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MULTIDIMENSIONAL SPECKLE NOISE, MODELLING AND FILTERING RELATED TO SAR DATA

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Chapter 3

Wavelet Analysis

3.1 Introduction

The wavelet transform has become an important tool of mathematical analysis, with a wide and ever increasing range of applications, in recent years. There exist various reasons behind the growing success of this tool. From the applications point of view, the wavelet transform has been employed to solve already existing problems in a wide range of scientific disciplines, as for instance, differential equations, numerical analysis, signal processing, to mention only a few. But perhaps, the main motive is the fact that the wavelet analysis theory represents the culmination of the attempts of workers in several fields to design new tools to solve problems in their areas. Consequently, the wavelet analysis theory has not to be understood as an isolated mathematical theory, but as a theory which collects and links studies originated in different scientific disciplines.

The aim of this chapter is not to present the complete wavelet analysis theory in a rigorous way. On the contrary, it shall be presented in an expository approach, with the sufficient level of precision to make the ideas and implications behind this theory easy to understand. References shall be given throughout, pointing to more details when needed. Overview texts can be found in [127, 128, 129, 130, 131, 132], whereas more precise and detailed descriptions of the wavelet analysis theory can be found in [133, 134, 135, 24, 132, 23].

There exist different ways in which the wavelet theory can be introduced, ranging from a rigorous and abstract mathematical form to a conceptual one. But, no matter how the wavelet theory is presented, one idea as to be kept in mind: *Divide et Vinces*, which can be translated as *Divide and Conquer*.

Wavelet analysis theory involves representing a general function f in terms of simpler ones, as follows: $f = \sum_{n\in\mathbb{Z}} c_n\psi_n$, where ψ_n stands for simple functions, also called elementary building blocks or atoms. Behind this basic idea, there are some important theories and techniques that have been proved of crucial importance in mathematical sciences and engineering. The most influential theory is by far, Fourier analysis. The existing amount of techniques based on the Fourier series or on the Fourier transform indicates its potential for functional analysis. However, it is well known in the mathematical and engineering communities, that Fourier analysis is not well suited for certain types of analysis, i.e., the properties of the set of elementary functions $\{\psi_n\}_{n\in\mathbb{Z}}$ are not appropriated for the problem under study. Time-frequency analysis of signals cannot be directly performed with Fourier analysis as it sacrifices time information in order to secure frequency information. The wavelet analysis theory was born as a response to find new tools for time-frequency analysis. The origins of the wavelet theory have to be found around the middle 1950s in the Calderón-Zygmund theory, as an area of harmonic analysis. This theory studies how to break complex functions into simpler ones. The functions $\{\psi_n\}_{n\in\mathbb{Z}}$ are designed in such a way that they provide time and frequency location information about the function under analysis.

In the early 1980s, J.O. Strömberg discovered the first orthogonal wavelet in the context to measure the size and smoothness of functions [136]. Independently, A. Grossmann and J. Morlet studied the wavelet transform in its continuous form [137, 138]. In the middle 1980s, several groups started to establish a bridge between the continuous and discrete worlds, making possible the idea to substitute the Fourier theory to analyze functions in a more suitable time-frequency frame. During the middle 1980s and the early 1990s, several scientist discovered new orthogonal wavelets. S. Mallat and Y. Meyer presented the concept of multiresolution analysis [139], offering a systematic framework to understand these orthogonal expansions. At the same time, they introduced the fast wavelet transform to perform the Discrete Wavelet Transform, leading simultaneously to a link between the wavelet transform and the quadrature mirror filtering theory. Finally, I. Daubechies provided the final link between the continuous and discrete wavelet transforms [135].

As stated at the introduction of this thesis, SAR imagery is characterized by being non-stationary signals with a high-spatial resolution. Another important characteristic is the presence of speckle noise. Consequently, the reduction of this noise term is highly constrained by the necessity to maintain the spatial properties of the SAR images. The wavelet theory presented in this chapter represents a complete different topic from SAR imagery, but as it will be demonstrated in the following chapters, both can be considered as complementary. The reason behind this complementariness is that the wavelet analysis theory represents an optimum vehicle to deal with the SAR imagery spatial features is an simple and efficient way. The aim of this chapter is to present the wavelet analysis theory with the sufficient level of detail to exploit its potential for SAR imagery processing.

In the following, f and g represent general functions. These functions depend on x which can be understood as time or space.

3.2 Fourier Analysis

In 1822 J. Fourier introduced in his mathematical theory of heat the concept that any periodic function can be decomposed in a series of harmonically related sinusoids. This work opened the door to analyze any periodic function on the basis of its frequency content. A clear application of the Fourier analysis can be found in signal analysis, in which is common to come across the problem of separating noise from useful information. In most of the cases, noise is due to some high frequency process whereas useful information has its support within the low frequency region of the spectra. Fourier analysis allows to decrease, or even to reject, high frequency components due to noise, without hardly damaging useful information.

The Fourier analysis theory is based on decomposing any periodical function in the set of complex exponentials $\{e^{jnx}\}_{n\in\mathbb{Z}}$. The properties of the Fourier series, as in the case of any other function analysis theory, are directly linked to the properties of the set of simple functions in which any complex function is decomposed.

3.2.1 The Spaces $\mathbf{L}^1(\mathbb{R})$ and $\mathbf{L}^2(\mathbb{R})$

Before to formally describe the concepts involved in the Fourier, as well as in the wavelet analysis theories, it is convenient to delimitate the class of functions or signals of interest. Most of the physical processes in nature are finite energy phenomena. Therefore, any function or signal associated to these phenomena shall be characterized by some constrains. Finite energy signals are represented by square integrable functions. In the following, two important classes of functions shall be formulated.

Let (χ, S, μ) be a measure space and consider the set of functions Γ^1 of all complex valued functions f whose absolute values are integrable over χ

$$\int |f| \, \mathrm{d}\mu < \infty. \tag{3.1}$$

Eq. (3.1) is not a norm on Γ^1 [140, 24]. Given the subset Γ_0 of Γ_1, defined as $\Gamma_0 = \{f \in \Gamma^1 | f = 0\}$ almost everywhere, the factor space Γ^1/Γ_0 is the set of all equivalence classes under the assumption that f is equivalent to g if $f = g$ almost everywhere. Consequently, Eq. (3.1) is a norm in Γ^1/Γ_0 [140, 24], which is denoted as $L^1(\chi, \mu)$ or simply $L^1(\chi)$. This space, with the norm defined by Eq. (3.1), is not an Euclidean space since the norm cannot be defined from a scalar product. The extension to $L^2(\chi)$ will address this issue.

Let (χ, S, μ) be a measure space and consider the set of functions Γ^2 of all complex valued functions f whose squares are integrable over χ

$$\int |f|^2 \, \mathrm{d}\mu < \infty. \tag{3.2}$$

As in the case of $L^1(\chi)$, $L^2(\chi)$ is defined as the factor space Γ^2/Γ_0. This measure space taken together with the inner product defined as

$$\langle f, g \rangle = \int f g^* \, \mathrm{d}\mu \tag{3.3}$$

where g^* stands for the complex conjugate of g, forms a Hilbert space. As a result, the space $L^2(\chi)$ has an orthogonal basis. Let $f \in L^2(\chi)$ and $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $L^2(\chi)$, then the Fourier coefficients of f with respect to this basis are given by $\widehat{f}(n) = \langle f, e_n \rangle$ $n = 1, 2, \ldots$. Moreover

$$\sum_{n \in \mathbb{Z}} |c_n|^2 = \|f\|^2 \tag{3.4}$$

where $\|f\|$ is the norm of f, defined as $\sqrt{\langle f, f \rangle}$, see Eq. (3.3). Eq. (3.4) is also known as Parseval's identity.

Throughout all this work, interest will be focused on real and complex valued functions in time. Consequently, the spaces $L^1(\chi)$ and $L^2(\chi)$ are particularized to $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. Another important case is when the field χ is formed by the positive integers. In this case, the spaces are represented by $\ell^1\{\mathbb{Z}\}$ and $\ell^2\{\mathbb{Z}\}$, which respectively denote absolute value summable and square summable series. Finally, SAR images are defined within a two-dimensional space. In this case, they will be represented by finite energy functions or series in two-dimensional spaces, respectively denoted by $L^2(\mathbb{R}^2)$ and $\ell^2(\mathbb{Z}^2)$.

