^n)$ such that $\Omega = \{\Phi > 0\}$ and $\nabla\Phi \ne 0$ on $\partial\Omega$. If Ω is smooth, then the function Φ can be chosen to be C^∞. The main purpose of the function Φ is to quantify the dependence of our bounds on the domain. In the proof below, $\mathrm{diam}(\Omega)$ denotes the diameter of Ω, that is, $\mathrm{diam}(\Omega) = \sup_{x,y\in\Omega}|x-y|$.

Step 1: *Under the additional assumptions that Ω is smooth, L satisfies* (3.1.5)*, and $u \in W^{3,p}(\Omega)$ for some $p > n$, we prove that*

$$\|\nabla u\|_{L^{2+\gamma}(\Omega)} \le C\|u\|_{L^1(\Omega)},$$

where $\gamma > 0$ is dimensional and C is a constant depending only on n, c_0, C_0, $\|DA\|_{L^\infty(\Omega)}$, $\|b\|_{L^\infty(\Omega)}$, $\|\nabla\Phi\|_{C^{0,1}(\mathbb{R}^n)}$, $\||\nabla\Phi|^{-1}\|_{L^\infty(\partial\Omega)}$, and $\mathrm{diam}(\Omega)$*. In addition,*

$$\|u\|_{C^\alpha(\overline{\Omega})} \le C\|u\|_{L^1(\Omega)} \quad \text{if } n \le 9,$$

where $\alpha > 0$ is universal and C is a constant depending only on n, c_0, C_0, $\|DA\|_{L^\infty(\Omega)}$, $\|b\|_{L^\infty(\Omega)}$, $\|\nabla\Phi\|_{C^{0,1}(\mathbb{R}^n)}$, and $\||\nabla\Phi|^{-1}\|_{L^\infty(\partial\Omega)}$.

Let $0 < R_1 < R_2$ be the functions of $\|\nabla\Phi\|_{C^{0,1}(\mathbb{R}^n)}$ and $\||\nabla\Phi|^{-1}\|_{L^\infty(\partial\Omega)}$ constructed in Lemma 3.5.1 above.

Let $\delta := R_1/3 > 0$. Since Ω is bounded, it is contained in a ball of radius $\mathrm{diam}(\Omega) < \infty$. Hence, we can cover $\overline{\Omega}$ by N balls $\{B_i\}_i$ of radius δ, where $N \le C\mathrm{diam}(\Omega)^n\delta^{-n}$ for some

[1] Indeed, since $\widetilde{A}(\widetilde{x})p\cdot p = A(x)D\Psi(x)p \cdot D\Psi(x)p$, by ellipticity $c_0|D\Psi(x)p|^2 \le \widetilde{A}(\widetilde{x})p\cdot p \le C_0|D\Psi(x)p|^2$. Moreover, since $|D\Psi(x)p|^2 = |p'|^2 + |\frac{\nabla'\Phi(x)}{\partial_n\Phi(x_0)}\cdot p' + \frac{\partial_n\Phi(x)}{\partial_n\Phi(x_0)}p_n|^2$, choosing $R_2 > 0$ smaller (as before, with the stated dependence) such that $\frac{9}{10} \le \frac{\partial_n\Phi(x)}{\partial_n\Phi(x_0)} \le \frac{|\nabla\Phi(x)|}{\partial\Phi(x_0)} \le \frac{11}{10}$ and $\frac{|\nabla'\Phi(x)|}{\partial\Phi(x_0)} \le \frac{1}{100}$, it is easy to check that $\frac{1}{2}|p|^2 \le |D\Psi(x)p|^2 \le \frac{3}{2}|p|^2$ as claimed.

dimensional C.[2] We write $2B_i$ to denote the ball centered at the same point but with twice the radius. We label the balls B_i in a way such that the first $N' < N$ are close to the boundary, in the sense that $2B_i \cap \partial\Omega \neq \emptyset$, and the remaining $N - N'$ are interior, i.e., they satisfy the inclusion $2B_i \subset \Omega$.

For each $i \leq N'$, by definition, there is a boundary point $x_i \in 2B_i \cap \partial\Omega$ and hence $B_i \subset B_{3\delta}(x_i) = B_{R_1}(x_i)$. In particular, flattening the boundary as explained above and applying the energy estimate (3.1.11) in Theorem 3.1.2 rescaled, we deduce

$$\|\nabla u\|_{L^{2+\gamma}(B_i\cap\Omega)} \leq \|\nabla u\|_{L^{2+\gamma}(B_{R_1}(x_i)\cap\Omega)} \leq C\|u\|_{L^1(B_{R_2}(x_i)\cap\Omega)} \quad \text{for all } i \leq N', \tag{3.5.8}$$

where $C = C(n, c_0, C_0, \|DA\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Omega)}, \|\nabla\Phi\|_{C^{0,1}(\mathbb{R}^n)}, \||\nabla\Phi|^{-1}\|_{L^\infty(\partial\Omega)})$. Therefore, by (3.5.8) and interior estimates (Theorem 1.1.1 above), we deduce

$$\begin{aligned}
\|\nabla u\|_{L^{2+\gamma}(\Omega)} &\leq \sum_{i\leq N'} \|\nabla u\|_{L^{2+\gamma}(B_i\cap\Omega)} + \sum_{i>N'} \|\nabla u\|_{L^{2+\gamma}(B_i)} \\
&\leq C\sum_{i\leq N'} \|u\|_{L^1(B_{R_2}(x_i)\cap\Omega)} + C\sum_{i>N'} \|u\|_{L^1(2B_i)} \\
&\leq C\|u\|_{L^1(\Omega)},
\end{aligned}$$

where C depends only on n, c_0, C_0, $\|DA\|_{L^\infty(\Omega)}$, $\|b\|_{L^\infty(\Omega)}$, $\|\nabla\Phi\|_{C^{0,1}(\mathbb{R}^n)}$, $\||\nabla\Phi|^{-1}\|_{L^\infty(\partial\Omega)}$, and $\operatorname{diam}(\Omega)$. The first claim follows.

Assume now that $n \leq 9$.

We prove the L^∞ estimate first. Let $x \in \Omega$. If $\operatorname{dist}(x, \partial\Omega) < R_1$, then $x \in B_{R_1}(x_0) \cap \Omega$ for some $x_0 \in \partial\Omega$. Hence, flattening the boundary, by Theorem 3.1.2 we deduce

$$|u(x)| \leq \|u\|_{L^\infty(B_{R_1}(x_0)\cap\Omega)} \leq C\|u\|_{L^1(B_{R_2}\cap\Omega)} \leq C\|u\|_{L^1(\Omega)},$$

where $C = C(n, c_0, C_0, \|DA\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Omega)}, \|\nabla\Phi\|_{C^{0,1}(\mathbb{R}^n)}, \||\nabla\Phi|^{-1}\|_{L^\infty(\partial\Omega)})$. Otherwise, if $\operatorname{dist}(x, \partial\Omega) \geq R_1$, then by interior estimates (rescaled) we have

$$|u(x)| \leq \|u\|_{L^\infty(B_{R_1/2}(x)} \leq C\|u\|_{L^1(B_{R_1}(x))} \leq C\|u\|_{L^1(\Omega)},$$

where again $C = C(n, c_0, C_0, \|DA\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Omega)}, \|\nabla\Phi\|_{C^{0,1}(\mathbb{R}^n)}, \||\nabla\Phi|^{-1}\|_{L^\infty(\partial\Omega)})$. The two inequalities yield the desired L^∞ bound $\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{L^1(\Omega)}$.

Next, we estimate the C^α seminorm. Let $x, y \in \Omega$ with $x \neq y$ and let $\widetilde{\delta} := 2R_1/3 > 0$. We distinguish three cases:

- If $|x - y| \geq \widetilde{\delta}/2$, then

  $$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq 2^\alpha\widetilde{\delta}^{-\alpha}\left(|u(x)| + |u(y)|\right) \leq 2^{\alpha+1}\widetilde{\delta}^{-\alpha}\|u\|_{L^\infty(\Omega)}$$

  and we may apply the L^∞ estimate $\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{L^1(\Omega)}$.

[2]To see this, given $k \in \mathbb{N}$ and $R > 0$, consider the set $A_{k,R} = \{Rl/k \colon l \in \mathbb{Z} \text{ with } -k \leq l \leq k\}^n \subset \mathbb{R}^n$. Notice that $A_{k,R}$ is a discrete set with $N = (2k+1)^n$ elements. For each $x \in B_R$, there is a $y \in A_{k,R}$ such that $|x - y| \leq \sqrt{n}\frac{R}{2k}$. Hence, if $\sqrt{n}\frac{R}{2\delta} < k \leq \sqrt{n}\frac{R}{2\delta} + 1$, then $B_R \subset \cup_{y\in A_{k,R}} B_\delta(y)$ and the number of balls can be estimated by $N \leq (\sqrt{n}\frac{R}{\delta} + 3)^n \leq (2\sqrt{n})^n\left(\frac{R}{\delta}\right)^n$ by taking δ smaller in terms of R and n.

- If $|x-y| < \widetilde{\delta}/2$ and (say) $\mathrm{dist}(x, \partial\Omega) < \widetilde{\delta}$, then $x, y \in B_{\widetilde{\delta}/2}(x) \subset B_{\frac{3}{2}\widetilde{\delta}}(x_0) \subset B_{R_1}(x_0)$ for some $x_0 \in \partial\Omega$. Hence, flattening the boundary, by Theorem 3.1.2 we deduce

$$\begin{aligned}\frac{|u(x)-u(y)|}{|x-y|^\alpha} &\le [u]_{C^\alpha(\overline{B_{\widetilde{\delta}/2}(x)\cap\Omega})} \le [u]_{C^\alpha(\overline{B_{R_1}(x_0)\cap\Omega})} \le C\|u\|_{L^1(B_{R_2}(x_0)\cap\Omega)} \\ &\le C\|u\|_{L^1(\Omega)},\end{aligned}$$

where C has the same dependence as above.

- If $|x-y| < \widetilde{\delta}/2$ and $\min\{\mathrm{dist}(x, \partial\Omega), \mathrm{dist}(y, \partial\Omega)\} \ge \widetilde{\delta}$, then $x, y \in B_{\widetilde{\delta}/2}(x) \subset B_{\widetilde{\delta}}(x) \subset \Omega$ and by interior estimates (rescaled) we deduce

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \le [u]_{C^\alpha(\overline{B}_{\widetilde{\delta}/2}(x))} \le C\|u\|_{L^1(B_{\widetilde{\delta}}(x))} \le C\|u\|_{L^1(\Omega)},$$

where C has the same dependence as above.

The three inequalities yield the bound

$$[u]_{C^\alpha(\overline{\Omega})} \le C\|u\|_{L^1(\Omega)},$$

where $C = C(n, c_0, C_0, \|DA\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Omega)}, \|\nabla\Phi\|_{C^{0,1}(\mathbb{R}^n)}, \||\nabla\Phi|^{-1}\|_{L^\infty(\partial\Omega)})$. This concludes the proof of Step 1.

Step 2: *Conclusion: Approximation argument.*

Let $\Omega_k = \{\Phi_k > 0\}$ be an exhaustion of $\Omega = \{\Phi > 0\}$ by C^∞ sets, with norms satisfying

$$\|\nabla\Phi_k\|_{C^{0,1}(\mathbb{R}^n)} + \||\nabla\Phi_k|^{-1}\|_{L^\infty(\partial\Omega_k)} \le C \tag{3.5.9}$$

for some constant C depending only on Φ and Ω; see Appendix E.

For each k, let $b_i^k := b_i * \eta_k$, where $(\eta_k)_k$ is a regularizing sequence such that $b_i^k \in C^\infty(\overline{\Omega_k})$. In particular, $b_i^k \to b_i$ locally uniformly in Ω. We define the operator

$$L_k := a_{ij}(x)\partial_{ij} + b_i^k(x)\partial_i,$$

where $a_{ij} \in C^{0,1}(\overline{\Omega})$ is the same coefficient as in the statement of the theorem. By elliptic regularity, all bounded strong solutions of the problem $-L_k u_k = f(u_k)$ in Ω_k, $u_k = 0$ on $\partial\Omega_k$, belong to $W^{3,p}(\Omega_k)$.

We will distinguish the two cases $f(0) > 0$ and $f(0) = 0$.

Case $f(0) > 0$. Let $\varepsilon_k \in (0,1)$ with $\varepsilon_k \downarrow 0$. For each k, we will construct a stable solution $u_k \in W^{3,p}(\Omega_k)$ to the problem

$$\begin{cases} -L_k u_k = (1-\varepsilon_k) f(u_k) & \text{in } \Omega_k \\ \quad u_k > 0 & \text{in } \Omega_k \\ \quad u_k = 0 & \text{on } \partial\Omega_k \end{cases} \tag{3.5.10}$$

by monotone iteration starting at 0. This is the so called *minimal solution* of (3.5.10), that is, the smallest positive supersolution of (3.5.10), and it is well-known to be stable.[3]

[3]To prove that it is stable one argues by contradiction, considering ϕ, the principal eigenfunction of $L + (1-\varepsilon_k) f'(u_k)$ in Ω_k, and showing that $u_k - \delta\phi$ would be a positive supersolution for $\delta > 0$ sufficiently small; see [8].

For the monotone iteration to converge, we need a barrier function. We claim that $u \in C^0(\overline{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ is a barrier for (3.5.10), in fact, we have $u > 0$ on $\overline{\Omega_k} \subset \Omega$ and

$$-L_k u \geq (1 - \varepsilon_k) f(u) \quad \text{ in } \Omega_k.$$

Indeed, using the equation satisfied by u and by monotonicity of f, we have

$$-L_k u - (1 - \varepsilon_k) f(u) = \varepsilon_k f(u) + (L - L_k) u \geq \varepsilon_k f(0) - \|b - b^k\|_{L^\infty(\Omega_k)} \|\nabla u\|_{L^\infty(\Omega_k)},$$

and the right-hand side is nonnegative by choosing the regularizing sequence η_k in terms of ε_k, $f(0)$, and $\|\nabla u_k\|_{L^\infty(\Omega_k)}$ so that $\|b - b^k\|_{L^\infty(\Omega_k)}$ is sufficiently small. Here, recall that by L^p estimates we have $u \in C^0(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ for all $p < \infty$, and hence $u \in C^{1,\alpha}(\overline{\Omega_k})$ for all $\alpha \in (0,1)$ and $k \in \mathbb{N}$.

Next, we carry out the monotone iteration. Let $k \in \mathbb{N}$ and $u_k^{(0)} = 0$. For each $l \in \mathbb{N}$, we consider the unique solution $u_k^{(l)}$ to the problem $-L_k u_k^{(l)} = (1 - \varepsilon_k) f(u_k^{(l-1)})$ in Ω_k, $u_k^{(l)} = 0$ on $\partial\Omega_k$. Since u is a barrier, by maximum principle, it is not hard to show that $0 \leq u_k^{(l-1)} \leq u_k^{(l)} \leq u$ in Ω_k, and by global regularity the monotone limit $u_k := \lim_{l \uparrow \infty} u_k^{(l)}$ converges uniformly in $C^2(\overline{\Omega_k})$ norm and solves (3.5.10). By construction, u_k is below any supersolution of (3.5.10) and hence it is the minimal solution.

Since $\overline{\Omega_k} \subset \Omega_{k+1} \subset \Omega$, by maximum principle we have

$$0 \leq u_k \leq u_{k+1} \leq u \quad \text{ in } \Omega_k.$$

Let $u^\star(x) := \lim_{k\to\infty} u_k(x)$ for $x \in \Omega$. Using that $u \in C^0(\overline{\Omega})$, by L^p estimates we have $u_k \to u^\star$ weakly in $W^{2,p}_{\mathrm{loc}}(\Omega)$ for all $p < \infty$. Since $u^\star \leq u$ in Ω, we can extend $u^\star$ up to the boundary to a function $u^\star \in C^0(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$. By weak convergence, it follows that $u^\star$ is a strong solution to $-Lu^\star = f(u^\star)$ in Ω, $u^\star = 0$. Moreover, $u^\star$ is a stable solution. To see this, taking a positive function $\varphi \in W^{2,n}_{\mathrm{loc}}(\Omega)$ as in the stability inequality (3.1.6) for u, using that $u^\star \leq u$ and by the convexity of f, we have

$$J_{u^\star}\varphi = (L + f'(u^\star))\varphi \leq (L + f'(u))\varphi = J_u \varphi \leq 0 \quad \text{ in } \Omega.$$

Finally, by the uniqueness of stable solutions for convex nonlinearities (see Appendix F), it follows that $u = u^\star$.

Applying Step 1 to the minimal solutions u_k, by the bounds $\|b^k\|_{L^\infty(\Omega_k)} \leq \|b\|_{L^\infty(\Omega)}$ and (3.5.9), and by the monotonicity of the sequence, for $k \geq l+1$ we obtain

$$\|\nabla u_k\|_{L^{2+\gamma}(\Omega_l)} \leq \|\nabla u_k\|_{L^{2+\gamma}(\Omega_k)} \leq C\|u_k\|_{L^1(\Omega_k)} \leq C\|u\|_{L^1(\Omega)}.$$

Now, since $\nabla u_k \to \nabla u$ uniformly on compacts, letting $k \to \infty$ and $l \to \infty$ in this last estimate and by monotone convergence, we deduce the higher integrability in $C^{1,1}$ domains.

Assuming moreover that $n \leq 9$, by Step 1 applied to u_k, for $k \geq l+1$ we have

$$\|u_k\|_{C^\alpha(\overline{\Omega_l})} \leq \|u_k\|_{C^\alpha(\overline{\Omega_k})} \leq C\|u_k\|_{L^1(\Omega_k)} \leq C\|u^\star\|_{L^1(\Omega)} = C\|u\|_{L^1(\Omega)}.$$

Hence, letting $k \to \infty$ and $l \to \infty$, we deduce the Hölder estimate in $C^{1,1}$ domains.

Case $f(0) = 0$**.** Without loss of generality, we may assume that $u > 0$ in Ω. Since 0 is a stable solution, by Proposition F.1 we deduce that $f(u) = \mu_1[L,\Omega]u$,[4] and hence u is a principal eigenfunction of L.

[4]Here we are following the notation in Appendix F, namely, $\mu_1[L,\Omega]$ denotes the principal eigenvalue of L in Ω with the sign convention $L\varphi = -\mu\varphi$.

Since $a_{ij} \in C(\overline{\Omega_k})$ and $b_i^k \in L^\infty(\Omega_k)$, by standard existence theory, there is a principal eigenvalue $\mu_k := \mu_1[L_k, \Omega_k]$ and eigenfunction $\varphi_k \in W^{2,p}(\Omega_k)$ for all $p < \infty$, satisfying $\varphi_k > 0$ in Ω_k, $\|\varphi_k\|_{L^1(\Omega_k)} = \|u\|_{L^1(\Omega)}$, $-L_k\varphi_k = \mu_k\varphi_k$ in Ω_k, and $\varphi_k = 0$ on $\partial\Omega_k$. Moreover, recalling that $a_{ij} \in C^{0,1}(\overline{\Omega_k})$ and $b_i^k \in C^\infty(\overline{\Omega_k})$, we further have $\varphi_k \in W^{3,p}(\Omega_k)$ for $p < \infty$. In particular, since φ_k are stable, by Step 1 we have the bounds

$$\|\nabla\varphi_k\|_{L^{2+\gamma}(\Omega_k)} \leq C\|u\|_{L^1(\Omega)}, \tag{3.5.11}$$

and

$$\|\varphi_k\|_{C^\alpha(\overline{\Omega_k})} \leq C\|u\|_{L^1(\Omega)} \quad \text{if } n \leq 9. \tag{3.5.12}$$

To deduce the final estimates it suffices to extract a subsequence converging to u. For this, we essentially follow the proof of Theorem 2.1 in [9]. Namely, by Harnack inequality, for $k \geq l+1$ we have $\|\varphi_k\|_{L^\infty(\Omega_l)} \leq C_l \inf_{\Omega_l}\varphi_k \leq C_l\|\varphi_k\|_{L^1(\Omega_l)} \leq C_l\|u\|_{L^1(\Omega)}$, hence, by interior estimates (up to a subsequence) $\varphi_k \to \varphi$ weakly in $W^{2,p}_{\mathrm{loc}}(\Omega)$ for some positive $\varphi \in W^{2,p}_{\mathrm{loc}}(\Omega)$. Moreover, since $\varphi_k \in W_0^{1,2+\gamma}(\Omega_k)$, the extension $\varphi_k\chi_{\Omega_k}$ is bounded in $W_0^{1,2+\gamma}(\Omega)$ and by compactness $\varphi_k\chi_{\Omega_k} \to \varphi$ weakly in $W_0^{1,2+\gamma}(\Omega)$, hence strongly in $L^{2+\gamma}(\Omega)$. In particular, by strong convergence $\|\varphi\|_{L^1(\Omega)} = \lim_k \|\varphi_k\|_{L^1(\Omega_k)} = \|u\|_{L^1(\Omega)}$ and by weak lower semicontinuity, from (3.5.11), we deduce

$$\|\nabla\varphi\|_{L^{2+\gamma}(\Omega)} \leq C\|u\|_{L^1(\Omega)}. \tag{3.5.13}$$

By (3.5.12), using that $\varphi_k \to \varphi$ converges locally uniformly in Ω, it is also clear that

$$\|\varphi\|_{C^\alpha(\overline{\Omega})} \leq C\|u\|_{L^1(\Omega)} \quad \text{if } n \leq 9. \tag{3.5.14}$$

Passing to the limit in the equation, we see that $\varphi \in W_0^{1,2+\gamma}(\Omega) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ solves $-L\varphi = \mu^\star\varphi$ where $\mu^\star = \lim_k \mu_1[L_k, \Omega_k]$. In fact, $\mu^\star = \mu_1[L, \Omega]$ by the characterization of the principal eigenfunction and the maximum principle.

It follows that both u and φ are positive principal eigenfunctions of L in Ω, with $\|u\|_{L^1(\Omega)} = \|\varphi\|_{L^1(\Omega)}$, hence $u = \varphi$ and (3.5.13), (3.5.14) are already the claimed estimates. □

Appendix A

Stability is not equivalent to the integral inequality

Let $u \in C^2(\overline{\Omega})$ be a solution to $-Lu = f(u)$ in Ω with $u = 0$ on $\partial\Omega$. Recall that u is a stable solution if

$$J_u\varphi = L\varphi + f'(u)\varphi \leq 0 \quad \text{in } \Omega, \tag{A.1}$$

for some function $\varphi \in C^2(\overline{\Omega})$ with $\varphi > 0$ in Ω and $\varphi = 0$ on $\partial\Omega$. This is the stability condition (1.1.3) presented in the Introduction and is equivalent to the nonnegativity of the first Dirichlet eigenvalue of J_u (with the sign convention $J_u\varphi = -\mu\varphi$). There, we also showed that stable solutions satisfy the integral inequality (1.1.8), which reads

$$\int_\Omega f'(u)\xi^2 \,\mathrm{d}x \leq \int_\Omega |\nabla\xi - \tfrac{1}{2}A^{-1}(x)\widehat{b}(x)\xi|^2_{A(x)} \quad \text{for all } \xi \in C^\infty_c(\Omega). \tag{A.2}$$

Our goal in this appendix is to show that the integral inequality (A.2) does not imply the stability condition (A.1) in general. The main reason is that the problem is not variational, due to the drift in L. We also give conditions under which the equivalence holds. Namely, writing the operator in divergence form $Lu = \operatorname{div}(A(x)\nabla u) + \widehat{b}(x) \cdot \nabla u$, we show that if $A^{-1}(x)\widehat{b}(x)$ is the gradient of a scalar function, then the problem is variational and the two conditions are equivalent.

First we write the integrals in (A.2) as the quadratic form associated to a linear self-adjoint operator. Integrating by parts, we have

$$\int_\Omega \left(|\nabla\xi - \tfrac{1}{2}\xi A^{-1}(x)\widehat{b}(x)|^2_{A(x)} - f'(u)\xi^2\right) \mathrm{d}x = -\int_\Omega \xi\widetilde{J}_u\xi \,\mathrm{d}x, \tag{A.3}$$

where $\widetilde{J}_u$ is the operator

$$\widetilde{J}_u\xi := \operatorname{div}(A(x)\nabla\xi) - \left\{\tfrac{1}{2}\operatorname{div}(\widehat{b}(x)) + \tfrac{1}{4}|\widehat{b}(x)|^2_{A^{-1}(x)}\right\}\xi + f'(u)\xi. \tag{A.4}$$

Hence, by the variational characterization of eigenvalues, (A.2) amounts to the nonnegativity of the principal eigenvalue of $\widetilde{J}_u$.

We can now state our example of a solution satisfying (A.2) but not (A.1):

Example A.1. Consider the operator $Lv = \Delta v + \widehat{b}(x) \cdot \nabla v$ with vector field

$$\widehat{b}(x) = \frac{-x_2e_1 + x_1e_2}{\sqrt{x_1^2 + x_2^2}}.$$

For each constant $c > 0$, we let $f(u) = (\lambda_1 + c)u + 1$, where λ_1 denotes the least Dirichlet eigenvalue of the Laplacian in the unit ball B_1.

If $c > 0$ is sufficiently small, then the unique solution u to the boundary value problem

$$\begin{cases} -Lu = f(u) & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases}$$

satisfies the integral stability condition (A.2) but is not a stable solution, i.e., the stability condition (A.1) does not hold.[1]

Proof. The problem for u is equivalent to solving

$$\begin{cases} -\Delta u - \widehat{b}(x) \cdot \nabla u - (\lambda_1 + c)u = 1 & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{A.5}$$

Notice that the drift $\widehat{b} \in L^\infty(B_1)$ has a weak derivative $D\widehat{b} \in L^p(B_1)$ for $1 \le p < 2$, and satisfies the identities $|\widehat{b}(x)| = 1$ and $\operatorname{div} \widehat{b}(x) = 0$ for a.e. $x \in B_1$. Moreover, since $\widehat{b}$ is tangent to spheres, the derivative $\widehat{b}(x) \cdot \nabla$ vanishes on radial functions. In particular, the principal eigenfunction of the Laplacian is also an eigenfunction of the adjoint operator $L^T = \Delta - \operatorname{div}\big(\widehat{b}(x) \cdot\big) = \Delta - \widehat{b}(x) \cdot \nabla$, with eigenvalue λ_1. Since the point spectrum of L^T is discrete, for $c > 0$ small, we deduce that $\lambda_1 + c$ is not an eigenvalue of the adjoint operator. The Fredholm alternative now gives that (A.5) has a unique solution.

Let φ_1 and ξ_1 be positive principal eigenfunctions of J_u and $\widetilde{J}_u$, respectively. Since φ_1 and ξ_1 are positive in B_1, they must be radial. It follows that

$$J_u \varphi_1 = \Delta \varphi_1 + (\lambda_1 + c)\varphi_1 = -\mu_1 \varphi_1 \quad \text{and} \quad \widetilde{J}_u \xi_1 = \Delta \xi_1 + (\lambda_1 + c - 1/4)\xi_1 = -\widetilde{\mu_1}\xi_1,$$

where μ_1 and $\widetilde{\mu_1}$ are the least eigenvalues of each operator. By uniqueness, the functions are multiples of the principal eigenfunction of the Laplacian. Therefore, we have $\mu_1 = -c < 0$ and $\widetilde{\mu_1} = 1/4 - c > 0$ for c sufficiently small. This means that u is not stable but (A.2) holds, which was the claim. $\square$

Next we investigate the relation between the failure of the equivalence and the form of the drift $\widehat{b}$. Let $\varphi_1 \in C^2(\overline{\Omega})$ be the unique positive principal eigenfunction of J_u with $\int \varphi_1^2 \, \mathrm{d}x = 1$. In particular, the function satisfies $\varphi_1 > 0$ in Ω, $\varphi_1 = 0$ on $\partial\Omega$, and $J_u \varphi_1 = -\mu_1 \varphi_1$, where $\mu_1 \in \mathbb{R}$ is the least eigenvalue of J_u. Consider a test function $\xi \in C_c^\infty(\Omega)$. Multiplying $J_u \varphi_1$ by ξ^2/φ_1 and integrating by parts in Ω, we have

$$\begin{aligned} -\mu_1 &= \int_\Omega (J_u \varphi_1) \frac{\xi^2}{\varphi_1} \, \mathrm{d}x = \int_\Omega \left(-A(x) \nabla \varphi_1 \cdot \nabla \left(\frac{\xi^2}{\varphi_1} \right) + \widehat{b}(x) \cdot \frac{\xi^2}{\varphi_1} \nabla \varphi_1 + f'(u)\xi^2 \right) \mathrm{d}x \\ &= \int_\Omega \Big(|\xi \nabla \log \varphi_1|^2_{A(x)} - 2A(x)\xi \, \nabla \log \varphi_1 \cdot \nabla \xi + \xi \, \widehat{b}(x) \cdot \xi \, \nabla \log \varphi_1 \Big) \, \mathrm{d}x \\ &\qquad\qquad + \int_\Omega f'(u)\xi^2 \, \mathrm{d}x. \end{aligned}$$

[1] The function u can be given explicitly in terms of Bessel functions of the first kind $\mathcal{J}_\alpha$ as

$$u(x) = \frac{1}{(\lambda_1 + c)\mathcal{J}_{\frac{n-2}{2}}\left(\sqrt{\lambda_1 + c}\right)} |x|^{\frac{2-n}{2}} \mathcal{J}_{\frac{n-2}{2}}\left(\sqrt{\lambda_1 + c}\,|x|\right) - \frac{1}{\lambda_1 + c}.$$

Using that

$$
\begin{aligned}
&|\xi \nabla \log \varphi_1|^2_{A(x)} - 2A(x)\xi\, \nabla \log \varphi_1 \cdot \Big(\nabla \xi - \tfrac{1}{2}\xi A^{-1}(x)\widehat{b}(x)\Big) \\
&\quad = \Big|\nabla \xi - \tfrac{1}{2}\xi A^{-1}(x)\widehat{b}(x) - \xi \nabla \log \varphi_1\Big|^2_{A(x)} - \Big|\nabla \xi - \tfrac{1}{2}\xi A^{-1}(x)\widehat{b}(x)\Big|^2_{A(x)},
\end{aligned}
$$

in the integral above, by (A.3) we obtain the identity

$$
-\mu_1 = \int_\Omega \xi \widetilde{J}_u \xi \,\mathrm{d}x + \int_\Omega \Big|\nabla \xi - \tfrac{1}{2}\xi A^{-1}(x)\widehat{b}(x) - \xi \nabla \log \varphi_1\Big|^2_{A(x)} \mathrm{d}x. \tag{A.6}
$$

Now, assuming the integral stability inequality (A.2), we can minimize (A.3) among smooth functions ξ with $\xi = 0$ on $\partial\Omega$ and $\int_\Omega \xi^2 \,\mathrm{d}x = 1$. The unique positive minimizer ξ_1 satisfies $\widetilde{J}_u \xi_1 = -\widetilde{\mu_1}\xi_1$, where $\widetilde{\mu_1} \geq 0$ is the least eigenvalue of $\widetilde{J}_u$. Letting $\xi = \xi_1$ in (A.6) yields

$$
-\mu_1 = -\widetilde{\mu_1} + \int_\Omega \big|\nabla\big(\log \xi_1 - \log \varphi_1\big) - \tfrac{1}{2}A^{-1}(x)\widehat{b}(x)\big|^2_{A(x)} \xi_1^2 \,\mathrm{d}x, \tag{A.7}
$$

and from (A.7) we see that we always have $\mu_1 \leq \widetilde{\mu_1}$, with equality if and only if

$$
\nabla \log \left(\frac{\xi_1}{\varphi_1}\right) = \frac{1}{2} A^{-1}(x)\widehat{b}(x). \tag{A.8}
$$

This can only happen when the drift $\widehat{b}$ is of a special form. Notice that the vector field from Example A.1 is the curl of $\sqrt{x_1^2 + x_2^2}\, e_3$ and so, by the Helmholtz decomposition, cannot be written as the gradient of a function.

Conversely, assume that $\widehat{b}(x) = A(x)\nabla w(x)$ for some function $w \in C^2(\overline{\Omega})$. In this case, the problem can be cast in variational form and conditions (A.1) and (A.2) are equivalent. Indeed, the solutions of $-Lu = f(u)$ in Ω are critical points of the functional $\mathcal{E}(u) = \int_\Omega e^{w(x)} \Big(\frac{1}{2}|\nabla u|^2_{A(x)} - F(u)\Big) \,\mathrm{d}x$, where $F(u) = \int_0^u f(t)\,\mathrm{d}t$. The integral stability inequality (A.2) amounts to the nonnegativity of the second variation

$$
\frac{\mathrm{d}^2}{\mathrm{d}^2 t}\Big|_{t=0} \mathcal{E}\big(u + t\varphi\big) = \int_\Omega e^{w(x)} \Big(|\nabla \varphi|^2_{A(x)} - f'(u)\varphi^2\Big) \,\mathrm{d}x = -\int_\Omega e^{w(x)} \varphi J_u \varphi \,\mathrm{d}x
$$

since, letting $\varphi = e^{-w/2}\xi$ in this expression, we have

$$
\frac{\mathrm{d}^2}{\mathrm{d}^2 t}\Big|_{t=0} \mathcal{E}\big(u + te^{-w/2}\xi\big) = \int_\Omega \Big(|\nabla \xi - \tfrac{1}{2}\xi \nabla w(x)|^2_{A(x)} - f'(u)\xi^2\Big) \,\mathrm{d}x = -\int_\Omega \xi \widetilde{J}_u \xi \,\mathrm{d}x.
$$

In particular, since $-\int_\Omega \xi \widetilde{J}_u \xi \,\mathrm{d}x \geq \widetilde{\mu_1}\|\xi\|^2_{L^2(\Omega)}$ and taking $\varphi = \varphi_1$ to be the principal eigenfunction of J_u above, we have

$$
\mu_1 \int_\Omega e^{w(x)} \varphi_1^2 \,\mathrm{d}x = -\int_\Omega (e^{w/2}\varphi_1)\widetilde{J}_u(e^{w/2}\varphi_1) \,\mathrm{d}x \geq \widetilde{\mu_1} \int_\Omega e^{w(x)} \varphi_1^2 \,\mathrm{d}x
$$

and we obtain the reverse inequality $\mu_1 \geq \widetilde{\mu_1}$.

Appendix B

A trace inequality

First we prove a simple lemma to control the L^p norm in the ball by the L^p norms of the trace and the gradient:

Lemma B.1. *For $p \geq 1$ and $u \in W^{1,p}(B_1)$, we have*

$$\|u\|_{L^p(B_1)}^p \leq 2^{p-1} \left(\|u\|_{L^p(\partial B_1)}^p + \|\nabla u\|_{L^p(B_1)}^p \right).$$

Proof. By approximation, we may assume that $u \in C^\infty(\overline{B}_1)$. For $r \in (0,1)$ and $\sigma \in \partial B_1$, we have $u(r\sigma) = u(\sigma) - \int_r^1 \sigma \cdot \nabla u(t\sigma)\,\mathrm{d}t$ and hence

$$\begin{aligned} r^{n-1}|u(r\sigma)|^p &\leq 2^{p-1} r^{n-1} |u(\sigma)|^p + 2^{p-1} r^{n-1} \int_r^1 |\nabla u(t\sigma)|^p \,\mathrm{d}t \\ &\leq 2^{p-1}|u(\sigma)|^p + 2^{p-1} \int_0^1 t^{n-1} |\nabla u(t\sigma)|^p \,\mathrm{d}t. \end{aligned} \tag{B.1}$$

Integrating (B.1) in $\int_0^1 \mathrm{d}r \int_{\partial B_1} \mathrm{d}\mathcal{H}^{n-1}(\sigma)$ now yields the claim. □

We prove a Sobolev trace inequality with best exponent:

Proposition B.2. *For $1 < p < n$, let $p^\star := \frac{n-1}{n-p} p$. Then*

$$\|u\|_{L^{p^\star}(\partial B_1)}^p \leq C \left(\|u\|_{L^p(\partial B_1)}^p + \|\nabla u\|_{L^p(B_1)}^p \right)$$

for all $u \in W^{1,p}(B_1)$, where C is a constant depending only on n and p.

Proof. By approximation, we may assume that $u \in C^\infty(\overline{B}_1)$. Recall the standard Sobolev inequality

$$\|u\|_{L^{p_S}(B_1)}^p \leq C\big(\|u\|_{L^p(B_1)}^p + \|\nabla u\|_{L^p(B_1)}^p\big), \tag{B.2}$$

where $p_S := \frac{n}{n-p} p$ is the Sobolev exponent and C depends only on n and p.

By the divergence theorem we have

$$\int_{\partial B_1} |u|^{p^\star} \,\mathrm{d}\mathcal{H}^{n-1} = \int_{B_1} \operatorname{div}(x|u|^{p^\star})\,\mathrm{d}x = n \int_{B_1} |u|^{p^\star}\,\mathrm{d}x + p^\star \int_{B_1} |u|^{p^\star - 2} u (x \cdot \nabla u)\,\mathrm{d}x,$$

whence

$$\int_{\partial B_1} |u|^{p^\star} \,\mathrm{d}\mathcal{H}^{n-1} \leq n \int_{B_1} |u|^{p^\star}\,\mathrm{d}x + p^\star \int_{B_1} |u|^{p^\star - 1} |\nabla u|\,\mathrm{d}x. \tag{B.3}$$

The last term in (B.3) can be bounded by the Hölder inequality as

$$\int_{B_1} |u|^{p^\star-1}|\nabla u|\,\mathrm{d}x \le \left(\int_{B_1} |u|^{(p^\star-1)\frac{p}{p-1}}\,\mathrm{d}x\right)^{\frac{p-1}{p}} \|\nabla u\|_{L^p(B_1)},$$

and noticing that $(p^\star - 1)\frac{p}{p-1} = p_S$ we deduce

$$\|u\|^{p^\star}_{L^{p^\star}(\partial B_1)} \le n\|u\|^{p^\star}_{L^{p^\star}(B_1)} + p^\star\|\nabla u\|_{L^p(B_1)}\|u\|^{p^\star-1}_{L^{p_S}(B_1)}. \tag{B.4}$$

Since $p^\star < p_S$, by Hölder we have $\|u\|_{L^{p^\star}(B_1)} \le C\|u\|_{L^{p_S}(B_1)}$, and applying the Sobolev inequality (B.2) in (B.4), we obtain the trace Sobolev inequality

$$\|u\|^p_{L^{p^\star}(\partial B_1)} \le C\big(\|u\|^p_{L^p(B_1)} + \|\nabla u\|^p_{L^p(B_1)}\big), \tag{B.5}$$

where C depends only on n and p. Applying Lemma B.1 in (B.5) now yields the claim. □

Appendix C

Two interpolation inequalities

We recall two interpolation inequalities in cubes by Cabré [19] (with elementary proofs in that paper). In the first one, the L^p norm of the gradient is bounded by a weighted L^1 norm of the Hessian and the L^p norm of the function:

Proposition C.1 ([19])**.** *Let $Q = (0,1)^n \subset \mathbb{R}^n$, $p \geq 1$, and $u \in C^2(\overline{Q})$.*
Then, for every $\delta \in (0,1)$,

$$\|\nabla u\|^p_{L^p(Q)} \leq C_p \left(\delta \| \, |\nabla u|^{p-1} \, D^2 u \, \|_{L^1(Q)} + \delta^{-p} \|u\|^p_{L^p(Q)} \right),$$

where C_p is a constant depending only on n and p.

Proof. We first prove the claim for $n = 1$. In this case, we have $|u'(x_0)| = \min_{x \in [0,1]} |u'(x)|$ for some $x_0 \in [0,1]$. For $0 < y < \frac{1}{3} < \frac{2}{3} < z < 1$, since $(u(y) - u(z))/(y - z)$ is equal to u' at some point, we have $|u'(x_0)| \leq 3(|u(y)| + |u(z)|)$ and integrating in y and z, we deduce

$$|u'(x_0)| \leq 6\|u\|_{L^1(0,1)}.$$

Integrating $\frac{\mathrm{d}}{\mathrm{d}x}(|u'|)^p$ from x_0 to $x \in (0,1)$, we obtain

$$|u'(x)|^p \leq p \int_0^1 |u'|^{p-1} |u''| \, \mathrm{d}x + |u'(x_0)|^p.$$

Now, applying the previous inequality, integrating in $x \in (0,1)$, and by Hölder, we deduce

$$\|u'\|^p_{L^p(0,1)} \leq p \| |u'|^{p-1} u'' \|_{L^1(0,1)} + 6^p \|u\|^p_{L^p(0,1)}.$$

Rescaling and covering, it is then easy to conclude the claim with a parameter $\delta > 0$.

For $n > 1$, we apply the last inequality to $x_n \mapsto u(x', x_n)$ for each $x' \in (0,1)^{n-1}$ and then integrate in x'. This leads to the claimed inequality with $\|u_{x_n}\|^p_{L^p(Q)}$ on the left-hand side. Applying this to the remaining variables and summing yields the claim. □

By a simple covering argument, we also have the analogous estimate in half-annuli. We state it for $p = 1$, since this is the form in which we will use it:

Corollary C.2. *Let $u \in C^2(\overline{B_1^+})$ and $0 < \rho_1 < \rho_2 < \rho_3 < \rho_4 \leq 1$.*
Then, for every $\widetilde{\delta} \in (0,1)$, we have

$$\|\nabla u\|_{L^1(B^+_{\rho_3} \setminus B^+_{\rho_2})} \leq C_{\rho_i} \left(\widetilde{\delta} \|D^2 u\|_{L^1(B^+_{\rho_4} \setminus B^+_{\rho_1})} + \widetilde{\delta}^{-1} \|u\|_{L^1(B^+_{\rho_4} \setminus B^+_{\rho_1})} \right),$$

where C_{ρ_i} is a constant depending only on n, ρ_1, ρ_2, ρ_3, and ρ_4.

Proof. We cover $\overline{B^+_{\rho_3}} \setminus B^+_{\rho_2}$ by disjoint cubes Q_j of side-length $\varepsilon > 0$, with ε sufficiently small such that $Q_j \subset B^+_{\rho_4} \setminus \overline{B^+_{\rho_1}}$. Since ε and the number of cubes depend only on n and ρ_i, applying Proposition C.1 in each cube and summing in j yields the claim. $\square$

The second inequality controls the L^p norm of the function by the L^p norm of the gradient and the L^1 norm of the function:

Proposition C.3 ([19])**.** *Let $Q = (0,1)^n \subset \mathbb{R}^n$, $p \geq 1$, and $u \in C^2(\overline{Q})$.*
Then, for every $\widetilde{\delta} \in (0,1)$,

$$\|u\|^p_{L^p(Q)} \leq C \left(\widetilde{\delta}^p \|\nabla u\|^p_{L^p(Q)} + \widetilde{\delta}^{-n(p-1)} \|u\|^p_{L^1(Q)} \right),$$

where C is a constant depending only on n.

Proof. Let $u_Q = \frac{1}{|Q|} \int_Q u \, \mathrm{d}x$. By Poincaré's inequality $\|u - u_Q\|_{L^p(Q)} \leq C_p \|\nabla u\|_{L^p(Q)}$ and hence

$$\|u\|_{L^p(Q)} \leq \|u - u_Q\|_{L^p(Q)} + \|u_Q\|_{L^p(Q)} \leq C_p(\|\nabla u\|_{L^p(Q)} + \|u\|_{L^1(Q)}).$$

A scaling and covering argument then leads to the result. $\square$

Appendix D

Absorbing errors in larger balls

We recall a celebrated device of Simon [89] which allows to absorb errors in large balls when controlling quantities in smaller balls:

Lemma D.1 ([89])**.** *Let $\beta \geq 0$ and $C_0 > 0$. Let $\mathcal{B}$ be the class of all open balls B contained in the unit ball B_1 of $\mathbb{R}^n$ and let $\sigma \colon \mathcal{B} \to [0, +\infty)$ satisfy the following subadditivity property:*

$$\sigma(B) \leq \sum_{j=1}^{N} \sigma(B^j) \quad \text{whenever } N \in \mathbb{Z}^+, \{B^j\}_{j=1}^N \subset \mathcal{B}, \text{ and } B \subset \bigcup_{j=1}^{N} B^j.$$

It follows that there exists a constant $\delta > 0$, which depends only on n and β, such that if

$$\rho^\beta \sigma\left(B_{\rho/2}(y)\right) \leq \delta \rho^\beta \sigma\left(B_\rho(y)\right) + C_0 \quad \text{whenever } B_\rho(y) \subset B_1,$$

then

$$\sigma(B_{1/2}) \leq C C_0$$

for some constant C which depends only on n and β.

Proof. The idea is to obtain an inequality for the quantity

$$S := \sup\{\rho^\beta \sigma(B_{\rho/2}(y)) \colon B_\rho(y) \subset B_1\}.$$

Given $\varepsilon \in (0, 1/2)$, there are $N = N(\varepsilon)$ points $y_i \in \overline{B_{1/2}}$ such that $B_{1/2} \subset \cup_{i=1}^N B_{\varepsilon/4}(y_i)$. Notice that $B_\varepsilon(y_i) \subset B_1$.

Let $\rho > 0$ and $y \in \mathbb{R}^n$ be such that $B_\rho(y) \subset B_1$, and consider $\varepsilon \in (0, 1/2)$ as above. Since $B_{\rho/2}(y) \subset \cup_i B_{\varepsilon\rho/4}(y + y_i)$ and $B_{\varepsilon\rho}(y + y_i) \subset B_\rho(y) \subset B_1$, by assumption we have

$$\begin{aligned}
\rho^\beta \sigma(B_{\rho/2}(y)) &\leq \rho^\beta \sum_{i=1}^{N} \sigma(B_{\varepsilon\rho/4}(y + y_i)) \\
&\leq \delta \varepsilon^{-\beta} \sum_{i=1}^{N} \rho^\beta \varepsilon^\beta \sigma(B_{\varepsilon\rho/2}(y + y_i)) + 2^\beta \varepsilon^{-\beta} N C_0 \\
&\leq \delta \varepsilon^{-\beta} N S + 2^\beta \varepsilon^{-\beta} N C_0
\end{aligned}$$

and hence, taking the supremum in ρ and y we deduce

$$S(1 - \delta \varepsilon^{-\beta} N) \leq 2^\beta \varepsilon^{-\beta} N C_0.$$

Taking $\delta > 0$ such that $\delta \varepsilon^{-\beta} N = 1/2$, it follows that $S \leq 2^{\beta+1} \varepsilon^{-\beta} N C_0$, and hence the claim. □

Appendix E

Approximating $C^{1,1}$ domains by smooth ones from the interior

In this appendix, we show that bounded domains of class $C^{1,1}$ can be approximated by smooth sets from the interior, satisfying certain uniform bounds. This is a well-known result in the literature, and is valid more generally for domains of class $C^{k,\alpha}$ with $k \geq 1$ and $\alpha \in [0,1]$. We have included an elementary proof for the sake of completeness. Our proof follows the approach suggested by Gilbarg and Trudinger in [63, Problem 6.9].

Recall the definition of $C^{1,1}$ domains:

Definition E.1. A bounded domain $\Omega \subset \mathbb{R}^n$ is of class $C^{1,1}$ if at each point $x_0 \in \partial\Omega$ there is a ball $B = B_\rho(x_0)$ and a one-to-one mapping Ψ of B onto $U \subset \mathbb{R}^n$ such that:

(i) $\Psi(B \cap \Omega) \subset \mathbb{R}^n_+$

(ii) $\Psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$

(iii) $\Psi \in C^{1,1}(\overline{B})$ and $\Psi^{-1} \in C^{1,1}(\overline{U})$

Equivalently, Ω is of class $C^{1,1}$ if each point of $\partial\Omega$ has a neighborhood in which $\partial\Omega$ is the graph of a $C^{1,1}$ function of $n-1$ of the coordinates.

Every such domain can be written as the positive set of a $C^{1,1}$ function. Namely, we have:

Lemma E.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{1,1}$. Then there is a function $\Phi \in C^{1,1}(\mathbb{R}^n)$ such that $\Omega = \{\Phi > 0\}$, $\Phi = 0$ on $\partial\Omega$, and $\nabla\Phi(x) \neq 0$ for all $x \in \partial\Omega$.*

Proof. By compactness, $\partial\Omega$ may be covered by finitely many balls $\{B_j = B_{\rho_j}(x_j)\}_{j=1}^N$, with $x_j \in \partial\Omega$ and $\rho_j > 0$, such that there are flattening maps $\Psi_j \in C^{1,1}(\overline{B_j})$ as in Definition E.1.

Let $\rho > 0$ be sufficiently small so that the set $B_0 := \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) > \rho\}$ satisfies $\overline{\Omega} \subset \cup_{j=0}^N B_j$, and consider a partition of unity $\{\eta_j\}_{j=0}^N$ subordinated to the covering $\{B_j\}_{j=0}^N$. The function

$$\Phi = \eta_0 + \sum_{j=1}^N \eta_j \Psi_j^n$$

now satisfies the desired properties. □

Remark E.3. Notice that, by construction, Φ is compactly supported and takes negative values in a bounded neighborhood of $\partial\Omega$ outside $\overline{\Omega}$.

Regularizing Φ and taking appropriate superlevel sets, we obtain the following approximation result:

Lemma E.4. *Let $\Omega = \{\Phi > 0\} \subset \mathbb{R}^n$ be a bounded domain of class $C^{1,1}$, with $\Phi \in C^{1,1}(\mathbb{R}^n)$ as in Lemma E.2 above. Then, there is an exhaustion of Ω by smooth sets $\Omega_k = \{\Phi_k > 0\}$,[1] where the functions $\Phi_k \in C^\infty(\mathbb{R}^n)$ satisfy*

$$\|\nabla\Phi_k\|_{C^1(\mathbb{R}^n)} + \||\nabla\Phi_k|^{-1}\|_{L^\infty(\partial\Omega_k)} \le C,$$

for some constant C depending only on Φ and Ω. Moreover, we have that $\partial\Omega_k \to \partial\Omega$ in the sense of the Hausdorff distance.[2]

Proof. Since $\Omega = \{\Phi > 0\}$ and $\nabla\Phi \neq 0$ on $\partial\Omega$, by continuity it follows that Φ is comparable to the distance function $d(x) = \operatorname{dist}(x, \partial\Omega)$ in $\overline{\Omega}$, say

$$L^{-1}d(x) \le \Phi(x) \le Ld(x) \quad \text{for } x \in \overline{\Omega}, \tag{E.1}$$

for some $L \ge 1$. From now on, for x outside Ω, we let $d(x) = -\operatorname{dist}(x, \partial\Omega)$.

Consider a mollifying sequence $(\eta^\varepsilon)_{\varepsilon>0}$ with $\operatorname{supp}\eta^\varepsilon \subset B_\varepsilon$. We will choose the functions

$$\Phi_k := \Phi * \eta^{\varepsilon_k} - 2L\varepsilon_k \tag{E.2}$$

for some appropriate sequence $\varepsilon_k \downarrow 0$. Since $\|\nabla\Phi_k\|_{C^1} = \|\nabla\Phi_k\|_{L^\infty} + \|D^2\Phi_k\|_{L^\infty}$, recalling that $\|D^2\Phi_k\|_{L^\infty(\mathbb{R}^n)} = [\nabla\Phi_k]_{C^{0,1}(\mathbb{R}^n)}$ (because Φ_k is C^∞) and $\Phi \in C^{1,1}$, the stated uniform bounds for $\|\nabla\Phi_k\|_{C^1}$ hold by the standard properties of convolutions.

Given $\delta > 0$, from (E.1) we see that

$$\Phi(x) > \delta/L \quad \text{for } x \in \{x \in \mathbb{R}^n : d(x) > \delta\} = \{d > \delta\},$$

and taking the convolution with η^ε, we deduce

$$\Phi * \eta^\varepsilon(x) > \delta/L \quad \text{for } x \in \{d > \delta + \varepsilon\}. \tag{E.3}$$

Similarly, for $\widetilde{\delta} > 0$ we have

$$\Phi(x) \le L\widetilde{\delta} \quad \text{for } x \in \{d \le \widetilde{\delta}\},$$

and regularizing we obtain

$$\Phi * \eta^\varepsilon(x) \le L\widetilde{\delta} \quad \text{for } x \in \{d \le \widetilde{\delta} - \varepsilon\}. \tag{E.4}$$

Letting $\widetilde{\delta} = \delta/L^2$, since $L\widetilde{\delta} = \delta/L$, by (E.3) and (E.4) we have

$$\{d > \delta + \varepsilon\} \subset \{\Phi * \eta^\varepsilon > \delta/L\} \subset \{d > \delta/L^2 - \varepsilon\},$$

and the choice $\delta = 2L^2\varepsilon$ now yields the inclusions

$$\{d > \varepsilon(2L^2 + 1)\} \subset \{\Phi * \eta^\varepsilon > 2L\varepsilon\} \subset \{d > \varepsilon\}. \tag{E.5}$$

Next, we construct the sequence ε_k in (E.2) inductively. Given $\varepsilon_1 > 0$ small, we define $\varepsilon_{k+1} := \varepsilon_k/\{2(2L^2+1)\}$. Hence, by (E.5), the sets $\Omega_k := \{\Phi_k > 0\} = \{\Phi * \eta^{\varepsilon_k} > 2L\varepsilon_k\}$ satisfy

$$\overline{\Omega_k} \subset \{d \ge \varepsilon_k\} \subset \{d > \varepsilon_{k+1}(2L^2 + 1)\} \subset \Omega_{k+1} \subset \Omega. \tag{E.6}$$

[1] By an exhaustion we mean that $\overline{\Omega_k} \subset \Omega_{k+1} \subset \Omega$ and $\Omega = \cup_k \Omega_k$.

[2] By this we mean that $\max\{\sup_{x\in\partial\Omega}\operatorname{dist}(x, \partial\Omega_k), \sup_{x\in\partial\Omega_k}\operatorname{dist}(x, \partial\Omega)\} \to 0$ as $k \to \infty$.

They clearly exhaust Ω, since $\varepsilon_k \downarrow 0$ and thus $\Omega = \cup_k \{d \geq \varepsilon_{k-1}\} \subset \cup_k \Omega_k$. Furthermore, the inclusions (E.6) show that $\partial\Omega_k$ is at a Hausdorff distance of at most $\varepsilon_k(2L^2+1)$ from $\partial\Omega$, and hence $\partial\Omega_k \to \partial\Omega$ with respect to this distance.

It remains to prove the lower bound for $|\nabla\Phi_k|$ on $\partial\Omega_k$, which will also show that the boundary $\partial\Omega_k$ is smooth. Let $x \in \mathbb{R}^n$ with $d(x) = |x - x_0|$ for some $x_0 \in \partial\Omega$. Since

$$\nabla\Phi(x) \cdot \tfrac{\nabla\Phi(x_0)}{|\nabla\Phi(x_0)|} \geq |\nabla\Phi(x_0)| - [\nabla\Phi]_{C^{0,1}(\mathbb{R}^n)} d(x),$$

we have the lower bound

$$\nabla\Phi(x) \cdot \tfrac{\nabla\Phi(x_0)}{|\nabla\Phi(x_0)|} \geq \tfrac{1}{2} \| |\nabla\Phi|^{-1} \|_{L^\infty(\partial\Omega)}^{-1} \qquad \text{for } x \in \{-\rho < d < \rho\},$$

where $\rho > 0$ depends only on $\| |\nabla\Phi|^{-1} \|_{L^\infty(\partial\Omega)}$ and $[\nabla\Phi]_{C^{0,1}(\mathbb{R}^n)}$. Taking the convolution with η^{ε_k} in this last inequality, we obtain

$$\nabla\Phi_k(x) \cdot \tfrac{\nabla\Phi(x_0)}{|\nabla\Phi(x_0)|} \geq \tfrac{1}{2} \| |\nabla\Phi|^{-1} \|_{L^\infty(\partial\Omega)}^{-1} \qquad \text{for } x \in \{-(\rho - \varepsilon_k) < d < \rho - \varepsilon_k\}. \tag{E.7}$$

By (E.6), the boundary $\partial\Omega_k$ is at a distance of at most $\varepsilon_k(2L^2+1)$ from $\partial\Omega$. Hence, choosing $\varepsilon_1 > 0$ sufficiently small such that $\rho - \varepsilon_1 > \varepsilon_1(2L^2+1)$, from (E.7) and the definition of $\varepsilon_k = \varepsilon_1 / \{2(2L^2+1)\}^{k-1}$ we deduce

$$|\nabla\Phi_k(x)| \geq \nabla\Phi_k(x) \cdot \tfrac{\nabla\Phi(x_0)}{|\nabla\Phi(x_0)|} \geq \tfrac{1}{2} \| |\nabla\Phi|^{-1} \|_{L^\infty(\partial\Omega)}^{-1} \qquad \text{on } \partial\Omega_k,$$

and therefore $\| |\nabla\Phi_k|^{-1} \|_{L^\infty(\partial\Omega_k)} \leq 2 \| |\nabla\Phi|^{-1} \|_{L^\infty(\partial\Omega)}$, which concludes the proof. $\square$

Appendix F

On the uniqueness of stable solutions

Here, we prove the uniqueness of stable solutions to nonvariational equations involving convex nonlinearities. For this, we employ some fundamental results of Berestycki, Nirenberg, and Varadhan [9] on the principal eigenfunction. Compare the following statement with Proposition 1.3.1 in Dupaigne's book [48]:

Proposition F.1. *Given $\Omega \subset \mathbb{R}^n$ a bounded domain, let u_1, $u_2 \in C^0(\overline{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ be two stable solutions of the equation $-Lu = f(u)$ in Ω, with $u = 0$ on $\partial\Omega$.*

Assume that $f \in C^1(\mathbb{R})$ is convex.

Then either $u_1 = u_2$ or $f(u) = \mu_1[L, \Omega]u$ on the ranges of u_1 and u_2.

Remark F.2. Here $\mu_1[L, \Omega]$ denotes the principal (or smallest) eigenvalue of L in Ω, with the sign convention $-L\varphi = \mu_1\varphi$. It is characterized by (see [9])

$$\mu_1[L, \Omega] = \sup\left\{\mu\colon \text{ there is a function } \varphi > 0 \in W^{2,n}_{\mathrm{loc}}(\Omega) \text{ satisfying } L\varphi + \mu\varphi \leq 0 \text{ in } \Omega\right\}.$$

Moreover, L can be any uniformly elliptic second order operator. In particular, we allow zero order terms.

Remark F.3. In the proof of Theorem 3.1.1 above, we only need a weaker version of Proposition F.1. Namely, we could assume additionally that $u_1 \leq u_2$, which admits a shorter proof. However, the present statement might be more useful in applications.

Proof. Assume $u_1 \neq u_2$ and consider the difference $w := u_2 - u_1$. Let

$$\Omega^+ := \{x \in \Omega\colon u_2(x) > u_1(x)\} = \{w > 0\},$$

and assume $\Omega^+ \neq 0$ (otherwise we exchange the roles of u_1 and u_2). By convexity

$$-Lw = f(u_2) - f(u_1) \leq f'(u_2)w,$$

whence

$$J_{u_2}w = Lw + f'(u_2)w \geq 0 \quad \text{ in } \Omega. \tag{F.1}$$

By the monotonicity of the principal eigenvalue with respect to the domain, since u_2 is stable, we have

$$\mu_1[J_{u_2}, \Omega^+] \geq \mu_1[J_{u_2}, \Omega] \geq 0. \tag{F.2}$$

Since $w > 0$ in Ω^+, by (F.1) and (F.2) it follows that

$$\begin{cases} J_{u_2}w + \mu_1[J_{u_2}, \Omega^+]w \geq 0 & \text{in } \Omega^+ \\ w = 0 & \text{on } \partial\Omega^+. \end{cases}$$

Applying [9, Corollary 2.2], we deduce that w is a positive principal eigenfunction of J_{u_2} in Ω^+, that is,

$$J_{u_2}w + \mu_1[J_{u_2}, \Omega^+]w = 0 \quad \text{in } \Omega^+. \tag{F.3}$$

Using the equation $-Lw = f(u_2) - f(u_1)$ in Ω^+, from (F.3) we see that

$$f(u_2) - f(u_1) - f'(u_2)(u_2 - u_1) = \mu_1[J_{u_2}, \Omega^+](u_2 - u_1) \geq 0 \quad \text{in } \Omega^+. \tag{F.4}$$

By convexity we also have $f(u_2) - f(u_1) - f'(u_2)(u_2 - u_1) \leq 0$ and hence, by (F.4), it follows that $\mu_1[J_{u_2}, \Omega^+] = 0$ and f is affine in the union of the intervals $[u_1(x), u_2(x)]$ when $x \in \Omega^+$.

For instance, if f is of the form $f(u) = au + b$ in the ranges above, since $-Lw = aw$ in Ω^+ with $w > 0$, we must have $a = \mu_1[L, \Omega^+]$. Moreover, to have nontrivial solutions of $-Lu = \mu_1[L, \Omega^+]u + b$, the Fredholm alternative forces $b = 0$. Therefore, we see that $f(u) = \mu_1[L, \Omega^+]u$ in the ranges of u_1 and u_2 in Ω^+.

If $\Omega^- := \{w < 0\} \neq \emptyset$, then, arguing as above with $-w$ in place of w, we deduce

$$(J_{u_1} + \mu_1[J_{u_1}, \Omega^-])w = 0 \quad \text{in } \Omega^-,$$

with $\mu_1[J_{u_1}, \Omega^-] = 0$ and $f(u) = \mu_1[L, \Omega^-]u$ in the ranges of u_2 and u_1 in Ω^-.

The regularity of f and the continuity of the solutions forces $\mu_1[L, \Omega^+] = \mu_1[L, \Omega^-]$. However, by the stability of u_2

$$0 \leq \mu_1[J_{u_2}, \Omega] = \mu_1\big[L + \mu_1[L, \Omega^+], \Omega\big] = \mu_1[L, \Omega] - \mu_1[L, \Omega^+],$$

but

$$\mu_1[L, \Omega] - \mu_1[L, \Omega^+] < 0$$

by the strict monotonicity of μ_1, since $\Omega^+ \subsetneq \Omega$ with $|\Omega \setminus \Omega^+| = |\Omega^-| > 0$ by assumption.[1] This contradiction forces either Ω^+ or Ω^- to be empty, hence $f(u) = \mu_1[L, \Omega]$ in the ranges of u_1 and u_2, as we claimed. $\square$

[1]To show that $\mu_1[L, \Omega] < \mu_1[L, \Omega^+]$, first notice that we already have $\mu_1[L, \Omega] \leq \mu_1[L, \Omega^+]$ by definition of μ_1. Suppose $\mu_1[L, \Omega] = \mu_1[L, \Omega^+]$ and consider φ_1 a positive principal eigenfunction of L in Ω. Hence $L\varphi_1 + \mu_1[L, \Omega^+]\varphi_1 \leq 0$ in Ω^+ and $\varphi_1 > 0$ in Ω^+. By Corollary 2.1 in [9] we must have $\varphi_1 = 0$ on $\partial\Omega^+ \cap \Omega \neq \emptyset$, contradicting the positivity in Ω.

Part II

A nonlocal Weierstrass extremal field theory

Chapter 4

A Weierstrass extremal field theory for the fractional Laplacian

In this chapter, we extend part of the Weierstrass extremal field theory in the Calculus of Variations to a nonlocal framework. Our model case is the energy functional for the fractional Laplacian (the Gagliardo-Sobolev seminorm), for which such a theory was still unknown until our work [23].

We build a null-Lagrangian and a calibration for nonlinear equations involving the fractional Laplacian in the presence of a field of extremals. Thus, our construction assumes the existence of a family of solutions to the Euler-Lagrange equation whose graphs produce a foliation. Then, the minimality of each leaf in the foliation follows from the existence of the calibration. As an application, we show that monotone solutions to fractional semilinear equations are minimizers.

4.1 Introduction

The Weierstrass extremal field theory, a classical tool from the Calculus of Variations, provides a sufficient condition for the minimality of critical points. Namely, if an extremal of an elliptic functional can be embedded in a family of critical points whose graphs produce a foliation (in particular, the graphs do not intersect each other), then the given extremal is a minimizer. The proof of this result is based on the construction of a *calibration*, that is, an auxiliary functional satisfying certain properties (see Definition 4.1.1). This theory has found important applications in the context of minimal surfaces, among others.

The purpose of this chapter is to extend the classical Weierstrass field theory to the setting of functionals associated to nonlocal equations, starting here with the simplest one. Our main result is the construction of a calibration for the fractional functional

$$\mathcal{E}_{s,F}(w) := \frac{c_{n,s}}{4} \iint_{Q(\Omega)} \frac{|w(x) - w(y)|^2}{|x - y|^{n+2s}} \,\mathrm{d}x \,\mathrm{d}y - \int_\Omega F(w(x)) \,\mathrm{d}x,$$

where $s \in (0, 1)$, $c_{n,s}$ is a positive normalizing constant, $F \in C^1(\mathbb{R})$,

$$Q(\Omega) := (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c) = (\Omega \times \Omega) \cup (\Omega \times \Omega^c) \cup (\Omega^c \times \Omega), \tag{4.1.1}$$

and $\Omega \subset \mathbb{R}^n$ is a given bounded domain. Here and throughout the paper, $\Omega^c = \mathbb{R}^n \setminus \Omega$.

The Euler-Lagrange equation for the functional $\mathcal{E}_{s,F}$ is the semilinear equation

$$(-\Delta)^s u = F'(u) \quad \text{in } \Omega,$$

where

$$(-\Delta)^s u\,(x) = c_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y$$

is the fractional Laplacian and P.V. stands for the principal value.

Our construction does not use the Caffarelli-Silvestre extension problem for the fractional Laplacian. This is relevant, since in the next chapter, it allows us to treat more general nonlocal functionals of the form[1]

$$\mathcal{E}_{\mathrm{N}}(w) := \frac{1}{2}\iint_{Q(\Omega)} G_{\mathrm{N}}(x,y,w(x),w(y))\,\mathrm{d}x\,\mathrm{d}y,$$

where the Lagrangian $G_{\mathrm{N}}(x,y,a,b)$ is required to satisfy the natural ellipticity condition

$$\partial^2_{ab} G_{\mathrm{N}}(x,y,a,b) + \partial^2_{ab} G_{\mathrm{N}}(y,x,b,a) \le 0. \tag{4.1.2}$$

As in the classical local theory, our calibration is built in the presence of a field of extremals, namely, a one-parameter family of critical points of $\mathcal{E}_{s,F}$ (or of $\mathcal{E}_{\mathrm{N}}$) whose graphs form a foliation (see Definition 4.1.2). In particular, the graphs do not intersect each other. For the construction, it suffices to have subsolutions, respectively supersolutions, on each respective side of a given extremal —something very useful for some applications.

As a first application of our calibration, we establish that monotone solutions to translation invariant nonlocal equations are minimizers. This is related to a celebrated conjecture of De Giorgi for the Allen-Cahn equation. More precisely, if u is a solution satisfying $\partial_{x_n} u > 0$ in $\mathbb{R}^n$, then it is a minimizer[2] among functions w such that

$$\lim_{\tau\to-\infty} u(x',\tau) \le w(x',x_n) \le \lim_{\tau\to+\infty} u(x',\tau)$$

for $x = (x', x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}$. This result was only known for those nonlocal functionals for which an existence and regularity theory of minimizers is available. We elaborate on this further in Subsection 4.1.4.

As a second application, in Chapter 5 below, we establish that minimizers of nonlocal elliptic functionals are viscosity solutions. Although this was previously known for problems where a weak comparison principle is available (see [5, 66, 87]), we can prove it in more general scenarios by using the calibration technique. This has been motivated by the theory of nonlocal minimal surfaces, where the calibration argument of Cabré [18] greatly simplified the original proof (that minimizers are viscosity solutions) from [32].

4.1.1 The notion of calibration

A fundamental problem in the Calculus of Variations consists of finding conditions for a function to be a minimizer of a given functional. More precisely, given a functional $\mathcal{E}\colon \mathcal{A}\to\mathbb{R}$ defined on some set of admissible functions $\mathcal{A}$, and given $u\in\mathcal{A}$, one wishes to know whether u minimizes $\mathcal{E}$ among competitors in $\mathcal{A}$ having the same Dirichlet condition as u.

[1] The subindices L and N will be used throughout the work to denote local and nonlocal objects, respectively.

[2] Monotone solutions are easily seen to be strictly stable solutions and, as a result, to be minimizers with respect to small compactly supported perturbations. Our result gives a more precise neighborhood in which the solution is minimizing.

In classical local problems, the Dirichlet condition refers to the value of u on the boundary of the domain Ω, while in nonlocal problems one prescribes the value in all the exterior of Ω, namely, in $\Omega^c = \mathbb{R}^n \setminus \Omega$.

One effective strategy to establish the minimality of a given function $u \in \mathcal{A}$ consists of constructing a calibration:

Definition 4.1.1. A functional $\mathcal{C} \colon \mathcal{A} \to \mathbb{R}$ is a *calibration* for the functional $\mathcal{E}$ and the admissible function $u \in \mathcal{A}$ if the following conditions hold:

($\mathcal{C}1$) $\mathcal{C}(u) = \mathcal{E}(u)$.

($\mathcal{C}2$) $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all $w \in \mathcal{A}$ with the same Dirichlet condition as u.

($\mathcal{C}3$) $\mathcal{C}(w) = \mathcal{C}(\widetilde{w})$ for all $w, \widetilde{w} \in \mathcal{A}$ with the same Dirichlet condition as u.

Functionals satisfying ($\mathcal{C}3$) are known as *null-Lagrangians* (see, for instance, [54, Chapter 8] and [61, Section 1.4]). It is, however, convenient to relax this last condition to the less stringent

($\mathcal{C}3'$) $\mathcal{C}(u) \leq \mathcal{C}(w)$ for all $w \in \mathcal{A}$ with the same Dirichlet condition as u.

In this work we still refer to functionals satisfying ($\mathcal{C}1$), ($\mathcal{C}2$), and ($\mathcal{C}3'$) as calibrations.[3]

Recall the meaning of the Dirichlet condition for local and nonlocal problems given right before Definition 4.1.1.

Once a calibration is available, the minimality of u follows immediately both in the local and nonlocal cases. Indeed, if $\mathcal{C}$ is a calibration for $\mathcal{E}$ and $u \in \mathcal{A}$, then, for every $w \in \mathcal{A}$ with the same Dirichlet condition as u, applying ($\mathcal{C}1$), ($\mathcal{C}3'$), and ($\mathcal{C}2$) (in this order) we obtain

$$\mathcal{E}(u) = \mathcal{C}(u) \leq \mathcal{C}(w) \leq \mathcal{E}(w).$$

Therefore, u is a minimizer.

4.1.2 The classical theory of calibrations

Calibrations arose in the development of the classical theory of the Calculus of Variations. Historically, a fundamental question was to determine necessary and sufficient conditions for a function to be a minimizer. A satisfactory answer has been obtained for functionals —that we often call "energies", following PDE terminology— of the form

$$\mathcal{E}_{\mathrm{L}}(w) := \int_\Omega G_{\mathrm{L}}\big(x, w(x), \nabla w(x)\big) \,\mathrm{d}x. \tag{4.1.3}$$

In this framework, the function $G_{\mathrm{L}}(x, \lambda, q)$ is called the Lagrangian of $\mathcal{E}_{\mathrm{L}}$.

A necessary condition for minimality is the vanishing of the first variation of $\mathcal{E}_{\mathrm{L}}$. That is, every minimizer is a critical point of $\mathcal{E}_{\mathrm{L}}$ (an *extremal*) and must satisfy the associated Euler-Lagrange equation. If the Lagrangian $G_{\mathrm{L}}(x, \lambda, q)$ is convex in the variables (λ, q), then the functional $\mathcal{E}_{\mathrm{L}}$ is convex and, in this case, every extremal is a minimizer. Although many models from Physics exhibit such a convexity property, it is well known that very relevant nonconvex energies appear in applications. This is the case of the Allen-Cahn energy, among many others. For such energy functionals, the Dirichlet problem may admit

[3]In the literature, functionals satisfying ($\mathcal{C}1$), ($\mathcal{C}2$), and ($\mathcal{C}3'$) are sometimes called *subcalibrations*.

several extremals, not all of them being minimizers. Still, if G_{L} is not convex in (λ, q), one often has that the Lagrangian $G_{\mathrm{L}}(x, \lambda, q)$ is convex with respect to the gradient variable q, which amounts to the *ellipticity* of the problem.