3.2.2 Fourier Series

The remarkable result behind the Fourier analysis theory is that any periodic function, or any compactly supported function (finite support) can be expressed as an infinite series of complex exponentials. A periodic, one-dimensional, function f is defined as a function such that $f(x+T) = f(x)$ $\forall x \in \mathbb{R}$, where T stands for the function's period. Without loss of generality, it is possible to derive a 2π-periodic function, $f(Tx/2\pi)$, as a way to analyze T-periodic functions. Similarly, any compactly supported function f can be transformed into a 2π-periodic function, $f_{2\pi} = \sum_{k=-\infty}^{\infty} f(\frac{Tx}{2\pi} - Tk)$.

The space of square integrable functions within the interval $[0, 2\pi)$, denoted by $L^2([0, 2\pi))$ is a Hilbert space with the inner product defined as given by Eq. (3.3). The functions $\cos(x)$ and $\sin(x)$ belong to $L^2([0, 2\pi))$. If e^{jx} is defined as $e^{jx} = \cos(x) + j\sin(x)$, it is clear that e^{jx} also belongs to $L^2([0, 2\pi))$. Let $e_n(x) = e^{jnx}$, then $e_n(x) \in L^2([0, 2\pi))$ $\forall n \in \mathbb{Z}$. It is easy to verify that

$$\langle e_n, e_m \rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}. \tag{3.5}$$

Consequence of that, the family of functions $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the space $L^2([0,2\pi))$. Hence, any function $f \in L^2([0,2\pi))$ can be expanded into a Fourier series as

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e_n \tag{3.6}$$

where the series coefficients c_n are obtained in the following way

$$c_n = \langle f, e_n\rangle = \int_{\mathbb{R}} f e_n^* \,\mathrm{d}\mu = \frac{1}{2\pi}\int_0^{2\pi} f \exp(-jnx)\,\mathrm{d}x. \tag{3.7}$$

An important aspect that arises in the representation given by Eq. (3.6) is wether or not the series converges to f for any value of x. The Dirichlet conditions guarantee the uniform convergence of the series to f, except for those points in which f is discontinuous [24, 141]. At these points, the series converges to the midpoint of the discontinuity. In summary, if f is a periodic function and satisfies the Dirichlet conditions, it can be represented by the Fourier series, where the series coefficients are given by Eq. (3.7).

The map $f \to c_n$ creates an isomorphism of $L^2([0,2\pi))$ onto $\ell^2(\mathbb{Z})$. Thus, both representations of the function f contain the same information. Whereas the function expressed in the space $L^2([0,2\pi))$ gives details about its time or space behavior, the function represented in the Fourier coefficients space $\ell^2(\mathbb{Z})$ gives information about its frequency content. Nevertheless, since the function is integrated over all the time domain, the expansion coefficients $\{c_n\}_{n\in\mathbb{Z}}$ do not contain time or space information. This prevents to use the Fourier series for those cases in which the function f has a variant frequency behavior in time or space.

Unlike the Fourier series of functions in $L^2([0,2\pi))$, the Fourier series of functions in $L^1([0,2\pi))$ does not always converge [24]. This issue also reappears when the Fourier transform is introduced.

3.2.3 Fourier Transform

In the previous section, interest was concentrated on periodic or compactly supported functions. Consider now, a compactly supported function f with a support of length T. As presented previously, a periodic function f_T can be constructed. Clearly, $f_T = f$ in the limit $T \to \infty$. If the set of complex exponentials $\{e_n\}_{n\in\mathbb{Z}}$ are now considered for a T-periodic function, i.e., $e_n(x) = e^{j2\pi nx/T}$, defining $\omega = 2\pi/T$, the expression of f as a Fourier series, Eq. (3.6), under proper conditions [24, 141], is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)\exp(-j\omega x)\,\mathrm{d}x\right)\exp(j\omega x)\,\mathrm{d}\omega. \tag{3.8}$$

The Fourier transform of a non-periodic function f, is defined consequently as

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\exp(-j\omega x)\,\mathrm{d}x. \tag{3.9}$$

As it can be deduced from Eq. (3.8), the original function f can be recovered from its Fourier transform $\widehat{f}(\omega)$ as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(w)\exp(j\omega x)\,\mathrm{d}\omega. \tag{3.10}$$

One requires uncountably many frequencies to describe the spectra of functions in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. Nevertheless, the quantity $\widehat{f}(\omega)$, as a function of ω, describes the spectral characteristics of f completely. For non-periodic functions, it makes no sense to talk about Fourier series in $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$, since $e^{j\omega x}$ is not in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, hence any nontrivial combination of such functions is also not in these spaces.

The Fourier transform of a function $f \in L^2(\mathbb{R})$ is defined as an isomorphism $L^2(\mathbb{R}) \to L^2(\mathbb{R})$. The inverse Fourier transform can be also seen as an isomorphism of $L^2(\mathbb{R})$ onto itself. This is not true for a function $f \in L^1(\mathbb{R})$, as the Fourier transform is not an isomorphism of $L^1(\mathbb{R})$ onto itself. This issue provokes that the inverse Fourier transform may not be always defined [24].

The Fourier transform presents the same handicap as the Fourier series of removing any time or space information of the function f in the Fourier transform $\widehat{f}(w)$. As it is clear from Eq. (3.9), any time or space detail is integrated over all the domain. The first consequence of this issue is that also the Fourier transform is not suitable to study any function particularity localized in time or space.

3.2.4 Time-Frequency Analysis

Let $f \in L^2(\mathbb{R})$ be an analog signal with finite energy $\|f\|^2$. As presented in the previous section, the Fourier transform $\widehat{f}(\omega)$ gives the spectrum of the signal. Besides, as it has been highlighted, it is necessary to have the signal defined over all the time or space domain in order to derive the information relative to a single frequency ω. Moreover, since all the time or space domain is integrated, no time or frequency information is available in the frequency domain. From a practical point of view, these issues make the Fourier analysis theory not suitable for non-stationary signals analysis.

For a spectral analysis to be useful it is necessary to go beyond the Fourier analysis theory. What it is desired is to decompose a function f in a way making possible to identify simultaneously transient phenomena in time, that is time localization, and the presence of particular frequencies, that is frequency localization. These new concepts of function analysis are covered by the so-called time-frequency analysis techniques.

The abstract ideas of time-frequency plane and time-frequency atoms are two useful idealizations of concepts in which time-frequency analysis is based on [128]. The time-frequency plane is a representation in which time and frequency are indicated along the horizontal and vertical axes, respectively. On the other hand, a time-frequency atom is a signal $\psi_\gamma \in L^2(\mathbb{R})$ (γ stands for a possible parameter) which is well concentrated both in time and frequency. A time-frequency atom is, hence, represented in the time-frequency plane as a rectangle with its sides finite and parallel to the axes. The idea of time-frequency transforms is to divide the time-frequency plane into small regions defined by a set of time-frequency atoms $\{\psi_\gamma\}_{\gamma\in\Gamma}$, where Γ represents the space of the parameter γ. Hence, each of the time-frequency atoms ψ_γ captures the time-frequency properties of a given signal f for a particular area of the time-frequency plane. Fig. 3.1 gives a representation of the time-frequency plane, as well as the time-frequency atoms associated with several signals. Some important issues arise at this point:

- Which have to be the properties of each particular signal ψ_γ?, but also, which have to be the characteristics of the set of functions $\{\psi_\gamma\}_{\gamma\in\Gamma}$ for a suitable tiling of the time-frequency phase?
- Does the tiling of the time-frequency plane by $\{\psi_\gamma\}_{\gamma\in\Gamma}$ characterizes completely the signal f?.
- Is it possible to recover f from the time-frequency plane tiling in a stable manner?

All these questions are answered by time-frequency analysis theories. Nevertheless, some preliminary answers can be stated. For a practical point of view, it is convenient the set $\{\psi_\gamma\}_{\gamma\in\Gamma}$ to have an internal structure, which facilitates calculations. Moreover, as it can be deduced, it exist an infinite number of possibilities to divide the time-frequency plane, i.e., an infinite number of possibilities to select the set $\{\psi_\gamma\}_{\gamma\in\Gamma}$. Consequently, it should be ideally selected by taking into consideration the properties of the function f.