For nonconvex elliptic problems, one is interested in having sufficient conditions for an extremal to be a minimizer. Two of such conditions, due to Jacobi and Weierstrass, are well known. First, if a solution is strictly stable,[4] then it is a *local minimizer* in a certain topology, that is, a minimizer in a small neighborhood. This is Jacobi's condition. The real difficulty is proving minimality in a larger, more interesting class of competitors (or perhaps even *absolute minimality*). In this direction, the Weierstrass sufficient condition yields minimality among functions taking values in a precise region. To further elaborate on this, we need to introduce the notion of *field*. Essentially, this is a collection of ordered functions $u^t : \overline{\Omega} \to \mathbb{R}$, with t in some interval $I \subset \mathbb{R}$, enjoying some regularity for the joint function $(x, t) \mapsto u^t(x)$. The key point is that the graphs of these functions produce a foliation of a certain region $\mathcal{G}$ in $\mathbb{R}^n \times \mathbb{R}$, which allows to carry out a subtle convexity argument to bound the nonconvex functional by below with a calibration.

While fields are a classical concept in local problems, we can extend their definition to include both the local and nonlocal settings, as follows.

Definition 4.1.2. Given a domain $D \subset \mathbb{R}^n$ (not necessarily bounded) and an interval $I \subset \mathbb{R}$ (not necessarily bounded, nor open), we say that a family $\{u^t\}_{t \in I}$ of functions $u^t \colon \overline{D} \to \mathbb{R}$ is a *field in D* if

- the function $(x, t) \mapsto u^t(x)$ is continuous in $\overline{D} \times I$;
- for each $x \in \overline{D}$, the function $t \mapsto u^t(x)$ is C^1 and increasing in I.

We say that $\{u^t\}_{t \in I}$ is a *C^2 field in D* if, additionally, the function $(x, t) \mapsto u^t(x)$ is C^2 in $\overline{D} \times I$.

Given a functional $\mathcal{E}$ acting on functions defined in $\overline{D}$, and given a subdomain $\Omega \subset D$, we say that $\{u^t\}_{t \in I}$ is a *field of extremals*[5] in Ω (roughly speaking, since we should refer to $\mathcal{E}$, D, and Ω) when it is a field in D and each of the functions u^t is a critical point of $\mathcal{E}$ in Ω.

In the local setting, we will take $D = \Omega$. For nonlocal Lagrangians we will set $D = \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$.

Given a field in D as above, the region

$$\mathcal{G} = \{(x, \lambda) \in \overline{D} \times \mathbb{R} \colon \lambda = u^t(x) \text{ for some } t \in I\} \subset \mathbb{R}^n \times \mathbb{R}$$

is foliated by the graphs of the functions u^t, which do not intersect each other (since $u^t(x)$ is increasing in t). In particular, we can uniquely define a *leaf-parameter function*

$$t \colon \mathcal{G} \to I, \quad (x, \lambda) \mapsto t(x, \lambda) \quad \text{determined by} \quad u^{t(x,\lambda)}(x) = \lambda. \tag{4.1.4}$$

The function t is continuous in $\mathcal{G}$ by the assumptions in Definition 4.1.2. We will often refer to the functions u^t (or their graphs) as the "*leaves*" of the field.

[4] A solution u is said to be *strictly stable* if the principal eigenvalue of the linearized equation at u is positive.

[5] The term *extremal field* is also often used in the literature, but we find it ambiguous.

Having defined what a field is, we can now state the classical theorem of Weierstrass, which was first proven for scalar ODEs:

Weierstrass sufficient condition. *For the functional $\mathcal{E}_{\mathrm{L}}$ in* (4.1.3)*, assuming ellipticity (i.e., that the Lagrangian $G_{\mathrm{L}}(x,\lambda,q)$ is convex in the gradient variable q), if a critical point is embedded in a field of extremals,*[6] *then the critical point is a minimizer among functions taking values in the foliated region $\mathcal{G}$ and having the same boundary values as the given critical point.* (4.1.5)

The proof of (4.1.5) is based on the construction of a calibration. For this, given an interval $I \subset \mathbb{R}$, a bounded domain $\Omega \subset \mathbb{R}^n$, and a C^2 field of extremals $\{u^t\}_{t\in I}$ in Ω, one considers the set of admissible functions $\mathcal{A}_{\mathrm{L}} = \{w \in C^1(\overline{\Omega}) : \operatorname{graph} w \subset \mathcal{G}\}$. Then, the functional $\mathcal{C}_{\mathrm{L}} \colon \mathcal{A}_{\mathrm{L}} \to \mathbb{R}$, defined through the Legendre transform of the Lagrangian G_{L} as

$$\begin{aligned}\mathcal{C}_{\mathrm{L}}(w) := \int_\Omega \Big\{ \partial_q G_{\mathrm{L}}(x, u^t(x), \nabla u^t(x)) \cdot \big(\nabla w(x) - \nabla u^t(x)\big) \Big\}\Big|_{t=t(x,w(x))} \, \mathrm{d}x \\ + \int_\Omega G_{\mathrm{L}}(x, u^t(x), \nabla u^t(x))\big|_{t=t(x,w(x))} \, \mathrm{d}x,\end{aligned} \tag{4.1.6}$$

is a calibration for $\mathcal{E}_{\mathrm{L}}$ and each critical point $u^{t_0}, t_0 \in I$. While condition ($\mathcal{C}1$) in Definition 4.1.1 follows directly from (4.1.6), and condition ($\mathcal{C}2$) amounts to ellipticity (i.e., the convexity of the Lagrangian in the gradient variable $q = \nabla w(x)$), it is a remarkable fact that the null-Lagrangian property ($\mathcal{C}3$) holds. Its proof will be recalled in Section 4.3.

As an illustrative example, in the presence of a field of extremals, the functional[7]

$$\mathcal{E}_{1,F}(w) = \frac{1}{2} \int_\Omega |\nabla w(x)|^2 \, \mathrm{d}x - \int_\Omega F(w(x)) \, \mathrm{d}x$$

(which is typically nonconvex), admits a calibration. It is given by

$$\mathcal{C}_{1,F}(w) = \int_\Omega \Big\{ \nabla u^t(x) \cdot (\nabla w(x) - \nabla u^t(x)) + \frac{1}{2} |\nabla u^t(x)|^2 \Big\}\Big|_{t=t(x,w(x))} \mathrm{d}x - \int_\Omega F(w(x)) \, \mathrm{d}x. \tag{4.1.7}$$

Notice that, although the functional $\mathcal{E}_{1,F}$ is not convex, its Lagrangian is elliptic.

The Weierstrass sufficient condition (4.1.5) naturally leads to the question of when it is possible to embed a solution of the Euler-Lagrange equation into a field of extremals in a large portion of space. An important case corresponds to functionals which are invariant with respect to some translations. A first example are those Lagrangians which do not depend on a direction of space and, at the same time, the extremal is monotone in that same direction. Such a solution can be translated along the invariant direction to produce a field of extremals. This applies to layer solutions of the Allen-Cahn equation; see Subsection 4.1.4 below. A second example are those Lagrangians $G_L(x,\lambda,q)$ which do not depend on the function variable λ. In this case, a field can be obtained by translating the solution in the vertical direction. This can be used, for instance, to show that minimal graphs are minimizing minimal surfaces; see Section 4.2.

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[6] In fact, it suffices that the leaves of the field above and below the graph of the given critical point are, respectively, super and subsolutions to the Euler-Lagrange equation. This, which is well known, will be easily seen within the proofs of our main results.

[7] The subindex 1 in the definition of the energy functional refers to the fractional parameter s in $\mathcal{E}_{s,F}$, which are the nonlocal analogues of $\mathcal{E}_{1,F}$ treated later. As s tends to 1, one recovers $\mathcal{E}_{1,F}$ from $\mathcal{E}_{s,F}$ after a suitable normalization; see, for instance, [47].

Fields of extremals can also be built in the presence of a concrete explicit solution. Here, using the precise solution and PDE at hand, one may be able to construct a field in a more or less explicit way. This approach has been applied in the theory of minimal surfaces to establish the minimality of Simons and Lawson cones, as explained also in Section 4.2.

4.1.3 Nonlocal calibrations

While the theory of calibrations for local equations is well understood, there are only two papers, to the best of our knowledge, dealing with nonlocal ones. In [18] the first author found an explicit calibration for the fractional perimeter, as explained in Section 4.2. Pagliari [80] investigated the abstract structure of calibrations for the fractional total variation.[8]

We now present our main result, which builds a calibration for the functional

$$\mathcal{E}_{s,F}(w) = \frac{c_{n,s}}{4} \iint_{Q(\Omega)} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x \tag{4.1.8}$$

in the presence of a field of extremals. Recall (4.1.1) for the meaning of $Q(\Omega)$, Definition 4.1.2 for the notion of field, and (4.1.4) for the leaf-parameter function t. The calibration properties ($\mathcal{C}1$), ($\mathcal{C}2$), ($\mathcal{C}3$), and ($\mathcal{C}3'$) have been introduced in Definition 4.1.1. As we will explain in Section 4.4, the regularity assumptions on the field can be significantly weakened.

Theorem 4.1.3. *Let $I \subset \mathbb{R}$ be an interval, $\Omega \subset \mathbb{R}^n$ a bounded domain, and $s \in (0,1)$. Let $\{u^t\}_{t\in I}$ be a C^2 field in $\mathbb{R}^n$ in the sense of Definition 4.1.2 satisfying*

$$|u^t(x)| + |\partial_t u^t(x)| \le C \quad \textit{for all } x \in \mathbb{R}^n \textit{ and } t \in I,$$

for some constant C. Consider the admissible functions

$$\mathcal{A}_s = \{w \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \,:\, \operatorname{graph} w \subset \mathcal{G}\},$$

where

$$\mathcal{G} = \{(x,\lambda) \in \mathbb{R}^n \times \mathbb{R} \colon \lambda = u^t(x) \textit{ for some } t \in I\}.$$

Given $t_0 \in I$ and $F \in C^1(\mathbb{R})$, let $\mathcal{C}_{s,F}$ be the functional

$$\begin{aligned}\mathcal{C}_{s,F}(w) := c_{n,s}\,\mathrm{P.V.}\iint_{Q(\Omega)} \int_{u^{t_0}(x)}^{w(x)} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x \\ + \frac{c_{n,s}}{4}\iint_{Q(\Omega)} \frac{|u^{t_0}(x) - u^{t_0}(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{4.1.9}$$

defined for $w \in \mathcal{A}_s$, where $c_{n,s}$ is the positive constant in (4.1.8).

Taking $\mathcal{C} = \mathcal{C}_{s,F}$ and $\mathcal{E} = \mathcal{E}_{s,F}$ as in (4.1.8)*, we have the following:*

(a) *$\mathcal{C}_{s,F}$ satisfies* ($\mathcal{C}1$) *and* ($\mathcal{C}2$) *with $u = u^{t_0}$.*

[8] This functional involves the fractional perimeter of each sublevel set of a given function. The author succeeded in constructing a calibration to prove that halfspaces are minimizers, but other fields of extremals are not mentioned in this work.

(b) *Assume in addition that the family* $\{u^t\}_{t\in I}$ *satisfies*

$$\begin{aligned} (-\Delta)^s u^t - F'(u^t) &\geq 0 \quad \text{in } \Omega \quad \text{for } t \geq t_0, \\ (-\Delta)^s u^t - F'(u^t) &\leq 0 \quad \text{in } \Omega \quad \text{for } t \leq t_0. \end{aligned}$$

Then, $\mathcal{C}_{s,F}$ *satisfies* $(\mathcal{C}3')$ *with* $u = u^{t_0}$. *In particular,* u^{t_0} *minimizes* $\mathcal{E}_{s,F}$ *among functions* w *in* $\mathcal{A}_s$ *such that* $w \equiv u^{t_0}$ *in* Ω^c.

(c) *Assume in addition that* $\{u^t\}_{t\in I}$ *is a field of extremals in* Ω, *that is, a field in* $\mathbb{R}^n$ *satisfying*

$$(-\Delta)^s u^t - F'(u^t) = 0 \quad \text{in } \Omega \quad \text{for all } t \in I.$$

Then, the functional $\mathcal{C}_{s,F}$ *satisfies* $(\mathcal{C}3)$ *with* $u = u^{t_0}$. *Therefore,* $\mathcal{C}_{s,F}$ *is a calibration for* $\mathcal{E}_{s,F}$ *and* u^{t_0}. *As a consequence, for every* $t \in I$, *the extremal* u^t *minimizes* $\mathcal{E}_{s,F}$ *among functions* w *in* $\mathcal{A}_s$ *such that* $w \equiv u^t$ *in* Ω^c.

The meaning of the principal value P.V. in the definition of the functional $\mathcal{C}_{s,F}$ will be made precise in Section 4.4; see Remark 4.4.3.

Even when the nonlocal energy functional is as simple as $\mathcal{E}_{s,F}$ (the energy functional associated to the fractional Laplacian) the form of a calibration, if any could exist, was not known prior to our work [23].

Our first attempts at constructing a calibration for $\mathcal{E}_{s,F}$ consisted on trying to "nonlocalize" the expression (4.1.7) for the local calibration, mainly by substituting gradients by fractional ones or double integrals of differences. This strategy seems to lead to functionals that are not calibrations. We comment on these attempts with more detail in Appendix H.

A second failed approach consisted of trying to find a satisfactory calibration using the extension problem for the fractional Laplacian. Indeed, applying the local theory in the extended space gives a calibration in terms of a certain field of extremals "upstairs", but it was not clear at all how to write it in terms of the given field "downstairs" (the reason being that the functional is too involved). Thus, the extension has not been useful to us; see Appendix G.

We were puzzled for a long time until we revisited the work of the first author [18], which found a calibration for the fractional perimeter. It was written in terms of the Euler-Lagrange and Neumann operators associated to the fractional perimeter. We then realized that such a structure was also present, but hidden, in the classical local calibration $\mathcal{C}_{\mathrm{L}}$ in (4.1.6). More precisely, for every $t_0 \in I$, in Theorem 4.3.1 we will see that

$$\begin{aligned} \mathcal{C}_{\mathrm{L}}(w) = \int_\Omega \int_{u^{t_0}(x)}^{w(x)} \mathcal{L}_{\mathrm{L}}(u^t)(x)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x \\ + \int_{\partial\Omega} \int_{u^{t_0}(x)}^{w(x)} \mathcal{N}_{\mathrm{L}}(u^t)(x)\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{H}^{n-1}(x) + \mathcal{E}_{\mathrm{L}}(u^{t_0}), \end{aligned} \tag{4.1.10}$$

where $\mathcal{L}_{\mathrm{L}}$ and $\mathcal{N}_{\mathrm{L}}$ are, respectively, the Euler-Lagrange and Neumann operators associated to the functional $\mathcal{E}_{\mathrm{L}}$ in (4.1.3). To the best of our knowledge, this is the first time that the local calibration has been written in this way. From this expression, the null-Lagrangian property follows easily.[9] Instead, traditionally one exhibited the null-Lagrangian property

[9]Indeed, if $\{u^t\}_{t\in I}$ is a field of extremals, then $\mathcal{L}_{\mathrm{L}}(u^t) \equiv 0$ and the calibration depends only on the value of w on $\partial\Omega$.

of $\mathcal{C}_{\mathrm{L}}$ by either expressing the functional as the flux of a divergence-free vector field in $\Omega\times\mathbb{R}$ or by certain straightforward although opaque analytic computations; see Section 4.3. Neither of these approaches reveals the exact role played by the Euler-Lagrange and Neumann operators in the calibration.

Once we found (4.1.10) for the local case, a simple extension of this expression easily led us to the nonlocal calibration of Theorem 4.1.3. The key point is that each of the terms in (4.1.10) has a clear nonlocal counterpart. In fact, the same procedure works for general nonlocal functionals $\mathcal{E}_{\mathrm{N}}$ of the form

$$\mathcal{E}_{\mathrm{N}}(w) = \frac{1}{2}\iint_{Q(\Omega)} G_{\mathrm{N}}(x, y, w(x), w(y))\,\mathrm{d}x\,\mathrm{d}y$$

with Lagrangian $G_{\mathrm{N}}(x, y, a, b)$ satisfying the natural ellipticity condition (4.1.2).

4.1.4 An application to monotone solutions

Our main motivation to find a calibration came from the study of *monotone solutions* to the fractional Allen-Cahn equation

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^n$$

(see [30, 31], for instance, and [42] for more general integro-differential operators). Note that when the operator is the classical Laplacian, these solutions are related to a famous conjecture of De Giorgi; see [26] for instance.

The following is an application of our main theorem to monotone solutions of translation invariant equations.

Corollary 4.1.4. *Given $s \in (0,1)$ and $F \in C^3(\mathbb{R})$, let $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded solution of*

$$(-\Delta)^s u = F'(u) \quad \text{in } \mathbb{R}^n. \tag{4.1.11}$$

Assume that u is increasing in x_n.

Then, for each bounded domain $\Omega \subset \mathbb{R}^n$, u is a minimizer of $\mathcal{E}_{s,F}$ among continuous functions $w : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\lim_{\tau\to-\infty} u(x', \tau) \le w(x', x_n) \le \lim_{\tau\to+\infty} u(x', \tau) \quad \text{for all } (x', x_n) \in \mathbb{R}^{n-1}\times\mathbb{R} \tag{4.1.12}$$

and such that $w \equiv u$ in Ω^c.

Let us mention that we only assume $F \in C^3$ for simplicity, to ensure that u is of class C^2 independently of s. We could weaken the regularity assumptions, but this is not the purpose of the thesis.

To prove Corollary 4.1.4, we define the one-parameter family of functions $u^t(x) := u(x', x_n + t)$, where $x = (x', x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}$. Thanks to the monotonicity assumption, $u_{x_n} > 0$, the family $\{u^t\}_{t\in\mathbb{R}}$ is a field in $\mathbb{R}^n$ in the sense of Definition 4.1.2. Moreover, it is a field of extremals on account of the translation invariance of the equation (4.1.11). Thus, Theorem 4.1.3 gives that, on each bounded domain $\Omega \subset \mathbb{R}^n$, u is a minimizer of the energy $\mathcal{E}_{s,F}$ among the admissible functions in the statement.

Let us point out that this minimality result was already known and can be proven without using calibrations. Nevertheless, this alternative proof (described in the next paragraph) requires an existence and regularity theorem for minimizers. Such a result is not available for many other nonlocal equations. In the next chapter, we construct calibrations in a general nonlocal setting, and thus the proof above will allow us to show minimality of monotone solutions for the fist time.

Now, we briefly discuss the proof of Corollary 4.1.4 which does not use calibrations; for the details, see [26]. One considers a minimizer within the region given by (4.1.12) and with the same exterior datum as the monotone solution. Its existence and regularity can be proved in the case of equation (4.1.11). Now, the monotone solution can be translated, starting from infinity, until it touches the minimizer from one side, something that must happen in Ω, not in the exterior. The strong comparison principle then yields that the translated solution and the minimizer must coincide. Moreover, by the exterior condition they must be equal to the original solution. In particular, this proves that the monotone solution is a minimizer. Furthermore, this proof gives uniqueness of solution with the exterior data of u.

4.1.5 Outline of the chapter

For the proofs of our main theorems, the reader may skip sections 4.2 and 4.3.

In Section 4.2 we briefly comment on the classical perimeter functional and review the work of the first author on the calibration for the nonlocal perimeter [18]. Section 4.3 is devoted to recalling some known facts from the classical theory of calibrations and proving the new expression (4.1.10); see Theorem 4.3.1. In Section 4.4 we prove Theorem 4.1.3 under weaker assumptions on the field.

4.2 The classical and nonlocal perimeters

In this section we recall different notions for the perimeter of a set. First we introduce the classical perimeter functional and its calibration. We will mention several results concerning fields of extremals in this setting. Later we revisit the work of the first author [18] on the construction of a calibration for the nonlocal perimeter. Here we will focus on identifying the key feature that leads to the calibration properties in this nonlocal framework. This will suggest a candidate structure to search for in local functionals, which will lead to (4.1.10) and then allow us to treat the fractional Laplacian case.

As mentioned in the Introduction, some relevant applications of calibrations concern the theory of minimal surfaces. In broad terms, a minimal surface $\Sigma \subset \mathbb{R}^n$ is a critical point of the $(n-1)$-dimensional area functional. Given a domain $\Omega \subset \mathbb{R}^n$, the classical perimeter of a (regular) set $F \subset \mathbb{R}^n$ inside Ω is defined by

$$\mathcal{P}_{\mathrm{L}}(F) := \mathcal{H}^{n-1}(\Omega \cap \partial F), \tag{4.2.1}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Here we interpret the boundary of F as the surface $\Sigma = \partial F$. The critical points E of $\mathcal{P}_{\mathrm{L}}$ (known as *minimal sets* in the literature) satisfy $H_{\mathrm{L}}[\partial E] = 0$ in Ω, where $H_{\mathrm{L}}[\Sigma]$ denotes the mean curvature of Σ. The variation of $\mathcal{P}_{\mathrm{L}}$ is taken with respect to perturbations preserving the boundary datum $E \cap \partial\Omega$.

We are interested in showing that certain minimal sets E minimize $\mathcal{P}_{\mathrm{L}}$ among sets F with the same boundary condition $F \cap \partial\Omega = E \cap \partial\Omega$. Assume that for a family of minimal sets $\{E^t\}_{t\in\mathbb{R}}$, the surfaces ∂E^t form a foliation of Ω. For $x \in \Omega$, we let $t(x)$ be the unique $t \in \mathbb{R}$ such that $x \in \partial E^t$. Denote the outward unit normal vector to ∂F by $\nu_{\partial F}$. Then, the perimeter functional admits as a calibration

$$\mathcal{C}_{\mathcal{P}_{\rm L}}(F) := \int_{\Omega\cap\partial F} X \cdot \nu_{\partial F}\, \mathrm{d}\mathcal{H}^{n-1}, \tag{4.2.2}$$

where

$$X(x) := \nu_{\partial E^t}\big|_{t=t(x)}(x)$$

is the vector field given by the normal vectors to the surfaces ∂E^t. Notice that Definition 4.1.1 can be easily modified to involve subsets of $\mathbb{R}^n$ instead of functions. Then, properties ($\mathcal{C}1$) and ($\mathcal{C}2$) are easy to check directly, while the null-Lagrangian property ($\mathcal{C}3$) follows from the divergence theorem and the fact that $\operatorname{div} X = 0$. As a consequence, each E^t minimizes $\mathcal{P}_{\rm L}$ and, therefore, each ∂E^t is a minimizing minimal surface.

This discussion leads to the question of when it is possible to embed a minimal surface in a field of extremals. The simplest situation is when the minimal surface is a graph. If $u\colon \Omega' \subset \mathbb{R}^{n-1} \to \mathbb{R}$ is a minimal graph, then the graphs of the translations $u^t = u + t$ give a field of extremals in $\Omega = \Omega' \times \mathbb{R}$. By the calibration $\mathcal{C}_{\mathcal{P}_{\rm L}}$ in (4.2.2), every minimal graph is a minimizing minimal surface.[10] We point out that the functional $\mathcal{C}_{\mathcal{P}_{\rm L}}$ can also be obtained by integrating a closed differential form; see Chapter 1 in [37].

Another interesting situation is when the minimal surface is not a graph but has an explicit expression. Here a field of extremals can still be obtained in some cases. For instance, this is done for the Simons cone and for the more general Lawson cones in Bombieri, De Giorgi, and Giusti [10] and in Davini [45]. The strategy here consists of using the symmetries of the cone to reduce the minimal surface equation to an ODE in the plane. It is then shown that the solutions of this ODE do not intersect each other, and thus give a foliation. We remark that, although the cone is an explicit extremal, the field of extremals itself is not explicit. An alternative is to build fields made of sub and supersolutions, which are easier to obtain and suffice to show the minimality (see footnote 6). Explicit examples of such fields have been found for Lawson cones, simplifying the proof of the minimality; see De Philippis and Paolini [46] for the Simons cone and Liu [71] for Lawson cones. We mention that the case of minimal surfaces of codimension greater than 1 can also be treated, where the appropriate notion of calibration involves the use of differential forms; see [77].

The perimeter functional $\mathcal{P}_{\rm L}$ in (4.2.1) has a nonlocal analogue. Given a nonnegative symmetric kernel $K = K(z)$, with $z \in \mathbb{R}^n$, the K-nonlocal perimeter of a set $F \subset \mathbb{R}^n$ inside Ω is defined by

$$\mathcal{P}_{\rm N}(F) := \frac{1}{2}\iint_{Q(\Omega)} |\mathbb{1}_F(x) - \mathbb{1}_F(y)|\, K(x-y)\, \mathrm{d}x\, \mathrm{d}y,$$

where $Q(\Omega)$ was defined in (4.1.1). It is well known that the Euler-Lagrange operator associated to $\mathcal{P}_{\rm N}$ is the nonlocal mean curvature H_K, which is defined for F at boundary points $x \in \partial F$ by

$$H_K[F](x) := \int_{\mathbb{R}^n} \big(\mathbb{1}_{F^c}(y) - \mathbb{1}_F(y)\big) K(x-y)\, \mathrm{d}y,$$

[10]Notice that restricting the area functional to the class of graphs yields a convex functional. In particular, every minimal graph minimizes area in this smaller class, but it is not a priori clear if they are minimizers with respect to all surfaces. The calibration is used to prove this stronger fact.

as introduced in [32]. In particular, if a (sufficiently regular) set E minimizes $\mathcal{P}_{\mathrm{N}}$ with respect to sets F with the same exterior values $F \setminus \Omega = E \setminus \Omega$, then $H_K[E](x) = 0$ for $x \in \partial E \cap \Omega$.

In [18], the first author showed that, given a measurable function $\phi\colon \mathbb{R}^n \to \mathbb{R}$, the functional[11]

$$\mathcal{C}_{\mathcal{P}_{\mathrm{N}}}(F) := \frac{1}{2} \iint_{Q(\Omega)} \operatorname{sign}\big(\phi(x) - \phi(y)\big)\big(\mathbb{1}_F(x) - \mathbb{1}_F(y)\big)\, K(x-y)\, \mathrm{d}x\, \mathrm{d}y \tag{4.2.3}$$

is a calibration for the nonlocal perimeter $\mathcal{P}_{\mathrm{N}}$ and each superlevel set

$$E^t := \{x \in \mathbb{R}^n \,:\, \phi(x) > t\},$$

assuming that these sets have zero nonlocal mean curvature. As a consequence, each E^t is a minimizer of $\mathcal{P}_{\mathrm{N}}$ with respect to sets that coincide with E^t outside Ω.

Properties ($\mathcal{C}1$) and ($\mathcal{C}2$) are easy to check directly from expression (4.2.3). However, showing the null-Lagrangian property ($\mathcal{C}3$) requires an alternative expression for $\mathcal{C}_{\mathcal{P}_{\mathrm{N}}}$. For this, [18] wrote (4.2.3) in terms of the sets E^t as follows. Assume for simplicity that ϕ is smooth and $\nabla\phi(x) \neq 0$ for all x. Then the level sets are smooth surfaces

$$\partial E^t = \{x \in \mathbb{R}^n \colon \phi(x) = t\},$$

which have zero Lebesgue measure in $\mathbb{R}^n$, and it can be readily checked that

$$\operatorname{sign}\big(\phi(x) - \phi(y)\big) = \big(\mathbb{1}_{(E^t)^c}(y) - \mathbb{1}_{E^t}(y)\big)\big|_{t=\phi(x)} \quad \text{for a.e. } (x,y) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{4.2.4}$$

By skew-symmetry of $\operatorname{sign}(\phi(x) - \phi(y))$, using (4.2.4) and splitting the integration domain into $\big(\Omega \times \mathbb{R}^n\big) \cup \big(\Omega^c \times \Omega\big)$, we arrive at the alternative expression

$$\begin{aligned} \mathcal{C}_{\mathcal{P}_{\mathrm{N}}}(F) = &\int_{\Omega \cap F} H_K[E^t](x)\big|_{t=\phi(x)}\, \mathrm{d}x \\ &+ \int_{F \setminus \Omega} \left\{ \int_{\Omega} \big(\mathbb{1}_{(E^t)^c}(y) - \mathbb{1}_{E^t}(y)\big) K(x-y)\, \mathrm{d}y \right\}\bigg|_{t=\phi(x)} \mathrm{d}x, \end{aligned} \tag{4.2.5}$$

see [18] for details. Thus, if for all t we have $H_K[E^t] = 0$ in Ω, then the quantity $\mathcal{C}_{\mathcal{P}_{\mathrm{N}}}(F)$ depends only on the exterior condition $F \setminus \Omega$, which makes it to be a null-Lagrangian.[12]

Passing from (4.2.3) to (4.2.5) is the crucial step in [18]. To our knowledge, the structure of the alternative expression (4.2.5) for $\mathcal{C}_{\mathcal{P}_{\mathrm{N}}}$ is the only way to prove the null-Lagrangian property. The two terms in (4.2.5) bring out the true dependence of $\mathcal{C}_{\mathcal{P}_{\mathrm{N}}}$ on the data:[13] the first one involves the Euler-Lagrange equation of $\mathcal{P}_{\mathrm{N}}$ at each superlevel set E^t, while the second one depends only on the set F outside Ω. The existence of such a structure for the nonlocal perimeter suggested that it could also be present, although hidden, in other calibrations, even in the local case, as we will see in next section.

[11]Let us point out that the idea of using the sign function comes from the Legendre transform of the absolute value that appears in the fractional perimeter functional.

[12]As mentioned in the Introduction, to show minimality one does not actually need the full null-Lagrangian property ($\mathcal{C}3$) but rather the weaker condition ($\mathcal{C}3'$). For instance, to prove that the set E^0 minimizes $\mathcal{P}_{\mathrm{N}}$, from (4.2.5) it can be shown that it suffices for the E^t "above" and "below" E^0 to be super and subsolutions, respectively. For more details see [18] and compare with Theorem 4.1.3 and Proposition 4.3.3 in the following sections.

[13]The structure in (4.2.5) also appears in the local framework, where the calibration $\mathcal{C}_{\mathcal{P}_{\mathrm{L}}}$ given by (4.2.2) can be written as

$$\mathcal{C}_{\mathcal{P}_{\mathrm{L}}}(F) = \int_{\Omega \cap F} H_{\mathrm{L}}[\partial E^t](x)|_{t=t(x)}\, \mathrm{d}x - \int_{\partial\Omega \cap F} X \cdot \nu_{\partial\Omega}\, \mathrm{d}\mathcal{H}^{n-1}.$$

Moreover, it is not difficult to see that the calibration for the *fractional perimeter*, i.e., the K-nonlocal perimeter with $K(z) = |z|^{-n-2s}$ for $s \in (0,1)$, recovers $\mathcal{C}_{\mathcal{P}_{\mathrm{L}}}$ in the limit when $s \to 1$.

4.3 The theory of calibrations for local equations, and a novelty

The purpose of this section is twofold: first, to review the classical theory of fields of extremals and calibrations for "local" functionals and, second, to give a new proof of the calibration properties in this setting. Inspired by the structure of the calibration (4.2.5) for the nonlocal perimeter, we will find an alternative expression for the classical calibration (4.3.8) (a new expression to the best of our knowledge), which involves only the Euler-Lagrange and Neumann operators acting on the field.

Consider an energy functional of the form

$$\mathcal{E}_{\mathrm{L}}(w) := \int_{\Omega} G_{\mathrm{L}}\big(x, w(x), \nabla w(x)\big)\, \mathrm{d}x, \tag{4.3.1}$$

where the Lagrangian $G_{\mathrm{L}}(x, \lambda, q)$ is of class C^2 in all arguments.

A function that plays an important role when studying minimality to nonconvex energy functionals of the form (4.3.1) is the so-called *Weierstrass excess function.* It is defined for $x \in \Omega$, $\lambda \in \mathbb{R}$, and q, $\widetilde{q} \in \mathbb{R}^n$ by

$$\mathrm{E}(x, \lambda, q, \widetilde{q}) := G_{\mathrm{L}}(x, \lambda, \widetilde{q}) - G_{\mathrm{L}}(x, \lambda, q) - \partial_q G_{\mathrm{L}}(x, \lambda, q) \cdot (\widetilde{q} - q). \tag{4.3.2}$$

It is well known (see [61]) that if $u \in C^1(\overline{\Omega})$ is a minimizer of $\mathcal{E}_{\mathrm{L}}$ with respect to small $C^0_c(\Omega)$ perturbations,[14] then it must satisfy the *Weierstrass necessary condition*

$$\mathrm{E}(x, u(x), \nabla u(x), \xi) \geq 0 \quad \text{ for all } x \in \Omega,\ \xi \in \mathbb{R}^n. \tag{4.3.3}$$

Note that condition (4.3.3) on the excess function is automatically satisfied by every function u whenever $G_{\mathrm{L}}(x, \lambda, q)$ is convex with respect to the variable q, i.e., when the problem is elliptic. The Dirichlet energy and more generally the Lagrangian associated to the p-Laplacian are important elliptic examples where (4.3.3) is thus automatically satisfied.

Given an interval $I \subset \mathbb{R}$ and $\{u^t\}_{t\in I}$ a C^2 field in Ω (in the sense of Definition 4.1.2), we let

$$\mathcal{G}_{\mathrm{L}} := \big\{(x, \lambda) \in \overline{\Omega} \times \mathbb{R} \,:\, \lambda = u^t(x) \text{ for some } t \in I\big\}$$

and consider the set of admissible functions

$$\mathcal{A}_{\mathrm{L}} := \big\{w \in C^1(\overline{\Omega}) \,:\, \operatorname{graph} w \subset \mathcal{G}_{\mathrm{L}}\big\}.$$

In the classical theory, one employs the Legendre transform of G_L to define the functional $\mathcal{C}_{\mathrm{L}} \colon \mathcal{A}_{\mathrm{L}} \to \mathbb{R}$ by

$$\begin{aligned} \mathcal{C}_{\mathrm{L}}(w) := &\int_{\Omega} \Big\{\partial_q G_{\mathrm{L}}(x, u^t(x), \nabla u^t(x)) \cdot \big(\nabla w(x) - \nabla u^t(x)\big)\Big\}\Big|_{t=t(x,w(x))} \mathrm{d}x \\ &+ \int_{\Omega} G_{\mathrm{L}}(x, u^t(x), \nabla u^t(x))\big|_{t=t(x,w(x))}\, \mathrm{d}x. \end{aligned} \tag{4.3.4}$$

Under the assumption that $\{u^t\}_{t\in I}$ is a field of extremals and every leaf u^t satisfies the Weierstrass necessary condition (4.3.3), it is well known that $\mathcal{C}_{\mathrm{L}}$ is a calibration for the functional $\mathcal{E}_{\mathrm{L}}$ and each u^t;[15] see [1,3,61]. As an illustrative example, the p-Dirichlet energy

$$\mathcal{E}_{p\text{-Dir}}(w) = \frac{1}{p} \int_{\Omega} |\nabla w(x)|^p \, \mathrm{d}x$$

[14] This type of local minimizers are often referred to as *strong minimizers* in the literature.