The selection of the function ψ_γ is not arbitrary. This function is selected in such a way that its energy is concentrated in a small region of the time-frequency plane, see Fig. 3.1. The dimensions of ψ_γ in the time and frequency axes, schematically represented by the length of the rectangle covering the same region in the time-frequency plane have to be chosen in such a way that they fulfill the Heisemberg's Uncertainty Principle [128,23,24]. Informally stated, this principle says that a signal's feature (frequency component) and the features's location (position at which that frequency component is found) cannot be

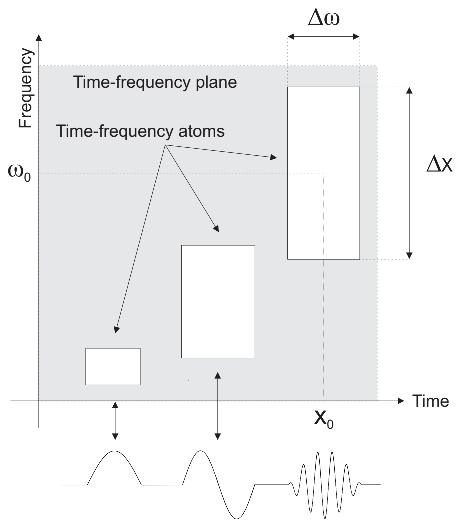

Figure 3.1: Representation of the time-frequency plane and the time-frequency atoms associated with three different functions.

measured to an arbitrary degree of precision simultaneously. Let x_0 and ω_0 be the average location of ψ_γ in the time and frequency axes of the time-frequency plane respectively defined as

$$x_0 = \frac{1}{\|\psi_\gamma\|^2} \int_{-\infty}^{\infty} x|\psi_\gamma(x)|^2 \, \mathrm{d}x \tag{3.11}$$

$$\omega_0 = \frac{1}{2\pi\|\psi_\gamma\|^2} \int_{-\infty}^{\infty} \omega|\widehat{\psi}_\gamma(\omega)|^2 \, \mathrm{d}\omega \tag{3.12}$$

and Δx and $\Delta\omega$ defined as the variances around these average values

$$\Delta x = \frac{1}{\|\psi_\gamma\|^2} \int_{-\infty}^{\infty} (x - x_0)^2|\psi_\gamma(x)|^2 \, \mathrm{d}x \tag{3.13}$$

$$\Delta\omega = \frac{1}{2\pi\|\psi_\gamma\|^2} \int_{-\infty}^{\infty} (\omega - \omega_0)^2|\widehat{\psi}_\gamma(\omega)|^2 \, \mathrm{d}\omega. \tag{3.14}$$

The Heisemberg's Uncertainty Principle states that $\Delta x \Delta\omega \geq 1/4$.

Windowed Fourier techniques are an attempt to overcome the drawbacks present in the classical Fourier theory [23, 24]. Let w be a window function, i.e., a compactly supported function, such its Fourier transform $\widehat{w}$ is also a compactly supported function. For w taken as a real symmetric function, w is translated by u and modulated by the frequency ω

$$w_{u,\omega} = \exp(j\omega x)w(x - u). \tag{3.15}$$

Thus, the resulting windowed Fourier transform for $f \in L^2(\mathbb{R})$ is

$$\mathcal{T}f(u, \omega) = \langle f, w_{u,\omega}\rangle = \int_{-\infty}^{\infty} f(x)w(x - u)\exp(j\omega x) \, \mathrm{d}x. \tag{3.16}$$

This transform is also called the short time Fourier transform, since the multiplication of f by the window $w(t - u)$ localizes the Fourier integral in the neighborhood of $x = u$. When the window function is a Gaussian function, the short time Fourier transform is called the Gabor transform [142], which attains the equality given by the Heisemberg's Uncertainty Principle. Fig. 3.2 presents the way the short time Fourier transform tiles the time-frequency plane. As it can be observed, this division is not adapted to the time-frequency plane. For low-frequencies, it would be desirable to have windows covering a wide time interval in order to capture this information. On the contrary, higher frequencies need a narrower time support to collect the desirable information. Consequently, the windows should have a narrow frequency support for low frequency and a wider one for high frequencies, as stated by the Heisemberg's Uncertainty Principle.

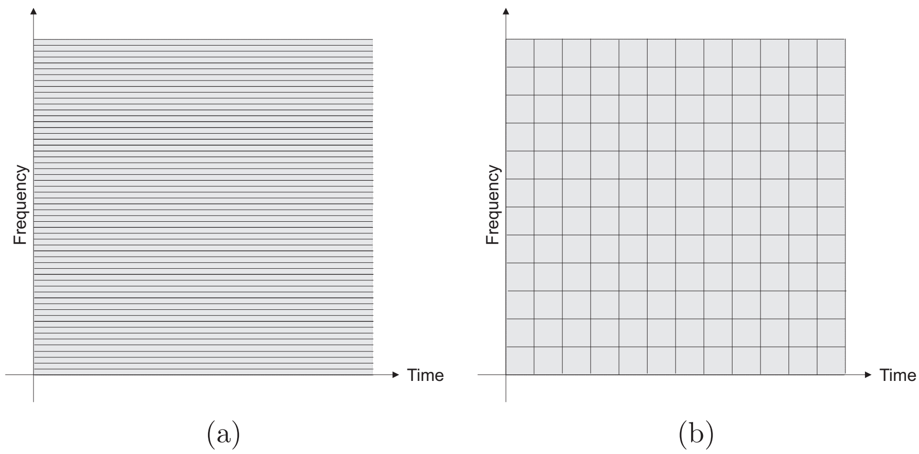

Figure 3.2: Tiling of the time-frequency plane carried out by the Fourier transform (a) and the short time Fourier transform (b).

3.3 Wavelet Analysis

Previous to introduce the wavelet transform, it will be helpful to revise, and to extend, some ideas already introduced. The Fourier series, the Fourier transform, but specially the short time Fourier transform allow to analyze a given function $f \in L^2(\mathbb{R})$ in the frequency domain. It is possible to affirm that the Fourier analysis theory permits to examine this function at different detail levels. The wavelet transform pretends to extend this idea in a way that a particular function feature can be analyzed at different levels of detail or resolutions, also called scales in the following. This analysis can be partially done by the short time Fourier transform, but as explained, it is not adapted to the time-frequency plane nature. The wavelet analysis theory gives an answer to this problem by decomposing a given function into a basis of simple functions which are well localized, both in time and frequency.

3.3.1 Continuous Wavelet Transform

Given a function $\psi \in L^2(\mathbb{R})$ which satisfies the condition

$$\int_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x = 0 \tag{3.17}$$

with $\|\psi\|$=1 and centered in $x = 0$, the function $\psi_{a,b}(x)$ is defined as

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right) \tag{3.18}$$

where $a, b \in \mathbb{R}$, $a \neq 0$. The function ψ is called the mother wavelet or simply wavelet, whereas the functions $\{\psi_{a,b}\}_{a,b\in\mathbb{R}}$ are called wavelets. The continuous wavelet transform of a function $f \in L^2(\mathbb{R})$, also denoted by CWT, is defined as

$$\mathcal{W}f(a,b) = \langle f, \psi_{a,b}\rangle = \int_{-\infty}^{\infty} f(x)\frac{1}{\sqrt{a}}\psi^*\left(\frac{x-b}{a}\right)\mathrm{d}x. \tag{3.19}$$

Eq. (3.19) can be also written as a convolution

$$\mathcal{W}f(a,b) = f * \overline{\psi}_a(b) \tag{3.20}$$

where $\overline{\psi}_a(x) = \frac{1}{\sqrt{a}}\psi^*(-x/a)$. The parameter b is called the translation parameter, whereas a is called the dilation parameter. This parameter can be also interpreted as the inverse of the frequency ω. Owing to Eq. (3.17), the Fourier transform of the function ψ, denoted by $\widehat{\psi}(\omega)$, satisfies $\widehat{\psi}(0) = 0$. Considering

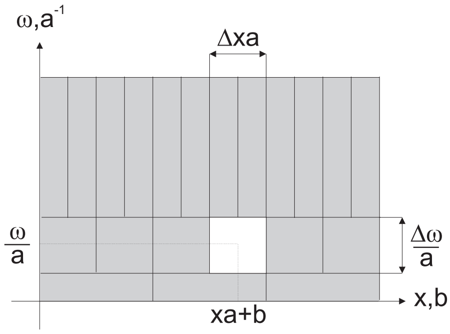

Figure 3.3: Time-frequency plane tiling done by the continuous wavelet transform at discrete positions of the translation parameter b and the dilation parameter a.