[15] The positivity of the excess function E for every leaf u^t is only required to show property ($\mathcal{C}2$). Properties ($\mathcal{C}1$) and ($\mathcal{C}3$) follow directly from the existence of the field of extremals.

admits the calibration

$$\mathcal{C}_{p\text{-Dir}}(w)=\int_\Omega\Big\{\left|\nabla u^t(x)\right|^{p-2}\nabla u^t(x)\cdot\big(\nabla w(x)-\nabla u^t(x)\big)+\frac{1}{p}\left|\nabla u^t(x)\right|^p\Big\}\Big|_{t=t(x,w(x))}\,\mathrm{d}x.$$

We will give a new proof that the functional $\mathcal{C}_{\mathrm{L}}$ is a calibration. As mentioned before, the key point in our approach is to rewrite $\mathcal{C}_{\mathrm{L}}$ in an alternative form involving only those operators which are of interest to the theory of PDE: the Euler-Lagrange and Neumann operators. These arise when computing the first variation of $\mathcal{E}_{\mathrm{L}}$ at $u\in C^2(\overline{\Omega})$ in a direction of $\eta\in C^\infty(\overline{\Omega})$, that is,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{E}_{\mathrm{L}}(w+\varepsilon\eta)\Big|_{\varepsilon=0}=\int_\Omega\mathcal{L}_{\mathrm{L}}(w)(x)\,\eta(x)\,\mathrm{d}x+\int_{\partial\Omega}\mathcal{N}_{\mathrm{L}}(w)(x)\,\eta(x)\,\mathrm{d}\mathcal{H}^{n-1}(x). \tag{4.3.5}$$

Here in (4.3.5), $\mathcal{L}_{\mathrm{L}}$ denotes the Euler-Lagrange operator

$$\mathcal{L}_{\mathrm{L}}(w)(x):=-\mathrm{div}\big(\partial_q G_{\mathrm{L}}\big(x,w(x),\nabla w(x)\big)\big)+\partial_\lambda G_{\mathrm{L}}\big(x,w(x),\nabla w(x)\big) \tag{4.3.6}$$

and $\mathcal{N}_{\mathrm{L}}$ denotes the Neumann operator

$$\mathcal{N}_{\mathrm{L}}(w)(x):=\partial_q G_{\mathrm{L}}\big(x,w(x),\nabla w(x)\big)\cdot\nu_{\partial\Omega}(x), \tag{4.3.7}$$

where $\nu_{\partial\Omega}$ is the outward unit normal vector to $\partial\Omega$.

The following identity is our new result.

Theorem 4.3.1. *Given an interval $I\subset\mathbb{R}$ and a bounded domain $\Omega\subset\mathbb{R}^n$, let $\{u^t\}_{t\in I}$ be a C^2 field in Ω in the sense of Definition 4.1.2. Let $G_{\mathrm{L}}=G_{\mathrm{L}}(x,\lambda,q)$ be a C^2 function.*

Then, for any $t_0\in I$, the functional $\mathcal{C}_{\mathrm{L}}$ defined in (4.3.4) *can be written as*

$$\begin{aligned}\mathcal{C}_{\mathrm{L}}(w)=&\int_\Omega\int_{u^{t_0}(x)}^{w(x)}\mathcal{L}_{\mathrm{L}}(u^t)(x)\Big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x\\&+\int_{\partial\Omega}\int_{u^{t_0}(x)}^{w(x)}\mathcal{N}_{\mathrm{L}}(u^t)(x)\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{H}^{n-1}(x)+\mathcal{E}_{\mathrm{L}}(u^{t_0}).\end{aligned} \tag{4.3.8}$$

The proof of the theorem follows a typical strategy for showing the null-Lagrangian property, as seen for instance in [1]. However, a new trick will allow us to identify in the expression the operators $\mathcal{L}_{\mathrm{L}}$ and $\mathcal{N}_{\mathrm{L}}$ acting on the leaves. This is the first time we have seen the calibration written this way.

Proof of Theorem 4.3.1. In order to prove the result it will be enough to show that

$$\begin{aligned}\mathcal{C}_{\mathrm{L}}(w)-\mathcal{C}_{\mathrm{L}}(\widetilde{w})=&\int_\Omega\int_{\widetilde{w}(x)}^{w(x)}\mathcal{L}_{\mathrm{L}}(u^t)(x)\Big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x\\&+\int_{\partial\Omega}\int_{\widetilde{w}(x)}^{w(x)}\mathcal{N}_{\mathrm{L}}(u^t)(x)\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{H}^{n-1}(x).\end{aligned} \tag{4.3.9}$$

for any given $w,\widetilde{w}\in\mathcal{A}_{\mathrm{L}}$. That is, we only need to take $\tilde{w}=u^{t_0}$ and use the easy equality $\mathcal{C}_{\mathrm{L}}(u^{t_0})=\mathcal{E}_{\mathrm{L}}(u^{t_0})$.

First, let us briefly describe the proof of identity (4.3.9). We consider $\mathcal{C}_{\mathrm{L}}$ acting on the convex combination $w_\theta:=(1-\theta)\widetilde{w}+\theta w$ and express the left-hand side of (4.3.9) as $\int_0^1\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{C}_{\mathrm{L}}(w_\theta)\,\mathrm{d}\theta$. While in the literature the functions w and $\widetilde{w}$ are assumed to have the same boundary conditions, here we do not impose such a restriction. Then, we compute the derivative in θ using the expression of $\mathcal{C}_{\mathrm{L}}(w_\theta)$ as an integral in x and integrating by parts. Finally, after applying Fubini's theorem to interchange the order of integration, the key point is to make the change of variables $\theta \mapsto w_\theta(x)$ for each x. This yields the final expression.

Next, let us proceed with the proof. We let $\zeta := w - \widetilde{w}$ and hence $w_\theta = \widetilde{w} + \theta\zeta$. Since

$$\mathcal{C}_{\mathrm{L}}(w) - \mathcal{C}_{\mathrm{L}}(\widetilde{w}) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{C}_{\mathrm{L}}(w_\theta)\,\mathrm{d}\theta \tag{4.3.10}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{C}_{\mathrm{L}}(w_\theta) = &\int_\Omega \frac{\mathrm{d}}{\mathrm{d}\theta}\Big\{G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\Big\}\,\mathrm{d}x \\ &+ \int_\Omega \frac{\mathrm{d}}{\mathrm{d}\theta}\bigg(\Big\{\partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\cdot\big(\nabla w_\theta - \nabla u^t\big)\Big\}\Big|_{t=t(x,w_\theta(x))}\bigg)\,\mathrm{d}x, \end{aligned} \tag{4.3.11}$$

we must compute each of the integrands in (4.3.11).

By the chain rule, the first integrand can be written as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\theta}\Big\{G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\Big\} \\ &\qquad = \partial_\lambda G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\frac{\mathrm{d}}{\mathrm{d}\theta}\big(u^{t(x,w_\theta(x))}\big) \\ &\qquad\qquad + \partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta}\big(\nabla u^t|_{t=t(x,w_\theta(x))}\big) \end{aligned}$$

and using that $\frac{\mathrm{d}}{\mathrm{d}\theta}\big(u^{t(x,w_\theta(x))}(x)\big) = \frac{\mathrm{d}}{\mathrm{d}\theta}w_\theta(x) = \zeta(x)$, we deduce

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\theta}\Big\{G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\Big\} \\ &\quad = \partial_\lambda G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\zeta + \partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta}\big(\nabla u^t|_{t=t(x,w_\theta(x))}\big). \end{aligned} \tag{4.3.12}$$

Similarly, the second integrand in (4.3.11) is

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\theta}\bigg(\Big\{\partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\cdot\big(\nabla w_\theta - \nabla u^t\big)\Big\}\Big|_{t=t(x,w_\theta(x))}\bigg) \\ &\qquad = \Big\{\partial_t\big[\partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big]\cdot\big(\nabla w_\theta - \nabla u^t\big)\Big\}\Big|_{t=t(x,w_\theta(x))}\frac{\mathrm{d}}{\mathrm{d}\theta}\big[t(x,w_\theta(x))\big] \\ &\qquad\qquad + \partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta}\nabla w_\theta \\ &\qquad\qquad - \partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta}\big(\nabla u^t|_{t=t(x,w_\theta(x))}\big) \\ &\qquad = \Big\{\partial_t\big[\partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big]\cdot\big(\nabla w_\theta - \nabla u^t\big)\Big\}\Big|_{t=t(x,w_\theta(x))}\partial_\lambda t(x,w_\theta(x))\zeta \\ &\qquad\qquad + \partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\cdot\nabla\zeta \\ &\qquad\qquad - \partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta}\big(\nabla u^t|_{t=t(x,w_\theta(x))}\big). \end{aligned} \tag{4.3.13}$$

Adding (4.3.12) and (4.3.13), substituting in (4.3.11), and rearranging terms, we see that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{C}_{\mathrm{L}}(w_\theta) \\
&\quad = \int_\Omega \partial_\lambda G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\zeta\,\mathrm{d}x + \int_\Omega \partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\cdot\nabla\zeta\,\mathrm{d}x \\
&\qquad\quad + \int_\Omega \Big\{\partial_t\big[\partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big]\cdot\big(\nabla w_\theta - \nabla u^t\big)\Big\}\Big|_{t=t(x,w_\theta(x))}\partial_\lambda t(x,w_\theta(x))\zeta\,\mathrm{d}x.
\end{aligned}
\tag{4.3.14}
$$

The second term in (4.3.14) can be integrated by parts as

$$
\begin{aligned}
&\int_\Omega \partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\cdot\nabla\zeta\,\mathrm{d}x \\
&\qquad = \int_{\partial\Omega} \partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\cdot\nu_{\partial\Omega}\,\zeta\,\mathrm{d}\mathcal{H}^{n-1} \\
&\qquad\qquad - \int_\Omega \operatorname{div}\big(\partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big)\big|_{t=t(x,w_\theta(x))}\zeta\,\mathrm{d}x \\
&\qquad\qquad - \int_\Omega \partial_t\big[\partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big]\big|_{t=t(x,w_\theta(x))}\cdot\nabla\big[t(x,w_\theta(x))\big]\zeta\,\mathrm{d}x.
\end{aligned}
\tag{4.3.15}
$$

We now claim that

$$
\nabla\big[t(x,w_\theta(x))\big] = \big(\nabla w_\theta - \nabla u^t\big)\Big|_{t=t(x,w_\theta(x))}\partial_\lambda t(x,w_\theta(x)),
\tag{4.3.16}
$$

which leads to the identity we wish to prove. Indeed, if (4.3.16) holds, then, substituting (4.3.15) in (4.3.14), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{C}_{\mathrm{L}}(w_\theta) &= \int_\Omega \big\{\partial_\lambda G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big) - \operatorname{div}\big(\partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big)\big\}\big|_{t=t(x,w_\theta(x))}\zeta\,\mathrm{d}x \\
&\qquad + \int_{\partial\Omega} \partial_q G_{\mathrm{L}}\big(x,u^t,\nabla u^t\big)\big|_{t=t(x,w_\theta(x))}\cdot\nu_{\partial\Omega}\,\zeta\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_\Omega \mathcal{L}_{\mathrm{L}}(u^t)\big|_{t=t(x,w_\theta(x))}\zeta\,\mathrm{d}x + \int_{\partial\Omega} \mathcal{N}_{\mathrm{L}}(u^t)\big|_{t=t(x,w_\theta(x))}\zeta\,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}
\tag{4.3.17}
$$

Thus, using (4.3.17) in (4.3.10), by Fubini's theorem we deduce

$$
\begin{aligned}
&\mathcal{C}_{\mathrm{L}}(w) - \mathcal{C}_{\mathrm{L}}(\widetilde{w}) \\
&\quad = \int_\Omega\int_0^1 \mathcal{L}_{\mathrm{L}}(u^t)\big|_{t=t(x,w_\theta(x))}\zeta(x)\,\mathrm{d}\theta\,\mathrm{d}x + \int_{\partial\Omega}\int_0^1 \mathcal{N}_{\mathrm{L}}(u^t)\big|_{t=t(x,w_\theta(x))}\zeta(x)\,\mathrm{d}\theta\,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad = \int_\Omega\int_{\widetilde{w}(x)}^{w(x)} \mathcal{L}_{\mathrm{L}}(u^t)\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x + \int_{\partial\Omega}\int_{\widetilde{w}(x)}^{w(x)} \mathcal{N}_{\mathrm{L}}(u^t)\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{H}^{n-1},
\end{aligned}
$$

where in the last line we have applied the change of variables $\lambda = \widetilde{w}(x) + \theta\zeta(x)$ for each x.

To show the claim (4.3.16), we use the definition of the leaf-parameter function $t(x,\lambda)$. Differentiating $u^t(x)\big|_{t=t(x,\lambda)} = \lambda$ with respect to λ, we have

$$
\partial_t u^t\big|_{t=t(x,\lambda)}\partial_\lambda t(x,\lambda) = 1.
\tag{4.3.18}
$$

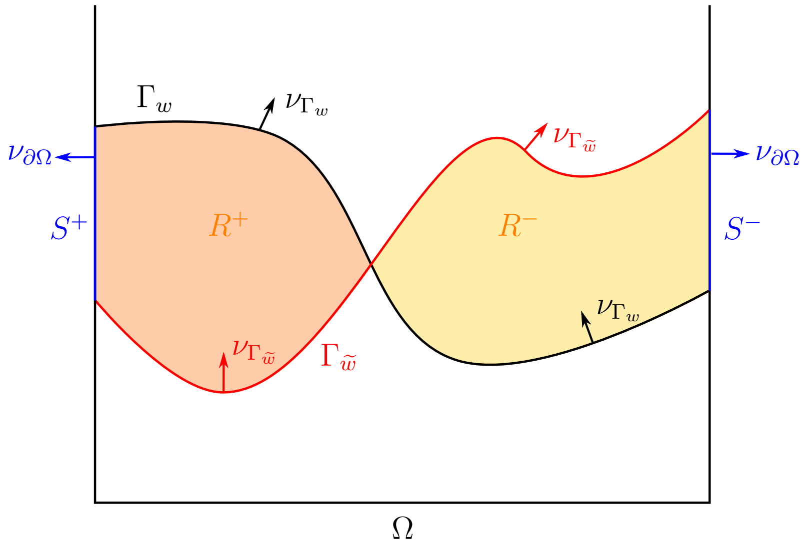

Figure 4.1

Moreover, taking the gradient of the identity $u^{t(x,w_\theta(x))}(x) = w_\theta(x)$, we obtain

$$\nabla u^t(x)\big|_{t=t(x,w_\theta(x))} + \partial_t u^t(x)\big|_{t=t(x,w_\theta(x))} \nabla\big[t(x, w_\theta(x))\big] = \nabla w_\theta(x). \tag{4.3.19}$$

Multiplying (4.3.19) by $\partial_\lambda t(x, w_\theta(x))$, applying (4.3.18) with $\lambda = w_\theta(x)$, and rearranging terms leads to (4.3.16) and concludes the proof. $\square$

Remark 4.3.2. The expression (4.3.8) can be deduced in a more geometric way using the divergence theorem in $\mathbb{R}^{n+1}$. As we see next, this gives an alternative proof of Theorem 4.3.1. Consider the vector field $X : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ given by

$$X(x, \lambda) = \big(X^x(x, \lambda), X^\lambda(x, \lambda)\big),$$

where

$$\begin{aligned} X^x(x,\lambda) &:= -\partial_q G_{\mathrm{L}}\big(x, u^t(x), \nabla u^t(x)\big)\big|_{t=t(x,\lambda)}, \\ X^\lambda(x,\lambda) &:= \Big\{ -\partial_q G_{\mathrm{L}}\big(x, u^t(x), \nabla u^t(x)\big) \cdot \nabla u^t(x) + G_{\mathrm{L}}\big(x, u^t(x), \nabla u^t(x)\big) \Big\}\Big|_{t=t(x,\lambda)}. \end{aligned}$$

Then, an easy computation from [1] shows

$$\mathrm{div} X(x, \lambda) = \mathcal{L}_{\mathrm{L}}(u^t)(x)\big|_{t=t(x,\lambda)},$$

where div is the divergence in $\mathbb{R}^{n+1}$, i.e., $\mathrm{div} X(x,\lambda) = \mathrm{div}_x X^x(x,\lambda) + \partial_\lambda X^\lambda(x,\lambda)$. From the definition of X, it can also be checked that $\mathcal{C}_{\mathrm{L}}$ can be written in the compact form

$$\mathcal{C}_{\mathrm{L}}(w) = \int_{\Gamma_w} X \cdot \nu_{\Gamma_w} \, \mathrm{d}\mathcal{H}^n,$$

where $\Gamma_w \subset \mathbb{R}^{n+1}$ is the graph of w and ν_{Γ_w} is the unit vector normal to Γ_w pointing "upwards". In coordinates, ν_{Γ_w} reads $\nu_{\Gamma_w}(x, w(x)) = \big(1 + |\nabla w(x)|^2\big)^{-1/2}(-\nabla w(x), 1)$.

Consider now the regions between the graphs of w and $\widetilde{w}$, distinguishing the parts above and below each function

$$R^+ = \{(x,\lambda) \in \Omega \times \mathbb{R} \colon \widetilde{w}(x) < \lambda < w(x)\},$$

$$R^- = \{(x,\lambda) \in \Omega \times \mathbb{R} \colon w(x) < \lambda < \widetilde{w}(x)\},$$

as well as their lateral boundaries on $\partial\Omega \times \mathbb{R}$, that is, $S^+ = \overline{R^+} \cap (\partial\Omega \times \mathbb{R})$ and $S^- = \overline{R^-} \cap (\partial\Omega \times \mathbb{R})$; see Figure 4.1. Applying the divergence theorem to the field X separately in each of the regions R^+ and R^-, we see that

$$\mathcal{C}_{\mathrm{L}}(w) = \mathcal{C}_{\mathrm{L}}(\widetilde{w}) + \int_{R^+} \operatorname{div} X \, \mathrm{d}\mathcal{H}^{n+1} - \int_{R^-} \operatorname{div} X \, \mathrm{d}\mathcal{H}^{n+1} - \int_{S^+} X \cdot \nu_{\partial\Omega} \, \mathrm{d}\mathcal{H}^n + \int_{S^-} X \cdot \nu_{\partial\Omega} \, \mathrm{d}\mathcal{H}^n,$$

where we have extended the outer normal $\nu_{\partial\Omega}$ parallel to the surface $\partial\Omega \times \mathbb{R}$; see Figure 4.1. It is also immediate to check that

$$X \cdot \nu_{\partial\Omega} = -\mathcal{N}_{\mathrm{L}}(u^t)\big|_{t=t(x,\lambda))}$$

on $S^+ \cup S^-$, with $\mathcal{N}_{\mathrm{L}}$ as in (4.3.7). Thus, we obtain the passage from (4.3.4) to (4.3.8) as an application of the divergence theorem.

Next we prove the key null-Lagrangian property ($\mathcal{C}3$) for the calibration, which follows readily from the new identity (4.3.8) for $\mathcal{C}_{\mathrm{L}}$:

Proposition 4.3.3. *Under the same hypotheses as in Theorem 4.3.1, assume that, for some $t_0 \in I$, the leaves of the field $\{u^t\}_{t \in I}$ satisfy*

$$\begin{aligned} \mathcal{L}_{\mathrm{L}}(u^t) &\geq 0 \quad \text{in } \Omega \quad \text{for } t \geq t_0, \\ \mathcal{L}_{\mathrm{L}}(u^t) &\leq 0 \quad \text{in } \Omega \quad \text{for } t \leq t_0, \end{aligned} \tag{4.3.20}$$

where $\mathcal{L}_{\mathrm{L}}$ is the Euler-Lagrange operator introduced in (4.3.6).

Then, for all w in $\mathcal{A}_{\mathrm{L}}$ such that $w \equiv u^{t_0}$ on $\partial\Omega$, the functional $\mathcal{C}_{\mathrm{L}}$ defined in (4.3.4) satisfies

$$\mathcal{C}_{\mathrm{L}}(u^{t_0}) \leq \mathcal{C}_{\mathrm{L}}(w).$$

Assume, moreover, that the leaves $\{u^t\}_{t \in I}$ satisfy the Euler-Lagrange equation in Ω, that is,

$$\mathcal{L}_{\mathrm{L}}(u^t) = 0 \quad \text{in } \Omega \quad \text{for all } t \in I. \tag{4.3.21}$$

Then, for all w as above, we have

$$\mathcal{C}_{\mathrm{L}}(w) = \mathcal{C}_{\mathrm{L}}(u^{t_0}).$$

Proof. Notice that, by (4.3.8), we have $\mathcal{C}_{\mathrm{L}}(u^{t_0}) = \mathcal{E}_{\mathrm{L}}(u^{t_0})$. Hence, assuming (4.3.20), since the boundary integral in (4.3.8) vanishes ($w \equiv u^{t_0}$ on $\partial\Omega$), it suffices to show that

$$\int_{u^{t_0}(x)}^{w(x)} \mathcal{L}_{\mathrm{L}}(u^t)(x)\big|_{t=t(x,\lambda)} \, \mathrm{d}\lambda \geq 0. \tag{4.3.22}$$

However, this is clear by (4.3.20) and the fact that u^t are increasing with respect to t. If we additionally have (4.3.21), then the integral in (4.3.22) is zero and the claim follows. □

The remaining calibration properties ($\mathcal{C}1$) and ($\mathcal{C}2$) can be directly obtained from the original definition (4.3.4) of $\mathcal{C}_{\mathrm{L}}$. First, we prove property ($\mathcal{C}1$):

Proposition 4.3.4. *Assume the same hypotheses of Theorem 4.3.1. Then, for all $t \in I$, the functional $\mathcal{C}_{\mathrm{L}}$ defined in (4.3.4) satisfies*

$$\mathcal{C}_{\mathrm{L}}(u^t) = \mathcal{E}_{\mathrm{L}}(u^t).$$

Proof. Given $t_0 \in I$, from the definition of the leaf-parameter function it follows that $t(x, u^{t_0}(x)) = t_0$. In particular, $\nabla u^{t_0}(x) = \nabla u^t(x)\big|_{t=t(x,u^{t_0}(x))}$, and substituting in the definition of $\mathcal{C}_{\mathrm{L}}$ in (4.3.4) we see that

$$\mathcal{C}_{\mathrm{L}}(u^{t_0}) = \int_\Omega G_{\mathrm{L}}\big(x, u^{t_0}(x), \nabla u^{t_0}(x)\big)\,\mathrm{d}x = \mathcal{E}_{\mathrm{L}}(u^{t_0}).$$

Since t_0 was arbitrary, the claim follows. □

Finally, we show property ($\mathcal{C}2$):

Proposition 4.3.5. *Under the same hypotheses as in Theorem 4.3.1, the energy $\mathcal{E}_{\mathrm{L}}$ can be decomposed in terms of $\mathcal{C}_{\mathrm{L}}$ and the excess function* E *as*

$$\mathcal{E}_{\mathrm{L}}(w) = \mathcal{C}_{\mathrm{L}}(w) + \int_\Omega \mathrm{E}(x, u^t(x), \nabla u^t(x), \nabla w(x))\big|_{t=t(x,w(x))}\,\mathrm{d}x,$$ [16]

for all w in $\mathcal{A}_{\mathrm{L}}$.

As a consequence, if each leaf of the field $\{u^t\}_{t\in I}$ satisfies the Weierstrass necessary condition (4.3.3)*, then*

$$\mathcal{C}_{\mathrm{L}}(w) \le \mathcal{E}_{\mathrm{L}}(w),$$

for all w in $\mathcal{A}_{\mathrm{L}}$.

Proof. For each $x \in \Omega$, take $\lambda = w(x) = u^t(x)|_{t=t(x,w(x))}$, $q = \nabla u^t(x)\big|_{t=t(x,w(x))}$, and $\widetilde{q} = \nabla w(x)$. Substituting in (4.3.2) and integrating in Ω, comparing this expression with the definition of $\mathcal{C}_{\mathrm{L}}$ in (4.3.4), the identity follows. □

Remark 4.3.6. If the energy functional includes reaction terms on a portion of the boundary $\Gamma_{\mathcal{N}} \subset \partial\Omega$, i.e.,

$$\widetilde{\mathcal{E}_{\mathrm{L}}}(w) = \int_\Omega G_{\mathrm{L}}\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x - \int_{\Gamma_{\mathcal{N}}} F(w(x))\,\mathrm{d}\mathcal{H}^{n-1}(x),$$

then we can still apply the methods from the local theory of calibrations to show that

$$\begin{aligned}\widetilde{\mathcal{C}_{\mathrm{L}}}(w) = &\int_\Omega \partial_q G_{\mathrm{L}}(x, u^t(x), \nabla u^t(x)) \cdot \big(\nabla w(x) - \nabla u^t(x)\big)\big|_{t=t(x,w(x))}\,\mathrm{d}x \\ &+ \int_\Omega G_{\mathrm{L}}(x, u^t(x), \nabla u^t(x))\big|_{t=t(x,w(x))} - \int_{\Gamma_{\mathcal{N}}} F(w(x))\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}x\end{aligned}$$

is a calibration. Hence, one can establish the minimality of the leaves among competitors with the same boundary data only on $\partial\Omega \setminus \Gamma_{\mathcal{N}}$. Note that in this scenario, extremals satisfy the equation

$$\begin{cases} \mathcal{L}_{\mathrm{L}}(u) = 0 & \text{in } \Omega, \\ \mathcal{N}_{\mathrm{L}}(u) = F'(u) & \text{in } \Gamma_{\mathcal{N}}. \end{cases}$$

In particular, this allows to treat the extension problem for the fractional Laplacian as explained in Appendix G. In this setting, one considers the Dirichlet energy in a domain of the extended space $\mathbb{R}^{n+1}_+$ with an additional potential energy on the part of its boundary lying on $\mathbb{R}^n = \partial\mathbb{R}^{n+1}_+$.

[16] This identity is known as the *Weierstrass representation formula*

4.4 The calibration for the fractional Laplacian

In this section we construct a calibration for the functional

$$\mathcal{E}_{s,F}(w) = \frac{c_{n,s}}{4} \iint_{Q(\Omega)} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x,$$

where $s \in (0,1)$, $c_{n,s}$ is a positive normalizing constant, and $F \in C^1(\mathbb{R})$. It involves the Gagliardo-Sobolev seminorm and a nonlinear potential term.

If u is a critical point of $\mathcal{E}_{s,F}$ with respect to functions with the same exterior data as u, then u satisfies the nonlocal semilinear equation

$$(-\Delta)^s u - F'(u) = 0 \quad \text{in } \Omega. \tag{4.4.1}$$

In this setting, recall the standard subspace of locally integrable functions given by

$$L^1_s(\mathbb{R}^n) := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \colon \|u\|_{L^1_s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{n+2s}} \, \mathrm{d}y < +\infty \right\}.$$

If $u \in L^1_s(\mathbb{R}^n)$ is C^2 in a neighborhood of $x \in \mathbb{R}^n$, then the fractional Laplacian $(-\Delta)^s u(x)$ is well defined. More generally, we only need u to be $C^{2s+\alpha}$ in a neighborhood of x for some small $\alpha > 0$ such that $2s + \alpha$ is not an integer. Here C^β denotes the space $C^{k,\gamma}$ of functions with Hölder continuous k-th order derivatives, where $k = \lfloor \beta \rfloor$, $\gamma = \beta - k$. In particular, for $(-\Delta)^s u$ to be well defined in a domain Ω, we just need that the function $u \in L^1_s(\mathbb{R}^n)$ is smooth in a neighborhood of Ω. The function could be extremely wild outside the domain, as long as it satisfies the growth assumption defining $L^1_s(\mathbb{R}^n)$.

As explained in the Introduction, to build the calibration we will assume the existence of a field in $\mathbb{R}^n$. In particular, we are given a family of functions $u^t \colon \mathbb{R}^n \to \mathbb{R}$, with $t \in I$ for some interval $I \subset \mathbb{R}$, satisfying certain regularity assumptions. For clarity reasons, in the statement of Theorem 4.1.3 we have assumed that the function $(x,t) \mapsto u^t(x)$ belongs to $C^2(\mathbb{R}^n \times I) \cap L^\infty(\mathbb{R}^n \times I)$ and $(x,t) \mapsto \partial_t u^t(x)$ is in $L^\infty(\mathbb{R}^n \times I)$. We have also assumed that $t \mapsto u^t(x)$ is increasing in I for all $x \in \mathbb{R}^n$. Thus, the graphs of u^t produce a foliation of a certain region in $\mathbb{R}^n \times \mathbb{R}$. Nevertheless, these conditions can be weakened, as presented next, to yield a more satisfactory theory for the fractional Laplacian.

The following is a weaker definition of field than the one in the statement of Theorem 4.1.3, but which suffices to establish the result. On the one hand, we allow the functions u^t in the field to be "wild" outside a neighborhood of Ω, as long as their fractional Laplacian is under control. On the other hand, the leaves u^t can touch each other (but not cross) outside Ω. That is, we need u^t to be increasing in t in $\overline{\Omega}$, but only nondecreasing outside. In particular, the graphs of u^t will only produce a foliation in a certain region of $\overline{\Omega} \times \mathbb{R}$.

Definition 4.4.1. Given an interval $I \subset \mathbb{R}$, a bounded domain $\Omega \subset \mathbb{R}^n$, and $s \in (0,1)$, we say that a family $\{u^t\}_{t \in I}$ of functions $u^t \colon \mathbb{R}^n \to \mathbb{R}$ is a *field* (to be precise, we should say a *field associated to the s-fractional Laplacian*, not to be inconsistent with Definition 4.1.2) when the following conditions hold:

(i) The function $(x,t) \mapsto u^t(x)$ is continuous in $\overline{\Omega} \times I$.

(ii) The function $t \mapsto u^t(x)$ is

- increasing in I for all $x \in \overline{\Omega}$

- nondecreasing in I for a.e. $x \in \Omega^c$
- C^1 in I for a.e. $x \in \mathbb{R}^n$

(iii) For each compact interval $J \subset I$, we have

$$\sup_{t \in J} \left\{ \|\partial_t u^t\|_{L^\infty(\mathbb{R}^n)} + \|u^t\|_{L^1_s(\mathbb{R}^n)} + \|u^t\|_{C^{2s+\alpha}(N)} \right\} < \infty,$$

for some bounded domain $N \subset \mathbb{R}^n$, with $\overline{\Omega} \subset N$, and some $\alpha > 0$.

Essentially, one needs a reasonable regularity of the joint function, as well as some further regularity separately in each of the variables, locally uniformly in the parameter t.

By properties (i) and (ii), the leaf-parameter function $t = t(x, \lambda)$ from (4.1.4) is well-defined and continuous in the region

$$\mathcal{G} := \{(x, \lambda) \in \overline{\Omega} \times \mathbb{R} \colon \lambda = u^t(x) \text{ for some } t \in I\}. \tag{4.4.2}$$

The potential F will play no role in the construction of the calibration for $\mathcal{E}_{s,F}$ and, hence, we can focus on the first term. Let

$$\mathcal{E}_s(u) := \frac{c_{n,s}}{4} \iint_{Q(\Omega)} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y$$

and consider the energy space $\dot{H}^s(\Omega) := \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \colon \mathcal{E}_s(u) < \infty\}$.

Let $\{u^t\}_{t \in I}$ be a field in the sense of Definition 4.4.1, and let $t_0 \in I$. We consider the functional

$$\mathcal{C}_s(w) := \int_\Omega \int_{u^{t_0}(x)}^{w(x)} (-\Delta)^s u^t(x)\big|_{t = t(x,\lambda)} \, \mathrm{d}\lambda \, \mathrm{d}x + \frac{c_{n,s}}{4} \iint_{Q(\Omega)} \frac{|u^{t_0}(x) - u^{t_0}(y)|^2}{|x - y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y \tag{4.4.3}$$

acting on continuous functions $w \in C^0(\overline{\Omega})$ with the same exterior datum as u^{t_0}, and such that their graph is contained in $\mathcal{G} \subset \overline{\Omega} \times \mathbb{R}$ when restricted to $\overline{\Omega}$. Here $\mathcal{G}$ has been introduced in (4.4.2). We denote this set of admissible functions by $\mathcal{A}_{s,t_0}$, that is,

$$\mathcal{A}_{s,t_0} := \left\{ w \in C^0(\overline{\Omega}) \colon w = u^{t_0} \text{ on } \partial\Omega, \quad w = u^{t_0} \text{ a.e. in } \Omega^c, \text{ graph}\,(w_{|\overline{\Omega}}) \subset \mathcal{G} \right\}. \tag{4.4.4}$$

Remark 4.4.2. The functional $\mathcal{C}_s$ is well defined in the set $\mathcal{A}_{s,t_0}$. Let us check this. For $x \in \Omega$ and λ between $u^{t_0}(x)$ and $w(x)$, we have that

$$t(x, \lambda) \in [t_{\min}, t_{\max}] \subset I,$$

where

$$t_{\min} = \min_{x \in \overline{\Omega}} t(x, w(x)) \quad \text{and} \quad t_{\max} = \max_{x \in \overline{\Omega}} t(x, w(x)).^{17} \tag{4.4.5}$$

Then, on the one hand, since the fractional Laplacians $(-\Delta)^s u^t(x)$ are uniformly bounded in $x \in \Omega$ and $t \in [t_{\min}, t_{\max}]$ by (iii), the iterated integral in the first term in (4.4.3) is finite. On the other hand, taking into account the identity

$$\frac{c_{n,s}}{2} \iint_{Q(\Omega)} \frac{|u^t(x) - u^t(y)|^2}{|x - y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y = \int_\Omega u^t(x) (-\Delta)^s u^t(x) \, \mathrm{d}x,$$

the second integral in (4.4.3) is finite thanks to the uniform boundedness in $x \in \Omega$ and $t \in [t_{\min}, t_{\max}]$ of each $u^t(x)$ and of the fractional Laplacians $(-\Delta)^s u^t(x)$.

[17] Note here that $t(x, u^{t_0}(x)) \equiv t_0$ and that $t_{min} \leq t_0 \leq t_{max}$ since $w \equiv u^{t_0}$ on $\partial\Omega$.

Remark 4.4.3. The functional $\mathcal{C}_s$ coincides, in the set $\mathcal{A}_{s,t_0}$, with the functional $\mathcal{C}_{s,F}$ appearing in Theorem 4.1.3 when $F = 0$. Indeed, we can write

$$\begin{aligned}
&\int_\Omega \int_{u^{t_0}(x)}^{w(x)} (-\Delta)^s u^t(x)\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x \\
&\qquad = \int_\Omega \mathrm{d}x \int_{u^{t_0}(x)}^{w(x)} \mathrm{d}\lambda \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n\setminus\{|x-y|>\varepsilon\}} \mathrm{d}y\, c_{n,s} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)} \\
&\qquad = \lim_{\varepsilon\downarrow 0} \iint_{(\Omega\times\mathbb{R}^n)\setminus\{|x-y|>\varepsilon\}} \mathrm{d}x\,\mathrm{d}y \int_{u^{t_0}(x)}^{w(x)} \mathrm{d}\lambda\, c_{n,s} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)} \\
&\qquad = c_{n,s} \lim_{\varepsilon\downarrow 0} \iint_{Q(\Omega)\setminus\{|x-y|>\varepsilon\}} \mathrm{d}x\,\mathrm{d}y \int_{u^{t_0}(x)}^{w(x)} \mathrm{d}\lambda\, \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)}
\end{aligned}$$

by the regularity of the field and where the last equality follows from the fact that $u^{t_0}(x) = w(x)$ for almost every $x \in \Omega^c$. This last expression gives meaning to the principal value in the definition of $\mathcal{C}_{s,F}$ in Theorem 4.1.3 in the Introduction.

Notice that the expression of $\mathcal{C}_s$ in (4.4.3) only involves the Euler-Lagrange equation of the field $\{u^t\}_{t\in I}$ and the energy of the leaf u^{t_0}. This will give both $(\mathcal{C}1)$ and the null-Lagrangian property $(\mathcal{C}3)$. To prove $(\mathcal{C}2)$, we next show, in Lemma 4.4.4, that the functional (4.4.3) can be recast in a useful alternative form. We include Figure 4.2 for the convenience of the reader, to better identify the terms involved in the new expression (of Lemma 4.4.4).

To simplify the statements and proofs below, for $\varepsilon > 0$ we use the truncated kernel $K_\varepsilon(z) = c_{n,s}|z|^{-n-2s}\,\mathbb{1}_{B_\varepsilon^c}(z)$, and for $u \in L^1_s(\mathbb{R}^n)$ we let

$$(-\Delta)^s_\varepsilon u(x) = \int_{\mathbb{R}^n} (u(x)-u(y))K_\varepsilon(x-y)\,\mathrm{d}y.$$

In particular, when ε goes to zero we recover the fractional Laplacian $(-\Delta)^s u^t(x) = \lim_{\varepsilon\downarrow 0}(-\Delta)^s_\varepsilon u^t(x)$. We also write

$$\begin{aligned}
\mathcal{C}_s^\varepsilon(w) := \int_\Omega \int_{u^{t_0}(x)}^{w(x)} &(-\Delta)^s_\varepsilon u^t(x)\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x \\
&+ \frac{1}{4}\iint_{Q(\Omega)} |u^{t_0}(x)-u^{t_0}(y)|^2 K_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{4.4.6}$$

Lemma 4.4.4. *Given an interval $I \subset \mathbb{R}$, a bounded domain $\Omega \subset \mathbb{R}^n$, and $s \in (0,1)$, let $\{u^t\}_{t\in I}$ be a field in the sense of Definition 4.4.1. Consider the set of admissible functions $\mathcal{A}_{s,t_0}$ defined in* (4.4.4).

Then, for each $\varepsilon > 0$ and $w \in \mathcal{A}_{s,t_0}$, with $t_0 \in I$, the functional $\mathcal{C}_s^\varepsilon$ defined in (4.4.6) *satisfies*

$$\begin{aligned}
\mathcal{C}_s^\varepsilon(w) = &-\frac{c_{n,s}}{2}\iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}} \mathrm{d}x\,\mathrm{d}y \int_{t(x,w(x))}^{t(y,w(y))} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\partial_t u^t(y)\,\mathrm{d}t \\
&+ \frac{c_{n,s}}{4}\iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}} \frac{|w(x)-u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

where we have extended the leaf-parameter function $x \mapsto t(x,w(x))$ continuously outside Ω by letting $t(x,w(x)) = t_0$ for $x \in \Omega^c$.

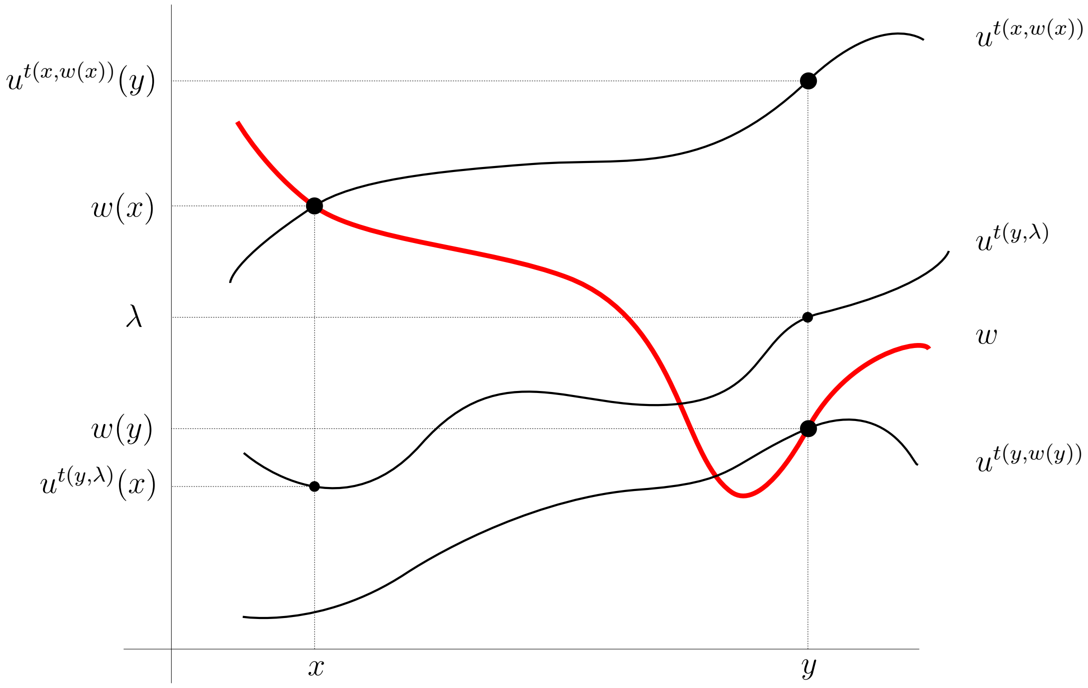

Figure 4.2: The function w (in red) and the leaves u^t.

Proof. Throughout the proof, C_ε denotes a generic positive constant depending only on n, s, $|\Omega|$, and ε. It is easy to check that

$$\left(1+|y|^{n+2s}\right) K_\varepsilon(x-y) \le C_\varepsilon \quad \text{for all } x \in \Omega \text{ and } y \in \mathbb{R}^n. \tag{4.4.7}$$

By (4.4.7) we have

$$K_\varepsilon(x-y)\,|u^t(x)-u^t(y)| \le C_\varepsilon \left(\|u^t\|_{C^0(\overline{\Omega})} + |u^t(y)|\right) \left(1+|y|^{n+2s}\right)^{-1}$$

for all $x \in \Omega$, a.e. $y \in \mathbb{R}^n$ and all $t \in I$. Hence, integrating in $y \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} K_\varepsilon(x-y)\,|u^t(x)-u^t(y)|\,\mathrm{d}y \le C_\varepsilon \left(\|u^t\|_{C^0(\overline{\Omega})} + \|u^t\|_{L^1_s(\mathbb{R}^n)}\right) \quad \text{for all } x \in \Omega,\ t \in I. \tag{4.4.8}$$

Consider now $t_{\min}$ and $t_{\max}$ given in (4.4.5). By properties (i) and (iii) in Definition 4.4.1, for $x \in \Omega$ we have

$$\begin{aligned}
&\left| \int_{u^{t_0}(x)}^{w(x)} \mathrm{d}\lambda \int_{\mathbb{R}^n} \mathrm{d}y\, K_\varepsilon(x-y)\,|u^t(x)-u^t(y)|\big|_{t=t(x,\lambda)} \right| \\
&\qquad\qquad \le C_\varepsilon \sup_{t\in[t_{\min},t_{\max}]} \left(\|u^t\|_{C^0(\overline{\Omega})} + \|u^t\|_{L^1_s(\mathbb{R}^n)}\right) \|w-u^{t_0}\|_{C^0(\overline{\Omega})} < \infty,
\end{aligned}$$

and we can apply Fubini's theorem to get

$$\begin{aligned}
&\int_\Omega \mathrm{d}x \int_{u^{t_0}(x)}^{w(x)} \mathrm{d}\lambda (-\Delta)^s_\varepsilon u^t(x)\big|_{t=t(x,\lambda)} \\
&\qquad = \int_\Omega \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}y K_\varepsilon(x-y) \int_{u^{t_0}(x)}^{w(x)} (u^t(x)-u^t(y))\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda.
\end{aligned} \tag{4.4.9}$$

Applying the change of variables $\lambda = u^t(x)$ for a.e. $x \in \Omega$ in (4.4.9), the integral becomes

$$\begin{aligned}\int_\Omega \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}y\ K_\varepsilon(x-y) \int_{u^{t_0}(x)}^{w(x)} (u^t(x) - u^t(y))\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \\ = \int_\Omega \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}y\ K_\varepsilon(x-y) \int_{t_0}^{t(x,w(x))} (u^t(x) - u^t(y))\, \partial_t u^t(x) \,\mathrm{d}t.\end{aligned} \tag{4.4.10}$$

Thanks to the extension of the leaf-parameter function by $t(x, w(x)) = t_0$ for $x \in \Omega^c$, using that $Q(\Omega) = (\Omega \times \mathbb{R}^n) \cup (\Omega^c \times \Omega)$, we can rewrite the right-hand side of (4.4.10) as

$$\begin{aligned}\int_\Omega \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}y\ K_\varepsilon(x-y) \int_{t_0}^{t(x,w(x))} (u^t(x) - u^t(y))\, \partial_t u^t(x) \,\mathrm{d}t \\ = \iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y\ K_\varepsilon(x-y) \int_{t_0}^{t(x,w(x))} (u^t(x) - u^t(y))\ \partial_t u^t(x) \,\mathrm{d}t.\end{aligned} \tag{4.4.11}$$

The idea now is to use the symmetry of the domain $Q(\Omega)$ to symmetrize the right-hand side of the previous identity. This will allow us to integrate an exact differential of t, and this will lead to the identity claimed in the lemma.

Symmetrizing (4.4.11), we have

$$\begin{aligned}\iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y) \int_{t_0}^{t(x,w(x))} (u^t(x) - u^t(y)) \partial_t u^t(x) \,\mathrm{d}t \\ = \frac{1}{2} \iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y) \int_{t_0}^{t(x,w(x))} (u^t(x) - u^t(y)) \partial_t u^t(x) \,\mathrm{d}t \\ - \frac{1}{2} \iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y) \int_{t_0}^{t(y,w(y))} (u^t(x) - u^t(y)) \partial_t u^t(y) \,\mathrm{d}t.\end{aligned} \tag{4.4.12}$$

Splitting the integral $\int_{t_0}^{t(y,w(y))} \mathrm{d}t$ into the sum $\int_{t_0}^{t(x,w(x))} \mathrm{d}t + \int_{t(x,w(x))}^{t(y,w(y))} \mathrm{d}t$ and rearranging terms, the right-hand side of (4.4.12) becomes

$$\begin{aligned}\frac{1}{2} \iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y) \int_{t_0}^{t(x,w(x))} (u^t(x) - u^t(y)) \partial_t u^t(x) \,\mathrm{d}t \\ - \frac{1}{2} \iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y) \int_{t_0}^{t(y,w(y))} (u^t(x) - u^t(y)) \partial_t u^t(y) \,\mathrm{d}t \\ = -\frac{1}{2} \iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y) \int_{t(x,w(x))}^{t(y,w(y))} (u^t(x) - u^t(y)) \partial_t u^t(y) \,\mathrm{d}t \\ + \frac{1}{2} \iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y) \int_{t_0}^{t(x,w(x))} (u^t(x) - u^t(y)) (\partial_t u^t(x) - \partial_t u^t(y)) \,\mathrm{d}t.\end{aligned} \tag{4.4.13}$$

Let us show that the integrals in the right-hand side of (4.4.13) are well defined. For the first integral, taking absolute values and using Fubini's theorem, we have

$$\begin{aligned}
&\frac{1}{2}\iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y)\left|\int_{t(x,w(x))}^{t(y,w(y))} |u^t(x)-u^t(y)||\partial_t u^t(y)|\,\mathrm{d}t\right| \\
&\quad\le \frac{1}{2}\iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y)\int_{t_{\min}}^{t_{\max}} |u^t(x)-u^t(y)|\|\partial_t u^t\|_{L^\infty(\mathbb{R}^n)}\,\mathrm{d}t \\
&\quad= \frac{1}{2}\int_{t_{\min}}^{t_{\max}} \mathrm{d}t\,\|\partial_t u^t\|_{L^\infty(\mathbb{R}^n)}\iint_{Q(\Omega)} K_\varepsilon(x-y)|u^t(x)-u^t(y)|\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le \int_{t_{\min}}^{t_{\max}} \mathrm{d}t\,\|\partial_t u^t\|_{L^\infty(\mathbb{R}^n)}\int_\Omega \mathrm{d}x\int_{\mathbb{R}^n}\mathrm{d}y K_\varepsilon(x-y)|u^t(x)-u^t(y)|,
\end{aligned} \tag{4.4.14}$$

where in the last line we have used that $Q(\Omega) = (\Omega\times\mathbb{R}^n)\cup(\mathbb{R}^n\times\Omega)$ (not a disjoint union) and the symmetry of K_ε. Applying the bound (4.4.8) in (4.4.14) and by property (iii) in Definition 4.4.1, we deduce the finiteness of (4.4.14). It follows that the first integral is well defined.

The second integral in the right-hand side of (4.4.13) can be integrated explicitly as

$$\begin{aligned}
&\frac{1}{2}\iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y)\int_{t_0}^{t(x,w(x))} (u^t(x)-u^t(y))(\partial_t u^t(x)-\partial_t u^t(y))\,\mathrm{d}t \\
&\qquad= \frac{1}{4}\iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y)\int_{t_0}^{t(x,w(x))} \frac{\mathrm{d}}{\mathrm{d}t}|u^t(x)-u^t(y)|^2\,\mathrm{d}t \\
&\qquad= \frac{1}{4}\iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y)|w(x)-u^{t(x,w(x))}(y)|^2 \\
&\qquad\quad - \frac{1}{4}\iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y)|u^{t_0}(x)-u^{t_0}(y)|^2.
\end{aligned} \tag{4.4.15}$$

Concatenating the equalities (4.4.9), (4.4.10), (4.4.11), (4.4.12), and (4.4.13), and using (4.4.15), we conclude

$$\begin{aligned}
&\int_\Omega \mathrm{d}x\int_{u^{t_0}(x)}^{w(x)} \mathrm{d}\lambda(-\Delta)^s_\varepsilon u^t(x)\big|_{t=t(x,\lambda)} \\
&\qquad= -\frac{1}{2}\iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y)\int_{t(x,w(x))}^{t(y,w(y))} (u^t(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t \\
&\qquad\quad + \frac{1}{4}\iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y)|w(x)-u^{t(x,w(x))}(y)|^2 \\
&\qquad\quad - \frac{1}{4}\iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y)|u^{t_0}(x)-u^{t_0}(y)|^2,
\end{aligned}$$

which is the claim of the lemma. □

Having Lemma 4.4.4 at hand, we can now prove the calibration property ($\mathcal{C}2$).

Lemma 4.4.5. *Given an interval $I\subset\mathbb{R}$, a bounded domain $\Omega\subset\mathbb{R}^n$, and $s\in(0,1)$, let $\{u^t\}_{t\in I}$ be a field in the sense of Definition 4.4.1. Consider the set of admissible functions $\mathcal{A}_{s,t_0}$ defined in* (4.4.4).

Then, for all $w\in\mathcal{A}_{s,t_0}$, with $t_0\in I$, we have

$$\mathcal{C}_s(w) \le \mathcal{E}_s(w).$$

Proof. Let $\varepsilon > 0$. By Lemma 4.4.4, we have

$$\begin{aligned} \mathcal{C}_s^\varepsilon(w) = &-\frac{1}{2}\iint_{Q(\Omega)} \mathrm{d}x\,\mathrm{d}y K_\varepsilon(x-y)\int_{t(x,w(x))}^{t(y,w(y))} (u^t(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t \\ &+\frac{1}{4}\iint_{Q(\Omega)} |w(x)-u^{t(x,w(x))}(y)|^2 K_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{4.4.16}$$

where $t(x,w(x)) = t_0$ for $x \in \Omega^c$. We claim that for a.e. $(x,y) \in Q(\Omega)$ we have

$$-\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (u^t(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t \le -\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (w(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t. \tag{4.4.17}$$

Indeed, first of all note that $\partial_t u^t(y) \ge 0$ for a.e. $y \in \mathbb{R}^n$ by property (ii) in Definition 4.4.1. Moreover, the quantities $u^t(x)$, $u^t(y)$, $w(x)$, and $\partial_t u^t(y)$ are finite for a.e. $(x,y) \in Q(\Omega)$ and all $t \in I$. For those $(x,y) \in Q(\Omega)$, if $t(x,u(x)) \le t(y,u(y))$ then by property (ii) we have $w(x) = u^{t(x,w(x))}(x) \le u^t(x)$ for $t \in [t(x,u(x)), t(y,u(y))]$, and the claim follows in this case. When $t(x,u(x)) \ge t(y,u(y))$ the argument is similar.

The right-hand side of (4.4.17) can be integrated explicitly as

$$\begin{aligned} -\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (w(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t &= \frac{1}{4}\int_{t(x,w(x))}^{t(y,w(y))} \frac{\mathrm{d}}{\mathrm{d}t}|w(x)-u^t(y)|^2\,\mathrm{d}t \\ &= \frac{1}{4}|w(x)-w(y)|^2 - \frac{1}{4}|w(x)-u^{t(x,w(x))}(y)|^2. \end{aligned} \tag{4.4.18}$$

Now, using (4.4.17) and (4.4.18) in (4.4.16), it follows that

$$\mathcal{C}_s^\varepsilon(w) \le \mathcal{E}_{K_\varepsilon}(w).$$

Finally, by property (iii) in Definition 4.4.1, $(-\Delta)_\varepsilon^s u^t$ converge to $(-\Delta)^s u^t$ in $L^1(\Omega)$, locally uniformly in t, as $\varepsilon \downarrow 0$. This is enough to pass to the limit in the inequality above and conclude the proof. □

We can finally give the proof of Theorem 4.1.3. We will show the identity

$$\mathcal{C}_{s,F}(w) = \int_\Omega \int_{u^{t_0}(x)}^{w(x)} \left((-\Delta)^s u^t(x) - F'(u^t(x))\right)\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x + \mathcal{E}_{s,F}(u^{t_0}) \tag{4.4.19}$$

for $w \in \mathcal{A}_{s,t_0}$, and that this functional is a calibration for $\mathcal{E}_{s,F}$ and each u^t when the family $\{u^t\}_{t\in I}$ is a field of extremals, that is, when each u^t solves the semilinear equation (4.4.1). In particular, each of the u^t will be a minimizer. More generally, we show that u^{t_0} minimizes $\mathcal{E}_{s,F}$ if the u^t above u^{t_0} are supersolutions of (4.4.1) and the u^t below are subsolutions.

Proof of Theorem 4.1.3. First of all, note that for $x \in \Omega$ we have

$$F(w(x)) - F(u^{t_0}(x)) = \int_{u^{t_0}(x)}^{w(x)} F'(\lambda)\,\mathrm{d}\lambda = \int_{u^{t_0}(x)}^{w(x)} F'(u^{t(x,\lambda)})\,\mathrm{d}\lambda$$

and, thanks to Remark 4.4.3, the functional $\mathcal{C}_{s,F}$ given by (4.1.9) can be written simply as

$$\mathcal{C}_{s,F}(w) = \mathcal{C}_s(w) - \int_\Omega F(w(x))\,\mathrm{d}x,$$

where $\mathcal{C}_s$ has been introduced in (4.4.3). This proves (4.4.19).

(a) From (4.4.19) it is clear that $\mathcal{C}_{s,F}(u^{t_0}) = \mathcal{E}_{s,F}(u^{t_0})$, which is condition ($\mathcal{C}1$). To obtain ($\mathcal{C}2$) we apply Lemma 4.4.5, which gives

$$\mathcal{C}_{s,F}(w) = \mathcal{C}_s(w) - \int_\Omega F(w(x))\,\mathrm{d}x \leq \mathcal{E}_s(u) - \int_\Omega F(w(x))\,\mathrm{d}x = \mathcal{E}_{s,F}(w).$$

(b) To show ($\mathcal{C}3'$), that is, $\mathcal{C}_{s,F}(w) \geq \mathcal{C}_{s,F}(u^{t_0}) = \mathcal{E}_{s,F}(u^{t_0})$, by (4.4.19) it suffices to show that

$$\int_{u^{t_0}(x)}^{w(x)} \big((-\Delta)^s u^t(x) - F'(u^t(x))\big)\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda \geq 0$$

for $x \in \Omega$. But this is clear from the monotonicity of the field u^t and the hypotheses in (b).

On the other hand, we have already seen in subsection 4.1.1 how properties ($\mathcal{C}1$), ($\mathcal{C}2$), and ($\mathcal{C}3'$) yield the minimality of $u = u^{t_0}$.

(c) By (4.4.19), using that each u^t satisfies the Euler-Lagrange equation (4.4.1), we have that

$$\mathcal{C}_{s,F}(w) - \mathcal{E}_{s,F}(u^{t_0}) = \int_\Omega \int_{u^{t_0}(x)}^{w(x)} \big((-\Delta)^s u^t(x) - F'(u^t(x))\big)\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x = 0.$$

Hence $\mathcal{C}_{s,F}(w) = \mathcal{E}_{s,F}(u^{t_0}) = \mathcal{C}_{s,F}(u^{t_0})$ for all $w \in \mathcal{A}_{s,t_0}$. In particular, the functional $\mathcal{C}_{s,F}$ satisfies all three properties ($\mathcal{C}1$),($\mathcal{C}2$), and ($\mathcal{C}3$), and thus it is a calibration. Choosing $t_0 = t$ for each $t \in I$, we deduce the minimality of u^t. $\square$

Proof of Corollary 4.1.4. First of all, by a simple argument from [1], it suffices to prove the corollary for the class of functions $w \in C^0$ satisfying the strict inequality

$$\lim_{\tau \to -\infty} u(x', \tau) < w(x', x_n) < \lim_{\tau \to +\infty} u(x', \tau) \quad \text{for all } (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{4.4.20}$$

Indeed, if w satisfies the non-strict inequality (4.1.12), then for $\theta \in (0,1)$ we consider $w_\theta := (1-\theta)u + \theta w$, which satisfies (4.4.20) by the strict monotonicity of u. Hence, applying the corollary in the strict case, we have $\mathcal{E}_{s,F}(u) \leq \mathcal{E}_{s,F}(w_\theta)$. Letting $\theta \to 1^-$ yields the result for w. Hence, we may assume (4.4.20).

Since u is bounded and $F \in C^3$, by regularity theory for the fractional Laplacian, we have that u is at least C^2 and $\nabla u \in L^\infty(\mathbb{R}^n)$; see [84]. For each $t \in \mathbb{R}$, we consider the family of translations $u^t(x) = u(x', x_n + t)$. By the monotonicity and the regularity properties of u, the family $\{u^t\}_{t \in \mathbb{R}}$ is a field in the sense of Definition 4.4.1. Moreover, by the translation invariance of the equation, it is a field of extremals. Hence, we can apply Theorem 4.1.3 to conclude that u is a minimizer in the set of admissible functions $\mathcal{A}_s$ with $w = u$ in Ω^c. $\square$

Chapter 5

Null-Lagrangians and calibrations for general nonlocal functionals and an application to the viscosity theory

In this chapter we build a null-Lagrangian and a calibration for general nonlocal elliptic functionals in the presence of a field of extremals. Thus, our construction assumes the existence of a family of solutions to the Euler-Lagrange equation whose graphs produce a foliation. Then, as a consequence of the calibration, we show the minimality of each leaf in the foliation. Our model case is the energy functional for the fractional Laplacian, for which such a null-Lagrangian was recently discovered by us.

As a first application of our calibration, we show that monotone solutions to translation invariant nonlocal equations are minimizers. Our second application is perhaps surprising, since here "minimality" is assumed instead of being concluded. We will see that the foliation framework is large enough to provide a proof showing that minimizers of nonlocal elliptic functionals are viscosity solutions.

5.1 Introduction

Null-Lagrangians and calibrations have played a prominent role in the Calculus of Variations, since they provide sufficient conditions for the minimality of critical points. Important examples are those calibrations constructed in the presence of a *field of extremals*, i.e., a foliation by critical points. These notions have their origin in the classical extremal field theory of Weierstrass and are a powerful tool to prove minimality of solutions to PDEs. Especially, they have found many relevant applications in the context of minimal surfaces.

In Chapter 4 we initiated the study of calibrations for nonlocal problems. There, we treated the simplest nonlocal model: the energy functional for the fractional Laplacian (the Gagliardo-Sobolev seminorm). In the present chapter, we extend the theory to a wide class of nonlocal functionals. Our main result is the construction of a calibration for the energy functional[1]

$$\mathcal{E}_{\mathrm{N}}(w) := \frac{1}{2} \iint_{Q(\Omega)} G_{\mathrm{N}}(x, y, w(x), w(y))\, \mathrm{d}x\, \mathrm{d}y,$$

[1]Consistent with the notation in [23], the subindices N and L are used throughout the text to denote nonlocal and local objects, respectively.

where, given a bounded domain $\Omega \subset \mathbb{R}^n$, we have written

$$Q(\Omega) := (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c) = (\Omega \times \Omega) \cup (\Omega \times \Omega^c) \cup (\Omega^c \times \Omega). \tag{5.1.1}$$

Throughout the chapter $\Omega^c = \mathbb{R}^n \setminus \Omega$, as above. The Lagrangian $G_{\mathrm{N}}(x, y, a, b)$ is required to satisfy the natural ellipticity condition

$$\partial^2_{ab} G_{\mathrm{N}}(x, y, a, b) + \partial^2_{ab} G_{\mathrm{N}}(y, x, b, a) \leq 0,$$

on which we elaborate below (see the comments before Theorem 5.1.3 and also Section 5.2).

As in the local theory, as well as in our fractional Laplacian theory developed in the previous chapter, our calibration for $\mathcal{E}_{\mathrm{N}}$ is built in the presence of a field of extremals. As mentioned above, this is a one-parameter family of critical points of $\mathcal{E}_{\mathrm{N}}$ whose graphs form a foliation (see Definition 5.1.2). For the construction, it suffices to have subsolutions and supersolutions on each respective side of a given extremal, a fact that is sometimes very useful.

A first application of our calibration concerns the minimality of monotone solutions to translation invariant nonlocal equations. More precisely, we prove that if u is a solution (with an appropriate regularity and growth at infinity, which will depend on the Lagrangian G_{N}) satisfying $\partial_{x_n} u > 0$ in $\mathbb{R}^n$, then it is a minimizer among functions w satisfying

$$\lim_{\tau \to -\infty} u(x', \tau) \leq w(x', x_n) \leq \lim_{\tau \to +\infty} u(x', \tau)$$

for $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. This result, which is related to a celebrated conjecture of De Giorgi for the Allen-Cahn equation, was only known for those nonlocal functionals for which an existence and regularity theory of minimizers is available. We explain this further in Subsection 5.1.4.

As a second application, we show that minimizers of nonlocal elliptic functionals are viscosity solutions. This type of result was previously known for problems where a weak comparison principle is available; see [5, 66, 87]. However, we can prove it in more general scenarios by using the calibration technique; see Subsection 5.1.5. This strategy was previously used by Cabré [18] in the context of nonlocal minimal surfaces.

5.1.1 Examples

Our theory covers several important elliptic functionals $\mathcal{E}_{\mathrm{N}}$ given by a Lagrangian G_{N} as above:

- The case
$$G_{\mathrm{N}}(x, y, a, b) = \frac{|a - b|^p}{2p|x - y|^{n+ps}}$$
with $p \in [1, \infty)$ and $s \in (0, 1)$, corresponds to the fractional p-Dirichlet Lagrangian, which gives rise to the fractional p-Laplace equation. More generally, considering
$$G_{\mathrm{N}}(x, y, a, b) = \frac{|a - b|^p}{2p|x - y|^{n+ps}} - \frac{1}{2|\Omega|} \mathbb{1}_{\Omega \times \Omega}(x, y)(F(a, x) + F(b, y)),$$
we can add a reaction term in the Euler-Lagrange equation. For instance, if we take $p = 2$ we recover (up to a multiplicative constant) the Lagrangian associated to the fractional

semilinear equation $(-\Delta)^s u = \partial_u F(u,x)$ in Ω, treated in our previous work [23]. Recall the expression for the fractional Laplacian:

$$(-\Delta)^s u(x) = c_{n,s}\ \text{P.V.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y,$$

where $c_{n,s}$ is a positive normalizing constant and P.V. stands for the principal value.

- The Lagrangian

$$G_{\mathrm{N}}(x,y,a,b) = \frac{G\left(\frac{a-b}{|x-y|}\right)}{|x-y|^{n+s-1}},$$

  where $s \in (0,1)$, $G''(\tau) = (1+\tau^2)^{-(n+s+1)/2}$, and $G(0) = G'(0) = 0$, recovers the fractional perimeter for subgraphs; see [41].

- The general structure

$$G_{\mathrm{N}}(x,y,a,b) = G(x-y, a-b)$$

  appears in the leading terms of the previous examples and gives rise to translation invariant equations. However, it is also of interest to treat functionals where the interactions occur only inside Ω, that is, when G_{N} is of the form

$$G_{\mathrm{N}}(x,y,a,b) = \mathbb{1}_{\Omega\times\Omega}(x,y)\ G(x-y,a-b).$$

  These Lagrangians appear, for instance, in the macroelastic energy from Peridynamics; see [88]. In this case, G might be compactly supported in the $(x-y)$-variable.

- The case

$$G_{\mathrm{N}}(x,y,a,b) = -\mathbb{1}_{\Omega\times\Omega}(x,y)K(x-y)\,ab + \frac{1}{2|\Omega|}\mathbb{1}_{\Omega\times\Omega}(x,y)(F(a)+F(b))$$

  corresponds to convolution-type operators. Functionals of this type appear in numerous problems, but most notably in the framework of constrained minimization (not treated in our setting); see, for instance, [4, 7, 35, 70] where the first term is the interaction energy and the second one is the entropy. Let us point out that, when the kernel K is even, the ellipticity condition boils down to the nonnegativity of K.

5.1.2 Calibrations and fields of extremals

As mentioned in the previous chapter, a fundamental problem in the Calculus of Variations is to find conditions for a function to be a minimizer of a given energy functional. More precisely, given a functional $\mathcal{E}\colon \mathcal{A} \to \mathbb{R}$ defined on some set of admissible functions $\mathcal{A}$, and given $u \in \mathcal{A}$, one wishes to know whether u minimizes $\mathcal{E}$ among competitors in $\mathcal{A}$ having the same Dirichlet condition as u. For nonlocal problems, given a bounded domain Ω, the Dirichlet condition refers to the value of the function in all the exterior of Ω, namely, in $\Omega^c = \mathbb{R}^n \setminus \Omega$.

One useful method to show the minimality of a given function $u \in \mathcal{A}$ consists of constructing a calibration. This is an auxiliary functional touching the energy $\mathcal{E}$ by below at u and satisfying a null-Lagrangian property.

Definition 5.1.1. A functional $\mathcal{C}\colon \mathcal{A} \to \mathbb{R}$ is a *calibration* for the functional $\mathcal{E}$ and the admissible function $u \in \mathcal{A}$ if the following conditions hold:

($\mathcal{C}1$) $\mathcal{C}(u) = \mathcal{E}(u)$.

($\mathcal{C}2$) $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all $w \in \mathcal{A}$ with the same Dirichlet condition as u.

($\mathcal{C}3$) $\mathcal{C}(w) = \mathcal{C}(\widetilde{w})$ for all $w, \widetilde{w} \in \mathcal{A}$ with the same Dirichlet condition as u.

Functionals satisfying ($\mathcal{C}3$) are known as *null-Lagrangians*. As in our previous work [23], it is convenient to relax this last condition to the less stringent

($\mathcal{C}3'$) $\mathcal{C}(u) \leq \mathcal{C}(w)$ for all $w \in \mathcal{A}$ with the same Dirichlet condition as u.

We still refer to functionals satisfying ($\mathcal{C}1$), ($\mathcal{C}2$), and ($\mathcal{C}3'$) as calibrations.

Once a calibration is available, the minimality of u among admissible functions with the same Dirichlet condition follows immediately. For this, simply apply ($\mathcal{C}1$), ($\mathcal{C}3'$), and ($\mathcal{C}2$), in this order.

Historically, motivated by classical problems in Mechanics and Geometry, significant efforts have been put into rigorously understanding minimizers of general functionals of the form

$$\mathcal{E}_{\mathrm{L}}(w) := \int_{\Omega} G_{\mathrm{L}}\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x. \tag{5.1.2}$$

It is well-known that every minimizer is a critical point of $\mathcal{E}_{\mathrm{L}}$ (an *extremal*) and must satisfy the associated Euler-Lagrange equation. Conversely, if the Lagrangian $G_{\mathrm{L}}(x, \lambda, q)$ is convex in the variables (λ, q), then the functional $\mathcal{E}_{\mathrm{L}}$ is convex and every critical point is a minimizer. This convexity assumption is too restrictive for many relevant applications, such as the Allen-Cahn energy. For these functionals, the Dirichlet problem may admit several extremals, not all of them being minimizers. Nevertheless, one often has that the Lagrangian $G_{\mathrm{L}}(x, \lambda, q)$ is convex with respect to the gradient variable q, which amounts to the *ellipticity* of the problem.

A systematic theory of calibrations has been developed for functionals $\mathcal{E}_{\mathrm{L}}$ of the form (5.1.2), namely, the *extremal field theory* going back to works of Weierstrass. The key idea is to assume the existence of a family of critical points $u^t \colon \overline{\Omega} \to \mathbb{R}$, with t in some interval $I \subset \mathbb{R}$, whose graphs do not intersect each other. Thus, the graphs of these functions produce a foliation of a certain region $\mathcal{G}$ in $\mathbb{R}^n \times \mathbb{R}$, which allows to carry out a subtle convexity argument to bound the nonconvex functional by below with a calibration.

Next, we recall our definition of field for nonlocal problems, as introduced in [23]:

Definition 5.1.2. Given an interval $I \subset \mathbb{R}$ (not necessarily bounded, nor open), we say that a family $\{u^t\}_{t\in I}$ of functions $u^t \colon \mathbb{R}^n \to \mathbb{R}$ is a *field in* $\mathbb{R}^n$ if

- the function $(x, t) \mapsto u^t(x)$ is continuous in $\mathbb{R}^n \times I$;
- for each $x \in \mathbb{R}^n$, the function $t \mapsto u^t(x)$ is C^1 and increasing in I.

Given a functional $\mathcal{E}$ acting on functions defined in $\mathbb{R}^n$, and given a bounded domain $\Omega \subset \mathbb{R}^n$, we say that $\{u^t\}_{t\in I}$ is a *field of extremals* in Ω (for $\mathcal{E}$) when it is a field in $\mathbb{R}^n$ and each of the functions u^t is a critical point of $\mathcal{E}$ in Ω.

Given a field in $\mathbb{R}^n$ as above, the region

$$\mathcal{G} = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} \colon \lambda = u^t(x) \text{ for some } t \in I\} \subset \mathbb{R}^n \times \mathbb{R}$$

is foliated by the graphs of the functions u^t, which do not intersect each other (since $u^t(x)$ is increasing in t). In particular, we can uniquely define a *leaf-parameter function*

$$t\colon \mathcal{G} \to I, \quad (x,\lambda) \mapsto t(x,\lambda) \quad \text{determined by} \quad u^{t(x,\lambda)}(x) = \lambda. \tag{5.1.3}$$

The function t is continuous in $\mathcal{G}$ by the assumptions in Definition 5.1.2. We will often refer to the functions u^t (or their graphs) as the "*leaves*" of the field.

Next, let us recall the fundamental result of the classical extremal field theory. Namely, given an elliptic Lagrangian[2] G_{L} and $\{u^t\}_{t\in I}$ a smooth field of extremals in Ω, the functional

$$\begin{aligned} \mathcal{C}_{\mathrm{L}}(w) := \int_\Omega \Big\{ \partial_q G_{\mathrm{L}}(x, u^t(x), \nabla u^t(x)) \cdot \big(\nabla w(x) - \nabla u^t(x)\big) \Big\} \Big|_{t=t(x,w(x))} \,\mathrm{d}x \\ + \int_\Omega G_{\mathrm{L}}(x, u^t(x), \nabla u^t(x)) \big|_{t=t(x,w(x))} \,\mathrm{d}x, \end{aligned} \tag{5.1.4}$$

is a calibration for the functional $\mathcal{E}_{\mathrm{L}}$ and each critical point u^{t_0}, $t_0 \in I$. In particular, each leaf u^{t_0} minimizes $\mathcal{E}_{\mathrm{L}}$ among competitors w satisfying $w = u^{t_0}$ on $\partial\Omega$ and whose graphs lie in the region $\mathcal{G}$. Moreover, in [23, Theorem 3.1] we found the following alternative expression for the calibration $\mathcal{C}_{\mathrm{L}}$. For each $t_0 \in I$, we have that $\mathcal{C}_{\mathrm{L}}$ in (5.1.4) can be written as

$$\begin{aligned} \mathcal{C}_{\mathrm{L}}(w) = \int_\Omega \int_{u^{t_0}(x)}^{w(x)} \mathcal{L}_{\mathrm{L}}(u^t)(x) \Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x \\ + \int_{\partial\Omega} \int_{u^{t_0}(x)}^{w(x)} \mathcal{N}_{\mathrm{L}}(u^t)(x) \big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{H}^{n-1}(x) + \mathcal{E}_{\mathrm{L}}(u^{t_0}), \end{aligned} \tag{5.1.5}$$

where $\mathcal{L}_{\mathrm{L}}$ and $\mathcal{N}_{\mathrm{L}}$ are, respectively, the Euler-Lagrange and Neumann operators associated to the functional $\mathcal{E}_{\mathrm{L}}$ in (5.1.2). As in the fractional Laplacian framework treated in [23], our new nonlocal calibration given in Theorem 5.1.3 below will be based on this identity.

While the theory of calibrations for local equations is well understood, there are very few papers prior to [23] dealing with nonlocal ones, which we mention next. In [18], Cabré gave a calibration for the fractional perimeter. Independently, Pagliari [80] investigated the abstract structure of calibrations for the fractional total variation. This last functional involves the fractional perimeter of each sublevel set of a given function. The author succeeded in constructing a calibration to prove that the characteristic functions of halfspaces are minimizers, but other fields of extremals are not mentioned in that work. Our present work provides, as a particular case, a calibration for the fractional total variation in the presence of a general field of extremals. Moreover, we can relate our construction with the calibration for the fractional perimeter in [18] applied to each superlevel set; see Appendix J.