Eq. (3.20), thus, the CWT can be seen as a filtering of f by dilated band-pass filters, whose impulse response are given by the wavelets, Eq. (3.18).

The issue arising at this point is wether or not it is possible to recover the function f from the transformed values $\mathcal{W}f(a,b)$. In other words, wether it is possible or not to define an inversion formula. If the wavelet function ψ satisfies the admissibility condition

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|} \,\mathrm{d}\omega < \infty \tag{3.21}$$

then, any $f \in L^2(\mathbb{R})$ satisfies

$$f(x) = \frac{1}{C_\psi} \int_0^{\infty} \int_{-\infty}^{\infty} \mathcal{W}f(a,b) \frac{1}{\sqrt{a}} \psi\left(\frac{x-b}{a}\right) \mathrm{d}b \, \frac{\mathrm{d}a}{a^2} \tag{3.22}$$

which is called the inverse continuous wavelet transform of f, denoted by ICWT. For a ψ satisfying the admissibility condition, Eq. (3.17) results.

The wavelet function ψ is defined in such a way that both, ψ and $\widehat{\psi}$ are window functions, allowing to define its centers and variances as shown by Eqs. (3.11), (3.12), (3.13) and (3.14). The wavelets $\{\psi_{a,b}\}_{a,b\in\mathbb{R}}$ are centered at the positions $xa+b$, with a time width equal to Δxa. The corresponding Fourier transforms have their centers at the positions ω/a whereas they have a frequency width of $\Delta\omega/a$. This means that the CWT obtains the information relative to f in the time dimension in a window of width Δxa centered at $xa+b$, and in the frequency dimension with a window of width $\Delta\omega/a$ centered at ω/a. Consequently, the frequency support of ψ is larger for high frequencies than for low frequencies, whereas its time support is low for high frequencies and larger for low frequencies. As it can be deduced, the region of influence of the functions $\{\psi_{a,b}\}_{a,b\in\mathbb{R}}$, i.e., the dimensions and position of the time-frequency atoms in the time-frequency plane, is adapted to the nature of this plane. Fig. 3.3 depicts the behavior of the functions $\{\psi_{a,b}\}_{a,b\in\mathbb{R}}$, schematically represented by rectangles.

The CWT presents two clear advantages in front of the Fourier transform. First of all, the CWT has better time-frequency location properties as a result of using a set of functions $\{\psi_{a,b}\}_{a,b\in\mathbb{R}}$, which have a compact support both in time and frequency. Moreover, these functions are in $L^2(\mathbb{R})$. The CWT is not suitable to analyze the properties of discrete series, as for instance, SAR imagery. For this reason, it is necessary to explore the possibility to obtain a discrete set of the support given by the parameters a and b, from which it was possible to completely characterize a given function.

3.3.2 Discrete Wavelet Transform

The discrete wavelet transform theory has its foundations within the frame theory [143, 144], which analyzes the completeness, stability and redundancy of linear discrete representations. Let $f \in L^2(\mathbb{R})$ be an arbitrary function, it can be expanded in a discrete basis of $L^2(\mathbb{R})$ as follows

$$f = \sum_{j,m\in\mathbb{Z}} \langle f, \psi_{a_j,b_m} \rangle \widetilde{\psi}_{a_j,b_m} \tag{3.23}$$

where $\{\psi_{a_j,b_m}\}_{a_j,b_m\in\mathbb{Z}}$ and $\{\widetilde{\psi}_{a_j,b_m}\}_{a_j,b_m\in\mathbb{Z}}$ are two bases of the space $L^2(\mathbb{R})$. Eq. (3.23) represents the most general way to obtain such a decomposition, in which the decomposition basis, $\{\psi_{a_j,b_m}\}_{a_j,b_m\in\mathbb{Z}}$, is different from the reconstruction one, $\{\widetilde{\psi}_{a_j,b_m}\}_{a_j,b_m\in\mathbb{Z}}$ and both bases are redundant. As Section 3.2.4 hinted at, two crucial issues referring the linear expansion depicted by Eq. (3.23) arise at this point:

- Does the sequence $\langle f, \psi_{a_j,b_m} \rangle$ completely characterizes the function f?, and
- It is possible to recover f from this sequence in a stable manner?.

The answer to the previous questions needs to introduce of the idea of frame. A sequence $\{\psi_{a_j,b_m}\}_{a_j,b_m\in\mathbb{Z}}$ in a Hilbert space H, is called a frame, if and only if, for all $f \in H$

$$A\|f\|^2 \leq \sum_{j,m\in\mathbb{Z}} |\langle f, \psi_{a_j,b_m} \rangle|^2 \leq B\|f\|^2 \tag{3.24}$$

where the frame bounds A and B are independent from f. In the frame theory, $\{\psi_{a_j,b_m}\}_{a_j,b_m\in\mathbb{Z}}$ receives the name of frame, whereas the basis $\{\widetilde{\psi}_{a_j,b_m}\}_{a_j,b_m\in\mathbb{Z}}$ is known as a dual basis. Consequence of Eq. (3.24), any function $f \in H$, can be expressed as the following linear expansions

$$f = \sum_{j,m\in\mathbb{Z}} \langle f, \psi_{a_j,b_m} \rangle \widetilde{\psi}_{a_j,b_m} = \sum_{j,m\in\mathbb{Z}} \langle f, \widetilde{\psi}_{a_j,b_m} \rangle \psi_{a_j,b_m}. \tag{3.25}$$

For $H = L^2(\mathbb{R})$, I. Daubechies [135] gave the necessary and sufficient conditions, on a wavelet ψ, under which the discrete set $\{\psi_{a_j,b_m}\}_{a_j,b_m\in\mathbb{Z}}$ satisfies Eq. (3.24), and therefore is a basis of $L^2(\mathbb{R})$. A particular case occurs for $A = B = 1$, hence, the frame basis and the dual basis are equal.

Having in mind the concept of the time-frequency plane, a general frame, as a basis of $L^2(\mathbb{R})$, should cover the complete plane. Therefore, any sampling of the continuous parameters a and b should be done in such a way that the discrete wavelet family resulting from this sampling forms a basis of $L^2(\mathbb{R})$. In the frequency dimension, these functions are centered at the positions ω_0/a with a spread equal to $\Delta\omega/a$. In order to obtain a full coverage, it is therefore necessary, to sample it at the positions $\{a_0^j\}_{j\in\mathbb{Z}}$, with $a_0 > 1$ [135, 23, 24]. The time domain is uniformly sampled at intervals proportional to the scale a_0^j. As a consequence, the components of the discrete set $\{\psi_{j,m}\}_{j,m\in\mathbb{Z}}$ have the expressions

$$\psi_{j,m}(x) = \frac{1}{a_0^{j/2}} \psi\left(\frac{x - a_0^j b_0 m}{a_0^j}\right) \quad j, m \in \mathbb{Z} \tag{3.26}$$

with $b_0 > 0$. Given the previous set $\{\psi_{j,m}\}_{j,m\in\mathbb{Z}}$, and provided that it is a frame of $L^2(\mathbb{R})$, any function $f \in L^2(\mathbb{R})$ can be decomposed as a series

$$f = \sum_{j,m\in\mathbb{Z}} \langle f, \psi_{j,m} \rangle \psi_{j,m} \tag{3.27}$$

where $\langle f, \psi_{j,m} \rangle$ is obtained as

$$\mathcal{W}f(j,m) = \langle f, \psi_{j,m} \rangle = \int_{-\infty}^{\infty} f(x) \frac{1}{a_0^{j/2}} \psi^*\left(\frac{x - a_0^j b_0 m}{a_0^j}\right) \mathrm{d}x. \tag{3.28}$$

This equation receives the name of discrete wavelet transform or DWT, which is very similar to the expression of the CWT, Eq. (3.19). It is important to notice that Eq. (3.28) represents a mapping from $L^2(\mathbb{R})$ onto $\ell^2(\mathbb{Z}^2)$. The DWT is obtained through a continuous process performed by the integration process. Moreover, the dimensionality of the space supporting the discrete wavelet values, referred in the following as transformed or wavelet domain, has increased with respect to the time or space domain (also referred as original domain of the function f). The conceptual reason which explains this issue is that, in the original domain only time or space information is available, whereas the transformed domain is able to give time or space information (within the parameter m) as well as frequency or scale information (within the parameter j).