In Chapter 4 we constructed a calibration for the energy associated to semilinear equations involving the fractional Laplacian, that is, for energies of the form

$$\mathcal{E}_{s,F}(w) = \frac{c_{n,s}}{4} \iint_{Q(\Omega)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x \,\mathrm{d}y - \int_\Omega F(w(x)) \,\mathrm{d}x.$$

[2]Recall that here ellipticity means that $G_{\mathrm{L}}(x,\lambda,q)$ is convex with respect to the gradient variable q. However, for (5.1.4) to be a calibration, a weaker condition than convexity in q suffices. One needs to assume that each u^t satisfies the *Weierstrass sufficient condition*, namely

$$G_{\mathrm{L}}(x, u^t(x), q) \geq G_{\mathrm{L}}(x, u^t(x), \nabla u^t(x)) + \partial_q G_{\mathrm{L}}(x, u^t(x), \nabla u^t(x)) \cdot \big(q - \nabla u^t(x)\big)$$

for all $x \in \Omega$, $q \in \mathbb{R}^n$, and $t \in I$; see [23] for more details.

Given $\{u^t\}_{t\in I}$ a field of extremals in Ω, we showed that

$$\mathcal{C}_{s,F}(w) = c_{n,s}\,\mathrm{P.V.}\iint_{Q(\Omega)} \int_{u^{t_0}(x)}^{w(x)} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x$$
$$+\frac{c_{n,s}}{4}\iint_{Q(\Omega)} \frac{|u^{t_0}(x)-u^{t_0}(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y$$

is a calibration for $\mathcal{E}_{s,F}$ and u^{t_0}, $t_0\in I$. We recall that the expression of $\mathcal{C}_{s,F}$ was obtained by replacing the operators $\mathcal{L}_{\mathrm{L}}$ and $\mathcal{N}_{\mathrm{L}}$ appearing in (5.1.5) by their nonlocal counterparts.

5.1.3 Main result

Next, we present our main result, which builds a calibration for the functional

$$\mathcal{E}_{\mathrm{N}}(w) = \frac{1}{2}\iint_{Q(\Omega)} G_{\mathrm{N}}(x,y,w(x),w(y))\,\mathrm{d}x\,\mathrm{d}y \tag{5.1.6}$$

when the Lagrangian $G_{\mathrm{N}}(x,y,a,b)$ satisfying the condition[3]

$$\partial^2_{ab}G_{\mathrm{N}}(x,y,a,b) + \partial^2_{ab}G_{\mathrm{N}}(y,x,b,a) \le 0. \tag{5.1.7}$$

We will see that (5.1.7) guarantees the ellipticity of the problem (or a strong comparison principle, see Appendix I). It will also ensure that the calibrating functional defined in Theorem 5.1.3 below satisfies property ($\mathcal{C}2$), thus mirroring the effect of ellipticity in the local case.

As in the classical theory, in this general nonlocal framework every extremal is a minimizer whenever the functional $\mathcal{E}_{\mathrm{N}}$ is convex. A sufficient condition to guarantee the convexity of $\mathcal{E}_{\mathrm{N}}$ is that the Lagrangian $(a,b)\mapsto G_{\mathrm{N}}(x,y,a,b)+G_{\mathrm{N}}(y,x,b,a)$ be convex. Contrary to the local case, this hypothesis does not guarantee the ellipticity assumption (5.1.7).[4] This seems to be due to the great generality of (5.1.6). Nevertheless, in most examples we have in mind, the Lagrangian has a leading term of the form $G(x,y,a-b)$ for which ellipticity does follow from convexity in the $(a-b)$-variable. For instance, consider the linear equation $(-\Delta)^s u = \lambda u$, with $\lambda\in\mathbb{R}$. This equation admits an energy functional

$$\frac{c_{n,s}}{4}\iint_{Q(\Omega)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y - \frac{\lambda}{2}\int_\Omega u(x)^2\,\mathrm{d}x$$

which is elliptic in the sense of (5.1.7) but not always convex (when λ is large enough). Notice that the equation satisfies the strong comparison principle, while the availability of the weak comparison principle depends on λ.[5]

[3]Compare it with (5.2.15) in Section 5.2, where the Lagrangian additionally satisfies the pairwise symmetry condition (5.2.1) (a condition that can always be assumed without loss of generality).

[4]For pairwise symmetric Lagrangians (see (5.2.1) below) ellipticity reduces to $\partial^2_{ab}G_{\mathrm{N}}\le 0$ while convexity amounts to the conditions $\partial^2_{aa}G_{\mathrm{N}}\ge 0$ and $\partial^2_{aa}G_{\mathrm{N}}\partial^2_{bb}G_{\mathrm{N}}\ge(\partial^2_{ab}G_{\mathrm{N}})^2$. The reader can check the non-equivalence with the simple quadratic examples $G_{\mathrm{N}}=\pm K(x-y)(a\pm b)^2$, with $K>0$.

[5]We say that an operator $\mathcal{L}$ satisfies the *strong comparison principle* if, given two functions u and v satisfying $\mathcal{L}u\le\mathcal{L}v$ in Ω, $u\le v$ in $\mathbb{R}^n$, and touching somewhere in Ω, then $u\equiv v$ in $\mathbb{R}^n$. By contrast, $\mathcal{L}$ satisfies the *weak comparison principle* if, given two functions u and v satisfying $\mathcal{L}u\le\mathcal{L}v$ in Ω and $u\le v$ in Ω^c, then $u\le v$ in Ω.

For the functional $\mathcal{E}_{\mathrm{N}}$ and its associated Euler-Lagrange operator

$$\mathcal{L}_{\mathrm{N}}(w)(x) := \frac{1}{2}\int_{\mathbb{R}^n} \Big(\partial_a G_{\mathrm{N}}(x,y,w(x),w(y)) + \partial_b G_{\mathrm{N}}(y,x,w(y),w(x))\Big)\,\mathrm{d}y \tag{5.1.8}$$

to be well defined for $x \in \Omega$, one needs to make growth and regularity assumptions on the Lagrangian G_{N}. These determine the class of admissible functions; see [15] for some examples of natural assumptions. In this respect, our main result (Theorem 5.1.3 below), which gives a calibration for general nonlocal Lagrangians satisfying the ellipticity hypothesis (5.1.7), is a formal result since it does not specify the precise class of admissible functions. In other words, the great generality of the functions does not allow for specifying the growth and regularity assumptions on G_{N} and on the admissible functions w. Thus, the theorem cannot take into account any integrability issues.[6] However, we could give completely rigorous results for some specific families of Lagrangians, adapting the admissible class of functions to the concrete problem. Indeed, within the proof of the next theorem, there are only a few points that must be justified, namely, the interchange of certain integrals and the convergence of some expressions. Hence, in the following statement we use the term "sufficiently regular for G_{N}" in the sense that those functions make all integrals to be well defined.

Recall (5.1.1) for the meaning of $Q(\Omega)$, Definition 5.1.2 for the notion of field, and (5.1.3) for the leaf-parameter function t. The calibration properties ($\mathcal{C}1$), ($\mathcal{C}2$), ($\mathcal{C}3$), and ($\mathcal{C}3'$) have been introduced in Definition 5.1.1.

Theorem 5.1.3. *Let $I \subset \mathbb{R}$ be an interval and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Given a function $G_{\mathrm{N}} = G_{\mathrm{N}}(x,y,a,b)$ satisfying the ellipticity condition* (5.1.7)*, let $\{u^t\}_{t\in I}$ be a field in $\mathbb{R}^n$ (in the sense of Definition 5.1.2) which is sufficiently regular for G_{N}.*

Given $t_0 \in I$, let $\mathcal{E}_{\mathrm{N}}$ be defined by (5.1.6) *and $\mathcal{C}_{\mathrm{N}}$ be the functional*

$$\begin{aligned}&\mathcal{C}_{\mathrm{N}}(w)\\ &\quad := \frac{1}{2}\iint_{Q(\Omega)}\int_{u^{t_0}(x)}^{w(x)} \Big\{\partial_a G_{\mathrm{N}}(x,y,u^t(x),u^t(y)) + \partial_b G_{\mathrm{N}}(y,x,u^t(y),u^t(x))\Big\}\Big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y\\ &\qquad + \mathcal{E}_{\mathrm{N}}(u^{t_0})\end{aligned}$$

defined in a set $\mathcal{A}_{\mathrm{N}}$ of sufficiently regular admissible functions $w\colon \mathbb{R}^n \to \mathbb{R}$ (for G_{N}) satisfying graph $w \subset \mathcal{G}$*, where*

$$\mathcal{G} = \big\{(x,\lambda) \in \mathbb{R}^n \times \mathbb{R} \colon \lambda = u^t(x) \quad \textit{for some } t \in I\big\}.$$

Taking $\mathcal{C} = \mathcal{C}_{\mathrm{N}}$ and $\mathcal{E} = \mathcal{E}_{\mathrm{N}}$ in Definition 5.1.1, we have the following:

(a) *$\mathcal{C}_{\mathrm{N}}$ satisfies ($\mathcal{C}1$) and ($\mathcal{C}2$) with $u = u^{t_0}$.*

(b) *Assume in addition that the family $\{u^t\}_{t\in I}$ satisfies*

$$\begin{aligned}\mathcal{L}_{\mathrm{N}}(u^t) \geq 0 \quad &\textit{in } \Omega \quad \textit{for } t \geq t_0,\\ \mathcal{L}_{\mathrm{N}}(u^t) \leq 0 \quad &\textit{in } \Omega \quad \textit{for } t \leq t_0,\end{aligned}$$

where $\mathcal{L}_{\mathrm{N}}$ is the Euler-Lagrange operator associated to $\mathcal{E}_{\mathrm{N}}$ given by (5.1.8)*. Then, $\mathcal{C}_{\mathrm{N}}$ satisfies ($\mathcal{C}3'$) with $u = u^{t_0}$. In particular, u^{t_0} minimizes $\mathcal{E}_{\mathrm{N}}$ among functions w in $\mathcal{A}_{\mathrm{N}}$ such that $w \equiv u^{t_0}$ in Ω^c.*

[6] This is in contrast with Theorem 1.3 in [23], where we gave a fully rigorous result for the fractional Laplacian.

(c) *Assume in addition that* $\{u^t\}_{t\in I}$ *is a field of extremals in* Ω*, that is, a field in* $\mathbb{R}^n$ *satisfying*

$$\mathcal{L}_{\mathrm{N}}(u^t) = 0 \quad \text{in } \Omega \quad \text{for all } t \in I.$$

Then, the functional $\mathcal{C}_{\mathrm{N}}$ *satisfies* ($\mathcal{C}3$) *with* $u = u^{t_0}$*. Therefore,* $\mathcal{C}_{\mathrm{N}}$ *is a calibration for* $\mathcal{E}_{\mathrm{N}}$ *and* u^{t_0}*. As a consequence, for every* $t \in I$*, the extremal* u^t *minimizes* $\mathcal{E}_{\mathrm{N}}$ *among functions* w *in* $\mathcal{A}_{\mathrm{N}}$ *such that* $w \equiv u^t$ *in* Ω^c*.*

As mentioned above, the class of functionals $\mathcal{E}_{\mathrm{N}}$ of the form (5.1.6) satisfying the ellipticity condition (5.1.7) includes the Gagliardo-Sobolev seminorm (for which we constructed a calibration in Chapter 4 above) as well as the fractional total variation (see Appendix J) and the examples in Subsection 5.1.1. Our calibration in Theorem 5.1.3 is a generalization of the one in the previous chapter. To guess the expression of $\mathcal{C}_{\mathrm{N}}$ above, we extrapolated our new identity (5.1.5) in the local theory. The key point is that each of the terms in (5.1.5) has a clear nonlocal counterpart; see (5.2.7) below.

An interesting feature of the calibrations considered in this chapter is their stability under the addition of functionals. Due to their special structure, calibrations given in terms of fields can be added together to obtain new ones. In particular, the local theory can be combined with the nonlocal one developed in this work to produce calibrations for energies involving both local and nonlocal interaction terms. We explain this further in Section 5.3.

5.1.4 An application to monotone solutions

Our interest in fields of extremals came from the study of *monotone solutions* to the fractional Allen-Cahn equation

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^n \tag{5.1.9}$$

(see [30, 31], as well as [42] for more general integro-differential operators). When the operator is the classical Laplacian, these solutions are related to a celebrated conjecture of De Giorgi; see [26].

In Corollary 4.1.4 above, we proved that monotone solutions of (5.1.9) are minimizers among competitors taking values in a precise region of space (the region specified in the next corollary). Thanks to Theorem 5.1.3 of the current chapter, the same proof allows to establish the minimality of monotone solutions to more general nonlocal translation invariant equations. More precisely, given a Lagrangian of the form $G_{\mathrm{N}} = G_{\mathrm{N}}(x-y, a, b)$, with associated energy functional $\mathcal{E}_{\mathrm{N}}$ defined by (5.1.6), the Euler-Lagrange operator $\mathcal{L}_{\mathrm{N}}$ given by (5.1.8) is translation invariant, that is, for all x and z in $\mathbb{R}^n$ the identity

$$\mathcal{L}_{\mathrm{N}}(w)(x+z) = \mathcal{L}_{\mathrm{N}}(w(\cdot + z))(x)$$

holds. We then have the following:

Corollary 5.1.4. *Let* $G_{\mathrm{N}} = G_{\mathrm{N}}(x-y, a, b)$ *be a function satisfying the ellipticity condition* (5.1.7)*. Let* u *be a sufficiently regular solution for* G_{N} *(see the comments before Theorem 5.1.3) of* $\mathcal{L}_{\mathrm{N}}(u) = 0$ *in* $\mathbb{R}^n$*. Assume that* u *is increasing in the* x_n*-variable, i.e.,*

$$\partial_{x_n} u > 0 \quad \text{in } \mathbb{R}^n. \tag{5.1.10}$$

Then, for each bounded domain $\Omega \subset \mathbb{R}^n$, u is a minimizer of $\mathcal{E}_{\mathrm{N}}$ among sufficiently regular admissible functions w satisfying

$$\lim_{\tau \to -\infty} u(x', \tau) \leq w(x', x_n) \leq \lim_{\tau \to +\infty} u(x', \tau) \quad \textit{for all} \;\; x = (x', x_n) \in \Omega$$

and such that $w \equiv u$ in Ω^c.

This minimality result was already known for the fractional Laplacian $\mathcal{L}_{\mathrm{N}} = (-\Delta)^s$. For such an operator, it can be proven with an alternative argument (described in the Introduction of [23]) which does not use any calibration. However, such an alternative proof requires an existence and regularity theorem for minimizers, as explained in Appendix I. Such a result is not available for many general Lagrangians of the form $G_{\mathrm{N}}(x-y, a, b)$. For these functionals, Corollary 5.1.4 allows to establish the minimality of monotone solutions for the first time.

Notice that, given a monotone solution, the translation invariance of the equation is all what is needed in order to produce a field of extremals (by sliding the solution in the x_n-variable). Therefore, Corollary 5.1.4 also holds for translation invariant equations involving both local and nonlocal terms; see Section 5.3.

5.1.5 An application to the viscosity theory

Here we are interested in conditions to ensure that minimizers, or more generally weak solutions, are viscosity solutions. These are different notions of solutions that have been formulated both for differential and for nonlocal equations. Within the Calculus of Variations, it is natural to work with *weak solutions* belonging to the energy space. On the contrary, when dealing with fully nonlinear equations, it is more suitable to work with *viscosity solutions*. Here, the equation is transferred to act on smooth functions touching the extremal from one side.

In the local framework, it has been shown in the literature that minimizers of many relevant functionals are viscosity solutions. For the p-Laplace equation (here every weak solution is a minimizer), Juutinen, Lindqvist, and Manfredi [65] obtained the result by using a weak comparison principle. This allows to compare the minimizer with a function touching it by below and which is later slid upwards, forcing the equation to have the correct sign. For functionals of the form (5.1.2), assuming convexity (a stronger condition than ellipticity), Barron and Jensen [6] found a simpler variational argument. We comment on their strategy at the end of the present subsection as well as in Remark 5.4.9. Showing that non-minimizing weak solutions are viscosity solutions has also been treated in the literature. For instance, this has been done by Medina and Ochoa [74] for semilinear equations driven by the p-Laplacian. Their proof again uses a comparison principle.

Concerning nonlocal problems, the first results in this direction appeared in the geometric setting. Caffarelli, Roquejoffre, and Savin [32] showed that minimizers to the nonlocal perimeter are viscosity solutions of the homogeneous nonlocal mean curvature equation. Their proof is quite involved and uses a comparison principle. Later, Cabré [18] was able to show the same result via a simpler calibration argument (here we will give the analogue of this result in the functional setting). The case of nonlocal minimal graphs has also been treated by Cozzi and Lombardini [41]. In the functional setting, as far as we know, the first nonlocal result appeared in the work of Servadei and Valdinoci [87] for linear equations involving the fractional Laplacian. There, the authors employ a regularization by convolution that is not available for other operators. For equations driven by the fractional

p-Laplacian, we mention the paper by Korvenpää, Kuusi, and Lindgren [66] where they treat the homogeneous problem, and the work by Barrios and Medina [5] for the semilinear one. In both cases, a comparison principle is needed.

Next, we state the main result of this subsection. We will show that every minimizer of our elliptic nonlocal functionals is a viscosity solution. In contrast with most of the previous works, the novelty of our result is that we do not need a weak comparison principle, allowing us to treat a bigger class of Lagrangians. This is achieved by a calibration argument. In a way, the information given by the weak comparison principle is already contained in the properties ($\mathcal{C}1$)-($\mathcal{C}3$) satisfied by the calibration. Recall, as explained at the beginning of Subsection 5.1.3, that the weak comparison principle does not follow from ellipticity. However, the ellipticity of the Lagrangian (condition (5.1.7) above) suffices for the calibration argument in our proof.

Our theorem applies to general nonlocal elliptic functionals of the form (5.1.6). Since we do not make any growth and regularity assumptions on the Lagrangian G_{N}, as in the main theorem above, our result is only formal. Nevertheless, again, we could give a fully rigorous statement for specific families of Lagrangians. In fact, this is what we do in Section 5.4 for fractional semilinear equations.

Theorem 5.1.5. *Let $G_{\mathrm{N}} = G_{\mathrm{N}}(x, y, a, b)$ be a function satisfying the ellipticity condition* (5.1.7) *and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let u be a sufficiently regular minimizer of the functional $\mathcal{E}_{\mathrm{N}}$ given by* (5.1.6)*.*

Then, the function u is a viscosity solution of the associated Euler-Lagrange equation $\mathcal{L}_{\mathrm{N}}(u) = 0$ in Ω.

Later in Section 5.4, we will give a more precise statement of this result, showing that minimizers by above (below) are viscosity supersolutions (subsolutions). Furthermore, while our theorem only applies to minimizers, we will explain how it can be used to prove that certain non-minimizing weak solutions are viscosity solutions. Here, the idea is to "freeze" the lower order terms; see Remark 5.4.10.

The proof of Theorem 5.1.5 is based on the following energy comparison result for ordered functions embedded in a *weak field* (that is, a "degenerate field" where the leaves are still ordered, but may touch each other; see Figure 5.1 and Definition 5.4.1). Thus, here we will need to extend the above theory of nonlocal calibrations to the more general setting of weak fields.

Theorem 5.1.6. *Let $G_{\mathrm{N}} = G_{\mathrm{N}}(x, y, a, b)$ be a function satisfying the ellipticity condition* (5.1.7)*. Given a bounded domain $\Omega \subset \mathbb{R}^n$, let u and v belong to $C(\overline{\Omega})$ and satisfy $u = v$ a.e. in Ω^c and $u \leq v$ in Ω.*

Assume that there exists a weak field $\{\varphi^t\}_{t\in[0,T]}$ for u and v (in the sense of Definition 5.4.1) which is sufficiently regular for G_{N} (see the comments at the beginning of Subsection 5.4.3).

Then, if $\mathcal{E}_{\mathrm{N}}(u) < \infty$, we have

$$\mathcal{E}_{\mathrm{N}}(v) \leq \mathcal{E}_{\mathrm{N}}(u) + \int_\Omega \int_{u(x)}^{v(x)} \mathcal{L}_{\mathrm{N}}\left(\varphi^t(x)\right)\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x.$$

Let us point out that, in Section 5.4, we will prove analogous results to Theorems 5.1.5 and 5.1.6 in the fractional semilinear setting, giving in this case fully rigorous statements under precise regularity assumptions (see Theorems 5.4.6 and 5.4.3 respectively). Furthermore, the same proof will allow us to prove Theorems 5.1.5 and 5.1.6 in the more general

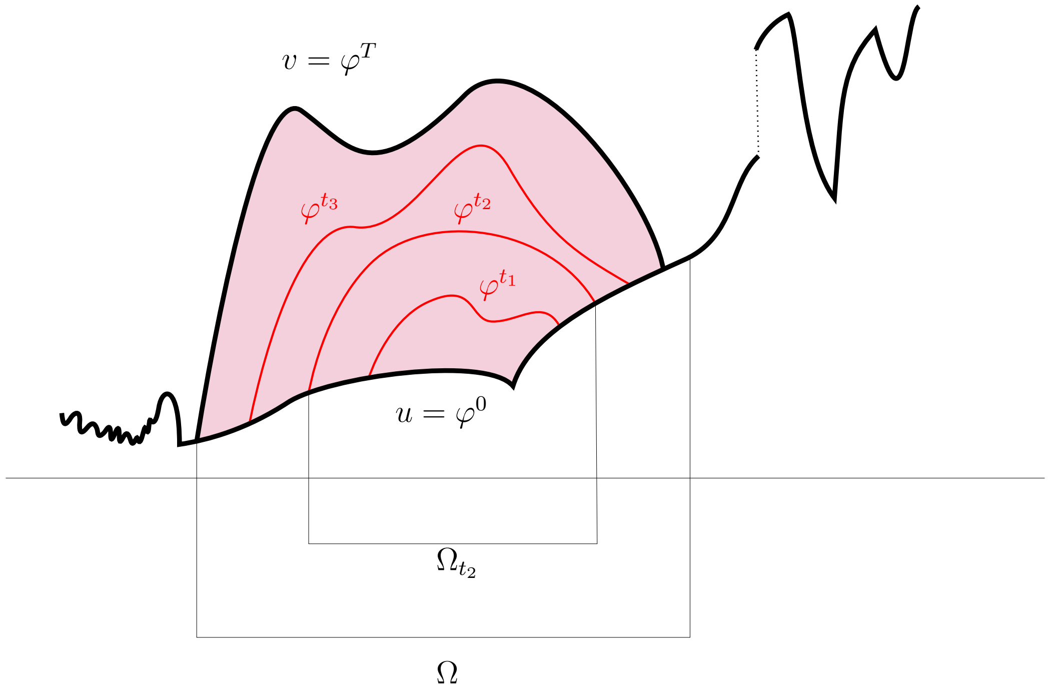

Figure 5.1: Example of a weak field for two functions u and v.

setting of "mixed" functionals involving both local and nonlocal terms (see Theorems 5.4.8 and 5.4.7 below).

The energy inequality in Theorem 5.1.6 is established as in our previous calibration arguments. Once available, we can prove Theorem 5.1.5. Indeed, assume that u is a minimizer and that a smooth function φ touches u by below at some contact point. Now, we slide φ upwards and take the maxima with u to obtain a weak field. The energy comparison with $v = \varphi^T$ will show that φ must be a supersolution at the contact point, since otherwise u would not be a minimizer. Applying the same procedure to smooth functions touching u by above, we will conclude that u is a viscosity solution.

Finally, let us mention that a simple variational proof of our result can be given without using the calibration argument. For this, in addition to the ellipticity condition (5.1.7), one needs to assume that the function $(a, b) \mapsto G_{\rm N}(x, y, a, b) + G_{\rm N}(y, x, b, a)$ is convex; see footnote 4. The proof is a nonlocal counterpart of the one by Barron and Jensen [6]. We explain this further in Remark 5.4.9.

5.1.6 Outline of the chapter

Section 5.2 contains the proofs of Theorem 5.1.3 and Corollary 5.1.4. In Section 5.3 we explain how to combine the local and nonlocal theory to obtain calibrations for mixed energy functionals. In Section 5.4 we apply the calibration formalism to the theory of viscosity solutions. First, we prove the fully rigorous results for the fractional Laplacian (Theorems 5.4.3 and 5.4.6) with all details in regularity and integrability issues. Then, we show Theorems 5.1.5 and 5.1.6 (contained, respectively, in the more general Theorems 5.4.8 and 5.4.7).

5.2 The calibration for general nonlocal functionals

Having obtained a calibration for the semilinear problem involving the fractional Laplacian in the previous chapter, we are now interested in extending this construction to a general class of nonlocal functionals. In this way, we plan to obtain a similar picture to that of the general local theory treated in Section 4.3 above. We find a functional $\mathcal{C}_{\mathrm{N}}$ that, at least at the formal level, is a calibration for the nonlocal energy functional $\mathcal{E}_{\mathrm{N}}$. We say at the formal level since the appropriate regularity assumptions on the field of extremals will depend on the concrete given functional $\mathcal{E}_{\mathrm{N}}$ and its associated nonlocal problem.

Consider the nonlocal energy functional $\mathcal{E}_{\mathrm{N}}$ of the form (5.1.6). Since $Q(\Omega)$ is invariant with respect to the reflection $(x,y) \mapsto (y,x)$, we may assume without loss of generality that the Lagrangian G_{N} is *pairwise symmetric*,[7] that is,

$$G_{\mathrm{N}}(y,x,b,a) = G_{\mathrm{N}}(x,y,a,b) \quad \text{for all } (x,y) \in Q(\Omega) \text{ and } (a,b) \in \mathbb{R}^2. \tag{5.2.1}$$

In particular, from the pairwise symmetry it follows that

$$\partial_b G_{\mathrm{N}}(x,y,\widetilde{a},\widetilde{b}) = \partial_a G_{\mathrm{N}}(y,x,\widetilde{b},\widetilde{a}) \quad \text{for all } (x,y) \in Q(\Omega) \text{ and } (\widetilde{a},\widetilde{b}) \in \mathbb{R}^2. \tag{5.2.2}$$

The first variation of $\mathcal{E}_{\mathrm{N}}$ at u in the direction of $\eta \in C_c^\infty(\mathbb{R}^n)$ (notice that η is not necessarily supported in Ω) is given by

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{E}_{\mathrm{N}}(u+\varepsilon\eta)\Big|_{\varepsilon=0} \\
&= \frac{1}{2}\iint_{Q(\Omega)} \partial_a G_{\mathrm{N}}(x,y,u(x),u(y))\,\eta(x)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\iint_{Q(\Omega)} \partial_b G_{\mathrm{N}}(x,y,u(x),u(y))\,\eta(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\iint_{Q(\Omega)} \partial_a G_{\mathrm{N}}(x,y,u(x),u(y))\,\eta(x)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\iint_{Q(\Omega)} \partial_b G_{\mathrm{N}}(y,x,u(y),u(x))\,\eta(x)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{Q(\Omega)} \partial_a G_{\mathrm{N}}(x,y,u(x),u(y))\,\eta(x)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

where we have used the symmetry of $Q(\Omega)$ and the identity (5.2.2).

Writing the domain $Q(\Omega)$ as the disjoint union $Q(\Omega) = (\Omega \times \mathbb{R}^n) \cup (\Omega^c \times \Omega)$, we can split the last integral to obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{E}_{\mathrm{N}}(u+\varepsilon\eta)\Big|_{\varepsilon=0} = \int_\Omega \mathcal{L}_{\mathrm{N}}(u)(x)\,\eta(x)\,\mathrm{d}x + \int_{\Omega^c} \mathcal{N}_{\mathrm{N}}(u)(x)\,\eta(x)\,\mathrm{d}x, \tag{5.2.3}$$

where we have introduced the nonlinear operators

$$\mathcal{L}_{\mathrm{N}}(u)(x) := \int_{\mathbb{R}^n} \partial_a G_{\mathrm{N}}(x,y,u(x),u(y))\,\mathrm{d}y$$

and

$$\mathcal{N}_{\mathrm{N}}(u)(x) := \int_{\Omega} \partial_a G_{\mathrm{N}}(x,y,u(x),u(y))\,\mathrm{d}y.$$

Consistent with the terminology in [23], we refer to $\mathcal{L}_{\mathrm{N}}$ as the *Euler-Lagrange operator* associated to $\mathcal{E}_{\mathrm{N}}$, while $\mathcal{N}_{\mathrm{N}}$ is its associated nonlocal *Neumann operator*.

[7] Here we follow the terminology of [49].

Since we are interested in minimization problems with respect to functions with the same exterior data, we only consider variations η that are compactly supported in Ω. Thus, an extremal u of $\mathcal{E}_{\mathrm{N}}$ will satisfy the Euler-Lagrange equation

$$\mathcal{L}_{\mathrm{N}}(u) = 0 \quad \text{in } \Omega. \tag{5.2.4}$$

Given an interval $I \subset \mathbb{R}$, let $u^t \colon \mathbb{R}^n \to \mathbb{R}$ be a field in $\mathbb{R}^n$ (in the sense of Definition 5.1.2), with $t \in I$, which covers the region

$$\mathcal{G} := \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : \ \lambda = u^t(x) \text{ for some } t \in I\}.$$

Let us also consider the class of admissible functions

$$\mathcal{A}_{\mathrm{N}} := \{w \colon \mathbb{R}^n \to \mathbb{R} : \ w \text{ is sufficiently regular for } G_{\mathrm{N}} \text{ and } \operatorname{graph} w \subset \mathcal{G}\}, \tag{5.2.5}$$

where "sufficiently regular" refers to the following issue. Since we are not making any growth or structure assumption on G_{N}, the class of functions w for which $\mathcal{E}_{\mathrm{N}}(w)$ makes sense must be chosen according to each nonlocal functional under investigation. This will be the functions considered in $\mathcal{A}_{\mathrm{N}}$, which may contain further regularity restrictions so that the operators $\mathcal{L}_{\mathrm{N}}$ and $\mathcal{N}_{\mathrm{N}}$, as well as all the integrals in the proofs are well defined.

Let $t_0 \in I$. Our goal is to construct a calibration for $\mathcal{E}_{\mathrm{N}}$ and u^{t_0}. We define the functional $\mathcal{C}_{\mathrm{N}}$ on $\mathcal{A}_{\mathrm{N}}$ by

$$\mathcal{C}_{\mathrm{N}}(w) := \iint_{Q(\Omega)} \int_{u^{t_0}(x)}^{w(x)} \partial_a G_{\mathrm{N}}(x, y, u^t(x), u^t(y))\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x \,\mathrm{d}y + \mathcal{E}_{\mathrm{N}}(u^{t_0}). \tag{5.2.6}$$

By the above considerations and splitting the domain into $Q(\Omega) = (\Omega \times \mathbb{R}^n) \cup (\Omega^c \times \Omega)$, we can rewrite (5.2.6) as

$$\begin{aligned} \mathcal{C}_{\mathrm{N}}(w) = & \int_{\Omega} \int_{u^{t_0}(x)}^{w(x)} \mathcal{L}_{\mathrm{N}}(u^t)(x)\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x \\ & \qquad + \int_{\Omega^c} \int_{u^{t_0}(x)}^{w(x)} \mathcal{N}_{\mathrm{N}}(u^t)(x)\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x + \mathcal{E}_{\mathrm{N}}(u^{t_0}). \end{aligned} \tag{5.2.7}$$

Notice that (5.2.7) is the "canonical" nonlocal analogue of Theorem 3.1 in [23], and thus of the classical local Weierstrass calibration $\mathcal{C}_{\mathrm{L}}$.

Next we show that if the field $\{u^t\}_{t\in I}$ is made up of supersolutions above u^{t_0} and subsolutions below, then u^{t_0} minimizes $\mathcal{C}_{\mathrm{N}}$ among functions in $\mathcal{A}_{\mathrm{N}}$ with the same exterior data. Furthermore, if *all* the functions u^t satisfy the Euler-Lagrange equation (i.e., u^t is a field of extremals), then $\mathcal{C}_{\mathrm{N}}$ is a null-Lagrangian and its value depends only on the exterior datum. The following result (properties ($\mathcal{C}3$) and ($\mathcal{C}3'$) of the calibration) follows readily from expression (5.2.7) for $\mathcal{C}_{\mathrm{N}}$. Note that here we do not need to assume the ellipticity of G_{N}.

Proposition 5.2.1. *Given an interval $I \subset \mathbb{R}$, a bounded domain $\Omega \subset \mathbb{R}^n$, and a function $G_{\mathrm{N}} = G_{\mathrm{N}}(x, y, a, b)$, let $\{u^t\}_{t\in I}$ be a field in $\mathbb{R}^n$ (in the sense of Definition 5.1.2) which is sufficiently regular for G_{N}. Assume that, for $t_0 \in I$, the leaves satisfy the inequalities*

$$\begin{aligned} \mathcal{L}_{\mathrm{N}}(u^t) \geq 0 \quad & \text{in } \Omega \quad \text{for } t \geq t_0, \quad \text{and} \\ \mathcal{L}_{\mathrm{N}}(u^t) \leq 0 \quad & \text{in } \Omega \quad \text{for } t \leq t_0. \end{aligned} \tag{5.2.8}$$

Consider the set of admissible functions $\mathcal{A}_{\mathrm{N}}$ *defined in* (5.2.5).

Then, for all w *in* $\mathcal{A}_{\mathrm{N}}$ *such that* $w \equiv u^{t_0}$ *in* Ω^c, *the functional* $\mathcal{C}_{\mathrm{N}}$ *defined in* (5.2.6) *satisfies*

$$\mathcal{C}_{\mathrm{N}}(u^{t_0}) \leq \mathcal{C}_{\mathrm{N}}(w).$$

Assume in addition that the leaves satisfy the Euler-Lagrange equation (5.2.4), *that is,*

$$\mathcal{L}_{\mathrm{N}}(u^t) = 0 \quad \text{in } \Omega \quad \text{for all } t \in I. \tag{5.2.9}$$

Then, for all w *in* $\mathcal{A}_{\mathrm{N}}$ *such that* $w \equiv u^{t_0}$ *in* Ω^c, *we have*

$$\mathcal{C}_{\mathrm{N}}(u^{t_0}) = \mathcal{C}_{\mathrm{N}}(w).$$

Proof. First, notice that $\mathcal{C}_{\mathrm{N}}(u^{t_0}) = \mathcal{E}_{\mathrm{N}}(u^{t_0})$. Since $w \equiv u^{t_0}$ in Ω^c, by (5.2.7) we have

$$\mathcal{C}_{\mathrm{N}}(w) - \mathcal{C}_{\mathrm{N}}(u^{t_0}) = \int_\Omega \int_{u^{t_0}(x)}^{w(x)} \mathcal{L}_{\mathrm{N}}(u^t)\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x.$$

Assuming (5.2.8), it suffices to show that for all $x \in \Omega$ we have

$$\int_{u^{t_0}(x)}^{w(x)} \mathcal{L}_{\mathrm{N}}(u^t)\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \geq 0. \tag{5.2.10}$$

If $w(x) \geq u^{t_0}(x)$, then, using that the functions $\{u^t\}_{t\in I}$ are increasing in t, we have $t(x,\lambda) \geq t_0$ for $\lambda \in [u^{t_0}(x), w(x)]$. Hence, by assumption (5.2.8), $\mathcal{L}_{\mathrm{N}}(u^t)\big|_{t=t(x,\lambda)} \geq 0$ and (5.2.10) follows in this case. The case $w(x) \leq u^{t_0}(x)$ is treated similarly.

If we further assume (5.2.9), then the integral in (5.2.10) vanishes and the claim follows. □

The functional $\mathcal{C}_{\mathrm{N}}$ can be rewritten in the following alternative form that we will use to verify the remaining calibration properties ($\mathcal{C}1$) and ($\mathcal{C}2$).