Of special importance is the case in which the parameter $a_0 = 2$ and $b_0 = 1$ [145,146]. This particular case is referred as dyadic discrete wavelet transform or simply discrete wavelet transform, in which the wavelets have the following expressions

$$\psi_{j,m}(x) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{x - 2^j m}{2^j}\right) \quad j, m \in \mathbb{Z}. \tag{3.29}$$

The advantage of this particular case lies in the fact that fast computations of the DWT are possible, but also a link between the DWT and certain filtering schemes is possible.

Before to continue, it is helpful to summarize the ideas behind sampling the CWT. Under the proper conditions, a sampling based on translations and dilatations of the wavelet $\psi \in L^2(\mathbb{R})$ makes possible to obtain a discrete basis for the space $L^2(\mathbb{R})$. Therefore, it is possible to express any function $f \in L^2(\mathbb{R})$ as a linear combination of the components of this basis. The properties of this linear expansion depend on the characteristics of the wavelet function ψ (properties as individual function), but also on how the continuous set of wavelet functions $\{\psi_{a,b}\}_{a,b\in\mathbb{R}}$, obtained from translations and dilations of the function ψ, see Eq. (3.18), is sampled to derive the discrete set $\{\psi_{j,m}\}_{j,m\in\mathbb{Z}}$ as basis of $L^2(\mathbb{R})$. As it has been shown, the wavelet theory is derived without assuming a particular wavelet function ψ, but only the properties a function should have to be a wavelet.

The next section is focused on the description of a large subclass of wavelets that arise from certain structures in $L^2(\mathbb{R})$ called multiresolution analysis. These wavelets yield discrete families of dilations and translations that are orthonormal basis for $L^2(\mathbb{R})$. Nevertheless, there exist other possibilities to derive such a basis. For instance, it is possible to select the basis to be redundant. One example of wavelets obtained through this process are those obtained through the *à trous algorithm* [147]. The principal feature of this algorithm is that the time or space dimension is redundant.

3.3.3 Multiresolution Analysis

As stated previously, the properties of the wavelet basis $\{\psi_{j,m}\}_{j,m\in\mathbb{Z}}$ depend on the properties of the wavelet function ψ as well as on the way the family of translations and dilations, $\{\psi_{a,b}\}_{a,b\in\mathbb{R}}$ is sampled. Consequently, it exist a wide range of possibilities to select a particular basis $\{\psi_{j,m}\}_{j,m\in\mathbb{Z}}$ for $L^2(\mathbb{R})$. The particular choice of $\{\psi_{j,m}\}_{j,m\in\mathbb{Z}}$, originated by the so-called multiresolution analysis structures in $L^2(\mathbb{R})$ [148], has a crucial importance as it allows to derive orthonormal wavelet bases for $L^2(\mathbb{R})$.

A multiresolution analysis of $L^2(\mathbb{R})$, called in the following MRA, is a sequence of closed subspaces $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ satisfying the following properties [149,150]:

(a) $\mathbf{V}_{j+1} \subset \mathbf{V}_j \ \forall j \in \mathbb{Z}$; (Nesting property).

(b) $\lim_{j\to-\infty} \mathbf{V}_j = \text{Closure}\left(\bigcup_{j=-\infty}^{\infty} \mathbf{V}_j\right) = L^2(\mathbb{R})$; (Density of the union in $L^2(\mathbb{R})$).

(c) $\lim_{j\to\infty} \mathbf{V}_j = \bigcap_{j=-\infty}^{\infty} \mathbf{V}_j = \{0\}$.

(d) $f(x) \in \mathbf{V}_j \Leftrightarrow f(x/2) \in \mathbf{V}_{j+1} \ \forall j \in \mathbb{Z}$; (Scaling property).

(e) $f(x) \in \mathbf{V}_j \Leftrightarrow f(x - 2^j m) \in \mathbf{V}_j \ \forall j \in \mathbb{Z}$; (Invariance under integral translations).

(f) $\exists \phi \in \mathbf{V}_0$ producing $\{\phi_{0,m}\}_{m\in\mathbb{Z}}$ to be an orthonormal basis of $\mathbf{V}_0$, where $\phi_{j,m}(x) = 2^{-j/2}\phi(2^{-j}x - m) \ \forall j, m \in \mathbb{Z}$; (Existence of a scaling function).

From the previous conditions, any space $\mathbf{V}_j$ is an scaled version of a central space $\mathbf{V}_0$. Moreover, as $\{\phi_{0,m}\}_{m\in\mathbb{Z}}$ is an orthonormal basis for $\mathbf{V}_0$, due to (e) and (f), $\{\phi_{j,m}\}_{j,m\in\mathbb{Z}}$ is an orthonormal basis for $\mathbf{V}_j$. The function ϕ receives the name of scaling function, as the family of translations and dilations of it generates orthonormal bases for the spaces $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$, which as it will be shown, represent spaces of different detail level or scale spaces. To avoid confusion in the following, 2^j shall be referred as the scale parameter whereas 2^{-j} shall denote the resolution.

Starting from the nesting and scaling properties of the spaces $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$, any component of $\mathbf{V}_{j+1}$ can be expressed as a linear combination of the elements of the orthonormal basis of $\mathbf{V}_j$

$$\phi_{j+1} = \sum_{m\in\mathbb{Z}} h_m \phi_{j,m} \tag{3.30}$$

where $h_m = \langle \phi_{j+1}, \phi_{j,m}\rangle$, and $\sum_{m\in\mathbb{Z}} |h_m|^2 = 1$ from the orthonormality condition. Taking the conditions (d) and (f), both ϕ_{j+1} and ϕ_j are related with the scaling function ϕ, allowing to rewrite Eq. (3.30) as a function of ϕ

$$\frac{1}{\sqrt{2^{j+1}}}\phi\left(\frac{x}{2^{j+1}}\right) = \sum_{m\in\mathbb{Z}} h_m \frac{1}{\sqrt{2^j}}\phi\left(\frac{x-2^j m}{2^j}\right). \tag{3.31}$$

Let $\{h_m\}_{m\in\mathbb{Z}}$ be a discrete filter with a Fourier transform $\widehat{h}(\omega) = \sum_{m=-\infty}^{\infty} h_m e^{jm\omega}$, then, one can obtain the Fourier transform of Eq. (3.31), for $j=1$, as

$$\widehat{\phi}(2\omega) = \frac{1}{\sqrt{2}}\widehat{h}(\omega)\widehat{\phi}(\omega). \tag{3.32}$$

As it can be deduced, the orthonormal basis for $\mathbf{V}_{j+1}$ is obtained by filtering the basis corresponding to the space $\mathbf{V}_j$ with a filter whose Fourier transform is $\widehat{h}(\omega)$. Eq. (3.32) can be obtained for a general scale j, assuming $j > 0$

$$\widehat{\phi}(2^{-j+1}\omega) = \frac{1}{\sqrt{2}}\widehat{h}(2^{-j}\omega)\widehat{\phi}(2^{-j}\omega). \tag{3.33}$$

By recursion, the following result can be obtained

$$\widehat{\phi}(\omega) = \left(\prod_{j=1}^{J} \frac{\widehat{h}(2^{-j}\omega)}{\sqrt{2}}\right)\widehat{\phi}(2^{-J}\omega). \tag{3.34}$$

If $\widehat{\phi}(\omega)$ is continuous at $\omega = 0$, then $\lim_{J\to\infty} = \widehat{\phi}(0)$, so

$$\widehat{\phi}(\omega) = \left(\prod_{j=1}^{\infty} \frac{\widehat{h}(2^{-j}\omega)}{\sqrt{2}}\right)\widehat{\phi}(0). \tag{3.35}$$

The previous recursion formula contains a key result for wavelet analysis theory. Eq. (3.35) states that the continuous scaling function $\widehat{\phi}(\omega)$, can be directly extracted from the discrete filter $\widehat{h}(\omega)$, establishing a first bridge between the continuous and discrete worlds. The convergence of the product given by Eq. (3.35) is studied in [151, 149].