Lemma 5.2.2. *Given an interval* $I \subset \mathbb{R}$, *a bounded domain* $\Omega \subset \mathbb{R}^n$, *and a pairwise symmetric function* $G_{\mathrm{N}} = G_{\mathrm{N}}(x, y, a, b)$ *in the sense of* (5.2.1), *let* $\{u^t\}_{t\in I}$ *be a field in* $\mathbb{R}^n$ *(in the sense of Definition 5.1.2) which is sufficiently regular for* G_{N}. *Consider the set of admissible functions* $\mathcal{A}_{\mathrm{N}}$ *defined in* (5.2.5).

Then, for all w *in* $\mathcal{A}_{\mathrm{N}}$, *the functional* $\mathcal{C}_{\mathrm{N}}$ *defined in* (5.2.6) *satisfies*

$$\begin{aligned} \mathcal{C}_{\mathrm{N}}(w) = {} & \frac{1}{2} \iint_{Q(\Omega)} G_{\mathrm{N}}(x, y, w(x), u^t(y))\big|_{t=t(x,w(x))} \,\mathrm{d}x \,\mathrm{d}y \\ & + \frac{1}{2} \iint_{Q(\Omega)} \int_{t(x,w(x))}^{t(y,w(y))} \partial_b G_{\mathrm{N}}(x, y, u^t(x), u^t(y)) \partial_t u^t(y) \,\mathrm{d}t \,\mathrm{d}x \,\mathrm{d}y. \end{aligned} \tag{5.2.11}$$

Proof. We will rewrite the integral term in the definition (5.2.6) of $\mathcal{C}_{\mathrm{N}}$. Applying the change of variables $\lambda \mapsto t$ with $u^t(x) = \lambda$ for each x, we have

$$\begin{aligned} & \iint_{Q(\Omega)} \int_{u^{t_0}(x)}^{w(x)} \partial_a G_{\mathrm{N}}(x, y, u^t(x), u^t(y))\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x \,\mathrm{d}y \\ & \qquad = \iint_{Q(\Omega)} \int_{t_0}^{t(x,w(x))} \partial_a G_{\mathrm{N}}(x, y, u^t(x), u^t(y)) \partial_t u^t(x) \,\mathrm{d}t \,\mathrm{d}x \,\mathrm{d}y. \end{aligned}$$

Symmetrizing this expression in (x,y) and using (5.2.2), we deduce

$$\begin{aligned}
&\iint_{Q(\Omega)} \int_{u^{t_0}(x)}^{w(x)} \partial_a G_{\mathrm{N}}(x,y,u^t(x),u^t(y))\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \frac{1}{2}\iint_{Q(\Omega)} \int_{t_0}^{t(x,w(x))} \partial_a G_{\mathrm{N}}(x,y,u^t(x),u^t(y))\partial_t u^t(x)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \qquad (5.2.12)\\
&\qquad + \frac{1}{2}\iint_{Q(\Omega)} \int_{t_0}^{t(y,w(y))} \partial_b G_{\mathrm{N}}(x,y,u^t(x),u^t(y))\partial_t u^t(y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Splitting the integral $\int_{t_0}^{t(y,w(y))} \cdot\,\mathrm{d}t$ in (5.2.12) into $\int_{t_0}^{t(x,w(x))} \cdot\,\mathrm{d}t + \int_{t(x,w(x))}^{t(y,w(y))} \cdot\,\mathrm{d}t$, we obtain

$$\begin{aligned}
&\iint_{Q(\Omega)} \int_{u^{t_0}(x)}^{w(x)} \partial_a G_{\mathrm{N}}(x,y,u^t(x),u^t(y))\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \frac{1}{2}\iint_{Q(\Omega)} \int_{t_0}^{t(x,w(x))} \frac{\mathrm{d}}{\mathrm{d}t}\big\{G_{\mathrm{N}}(x,y,u^t(x),u^t(y))\big\}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \qquad (5.2.13)\\
&\qquad + \frac{1}{2}\iint_{Q(\Omega)} \int_{t(x,w(x))}^{t(y,w(y))} \partial_b G_{\mathrm{N}}(x,y,u^t(x),u^t(y))\partial_t u^t(y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Integrating the derivative with respect to t in (5.2.13) and recalling, by definition of the leaf-parameter function, that $w(x) = u^{t(x,w(x))}(x)$, we have

$$\begin{aligned}
&\iint_{Q(\Omega)} \int_{u^{t_0}(x)}^{w(x)} \partial_a G_{\mathrm{N}}(x,y,u^t(x),u^t(y))\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \frac{1}{2}\iint_{Q(\Omega)} G_{\mathrm{N}}(x,y,w(x),u^{t(x,w(x))}(y))\,\mathrm{d}x\,\mathrm{d}y - \mathcal{E}_{\mathrm{N}}(u^{t_0}) \qquad (5.2.14)\\
&\qquad + \frac{1}{2}\iint_{Q(\Omega)} \int_{t(x,w(x))}^{t(y,w(y))} \partial_b G_{\mathrm{N}}(x,y,u^t(x),u^t(y))\partial_t u^t(y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Adding $\mathcal{E}_{\mathrm{N}}(u^{t_0})$ to both sides of (5.2.14) now yields the claim. □

In the next proposition we prove the calibration property ($\mathcal{C}1$). This follows directly from Lemma 5.2.2. Here, ellipticity of G_{N} is still not needed.

Proposition 5.2.3. *Given an interval $I \subset \mathbb{R}$, a bounded domain $\Omega \subset \mathbb{R}^n$, and a pairwise symmetric function $G_{\mathrm{N}} = G_{\mathrm{N}}(x,y,a,b)$ in the sense of (5.2.1), let $\{u^t\}_{t\in I}$ be a field in $\mathbb{R}^n$ (in the sense of Definition 5.1.2) which is sufficiently regular for G_{N}.*

Then, for all $t \in I$, the functional $\mathcal{C}_{\mathrm{N}}$ defined in (5.2.6) satisfies

$$\mathcal{C}_{\mathrm{N}}(u^t) = \mathcal{E}_{\mathrm{N}}(u^t).$$

Proof. Let $t_0 \in I$. Choosing $w = u^{t_0}$ in (5.2.11), since $t(x,w(x)) = t_0$ for all x, we have

$$\mathcal{C}_{\mathrm{N}}(u^{t_0}) = \frac{1}{2}\iint_{Q(\Omega)} G_{\mathrm{N}}(x,y,u^{t_0}(x),u^{t_0}(y))\,\mathrm{d}x\,\mathrm{d}y = \mathcal{E}_{\mathrm{N}}(u^{t_0}).$$

Since $t_0 \in I$ was arbitrary, this proves the proposition. □

It remains to prove the last calibration property ($\mathcal{C}2$). We will now need the natural ellipticity assumption on the Lagrangian G_{N}:

$$\partial^2_{ab} G_{\mathrm{N}}(x, y, a, b) \leq 0. \tag{5.2.15}$$

Notice that (5.2.15) is simply the ellipticity condition (5.1.7) from the Introduction written for a pairwise symmetric Lagrangian. Moreover, this condition is related to a strong comparison principle, as explained in Appendix I.

Proposition 5.2.4. *Given an interval $I \subset \mathbb{R}$, a bounded domain $\Omega \subset \mathbb{R}^n$, and a pairwise symmetric function $G_{\mathrm{N}} = G_{\mathrm{N}}(x, y, a, b)$ in the sense of* (5.2.1)*, let $\{u^t\}_{t \in I}$ be a field in $\mathbb{R}^n$ (in the sense of Definition 5.1.2) which is sufficiently regular for G_{N}. Consider the set of admissible functions $\mathcal{A}_{\mathrm{N}}$ defined in* (5.2.5)*. Assume that the ellipticity condition $\partial^2_{ab} G_{\mathrm{N}} \leq 0$ holds.*

Then, for all w in $\mathcal{A}_{\mathrm{N}}$, the functional $\mathcal{C}_{\mathrm{N}}$ defined in (5.2.6) *satisfies*

$$\mathcal{C}_{\mathrm{N}}(w) \leq \mathcal{E}_{\mathrm{N}}(w).$$

Proof. If we compute the difference $\mathcal{E}_{\mathrm{N}}(w) - \mathcal{C}_{\mathrm{N}}(w)$, using the alternative expression (5.2.11) for $\mathcal{C}_{\mathrm{N}}$, we obtain

$$\begin{aligned} &\mathcal{E}_{\mathrm{N}}(w) - \mathcal{C}_{\mathrm{N}}(w) \\ &\quad = \frac{1}{2} \iint_{Q(\Omega)} \left\{ G_{\mathrm{N}}(x, y, w(x), w(y)) - G_{\mathrm{N}}(x, y, w(x), u^{t(x,w(x))}(y)) \right\} \mathrm{d}x \, \mathrm{d}y \\ &\qquad - \frac{1}{2} \iint_{Q(\Omega)} \int_{t(x,w(x))}^{t(y,w(y))} \partial_b G_{\mathrm{N}}(x, y, u^t(x), u^t(y)) \partial_t u^t(y) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}y. \end{aligned} \tag{5.2.16}$$

Recalling that $u^{t(y,w(y))}(y) = w(y)$, we can write the first integral on the right-hand side of (5.2.16) as

$$\begin{aligned} &\frac{1}{2} \iint_{Q(\Omega)} \left\{ G_{\mathrm{N}}(x, y, w(x), w(y)) - G_{\mathrm{N}}(x, y, w(x), u^{t(x,w(x))}(y)) \right\} \mathrm{d}x \, \mathrm{d}y \\ &\qquad = \frac{1}{2} \iint_{Q(\Omega)} \int_{t(x,w(x))}^{t(y,w(y))} \frac{\mathrm{d}}{\mathrm{d}t} \left\{ G_{\mathrm{N}}(x, y, w(x), u^t(y)) \right\} \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}y \\ &\qquad = \frac{1}{2} \iint_{Q(\Omega)} \int_{t(x,w(x))}^{t(y,w(y))} \partial_b G_{\mathrm{N}}(x, y, w(x), u^t(y)) \partial_t u^t(y) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}y. \end{aligned} \tag{5.2.17}$$

Plugging (5.2.17) into (5.2.16), we see that

$$\begin{aligned} &\mathcal{E}_{\mathrm{N}}(w) - \mathcal{C}_{\mathrm{N}}(w) \\ &\quad = \iint_{Q(\Omega)} \int_{t(x,w(x))}^{t(y,w(y))} \left\{ \partial_b G_{\mathrm{N}}(x, y, w(x), u^t(y)) - \partial_b G_{\mathrm{N}}(x, y, u^t(x), u^t(y)) \right\} \partial_t u^t(y) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}y. \end{aligned}$$

Thus, it suffices to show that

$$\int_{t(x,w(x))}^{t(y,w(y))} \left\{ \partial_b G_{\mathrm{N}}(x, y, w(x), u^t(y)) - \partial_b G_{\mathrm{N}}(x, y, u^t(x), u^t(y)) \right\} \partial_t u^t(y) \, \mathrm{d}t \geq 0 \tag{5.2.18}$$

for all $(x, y) \in Q(\Omega)$.

Let $(x, y) \in Q(\Omega)$ and assume first that $t(x, w(x)) \leq t(y, w(y))$. By monotonicity of the leaves u^t in I, for $t \in [t(x, w(x)), t(y, w(y))]$ we have

$$w(x) = u^{t(x,w(x))}(x) \leq u^t(x),$$

and by ellipticity

$$\partial_b G_{\mathrm{N}}(x, y, w(x), u^t(y))\partial_t u^t(y) \geq \partial_b G_{\mathrm{N}}(x, y, u^t(x), u^t(y))\partial_t u^t(y).$$

Whence, (5.2.18) follows. The case $t(x, w(x)) \geq t(y, w(y))$ is treated similarly. □

Finally, combining Propositions 5.2.1, 5.2.3, and 5.2.4, we easily conclude Theorem 5.1.3.

Proof of Theorem 5.1.3. (a) Property $(\mathcal{C}1)$ follows from Proposition 5.2.3 and property $(\mathcal{C}2)$ follows from Proposition 5.2.4.

(b) This follows from the first part of Proposition 5.2.1.

(c) This follows from the second part of Proposition 5.2.1. □

Proof of Corollary 5.1.4. For each $t \in \mathbb{R}$ we define $u^t(x) := u(x', x_n + t)$, where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. By the monotonicity (5.1.10) of u and by translation invariance of the equation $\mathcal{L}_{\mathrm{N}}(u) = 0$, it follows that the family $\{u^t\}_{t\in\mathbb{R}}$ is a field of extremals in $\mathbb{R}^n$ in the sense of Definition 5.1.2. Hence, Theorem 5.1.3 yields the minimality of each u^t among competitors w with $w \equiv u$ in Ω^c and satisfying the assumption

$$\lim_{\tau\to-\infty} u(x', \tau) < w(x', x_n) < \lim_{\tau\to+\infty} u(x', \tau) \quad \text{for all} \quad x = (x', x_n) \in \Omega.$$

Finally, we can relax the previous strict inequalities by considering the competitor $(1 - \varepsilon)w + \varepsilon u$ and letting $\varepsilon \to 0$. In this way we recover the condition in the statement of Corollary 5.1.4 where the inequalities are not strict. □

5.3 The calibration for functionals involving both local and nonlocal terms

The results derived in Section 5.2 may be combined with the classical local ones to yield a theory that applies to functionals involving both local and nonlocal interactions. These functionals appear when dealing with symmetric Lévy processes, where the infinitesimal generators are given by the sum of a second order differential operator and an integro-differential one. Recently, mixed functionals have attracted great attention from different points of view; see [72, 93] and references therein.

The mixed energy[8]

$$\begin{aligned}\mathcal{E}_{\mathrm{T}}(w) &:= \mathcal{E}_{\mathrm{N}}(w) + \mathcal{E}_{\mathrm{L}}(w) \\ &= \frac{1}{2}\iint_{Q(\Omega)} G_{\mathrm{N}}(x, y, w(x), w(y))\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega G_{\mathrm{L}}(x, w(x), \nabla w(x))\,\mathrm{d}x\end{aligned} \tag{5.3.1}$$

admits a calibrating functional

$$\mathcal{C}_{\mathrm{T}}(w) := \mathcal{C}_{\mathrm{N}}(w) + \mathcal{C}_{\mathrm{L}}(w),$$

[8]Here in the notation we use the subscript M, which stands for "Mixed".

where $\mathcal{E}_{\mathrm{L}}$, G_{L}, $\mathcal{C}_{\mathrm{L}}$, and $\mathcal{E}_{\mathrm{N}}$, G_{N}, $\mathcal{C}_{\mathrm{N}}$ are defined as in the Introduction. By combining identities (5.1.5) and (5.2.7), the functional $\mathcal{C}_{\mathrm{T}}$ may be written equivalently as

$$\begin{aligned}\mathcal{C}_{\mathrm{T}}(w) = &\int_{\Omega}\int_{u^{t_0}(x)}^{w(x)} \mathcal{L}_{\mathrm{T}}(u^t)\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x\\ &+\int_{\Omega^c}\int_{u^{t_0}(x)}^{w(x)} \mathcal{N}_{\mathrm{N}}(u^t)\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x + \int_{\partial\Omega}\int_{u^{t_0}(x)}^{v(x)} \mathcal{N}_{\mathrm{L}}(u^t)\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{H}^{n-1}(x)\\ &+\mathcal{E}_{\mathrm{T}}(u^{t_0}),\end{aligned}$$

where the Euler-Lagrange operator of the mixed problem is

$$\mathcal{L}_{\mathrm{T}}(w) := \mathcal{L}_{\mathrm{N}}(w) + \mathcal{L}_{\mathrm{L}}(w),$$

and $\mathcal{L}_{\mathrm{L}}$, $\mathcal{N}_{\mathrm{L}}$, $\mathcal{L}_{\mathrm{N}}$, and $\mathcal{N}_{\mathrm{N}}$ are the operators introduced above.

Since $\mathcal{C}_{\mathrm{T}}$ shares the same structure as $\mathcal{C}_{\mathrm{L}}$ and $\mathcal{C}_{\mathrm{N}}$, a straightforward adaptation of the proofs in the sections above shows that $\mathcal{C}_{\mathrm{T}}$ satisfies all three calibration properties. We mention that property ($\mathcal{C}2$) requires both the local and nonlocal ellipticity conditions, that is, one must assume that both

$$\partial^2_{qq} G_{\mathrm{L}}(x,\lambda,q) \geq 0 \quad\text{and}\quad \partial^2_{ab} G_{\mathrm{N}}(x,y,a,b) \leq 0$$

hold.

As an application of this theory, we can prove the analogue of Corollary 5.1.4 for mixed functionals. Namely, if $\mathcal{L}_{\mathrm{T}}$ is translation invariant, i.e., $\mathcal{L}_{\mathrm{T}}(u(\cdot+y))(x) = \mathcal{L}_{\mathrm{T}}(u)(x+y)$ for all x and y in $\mathbb{R}^n$, then monotone solutions are minimizers among functions lying between the limits of the solution in the direction of monotonicity. The proof is identical to the one of Corollary 5.1.4.

Remark 5.3.1. Mixed energies appear in the following relevant frameworks. However, these minimization problems include constraints. Thus, one cannot directly apply the calibration theory developed above, since constrained minimizers need not be minimizers of the original functional and no foliation of extremals is expected. As examples of such frameworks, we mention the theory of aggregation equations [35], certain problems from astrophysics [70], the Thomas-Fermi theory [7], the Choquard-Pekar model [68], as well as the problem of finding the best constant in the Sobolev inequality [94].

5.4 Application to the viscosity theory

For the application of calibrations to prove that minimizers are viscosity solutions, we need to consider more general fields, namely, those which are not necessarily increasing in a bounded domain $\Omega \subset \mathbb{R}^n$, but only nondecreasing. The situation is that different leaves will coincide in certain subsets of Ω; see Figure 5.1 in the Introduction. Such a field will appear when sliding a touching test function and truncating it with the minimizer.

5.4.1 An energy comparison result for fractional functionals

Given $s \in (0,1)$ and a bounded domain $\Omega \subset \mathbb{R}^n$, let $u, v \in C(\overline{\Omega}) \cap L^1_s(\mathbb{R}^n)$ be two functions such that $u = v$ a.e. in Ω^c and $u \leq v$ in Ω. Assume we are given functions $\varphi^t\colon \mathbb{R}^n \to \mathbb{R}$, with $t \in [0,T]$ and nondecreasing in t, such that they "interpolate" between the two functions:

$\varphi^0 = u$, $\varphi^T = v$. When the family $\{\varphi^t\}_{t\in[0,T]}$ satisfies appropriate regularity assumptions, we will be able to construct a calibration involving the field.

Consider the region

$$\mathcal{G} = \big\{(x,\lambda) \in \Omega \times \mathbb{R} : u(x) < \lambda < v(x)\big\},$$

as well as the sections

$$\Omega_t := \{x \in \Omega \colon \varphi^t(x) > u(x)\} \quad \text{for each } t \in (0,T),$$

which will be increasing in t, and

$$I_x := \{t \in (0,T) \colon \varphi^t(x) > u(x)\} \quad \text{for each } x \in \Omega.$$

Definition 5.4.1. Given $s \in (0,1)$ and a bounded domain $\Omega \subset \mathbb{R}^n$, let $u, v \in C(\overline{\Omega}) \cap L^1_s(\mathbb{R}^n)$ be such that $u = v$ a.e. in Ω^c and $u \le v$ in Ω. A family $\{\varphi^t\}_{t\in[0,T]}$ of functions $\varphi^t \colon \mathbb{R}^n \to \mathbb{R}$ is said to be a *weak field for u and v* (see Figure 5.1) if the following conditions are satisfied:

(i) $\varphi^0 = u$ and $\varphi^T = v$.

(ii) $\varphi^t = u$ a.e. in Ω^c, for all $t \in [0,T]$.

(iii) The function $(x,t) \mapsto \varphi^t(x)$ is continuous in $\overline{\Omega} \times [0,T]$.

(iv) For each $x \in \Omega$, the function $t \mapsto \varphi^t(x)$ is $C^1(I_x)$ and increasing in $\overline{I_x}$. Moreover, there exists a constant $C_0 > 0$ such that

$$|\partial_t \varphi^t(x)| \le C_0 \quad \text{for all } x \in \Omega \text{ and } t \in I_x.$$

Moreover, the weak field is *regular by below*, if the following regularity condition holds:

(v) The functions $\{\varphi^t\}_{t\in(0,T)}$ are uniformly $C^{1,1}$ by below in Ω_t, uniformly in t, in the following sense. There exist a constant $C_0 > 0$ and a bounded domain $N \subset \mathbb{R}^n$, with $\overline{\Omega} \subset N$, such that, for each $t \in (0,T)$ and $x \in \Omega_t$, there is a function $\psi \in C^2(N)$ touching φ^t by below in N at x, that is, $\psi(x) = \varphi^t(x)$ and $\psi \le \varphi^t$ a.e. in N, satisfying

$$D^2\psi \ge -C_0 \quad \text{in } N.$$

Remark 5.4.2. The more technical assumption in Definition 5.4.1, condition (v), is needed for the calibration of the fractional Laplacian to be well defined. An important consequence of (v) is that the fractional Laplacian $(-\Delta)^s\varphi^t(x)$ is bounded by above uniformly in $t \in (0,T)$ and $x \in \Omega_t$.

If $\{\varphi^t\}_{t\in[0,T]}$ is a weak field for u and v, then for each $(x,\lambda) \in \mathcal{G}$ there exists a unique $t = t(x,\lambda)$ such that $\varphi^{t(x,\lambda)}(x) = \lambda$. The existence is a consequence of (iii), since $\varphi^0(x) < \lambda < \varphi^T(x)$, while the uniqueness follows from (iv).

Recall that, given $s \in (0,1)$ and $F \in C^1(\mathbb{R})$, we have

$$\mathcal{E}_{s,F}(w) = \frac{c_{n,s}}{4} \iint_{Q(\Omega)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x.$$

We also write

$$\mathcal{E}_s(w) = \frac{c_{n,s}}{4} \iint_{Q(\Omega)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y$$

following the notation introduced in Chapter 4. The weak field allows to compare the energies of u and v via the following theorem:

Theorem 5.4.3. *Given a bounded domain $\Omega \subset \mathbb{R}^n$ and $s \in (0,1)$, let $u, v \in C(\overline{\Omega}) \cap L^1_s(\mathbb{R}^n)$ be such that $u = v$ a.e. in Ω^c and $u \le v$ in Ω. Let $\{\varphi^t\}_{t\in[0,T]}$ be a weak field for u and v which is regular by below in the sense of Definition 5.4.1.*

Then, given $F \in C^1(\mathbb{R})$, if $\mathcal{E}_s(u) < \infty$ we have

$$\mathcal{E}_{s,F}(v) \le \mathcal{E}_{s,F}(u) + \int_{\Omega\cap\{v>u\}} \int_{u(x)}^{v(x)} \big((-\Delta)^s\varphi^t(x) - F'(\varphi^t(x))\big)\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x.$$

Proof. We proceed as in the proof of the calibration properties although giving fewer details. The idea is to consider the analogue of the fractional calibration $\mathcal{C}_{s,F}$ for $\mathcal{E}_{s,F}$ and v introduced in (4.1.9) above, and to use property ($\mathcal{C}2$), i.e., $\mathcal{C}_{s,F}(u) \le \mathcal{E}_{s,F}(u)$, which gives the energy comparison.

For $\varepsilon > 0$, we consider the kernel $K_\varepsilon = c_{n,s}|\cdot|^{-n-2s}\mathbb{1}_{\mathbb{R}^n\setminus B_\varepsilon}$ and the truncated fractional Laplacian $(-\Delta)^s_\varepsilon\varphi(x) = \int_{\mathbb{R}^n}(\varphi(x)-\varphi(y))K_\varepsilon(x-y)\,\mathrm{d}y$.

Since u and v differ only in the region $\Omega_T \subset \Omega$, it suffices to consider all functionals defined in $Q(\Omega_T)$ instead of on the larger set

$$Q(\Omega) = Q(\Omega_T) \cup ((\Omega\setminus\Omega_T)\times\Omega_T^c) \cup (\Omega^c \times (\Omega\setminus\Omega_T)).$$

By a slight modification of the proof of Lemma 4.4.4 above, we have the identity

$$\begin{aligned}
&\int_{\Omega_T}\int_{v(x)}^{u(x)} (-\Delta)^s_\varepsilon\varphi^t(x)\big|_{t=t(x,\lambda)}\,\mathrm{d}x + \frac14\iint_{Q(\Omega_T)} |v(x)-v(y)|^2K_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&= -\frac12\iint_{Q(\Omega_T)} \mathrm{d}x\,\mathrm{d}yK_\varepsilon(x-y)\int_{t(x,u(x))}^{t(y,u(y))} (\varphi^t(x)-\varphi^t(y))\,\partial_t\varphi^t(y)\,\mathrm{d}t \\
&\qquad + \frac14\iint_{Q(\Omega_T)} |u(x)-\varphi^{t(x,u(x))}(y)|^2K_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}\tag{5.4.1}$$

Moreover, since φ^t is nondecreasing in t, the first term in the right-hand side of (5.4.1) can be bounded by

$$\begin{aligned}
&-\frac12\iint_{Q(\Omega_T)} \mathrm{d}x\,\mathrm{d}yK_\varepsilon(x-y)\int_{t(x,u(x))}^{t(y,u(y))} (\varphi^t(x)-\varphi^t(y))\,\partial_t\varphi^t(y)\,\mathrm{d}t \\
&\qquad \le -\frac12\iint_{Q(\Omega_T)} \mathrm{d}x\,\mathrm{d}yK_\varepsilon(x-y)\int_{t(x,u(x))}^{t(y,u(y))} (u(x)-\varphi^t(y))\,\partial_t\varphi^t(y)\,\mathrm{d}t \\
&\qquad = \frac14\iint_{Q(\Omega_T)} \mathrm{d}x\,\mathrm{d}yK_\varepsilon(x-y)\big(|u(x)-u(y)|^2 - |u(x)-\varphi^{t(x,u(x))}(y)|^2\big),
\end{aligned}\tag{5.4.2}$$

where in the last line we have integrated $-(u(x)-\varphi^t(y))\partial_t\varphi^t(y) = \frac{\mathrm{d}}{\mathrm{d}t}|u(x)-\varphi^t(y)|^2$. Hence, writing $\mathcal{E}_{K_\varepsilon}(u) = \frac12\iint_{Q(\Omega)}|u(x)-u(y)|^2K_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y$, combining (5.4.1) and (5.4.2) we deduce

$$\begin{aligned}
\mathcal{E}_{K_\varepsilon}(u) - \mathcal{E}_{K_\varepsilon}(v) &= \frac14\iint_{Q(\Omega_T)} \big(|u(x)-u(y)|^2 - |v(x)-v(y)|^2\big)K_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\ge \int_{\Omega_T}\int_{v(x)}^{u(x)} (-\Delta)^s_\varepsilon\varphi^t(x)\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x.
\end{aligned}\tag{5.4.3}$$

Finally, thanks to property (v) and by Fatou's lemma, we can pass to the limit as $\varepsilon \downarrow 0$ inside the integrals in (5.4.3) to obtain

$$\mathcal{E}_s(v) \le \mathcal{E}_s(u) + \int_{Q(\Omega_T)}\int_{u(x)}^{v(x)} (-\Delta)^s\varphi^t(x)\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x.$$

Since $F(u) - F(v) = \int_{v(x)}^{u(x)} F'(\lambda)\, \mathrm{d}\lambda = \int_{v(x)}^{u(x)} F'(\varphi^t(x))\big|_{t=t(x,\lambda)}\, \mathrm{d}\lambda$, adding the potential term now yields the result. $\square$

5.4.2 Minimizers of fractional functionals are viscosity solutions

We recall the definition of viscosity solution in the nonlocal setting. The following is taken from Definition 2.2 in [34]:

Definition 5.4.4. Given bounded domain $\Omega \subset \mathbb{R}^n$, $s \in (0,1)$, and $F \in C^1(\mathbb{R})$, we say that $u \in C(\Omega) \cap L^1_s(\mathbb{R}^n)$ is a *viscosity supersolution* of the semilinear equation

$$(-\Delta)^s v = F'(v) \text{ in } \Omega,$$

if whenever the following happens

- x_0 is any point in Ω
- N is a neighborhood of x_0 in Ω
- φ is some C^2 function in $\overline{N}$
- $\varphi(x_0) = u(x_0)$
- $\varphi(x) < u(x)$ for every $x \in N \setminus \{x_0\}$,

then, the function

$$\overline{\varphi}(x) := \begin{cases} \varphi(x) & \text{for } x \in N \\ u(x) & \text{for } x \in \mathbb{R}^n \setminus N, \end{cases}$$

satisfies $(-\Delta)^s \overline{\varphi}(x_0) \geq F'(\overline{\varphi}(x_0))$.

We also have the analogous definition of viscosity subsolution. We say that u is a viscosity solution if it is both a viscosity supersolution and subsolution.

Our main results in this section deal with minimizers of the energy functional $\mathcal{E}_{s,F}$. In fact, it is enough to consider one-sided minimizers, which are defined as follows:

Definition 5.4.5. Given a bounded domain $\Omega \subset \mathbb{R}^n$, $s \in (0,1)$, and $F \in C^1(\mathbb{R})$, we say that a function $u\colon \mathbb{R}^n \to \mathbb{R}$ is a *one-sided minimizer by above* of the functional $\mathcal{E}_{s,F}$ if $\mathcal{E}_{s,F}(u) < \infty$ and for all functions v such that $v \geq u$ in Ω and $v = u$ in Ω^c we have

$$\mathcal{E}_{s,F}(v) \geq \mathcal{E}_{s,F}(u).$$

We also have the analogous definition of one-sided minimizer by below.

We will now prove that one-sided minimizers by above are viscosity supersolutions. This is a consequence of Theorem 5.4.3.

Theorem 5.4.6. *Given a bounded domain $\Omega \subset \mathbb{R}^n$, $s \in (0,1)$, and $F \in C^1(\mathbb{R})$, let $u\colon \mathbb{R}^n \to \mathbb{R}$ in $C(\Omega)$ be a one-sided minimizer by above of the functional $\mathcal{E}_{s,F}$.*

Then u is a viscosity supersolution.

Proof. We proceed by contradiction. Suppose that u is not a viscosity supersolution. Then there exist $x_0 \in \Omega$, a neighborhood $N \subset \Omega$ of x_0, and a function $\varphi \in C^2(\overline{N})$, with $\varphi(x_0) = u(x_0)$ and $\varphi(x) < u(x)$ for all $x \in \Omega$, such that the extension $\overline{\varphi}$ satisfies

$$(-\Delta)^s \overline{\varphi}(x_0) < F'(\overline{\varphi}(x_0)).$$

We will now construct a function above u which has less energy, thus violating the one-sided minimality by above. The idea of the proof is to raise the function φ to produce a local foliation whose leaves are strict subsolutions.

Recall the truncations introduced in the proof of Theorem 5.4.3 above. Namely, for $\varepsilon > 0$ we let $K_\varepsilon = c_{n,s}|\cdot|^{-n-2s}\mathbb{1}_{\mathbb{R}^n \setminus B_\varepsilon}$ and $(-\Delta)^s_\varepsilon \varphi(x) = \int_{\mathbb{R}^n}(\varphi(x) - \varphi(y))K_\varepsilon(x-y)\,\mathrm{d}y$.

Since $(-\Delta)^s\overline{\varphi}(x_0) - F'(\overline{\varphi}(x_0)) =: -4c_0 < 0$, by continuity of F' and of the fractional Laplacian, there is a smaller neighborhood $\widetilde{N}$ of x_0, with $\widetilde{N} \subset N$, such that

$$(-\Delta)^s_\varepsilon \overline{\varphi}(x) - F'(\overline{\varphi}(x)) < -2c_0$$

for all $x \in \widetilde{N}$ and $\varepsilon \in (0, \varepsilon_0)$, for some small $\varepsilon_0 > 0$.

For $0 \leq t \leq T$, where $0 < T \leq \min_{\partial N}(u - \varphi)$, we define the functions

$$\varphi^t(x) := \begin{cases} \max\{u(x), \varphi(x) + t\} & \text{for } x \in N \\ u(x) & \text{for } x \in \mathbb{R}^n \setminus N. \end{cases}$$

It is clear that the family $\{\varphi^t\}_{t\in[0,T]}$ is a weak field for u and φ^T which is regular by below, in the sense of Definition 5.4.1.

For $0 < t < T$, $\varepsilon > 0$, and $x \in N$ such that $\varphi^t(x) > u(x)$, we have

$$(-\Delta)^s_\varepsilon\, \varphi^t(x) \leq (-\Delta)^s_\varepsilon\, \overline{\varphi}(x) + T\int_{\mathbb{R}^n\setminus N} c_{n,s}|x-y|^{-n-2s}\,\mathrm{d}y.$$

From this inequality and the continuity of F', taking a sufficiently small T, we obtain

$$(-\Delta)^s_\varepsilon \varphi^t(x) - F'(\varphi^t(x)) < -c_0$$

for all $x \in \widetilde{N}$, $\varepsilon \in (0, \varepsilon_0)$, and $t \in (0, T)$ such that $\varphi^t(x) > u(x)$.

Letting $v = \varphi^T$, by Theorem 5.4.3, we conclude that

$$\begin{aligned} \mathcal{E}_{s,F}(v) &\leq \mathcal{E}_{s,F}(u) + \int_{\Omega\cap\{v>u\}}\int_{u(x)}^{v(x)} \Big\{(-\Delta)^s\varphi^t(x) - F'(\varphi^t(x))\Big\}\Big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x \\ &\leq \mathcal{E}_{s,F}(u) - c_0\big|\{(x,\lambda) \in \Omega\times\mathbb{R} : u(x) < \lambda < v(x)\}\big| \\ &< \mathcal{E}_{s,F}(u), \end{aligned}$$

which contradicts the minimality of u. □

5.4.3 More general nonlocal functionals

We now extend the previous approach to the more general setting of mixed functionals. First, we note that the notion of weak field needs to be adapted to arbitrary mixed functionals $\mathcal{E}_{\mathrm{T}}$ of the form (5.3.1). This class of energies is too big to allow for a general definition of weak field. However, it can be done for specific families of Lagrangians G_{L} and G_{N}, as the ones given in the Introduction. Under precise growth and regularity assumptions on the Lagrangians, it suffices to modify condition (v) in Definition 5.4.1 appropriately so that there are no integrability issues in our proofs below. Thus, in the next theorems, by a weak field which is "sufficiently regular for $G_{\rm L}$ and $G_{\rm N}$" we mean a weak field satisfying the required additional regularity conditions.

Next, we prove the following energy comparison result in the presence of a sufficiently regular weak field, which contains Theorem 5.1.6 in the Introduction (and will be proven as in Theorem 5.4.3 above):

Theorem 5.4.7. *Let $G_{\rm L} = G_{\rm L}(x, \lambda, q)$ be a function satisfying $\partial^2_{qq} G_{\rm L}(x, \lambda, q) \geq 0$, and let $G_{\rm N} = G_{\rm N}(x, y, a, b)$ be a pairwise symmetric function satisfying $\partial^2_{ab} G_{\rm N}(x, y, a, b) \leq 0$.*

Given a bounded domain $\Omega \subset \mathbb{R}^n$, let $u, v \in C(\overline{\Omega})$ such that $u = v$ a.e. in Ω^c and $u \leq v$ in Ω. Assume that there exists $\{\varphi^t\}_{t\in[0,T]}$, a weak field for u and v (in the sense of Definition 5.4.1) which is sufficiently regular for $G_{\rm L}$ and $G_{\rm N}$.

Then, if $\mathcal{E}_{\rm T}(u) = \mathcal{E}_{\rm L}(u) + \mathcal{E}_{\rm N}(u) < \infty$ (defined in Section 5.3), we have

$$\mathcal{E}_{\rm T}(v) \leq \mathcal{E}_{\rm T}(u) + \int_\Omega \int_{u(x)}^{v(x)} \mathcal{L}_{\rm T}\left(\varphi^t(x)\right)\Big|_{t=t(x,\lambda)} \,{\rm d}\lambda \,{\rm d}x,$$

where $\mathcal{L}_{\rm T} = \mathcal{L}_{\rm L} + \mathcal{L}_{\rm N}$ is the Euler-Lagrange operator associated to $\mathcal{E}_{\rm T}$.