From the condition (f) of an MRA in $L^2(\mathbb{R})$, the basis $\{\phi(x-m)\}_{m\in\mathbb{Z}}$ is an orthonormal basis, i.e., $\langle\phi(x-p), \phi(x-q)\rangle = \delta[p-q]$. The orthonormality condition can be also obtained within the Fourier domain as

$$\sum_{k=-\infty}^{\infty} |\widehat{\phi}(\omega + 2k\pi)|^2 = 1. \tag{3.36}$$

Introducing Eq. (3.32) within Eq. (3.36), $\widehat{h}(\omega)$ satisfies [152, 153, 24]

$$|\widehat{h}(\omega)|^2 + |\widehat{h}(\omega+\pi)|^2 = 2 \tag{3.37}$$

$$\widehat{h}(0) = \sqrt{2}. \tag{3.38}$$

One can also conclude that $\widehat{h}(\pi) = 0$. Hence, $\widehat{h}(\omega)$ has a low-pass filter behavior. As the scaling function Fourier transform $\widehat{\phi}(\omega)$ can be directly derived from $\widehat{h}(\omega)$, see Eq. (3.35), it can be though as a low-pass filter in the continuous domain

$$\widehat{\phi}(\omega) = \prod_{j=1}^{\infty} \widehat{h}\left(\frac{\omega}{2^j}\right). \tag{3.39}$$

At this point, the set of spaces $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$ can be seen as a set of approximation spaces at scales 2^j, in which as $j \to -\infty$, the spaces contain more details of $L^2(\mathbb{R})$. Moreover, these spaces are obtained through a recursive low-pass filtering, in such a way that the space $\mathbf{V}_{j+1}$ is a low-pass version of the space $\mathbf{V}_j$, that is, it contain less details of $L^2(\mathbb{R})$.

The MRA is now employed as starting point to obtain the wavelet function ψ leading to an orthonormal wavelet basis $\{\psi_{j,m}\}_{j,m\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ [135, 23, 139]. The procedure to obtain the wavelet function ψ is very similar to the one employed to derive the scaling function ϕ. The step of passing from the space $\mathbf{V}_j$ to the space $\mathbf{V}_{j+1}$ involves a loss of detail or resolution. A new space $\mathbf{W}_{j+1}$ can be introduced as the orthogonal complement of $\mathbf{V}_{j+1}$ in $\mathbf{V}_j$ as follows

$$\mathbf{V}_j = \mathbf{V}_{j+1} \oplus \mathbf{W}_{j+1} \tag{3.40}$$

where $\oplus$ stands for the direct sum. The space $\mathbf{W}_{j+1}$ contains the necessary details for going from the resolution space $\mathbf{V}_{j+1}$ to $\mathbf{V}_j$, hence for $j \neq j'$ and $j, j' \in \mathbb{Z}$, $\mathbf{W}_j$ is orthogonal to $\mathbf{W}_{j'}$. By iterating Eq. (3.40), one can arrive to

$$\bigoplus_{j=-\infty}^{\infty} W_j = L^2(\mathbb{R}). \tag{3.41}$$

Similarly as it was done for the space $\mathbf{V}_{j+1}$, as $\mathbf{W}_{j+1} \subset \mathbf{V}_j$, any function $\psi \in \mathbf{W}_{j+1}$ can be obtained as a linear combination of the basis elements of the space $\mathbf{V}_j$

$$\psi_{j+1} = \sum_{m\in\mathbb{Z}} g_m \phi_{j,m} \tag{3.42}$$

where $g_m = \langle \psi_{j+1}, \phi_{j,m}\rangle$ and $\sum_{m\in\mathbb{Z}} |g_m|^2 = 1$. On the other hand, as $\mathbf{W}_{j+1} \subset \mathbf{V}_j$ and $\mathbf{W}_{j+2} \subset \mathbf{V}_{j+1}$, a scaling equation can be also defined for the wavelet function ψ [152, 153, 24]

$$\psi_{j,m}(x) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{x - 2^j m}{2^j}\right). \tag{3.43}$$

By introducing Eq. (3.43) into (3.42), a similar equation to Eq. (3.31) can be written for the wavelet function

$$\frac{1}{\sqrt{2^{j+1}}}\psi\left(\frac{x}{2^{j+1}}\right) = \sum_{m\in\mathbb{Z}} g_m \frac{1}{\sqrt{2^j}}\phi\left(\frac{x - 2^j m}{2^j}\right). \tag{3.44}$$

The Fourier transform of the previous equation, for the particular case $j = 1$ is

$$\widehat{\psi}(2\omega) = \frac{1}{\sqrt{2}}\widehat{g}(\omega)\widehat{\phi}(\omega) \tag{3.45}$$

where $\widehat{g}(\omega) = \sum_{m=-\infty}^{\infty} g_m e^{jm\omega}$. Let's assume the coefficients $\{g_m\}_{m\in\mathbb{Z}}$ to be the coefficients of a discrete filter, therefore, Eq. (3.45) can be also though as corresponding to an equation of a filter with a frequency response equal to $\widehat{g}(\omega)$. For a general 2^j, Eq. (3.45) takes the form

$$\widehat{\psi}(2^{-j+1}\omega) = \frac{1}{\sqrt{2}}\widehat{g}(2^{-j}\omega)\widehat{\phi}(2^{-j}\omega). \tag{3.46}$$

In the same way as it was shown for the scaling function, it is possible to verify that the continuous wavelet function $\widehat{\psi}(\omega)$ can be derived as [152, 153, 24]

$$\widehat{\psi}(\omega) = \widehat{g}\left(\frac{\omega}{2}\right)\prod_{j=2}^{\infty} \widehat{h}\left(\frac{\omega}{2^j}\right). \tag{3.47}$$

The conditions under which the basis $\{\psi_{j,m}\}_{m\in\mathbb{Z}}$ of the space $\mathbf{W}_j$, for $\widehat{\psi}$ given by Eq. (3.47), is an orthonormal basis, but also is orthogonal to the basis of the space $\mathbf{V}_j$,$\{\phi_{j,m}\}_{j,m\in\mathbb{Z}}$, are

$$|\widehat{g}(\omega)|^2 + |\widehat{g}(\omega+\pi)|^2 = 2 \tag{3.48}$$
$$\widehat{g}(\omega)\widehat{h}^*(\omega) + \widehat{g}(\omega+\pi)\widehat{h}^*(\omega+\pi) = 0. \tag{3.49}$$

From the applications point of view, it is desired ψ and $\widehat{\psi}$ to have compact support [135, 131]. This is guaranteed if the filters h and g have a finite number of non-zero coefficients, which is equivalent with the fact that $\widehat{h}(\omega)$ and $\widehat{g}(\omega)$ are trigonometric polynomials. Assuming finite responses for h and g, together with the conditions given by Eqs. (3.37), (3.38), (3.48) and (3.49), one can prove that they are satisfied by the function $\widehat{g}(\omega) = e^{-j\omega}\widehat{h}^*(\omega+\pi)$ [152, 153, 24], whose inverse Fourier transform is

$$g_m = (-1)^{1-m}h_{1-m}. \tag{3.50}$$

One can now see the filter given by the coefficients g_m as a high-pass filter, and to demonstrate the conditions $\widehat{g}(0) = 0$ and $\widehat{g}(\pi) = \sqrt{2}$. Finally, as $\{\psi_{j,m}\}_{m\in\mathbb{Z}}$ is an orthonormal basis for the space $\mathbf{W}_j$, by Eq. (3.41), the basis $\{\psi_{j,m}\}_{j,m\in\mathbb{Z}}$ is a basis of $L^2(\mathbb{R})$, i.e., this basis spans $L^2(\mathbb{R})$. Consequently, any function $f \in L^2(\mathbb{R})$ can be expressed as a linear combination of the elements of the basis $\{\psi_{j,m}\}_{j,m\in\mathbb{Z}}$.

The results presented within this section can be considered as one of the cornerstones of the wavelet analysis theory. First, it is possible to define an orthonormal wavelet basis $\{\psi_{j,m}\}_{j,m\in\mathbb{Z}}$ which spans $L^2(\mathbb{R})$. A second important issue is the relation that has been established between the discrete and continuous worlds. The discrete filter given by g has been shown to be related with the orthonormal wavelet basis of the spaces $\mathbf{W}_j$. Hence, as it shall be presented in the following, it defines the linear expansion of a function $f \in L^2(\mathbb{R})$. The coefficients of this linear expansion are obtained by the so-called discrete wavelet transform (DWT). Moreover, the filter g also defines the continuous wavelet function ψ and thus, the continuous wavelet transform (CWT). As one can see, there is a clear and concise link between the continuous and discrete wavelet transforms. On the contrary, this links is not present in the Fourier analysis theory.

The MRA theory is applied in the following to analyze any function $f \in L^2(\mathbb{R})$. The MRA allows also to introduce a fast discrete wavelet transform algorithm, which can be related with the quadrature mirror filtering (QMF) theory.

3.3.4 Fast Discrete Wavelet Transform

The space of interest, $L^2(\mathbb{R})$, can be studied in two different ways. First, within it, a MRA can be defined as a set of closed subspaces $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$, which satisfies the conditions presented at the beginning of Section 3.3.3. These spaces represent different detail levels or resolutions of $L^2(\mathbb{R})$. A complementary set of subspaces $\{\mathbf{W}_j\}_{j\in\mathbb{Z}}$ can be obtained in such a way that each particular space $\mathbf{W}_j$ is the orthogonal complement of $\mathbf{V}_j$ in $\mathbf{V}_{j-1}$. These spaces represent the details contained in $\mathbf{V}_{j-1}$, but not in $\mathbf{V}_j$. As stated by Eq. (3.41), $L^2(\mathbb{R})$ is obtained as the direct sum of the spaces $\{\mathbf{W}_j\}_{j\in\mathbb{Z}}$. The spaces $\{\mathbf{W}_j\}_{j\in\mathbb{Z}}$ are spanned by orthonormal basis. Consequently, $L^2(\mathbb{R})$ is also spanned by an orthonormal basis.

Given a function $f \in L^2(\mathbb{R})$, let $P_{\mathbf{V}_j}f$ to represent its projection into the space $\mathbf{V}_j$ and $P_{\mathbf{W}_j}f$ to represent its projection into $\mathbf{W}_j$. By taking Eq. (3.40)

$$P_{\mathbf{V}_j}f = P_{\mathbf{V}_{j+1}}f + P_{\mathbf{W}_{j+1}}f. \tag{3.51}$$

As it has been demonstrated previously, an orthonormal basis $\{\phi_{j,m}\}_{m\in\mathbb{Z}}$ can be obtained for the space $\mathbf{V}_j$, and an orthonormal basis $\{\psi_{j,m}\}_{m\in\mathbb{Z}}$ can be defined for $\mathbf{W}_j$. Let's define the space projections as

$$P_{\mathbf{V}_j}f = \sum_{m\in\mathbb{Z}}\langle f,\phi_{j,m}\rangle\phi_{j,m} = \sum_{m\in\mathbb{Z}} a_j[m]\phi_{j,m} \tag{3.52}$$
$$P_{\mathbf{W}_j}f = \sum_{m\in\mathbb{Z}}\langle f,\psi_{j,m}\rangle\psi_{j,m} = \sum_{m\in\mathbb{Z}} d_j[m]\psi_{j,m} \tag{3.53}$$

where $a_j[m]$ and $d_j[m]$ represent the coefficients of these expansions. The coefficients $d_j[m]$ are indeed the values obtained by the DWT as defined in Eq. (3.28) for the dyadic case. Any $\phi_{j+1,p} \in \mathbf{V}_{j+1} \subset \mathbf{V}_j$ can be decomposed in the orthonormal basis $\{\phi_{j,n}\}_{n\in\mathbb{Z}}$, see Eq. (3.30)

$$\phi_{j+1,p} = \sum_{m\in\mathbb{Z}} \langle \phi_{j+1,p}, \phi_{j,m} \rangle \phi_{j,m} \tag{3.54}$$

where

$$\langle \phi_{j+1,p}, \phi_{j,m} \rangle = \left\langle \frac{1}{\sqrt{2}} \phi\left(\frac{x}{2}\right), \phi(x - m - 2p) \right\rangle = h[m - 2p]. \tag{3.55}$$

Similarly, since $\psi_{j+1,p} \in \mathbf{W}_{j+1} \subset \mathbf{V}_j$, see Eq. (3.42)

$$\psi_{j+1,p} = \sum_{m\in\mathbb{Z}} \langle \psi_{j+1,p}, \phi_{j,m} \rangle \phi_{j,m} \tag{3.56}$$

where

$$\langle \psi_{j+1,p}, \phi_{j,n} \rangle = \left\langle \frac{1}{\sqrt{2}} \psi\left(\frac{x}{2}\right), \phi(x - m - 2p) \right\rangle = g[m - 2p]. \tag{3.57}$$

Finally, if one computes the inner product of any function $f \in L^2(\mathbb{R})$ in each side of Eqs. (3.54) and (3.56), the following results are obtained [139, 23]

$$a_{j+1}[p] = \sum_{m\in\mathbb{Z}} h[m - 2p] a_j[m] = a_j * \overline{h}[2p] \tag{3.58}$$

$$d_{j+1}[p] = \sum_{m\in\mathbb{Z}}^{\infty} g[m - 2p] a_j[m] = a_j * \overline{g}[2p] \tag{3.59}$$

where $\overline{x}[n] = x[-n]$. Using the same arguments, the inner product of $f \in L^2(\mathbb{R})$ with the components of the basis of $\mathbf{V}_j$ is obtained as

$$a_j[p] = \sum_{m\in\mathbb{Z}} h[p - 2m] a_{j+1}[m] + \sum_{m\in\mathbb{Z}} g[p - 2m] d_{j+1}[m] = \check{a}_{j+1} * h[p] + \check{d}_{j+1} * g[p] \tag{3.60}$$

where $\check{x}$ is obtained from x by adding one zero between the successive values of x. The coefficients $a_j[p]$ and $d_j[p]$ represent the projection coefficients of a function $f \in L^2(\mathbb{R})$ in the spaces $\mathbf{V}_j$ and $\mathbf{W}_j$ respectively. Eqs. (3.58) and (3.59), called analysis or decomposition equations, state that the expansion coefficients can be obtained in a recursive way through a filtering process. Eq. (3.60), called reconstruction equation, shows that the analysis process is invertible. The most important issue at this point is that the discrete wavelet coefficients $d_j[p]$ can be obtained through a filtering process, without the necessity of the wavelet function ψ, and that the original function can be also reconstructed. Therefore, Eqs. (3.58), (3.59) and (3.60) represent a way to derive a fast discrete wavelet transform, also known as Mallat algorithm [139,23]. As Fig. 3.4 depicts, the analysis step splits the coarse approximation coefficients $a_j[p]$ at the scale 2^j into a coarse approximation given by $a_{j+1}[p]$, obtained by a low-pass filter and a down-sampling, and the difference details, or wavelet coefficients $g_{j+1}[p]$, obtained by a high-pass filter and a down-sampling. The original coefficients $a_j[p]$ can be recovered by the complementary process. Fig. 3.4 presents an scheme of the filtering processes performing a discrete wavelet transform.

The direct sum of the spaces $\{\mathbf{W}_j\}_{j\in\mathbb{Z}}$, as presented, leads to $L^2(\mathbb{R})$, i.e., they span the space of square integrable functions. The coefficients $d_j[p]$ are the projection of the function f in the spaces $\{\mathbf{W}_j\}_{j\in\mathbb{Z}}$, therefore, these coefficients are the coefficients of the discrete wavelet transform or DWT.