Proof. We consider the calibration functional constructed in Section 5.3, that is,

$$\mathcal{C}_{\rm T}(w) = \mathcal{E}_{\rm T}(v) + \int_\Omega \int_{v(x)}^{w(x)} \mathcal{L}_{\rm T}(\varphi^t)\Big|_{t=t(x,\lambda)} \,{\rm d}\lambda \,{\rm d}x.$$

Following the strategy there, one can show, by using the ellipticity conditions, that $\mathcal{C}_{\rm T}$ also satisfies property ($\mathcal{C}2$) in the new framework of weak fields. In particular,

$$\mathcal{E}_{\rm T}(u) \geq \mathcal{C}_{\rm T}(u) = \mathcal{E}_{\rm T}(v) + \int_\Omega \int_{v(x)}^{u(x)} \mathcal{L}_{\rm T}(\varphi^t)\Big|_{t=t(x,\lambda)} \,{\rm d}\lambda \,{\rm d}x,$$

which yields the desired result. □

With this result at hand, we can easily show that one sided minimizers by above are viscosity supersolutions of the Euler-Lagrange equation. Here it is clear how to adapt Definitions 5.4.4 and 5.4.5 to the case of mixed energy functionals. The following result includes Theorem 5.1.5 in the Introduction (and is proven as Theorem 5.4.6 above):

Theorem 5.4.8. *Let $G_{\rm L} = G_{\rm L}(x, \lambda, q)$ be a function satisfying $\partial^2_{qq} G_{\rm L}(x, \lambda, q) \geq 0$, and let $G_{\rm N} = G_{\rm N}(x, y, a, b)$ be a pairwise symmetric function satisfying $\partial^2_{ab} G_{\rm N}(x, y, a, b) \leq 0$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let u be a sufficiently regular one-sided minimizer by above of the functional $\mathcal{E}_{\rm T}$.*

Then, the function u is a viscosity supersolution of the associated Euler-Lagrange equation $\mathcal{L}_{\rm T}(w) = 0$ in Ω.

Proof. Proceeding as in the proof of Theorem 5.4.6, we slide the touching function φ upwards and take the maximum with u to obtain a weak field. Applying Theorem 5.4.7, we see immediately that φ cannot be a strict subsolution, otherwise, the leaves of the weak field would have smaller energy than the minimizer. □

We can finally give the proof of Theorem 5.1.5 in the Introduction:

Proof of Theorem 5.1.5. If u is a minimizer, then u and $-u$ are one-sided minimizers by above of the functionals $\mathcal{E}_{\mathrm{N}}(\cdot)$ and $\mathcal{E}_{\mathrm{N}}(-\cdot)$, respectively. By Theorem 5.4.8, the function u is both a viscosity supersolution and subsolution. In particular, u is a viscosity solution. $\square$

Remark 5.4.9. There is a more direct proof of Theorem 5.4.8 which does not use calibrations. In addition to the ellipticity $\partial_{ab} G_{\mathrm{N}}(x, y, a, b) \leq 0$, one further assumes that the functions $(\lambda, q) \mapsto G_{\mathrm{L}}(x, \lambda, q)$ and $(a, b) \mapsto G_{\mathrm{N}}(x, y, a, b)$ are both convex. Proceeding by contradiction as above, we consider the same weak field and pick one leaf φ^{t_0} with $t_0 > 0$. By continuity and the ellipticity assumption, this function will satisfy $\mathcal{L}_{\mathrm{T}}(\varphi^{t_0})(x) < 0$ whenever $\varphi^{t_0}(x) > u(x)$. Then, applying integration by parts in the local term, symmetrizing in the nonlocal one, and using the convexity assumptions, we obtain

$$\begin{aligned}
0 &< \int_\Omega (u(x) - \varphi^{t_0}(x))\, \mathcal{L}_{\mathrm{T}}(\varphi^{t_0})(x)\, \mathrm{d}x \\
&\leq \int_\Omega \Big(G_{\mathrm{L}}(x, u(x), \nabla u(x)) - G_{\mathrm{L}}(x, \varphi^{t_0}(x), \nabla \varphi^{t_0}(x)) \Big)\, \mathrm{d}x \\
&\quad + \frac{1}{2} \iint_{Q(\Omega)} \Big(G_{\mathrm{N}}(x, y, u(x), u(y)) - G_{\mathrm{N}}(x, y, \varphi^{t_0}(x), \varphi^{t_0}(y)) \Big)\, \mathrm{d}x\, \mathrm{d}y \\
&= \mathcal{E}_{\mathrm{T}}(u) - \mathcal{E}_{\mathrm{T}}(\varphi^{t_0}).
\end{aligned}$$

This contradicts the one-sided minimality by below.

Remark 5.4.10. The calibration approach allows us to prove that one-sided minimizers by above are viscosity supersolutions, but says nothing a priori about supersolutions that are not minimizers. Nevertheless, the strategy can be adapted to treat some of these cases. The idea consists on building an auxiliary functional for which the weak supersolution is a one-sided minimizer.

We briefly discuss the semilinear case for the sake of clarity. Let u be a weak supersolution, not necessarily a one-sided minimizer, of the equation $(-\Delta)^s v = f(v)$ in Ω, that is, u satisfies

$$\frac{c_{n,s}}{2} \iint_{Q(\Omega)} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y)))}{|x - y|^{n+2s}}\, \mathrm{d}x\, \mathrm{d}y \geq \int_\Omega f(u(x))\, \varphi(x)\, \mathrm{d}x$$

for all $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \geq 0$. Then, one can check that u is also a weak supersolution of the linear equation $(-\Delta)^s v = g$ in Ω, with $g(x) := f(u(x))$. In particular, u is a one-sided minimizer of the auxiliary convex energy functional

$$\widetilde{\mathcal{E}}(w) = \frac{c_{n,s}}{4} \iint_{Q(\Omega)} \frac{|w(x) - w(y)|^2}{|x - y|^{n+2s}} - \int_\Omega g(x)\, w(x).$$

Applying Theorem 5.4.8, we deduce that u is a viscosity supersolution of the linear equation $(-\Delta)^s w = g$. By definition of g, one clearly concludes that u is a viscosity supersolution of the original semilinear equation.

Appendix G

A calibration for the extension problem of the fractional Laplacian

In this appendix we study minimizers of the energy functional $\mathcal{E}_{s,F}$ by using the extension technique for the fractional Laplacian. The strategy is based on building a calibration for an auxiliary local energy $\widetilde{\mathcal{E}}_{s,F}$ in the extended space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$. We point out that this construction did not give us, during the conception of our work, any a priori information about the form of a calibration written "downstairs" (i.e., in $\mathbb{R}^n$) for the original energy functional $\mathcal{E}_{s,F}$. It was only after finding $\mathcal{C}_{s,F}$ by nonlocal arguments that we noticed how to deduce it, at least formally, from the extension problem.

We denote by $(x,z) \in \mathbb{R}^n \times \mathbb{R}$ points in $\mathbb{R}^{n+1}_+$. Given a bounded domain $\Omega \subset \mathbb{R}^n$, we say that a bounded set $\widetilde{\Omega} \subset \mathbb{R}^{n+1}_+$ is an extension of Ω if $\partial_0\widetilde{\Omega} := \partial\widetilde{\Omega} \cap \{z = 0\} \subset \Omega$. It is well known that there is a strong connection between the nonlocal energy functional $\mathcal{E}_{s,F}$ and the local one

$$\widetilde{\mathcal{E}}_{s,F}(W;\widetilde{\Omega}) := \frac{d_s}{2}\iint_{\widetilde{\Omega}} z^{1-2s}|\nabla W(x,z)|^2 \,\mathrm{d}x\,\mathrm{d}z - \int_{\partial_0\widetilde{\Omega}} F(W(x,0))\,\mathrm{d}x,$$

where d_s is a positive normalizing constant. For this, given a function u defined in $\mathbb{R}^n$ we consider $U\colon \mathbb{R}^{n+1}_+ \to \mathbb{R}$ the solution of

$$\begin{cases} \operatorname{div}(z^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ U = u & \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases}$$

Here, U is the so-called s-harmonic extension of u. In [32, Lemma 7.2], Caffarelli, Roquejoffre, and Savin showed that u is a minimizer of $\mathcal{E}_{s,F}$ among functions with the same exterior data as u in Ω^c if and only if, for every extension domain $\widetilde{\Omega}$, the s-harmonic extension U of u is a minimizer of $\widetilde{\mathcal{E}}_s(\cdot;\widetilde{\Omega})$ among functions with the same boundary condition as U on $\partial_L\widetilde{\Omega} := \partial\widetilde{\Omega} \cap \{z > 0\}$.

Taking into account this equivalence we can apply the classical theory of calibrations to the mixed Dirichlet-Neumann problem as explained in Remark 4.3.6. To do this, given a field $\{u^t\}_{t\in I}$ in $\mathbb{R}^n$, for some interval $I \subset \mathbb{R}$, it is clear by the maximum principle that we can define a new field $\{U^t\}_{t\in I}$ in $\mathbb{R}^{n+1}_+$ where each leaf U^t is the s-harmonic extension of u^t. Then, the functional

$$\begin{aligned}\widetilde{\mathcal{C}}_{s,F}(W;\widetilde{\Omega}) :=& d_s \iint_{\widetilde{\Omega}} z^{1-2s}\Big\{\nabla U^t(x,z)\cdot\nabla W(x,z) - \frac{1}{2}|\nabla U^t(x,z)|^2\Big\}_{t=t(x,z,W(x,z))}\,\mathrm{d}x\,\mathrm{d}z \\ &- \int_{\partial_0\widetilde{\Omega}} F(W(x,0))\,\mathrm{d}x \end{aligned} \tag{G.1}$$

can be proved to be a calibration for $\widetilde{\mathcal{E}}_{s,F}$ and U. Therefore, U is a minimizer of $\widetilde{\mathcal{E}}_{s,F}$, and by [32, Lemma 7.2] it follows that u is a minimizer of $\mathcal{E}_{s,F}$.

We point out that although in this way we easily found a calibration for the local energy $\widetilde{\mathcal{E}}_{s,F}(\cdot;\widetilde{\Omega})$, it was not clear at all how it translated into a calibration written "downstairs" for the original energy functional $\mathcal{E}_{s,F}$. It was only after building the calibration $\mathcal{C}_{s,F}$ by using purely nonlocal techniques that we discovered how to pass, at least formally, from $\widetilde{\mathcal{C}}_{s,F}(\cdot;\widetilde{\Omega})$ to $\mathcal{C}_{s,F}$. Let us explain this. First, as in Section 4.3, for $t_0 \in I$, we rewrite (G.1) in the alternative form[1]

$$
\begin{aligned}
\widetilde{\mathcal{C}}_{s,F}(W;\widetilde{\Omega}) = -d_s \iint_{\widetilde{\Omega}} \int_{U^{t_0}(x,z)}^{W(x,z)} \operatorname{div}\left(z^{1-2s}\nabla U^t(x,z)\right)\Big|_{t=t(x,z,\lambda)} \mathrm{d}\lambda \, \mathrm{d}z \, \mathrm{d}x \\
+ \int_{\partial_0\widetilde{\Omega}} \int_{U^{t_0}(x,0)}^{W(x,0)} \Big\{(-\Delta)^s u^t(x) - F'(u^t(x))\Big\}\Big|_{t=t(x,0,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x \\
+ d_s \int_{\partial_L\widetilde{\Omega}} \int_{U^{t_0}(x,z)}^{W(x,z)} z^{1-2s}\, \nu_{\partial_L\widetilde{\Omega}} \cdot \nabla U^t(x,z)\Big|_{t=t(x,z,\lambda)} \mathrm{d}\lambda \, \mathrm{d}\mathcal{H}^n(x,z) \\
+ \widetilde{\mathcal{E}}_{s,F}(U^{t_0};\widetilde{\Omega}),
\end{aligned}
$$

where $\nu_{\partial_L\widetilde{\Omega}}$ is the exterior normal vector to the lateral boundary $\partial_L\widetilde{\Omega}$. Notice here that in the second term we have used the identity

$$
\lim_{z\downarrow 0} \Big\{ -d_s z^{1-2s} U^t_z(x,z) - F'(U^t(x,z))\Big\} = (-\Delta)^s u^t(x) - F'(u^t(x)),
$$

which follows from the Caffarelli-Silvestre extension; see [33]. Finally, taking a sequence of extended domains $\widetilde{\Omega}_i$ converging to the half-space $\mathbb{R}^{n+1}_+$, we recover the functional $\mathcal{C}_{s,F}$ (up to an additive constant) as the formal limit of $\widetilde{\mathcal{C}}_{s,F}(\cdot;\widetilde{\Omega}_i)$.

[1] Here, the first term is the one associated to the Euler-Lagrange operator of the local energy functional and vanishes by the definition of the field U^t. On the other hand, the second and third terms are the ones involving the Neumann operator for the extended problem.

Appendix H

Other candidates for the fractional calibration

In this section we discuss three other natural candidates to be a calibration for the energy functional

$$\mathcal{E}_{s,F}(w) = \frac{c_{n,s}}{4} \iint_{Q(\Omega)} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x.$$

We will be able to discard two of them since some of the calibration properties fail in these cases. Nevertheless, there is still one candidate for which we cannot determine whether it is a calibration or not.

Let us recall that the local counterpart of $\mathcal{E}_{s,F}$ is the functional

$$\mathcal{E}_{1,F}(w) = \frac{1}{2} \int_\Omega |\nabla w(x)|^2 \, \mathrm{d}x - \int_\Omega F(w(x)) \, \mathrm{d}x,$$

which admits the calibration

$$\mathcal{C}_{1,F}(w) = \int_\Omega \Big(\nabla u^t(x) \cdot (\nabla w(x) - \nabla u^t(x)) + \frac{1}{2} \left|\nabla u^t(x)\right|^2 \Big)\Big|_{t=t(x,w(x))} \mathrm{d}x - \int_\Omega F(w(x)) \, \mathrm{d}x,$$

a functional that can also be written as

$$\mathcal{C}_{1,F}(w) = \int_\Omega \Big(\nabla u^t(x) \cdot \nabla w(x) - \frac{1}{2} \left|\nabla u^t(x)\right|^2 \Big)\Big|_{t=t(x,w(x))} \mathrm{d}x - \int_\Omega F(w(x)) \, \mathrm{d}x.$$

Inspired by the form of $\mathcal{C}_{1,F}$, the first natural calibration candidate for $\mathcal{E}_{s,F}$ can be built replacing the gradient terms by differences and double integrals. That is, we let

$$\begin{aligned} \mathcal{F}^1_{s,F}(w) := \frac{c_{n,s}}{2} \iint_{Q(\Omega)} & \frac{(u^t(x) - u^t(y))(w(x) - w(y))}{|x-y|^{n+2s}} \Big|_{t=t(x,w(x))} \mathrm{d}x \, \mathrm{d}y \\ & - \frac{c_{n,s}}{4} \iint_{Q(\Omega)} \frac{|u^t(x) - u^t(y)|^2}{|x-y|^{n+2s}} \Big|_{t=t(x,w(x))} \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x. \end{aligned}$$

By using Young's inequality and the definition of the leaf-parameter function, one can directly conclude that $\mathcal{F}^1_{s,F}$ satisfies properties ($\mathcal{C}1$) and ($\mathcal{C}2$). It remains to check whether the null-Lagrangian property ($\mathcal{C}3$) is satisfied, but we do not know how to answer this question. For the affirmative answer, the idea would be to use the usual nonlocal integration by parts technique to obtain the Euler-Lagrange equation on the leaves. However, since the leaf-parameter function t depends on the variable x, we get remainder terms that we do not know how to treat. It is then natural to look for a counterexample. We looked at cases where an explicit field is available. For the trivial potential $F = 0$, for which $u^t(x) = x + t$ are extremals (even if not bounded), property $(\mathcal{C}3)$ does not fail. Hence, this case does not discard the candidate $\mathcal{F}^1_{s,F}$. Another interesting example with explicit solutions is the Peierls-Nabarro model, corresponding to the case $n = 1$, $s = 1/2$, and $F(u) = 1-\cos(u)$. Here the equation $(-\Delta)^{1/2}u = \sin(u)$ in $\mathbb{R}$ admits the field of extremals $u^t(x) = 2\arctan(x+t)$. We do not know if the null-Lagrangian property holds for $\mathcal{F}^1_{s,F}$ in this concrete example.

It is also interesting to compare $\mathcal{F}^1_{s,F}$ with the calibration $\mathcal{C}_{s,F}$ constructed in Section 4.4. There, by the alternative expression for $\mathcal{C}_{s,F}$ derived in Lemma 4.4.4, we see that $\mathcal{F}^1_{s,F}(w)$ and $\mathcal{C}_{s,F}(w)$ would coincide if the following equality were true:

$$
\begin{aligned}
-\lim_{\varepsilon\downarrow 0}\iint_{Q(\Omega)\setminus\{|x-y|>\varepsilon\}} & \frac{\int_{t(x,w(x))}^{t(y,w(y))}(u^t(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{Q(\Omega)} \frac{(w(x)-u^{t(x,w(x))}(y))(u^{t(x,w(x))}(y)-w(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

However, we do not know how to prove or disprove this identity.

The functional $\mathcal{F}^1_{s,F}$ does not capture the symmetry in the variables x and y that has appeared in the two previous works on nonlocal calibrations [18, 80]. Hence, it is also natural to propose the following new candidate:

$$
\begin{aligned}
\mathcal{F}^2_{s,F}(w) := \frac{c_{n,s}}{2}\iint_{Q(\Omega)} & \left.\frac{(u^\tau(x)-u^t(y))(w(x)-w(y))}{|x-y|^{n+2s}}\right|_{\substack{t=t(x,w(x))\\ \tau=t(y,w(y))}}\mathrm{d}x\,\mathrm{d}y \\
&- \frac{c_{n,s}}{4}\iint_{Q(\Omega)} \left.\frac{|u^\tau(x)-u^t(y)|^2}{|x-y|^{n+2s}}\right|_{\substack{t=t(x,w(x))\\ \tau=t(y,w(y))}}\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x.
\end{aligned}
$$

As in the preceding case, we can apply Young's inequality and the definition of the leaf-parameter function to deduce that $\mathcal{F}^2_{s,F}$ satisfies properties $(\mathcal{C}1)$ and $(\mathcal{C}2)$. Nevertheless, in this case we can discard it as a calibration since the null-Lagrangian property fails even when $F = 0$ and $u^t(x) = x + t$.

One could also think of a calibration candidate constructed by replacing the gradient terms in the local theory by fractional ones. That is,

$$
\mathcal{F}^3_{s,F}(w) := \int_\Omega \left\{\nabla^s u^t(x)\cdot\nabla^s w(x)\,\mathrm{d}x - \frac{1}{2}\left|\nabla^s u^t(x)\right|^2\right\}\Big|_{t=t(x,w(x))}\mathrm{d}x - \int_\Omega F(w(x))\,\mathrm{d}x.
$$

Here, the fractional gradient is defined as

$$
\nabla^s w(x) = \widetilde{c}_{n,s}\int_{\mathbb{R}^n}\frac{w(x)-w(y)}{|x-y|^{n+s}}\frac{x-y}{|x-y|}\,dy.
$$

This last candidate would be motivated by the identity

$$
\int_{\mathbb{R}^n}\nabla^s v(x)\cdot\nabla^s w(x)\,\mathrm{d}x = \frac{c_{n,s}}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.
$$

Nevertheless, a similar equality does not hold when restricting to a domain Ω, i.e.,

$$
\int_\Omega \nabla^s v(x)\cdot\nabla^s w(x)\,\mathrm{d}x \neq \frac{c_{n,s}}{2}\iint_{Q(\Omega)}\frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.
$$

Hence, $\mathcal{F}^3_{s,F}$ does not satisfy property $(\mathcal{C}1)$ and thus it is not a calibration for $\mathcal{E}_{s,F}$.

Appendix I

Minimality via a comparison principle

Here in this appendix, we explain how to prove minimality for a function embedded in a field of extremals via a strong comparison principle. The proof will require an existence and regularity theorem for minimizers.

As in Section 5.2, we let $G_{\mathrm{N}}(x, y, a, b)$ be a nonlocal Lagrangian giving the energy functional

$$\mathcal{E}_{\mathrm{N}}(w) = \frac{1}{2} \iint_{Q(\Omega)} G_{\mathrm{N}}(x, y, w(x), w(y)) \, \mathrm{d}x \, \mathrm{d}y.$$

For the sake of clarity, we assume that G_{N} is pairwise symmetric, that is,

$$G_{\mathrm{N}}(y, x, b, a) = G_{\mathrm{N}}(x, y, a, b) \quad \text{for all } (x, y) \in Q(\Omega) \text{ and } a, b \in \mathbb{R},$$

which can always be done by the symmetry of the domain $Q(\Omega) = (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)$, see (5.2.1). The Euler-Lagrange operator associated to $\mathcal{E}_{\mathrm{N}}$ is given in terms of the integral

$$\mathcal{L}_{\mathrm{N}}(w)(x) = \int_{\mathbb{R}^n} \partial_a G_{\mathrm{N}}(x, y, w(x), w(y)) \, \mathrm{d}y,$$

see (5.2.3).

A sufficient condition for the operator $\mathcal{L}_{\mathrm{N}}$ to satisfy a strong comparison principle is the strict ellipticity condition $\partial^2_{ab} G_{\mathrm{N}} < 0$. Indeed, given two regular functions u, $v \colon \mathbb{R}^n \to \mathbb{R}$, if u touches v from below at some point x_0, then the monotonicity of $\partial_a G_{\mathrm{N}}$ leads to the inequality $\mathcal{L}_{\mathrm{N}}(u)(x_0) \geq \mathcal{L}_{\mathrm{N}}(v)(x_0)$. To see this, one must simply integrate $\partial_a G_{\mathrm{N}}(x, y, u(x_0), u(y)) \geq \partial_a G_{\mathrm{N}}(x, y, u(x_0), v(y))$ with respect to y and use that $u(x_0) = v(x_0)$. Moreover, when $u \neq v$ we have the strict inequality $\mathcal{L}_{\mathrm{N}}(u)(x_0) > \mathcal{L}_{\mathrm{N}}(v)(x_0)$.

Whenever a result on the existence of minimizers for $\mathcal{E}_{\mathrm{N}}$ is available, the comparison principle above can be used to show the minimality of solutions in a field of extremals. This method has been known for a long time for the fractional Laplacian. It does not need the construction of a calibration, but again requires an existence result which in general will not be available for nonlocal energy functionals.

To see this, let $\{u^t\}_{t \in I}$ be a field of extremals in Ω and suppose, for the sake of contradiction, that u^{t_0} with $t_0 \in I$ is not a minimizer. Let v be a minimizer of $\mathcal{E}_{\mathrm{N}}$ in the set of functions with graph $v \subset \mathcal{G} = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : \lambda = u^t(x) \text{ for some } t \in I\}$ satisfying the exterior condition $v = u^{t_0}$ in Ω^c. In particular, $v \neq u^{t_0}$ and by the monotonicity with respect to the leaf-parameter, there is a first leaf u^{t_1} touching v from (say) above at an interior point $x_0 \in \Omega$. The strong comparison principle now gives $0 = \mathcal{L}_{\mathrm{N}}(v)(x_0) > \mathcal{L}_{\mathrm{N}}(u^{t_1})(x_0) = 0$, which is a contradiction, and thus $u^{t_0} \equiv v$. Note that this argument gives the uniqueness of the minimizer (and even of the extremal) with the given exterior condition. It is also clear that the same argument works for fields made of super and subsolutions, that is, fields such that $\mathcal{L}_{\mathrm{N}}(u^t) \geq 0$ for $t \geq t_0$ and $\mathcal{L}_{\mathrm{N}}(u^t) \leq 0$ for $t \leq t_0$ in Ω.

Appendix J

The calibration for the nonlocal total variation

In this appendix, we relate our functional setting to the geometric calibrations for the nonlocal perimeter appearing in the works of Cabré [18] and Pagliari [80]. This is achieved through the *nonlocal total variation*, which amounts to the integral of the nonlocal perimeters of the levels sets of a function.

Let us recall that, given an even kernel $K : \mathbb{R}^n \setminus \{0\} \to [0, +\infty)$, the K-nonlocal total variation of a function $w \colon \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\mathcal{E}_{\mathrm{NTV}}(w) := \frac{1}{2} \iint_{Q(\Omega)} |w(x) - w(y)|\, K(x-y)\, \mathrm{d}x\, \mathrm{d}y,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. In particular, it is an energy functional of the form (5.1.6) with Lagrangian

$$G_{\mathrm{N}}(x, y, a, b) = |a - b|\, K(x-y),$$

which is elliptic and hence covered by our extremal field theory.

There is a strong connection between the nonlocal total variation and nonlocal minimal surfaces. Given a sufficiently regular set $E \subset \mathbb{R}^n$, its K-nonlocal perimeter is

$$\mathcal{P}_{\mathrm{N}}(E) := \frac{1}{2} \iint_{Q(\Omega)} |\mathbb{1}_E(x) - \mathbb{1}_E(y)|\, K(x-y)\, \mathrm{d}x\, \mathrm{d}y,$$

and E is called a K-nonlocal minimal surface if the first variation of $\mathcal{P}_{\mathrm{N}}$ at E vanishes. Notice that $\mathcal{P}_{\mathrm{N}}(E) = \mathcal{E}_{\mathrm{NTV}}(\mathbb{1}_E)$. It is well-known that the sublevel sets of minimizers of $\mathcal{E}_{\mathrm{NTV}}$ are K-nonlocal minimal surfaces. Moreover, one can recover the nonlocal total variation $\mathcal{E}_{\mathrm{NTV}}$ of any function w in terms of the nonlocal perimeter $\mathcal{P}_{\mathrm{N}}$ of its sublevel sets. Namely, we have the following nonlocal coarea formula

$$\mathcal{E}_{\mathrm{NTV}}(w) = \int_{\mathbb{R}} \mathcal{P}_{\mathrm{N}}(\{w < \lambda\})\, \mathrm{d}\lambda. \tag{J.1}$$

In [80], Pagliari studied minimality properties of the nonlocal total variation $\mathcal{E}_{\mathrm{NTV}}$ when acting on functions taking values in the interval $[0, 1]$. He showed that characteristic functions of halfspaces minimize $\mathcal{E}_{\mathrm{NTV}}$ among those functions by constructing a calibration. On the other hand, Cabré [18] gave a calibration (recalled in (J.2) below) for the K-nonlocal perimeter $\mathcal{P}_{\mathrm{N}}$ and an arbitrary set E whenever it is embedded in a family of nonlocal minimal surfaces. Thus, the author extended the classical extremal field theory to nonlocal minimal surfaces. Our thesis is in the spirit of this second result but applied to the nonlocal total variation considered in [80]. In particular, Theorem 5.1.3 provides a calibration for the nonlocal total variation $\mathcal{E}_{\rm NTV}$ and an arbitrary function whenever it is embedded in a field of extremals.

Let us recall the calibration for the perimeter $\mathcal{P}_{\rm N}$ obtained in [18]. Given a smooth function $\phi\colon \mathbb{R}^n \to \mathbb{R}$, for each $t \in \mathbb{R}$, we consider the superlevel sets $E^t = \{\phi(x) > t\}$. In [18] (see also [23, Section 2]), under the assumption that E^t are nonlocal minimal surfaces, it was shown that the functional

$$\mathcal{C}_{\mathcal{P}_{\rm N}}(F) = \frac{1}{2} \iint_{Q(\Omega)} \operatorname{sign}\left(\phi(x) - \phi(y)\right) \left(\mathbb{1}_F(x) - \mathbb{1}_F(y)\right) K(x-y)\, \mathrm{d}x\, \mathrm{d}y \tag{J.2}$$

is a calibration for $\mathcal{P}_{\rm N}$ and each E^{t_0}, $t_0 \in I$.

Finally, we show that the analogue of the nonlocal coarea formula (J.1) holds for the calibration functional. Namely, the calibration for the nonlocal total variation constructed in the present paper can be written in terms of the calibration for the nonlocal perimeter of each sublevel set. We point out that all identities in Proposition J.1 continue to hold if $\{u^t\}_{t\in I}$ is simply a field in $\mathbb{R}^n$, that is, if the u^t are not necessarily extremals of $\mathcal{E}_{\rm NTV}$.

Proposition J.1. *Let $\{u^t\}_{t\in I}$ be a field of extremals for $\mathcal{E}_{\rm NTV}$. Then, the associated calibration functional $\mathcal{C}_{\rm NTV}$ given by Theorem 5.1.3 can be written as*

$$\mathcal{C}_{\rm NTV}(w) = \frac{1}{2} \iint_{Q(\Omega)} \int_{w(y)}^{w(x)} \operatorname{sign}\left(u^{t(x,\lambda)}(x) - u^{t(x,\lambda)}(y)\right) \mathrm{d}\lambda \; K(x-y)\; \mathrm{d}x\, \mathrm{d}y.$$

Moreover, the functional $\mathcal{C}_{\rm NTV}$ can also be expressed as

$$\mathcal{C}_{\rm NTV}(w) = \int_{\mathbb{R}} \mathcal{C}_{\mathcal{P}_{\rm N},\lambda}(\{w < \lambda\})\, \mathrm{d}\lambda,$$

where $\mathcal{C}_{\mathcal{P}_{\rm N},\lambda}$ is the calibration for the K-nonlocal perimeter $\mathcal{P}_{\rm N}$ in (J.2) *constructed via the foliation given by the sublevel sets $\{u^t < \lambda\}_{t\in I}$.*

Remark J.2. Before we succeeded in constructing a calibration for general functionals (and even the quadratic one in Chapter 4), we were able to build one for the nonlocal total variation $\mathcal{E}_{\rm NTV}$. For this, we considered the second identity in Proposition J.1 as our definition of the calibration. This idea was motivated by the coarea formula (J.1). It is quite remarkable that our general construction in Theorem 5.1.3 (found by completely different means) recovers this natural calibration.

Notice that by the properties of the field $\{u^t\}_{t\in I}$, for each $\lambda \in \mathbb{R}$, the level sets $\{u^t = \lambda\}_{t\in I}$ give a foliation of $\mathbb{R}^n$. Moreover, since each u^t is an extremal of the nonlocal total variation, the sublevel sets

$$E^t_\lambda := \{x \in \mathbb{R}^n \colon u^t(x) < \lambda\}$$

are K-nonlocal minimal surfaces. By monotonicity, we have $u^t(x) < \lambda = u^{t(x,\lambda)}(x)$ if and only if $t < t(x,\lambda)$, and hence $E^t_\lambda = \{\phi^\lambda > t\}$ with $\phi^\lambda(x) := t(x,\lambda)$, consistently with the notation for $\mathcal{C}_{\mathcal{P}_{\rm N}}$ in (J.2).

Proof of Proposition J.1. Let $t_0 \in I$. First, letting $G_{\mathrm{N}}(x, y, a, b) = |a - b|\, K(x - y)$ in Theorem 5.1.3, the calibration functional associated to $\mathcal{C}_{\mathrm{NTV}}$ is

$$\mathcal{C}_{\mathrm{NTV}}(w) = \iint_{Q(\Omega)} \mathrm{d}x\, \mathrm{d}y \int_{u^{t_0}(x)}^{w(x)} \mathrm{d}\lambda \ \mathrm{sign}\left(u^{t(x,\lambda)}(x) - u^{t(x,\lambda)}(y)\right) K(x-y) + \mathcal{E}_{\mathrm{NTV}}(u^{t_0}). \quad \text{(J.3)}$$

It is easy to check that the sign term in (J.3) can be written as

$$\mathrm{sign}\left(u^{t(x,\lambda)}(x) - u^{t(x,\lambda)}(y)\right) = \mathrm{sign}\left(t(y, \lambda) - t(x, \lambda)\right)$$

for all $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Symmetrizing in the variables x and y, and using that $\int_{u^{t_0}(x)}^{w(x)} \cdot\, \mathrm{d}\lambda - \int_{u^{t_0}(y)}^{w(y)} \cdot\, \mathrm{d}\lambda = \int_{w(y)}^{w(x)} \cdot\, \mathrm{d}\lambda - \int_{u^{t_0}(y)}^{u^{t_0}(x)} \cdot\, \mathrm{d}\lambda$, we have

$$\begin{aligned}
&\iint_{Q(\Omega)} \mathrm{d}x\, \mathrm{d}y \int_{u^{t_0}(x)}^{w(x)} \mathrm{d}\lambda \ \mathrm{sign}\left(t(y, \lambda) - t(x, \lambda)\right)\ K(x-y) \\
&\qquad = \frac{1}{2} \iint_{Q(\Omega)} \mathrm{d}x\, \mathrm{d}y \int_{w(y)}^{w(x)} \mathrm{d}\lambda \ \mathrm{sign}\left(t(y, \lambda) - t(x, \lambda)\right)\ K(x-y) \\
&\qquad\qquad - \frac{1}{2} \iint_{Q(\Omega)} \mathrm{d}x\, \mathrm{d}y \int_{u^{t_0}(y)}^{u^{t_0}(x)} \mathrm{d}\lambda \ \mathrm{sign}\left(t(y, \lambda) - t(x, \lambda)\right)\ K(x-y).
\end{aligned} \tag{J.4}$$

On the other hand, by the nonlocal coarea formula (J.1) and the simple identity

$$|1_{\{u^{t_0}<\lambda\}}(x) - 1_{\{u^{t_0}<\lambda\}}(y)| = \mathrm{sign}\left(t(x, \lambda) - t(y, \lambda)\right) \left(1_{\{u^{t_0}<\lambda\}}(x) - 1_{\{u^{t_0}<\lambda\}}(y)\right),$$

it is not hard to show that

$$\begin{aligned}
\mathcal{E}_{\mathrm{NTV}}(u^{t_0}) &= \frac{1}{2} \iint_{Q(\Omega)} \mathrm{d}x\, \mathrm{d}y \int_{\mathbb{R}} \mathrm{d}\lambda \ |1_{\{u^{t_0}<\lambda\}}(x) - 1_{\{u^{t_0}<\lambda\}}(y)| K(x-y) \\
&= \frac{1}{2} \iint_{Q(\Omega)} \mathrm{d}x\, \mathrm{d}y \int_{u^{t_0}(y)}^{u^{t_0}(x)} \mathrm{d}\lambda \ \mathrm{sign}\left(t(y, \lambda) - t(x, \lambda)\right) K(x-y).
\end{aligned} \tag{J.5}$$

Combining (J.4) and (J.5), from (J.3) we deduce

$$\begin{aligned}
\mathcal{C}_{\mathrm{NTV}}(w) &= \frac{1}{2} \iint_{Q(\Omega)} \mathrm{d}x\, \mathrm{d}y \int_{w(y)}^{w(x)} \mathrm{d}\lambda \ \mathrm{sign}\left(t(y, \lambda) - t(x, \lambda)\right)\ K(x-y) \\
&= \frac{1}{2} \iint_{Q(\Omega)} \mathrm{d}x\, \mathrm{d}y \int_{w(y)}^{w(x)} \mathrm{d}\lambda \ \mathrm{sign}\left(u^{t(x,\lambda)}(x) - u^{t(x,\lambda)}(y)\right)\ K(x-y),
\end{aligned}$$

which was the first claim. Moreover, this expression can also be written as

$$\mathcal{C}_{\mathrm{NTV}}(w) = \frac{1}{2} \iint_{Q(\Omega)} \mathrm{d}x\, \mathrm{d}y \int_{\mathbb{R}} \mathrm{d}\lambda \ \mathrm{sign}\left(t(x, \lambda) - t(y, \lambda)\right) (1_{\{w<\lambda\}}(x) - 1_{\{w<\lambda\}}(y)) K(x-y).$$

Changing the order of integration, to finish the proof it remains to show that

$$\mathcal{C}_{\mathcal{P}_{\mathrm{N}},\lambda}(\{w < \lambda\}) = \frac{1}{2} \iint_{Q(\Omega)} \mathrm{sign}\left(t(x, \lambda) - t(y, \lambda)\right) (1_{\{w<\lambda\}}(x) - 1_{\{w<\lambda\}}(y)) K(x-y)\, \mathrm{d}x\, \mathrm{d}y,$$

but this is precisely the calibration $\mathcal{C}_{\mathcal{P}_{\mathrm{N}}}$ in (J.2) with $\phi(x) = \phi^{\lambda}(x) = t(x, \lambda)$, which yields the claim. $\square$

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