The process of obtaining the coefficients $d_j[p]$ starts with $j = -\infty$, that is, the space $L^2(\mathbb{R})$, and finishes with $j = \infty$. From a practical point of view, in the case of real signals, it is impossible to arrive to these limits. Let's assume that the transformation process starts in the coarse approximation space $\mathbf{V}_0$. In this case, the projection of the function f in this space is needed. Quite often, discrete signals

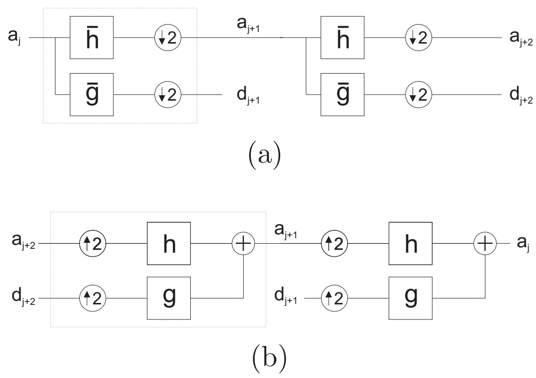

Figure 3.4: Iterated two branch filter bank to calculate the DWT. (a) Fast DWT. (b) Fast IDWT. This scheme calculates the wavelet transform with two scales: a_j represent the coarse approximation coefficients whereas d_j are the detail or wavelet coefficients.

are obtained through a finite resolution device that averages and samples a given continuous process. Therefore, without loss of generality, the available discrete signal can be considered as a discrete signal belonging to $\mathbf{V}_0$, that is, the coefficients $a_0[p]$ [24, 131]. On the other hand, the transformation process can not be iterated until $j = \infty$, as generally the processed signal has a finite support. For instance, for a signal of length N, only $\log_2(N)$ iterations can be performed. The last iteration contains a coefficient a_j, which contains the signal components, from $j = \log_2(N)$ to $j = -\infty$. Quite often, one stops the transformation process at a given scale 2^j. As it can be deduced in this case, the coefficients $a_j[p]$, which are approximation coefficients, play a support role as they allow to stop the transformation process at any scale 2^j.

As it is clear from Fig. 3.4 the fast discrete wavelet transform is based on iterating a basic cell of filters recursively over the low-frequency branch. This basic cell of filters is composed by a low-pass filter h and a high-pass filter g, both followed by a downsampling by 2. This scheme corresponds to the classical multirate filter banks with perfect reconstruction, which is possible by quadrature mirror filters (QMF) [23, 25, 154]. The so-called conjugate mirror filters (CMF) allow to perform such a decomposition with finite impulse response filters, leading to compactly supported orthonormal wavelets [25].

3.3.5 Two-dimensional Wavelet Transform

SAR imagery can be assumed to belong to the space $L^2(\mathbb{R}^2)$. Accordingly, it is necessary to define a two-dimensional wavelet transform algorithm. Despite non-separable wavelets exist in the space $L^2(\mathbb{R}^2)$ [155], separable schemes are preferable as computationally efficient algorithms can be defined to calculate the two-dimensional wavelet transform. A two-dimensional signal $f(x_1, x_2)$ is said to be separable if $f(x_1, x_2) = f(x_1)f(x_2)$, where x_1 and x_2 represent the two signal dimensions, assumed to be independent. Consequently, the two-dimension discrete wavelet transform can be obtained as an extension of the one-dimensional algorithm applied independently to each one of the signal dimensions. As it shall be shown in this section, the two-dimensional fast discrete wavelet transform is obtained by applying the Mallat algorithm independently to the image's rows and columns [139, 23].

A separable orthonormal wavelet basis of $L^2(\mathbb{R}^2)$ can be obtained by means of products of the one-dimensional scaling function ϕ and the one-dimensional wavelet function ψ. Therefore, the orthonormal wavelet basis of $L^2(\mathbb{R}^2)$ can be expressed as $\{\psi_{j_1,m_1}, \psi_{j_2,m_2}\}_{j_1,j_2,m_1,m_2 \in \mathbb{Z}}$. The previous basis depends on the scale parameters j_1 and j_2, and on the location parameters n_1 and n_2. Despite this type of bases exist, bases in which $j_1 = j_2$ are more suitable as information from different scales is not mixed.

A separable two-dimensional MRA of $L^2(\mathbb{R}^2)$ can be straightforwardly defined from a MRA of $L^2(\mathbb{R})$. For the one-dimensional MRA case, $L^2(\mathbb{R})$ is divided into a set of nested subspaces $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$, which represent different levels of detail or resolution. The space $\mathbf{V}_0$ is defined as the central space of the MRA.

A separable two-dimensional space $\mathbf{V}_0^2$ can be obtained as the tensor product of spaces

$$\mathbf{V}_0^2 = \mathbf{V}_0 \otimes \mathbf{V}_0. \tag{3.61}$$

Thus, the projection of a function $f(x_1, x_2) \in L^2(\mathbb{R}^2)$ into this space, denoted by $P_{\mathbf{V}_0^2}$ is

$$P_{\mathbf{V}_0^2} = \sum_{m_1, m_2 \in \mathbb{Z}} a_0[m_1, m_2] \phi_{0,m_1} \phi_{0,m_2} \tag{3.62}$$

where $a_0[m_1, m_2]$ stands for the linear expansion coefficients, whereas the scaling functions ϕ_{0,m_1} and ϕ_{0,m_2} belong to $\mathbf{V}_0$. Using the same argument, any approximation space of a MRA of $L^2(\mathbb{R}^2)$ $\mathbf{V}_j^2$, can be derived through the tensor product space

$$\mathbf{V}_j^2 = \mathbf{V}_j \otimes \mathbf{V}_j. \tag{3.63}$$

If $\{\mathbf{V}_j\}_{j \in \mathbb{Z}}$ is a multiresolution approximation of $L^2(\mathbb{R})$, then, $\{\mathbf{V}_j^2\}_{j \in \mathbb{Z}}$, obtained through a tensor product of spaces, is a separable multiresolution analysis of the space $L^2(\mathbb{R}^2)$ [23]. Since $\mathbf{V}_j^2 = \mathbf{V}_j \otimes \mathbf{V}_j$, and $\{\phi_{j,m}\}_{j,m \in \mathbb{Z}}$ is an orthonormal basis for $\mathbf{V}_j$, the separable basis for $\mathbf{V}_j^2$ is obtained as [23]

$$\left\{ \phi_{j,m}(x_1, x_2) = \phi_{j,m_1}(x_1) \phi_{j,m_2}(x_2) = \frac{1}{2^j} \phi\left(\frac{x_1 - 2^j m_1}{2^j} \right) \phi\left(\frac{x_2 - 2^j m_2}{2^j} \right) \right\}_{m_1, m_2 \in \mathbb{Z}}. \tag{3.64}$$

As for the one-dimensional case, a detail space $\mathbf{W}_{j+1}^2$ can be defined as the orthogonal complement of $\mathbf{V}_{j+1}^2$ in $\mathbf{V}_j^2$

$$\mathbf{V}_j^2 = \mathbf{V}_{j+1}^2 \oplus \mathbf{W}_{j+1}^2. \tag{3.65}$$

By using Eq. (3.63), Eq. (3.65) can be rewritten as follows

$$\mathbf{V}_j \otimes \mathbf{V}_j = (\mathbf{V}_{j+1} \otimes \mathbf{V}_{j+1}) \oplus \mathbf{W}_{j+1}^2. \tag{3.66}$$

The one-dimensional spaces $\mathbf{V}_j$ can be also decomposed as stated by Eq. (3.40). Therefore, by using the distributive property of the operator $\oplus$ with respect to $\otimes$, the space $\mathbf{W}_{j+1}^2$ is obtained as

$$\mathbf{W}_{j+1}^2 = (\mathbf{V}_{j+1} \otimes \mathbf{W}_{j+1}) \oplus (\mathbf{W}_{j+1} \otimes \mathbf{V}_{j+1}) \oplus (\mathbf{W}_{j+1} \otimes \mathbf{W}_{j+1}) = \mathbf{Q}_{j+1}^1 \oplus \mathbf{Q}_{j+1}^2 \oplus \mathbf{Q}_{j+1}^3. \tag{3.67}$$

Since $\{\phi_{j,m}\}_{m \in \mathbb{Z}}$ and $\{\psi_{j,m}\}_{m \in \mathbb{Z}}$ are orthonormal bases of $\mathbf{V}_j$ and $\mathbf{W}_j$, the basis

$$\{\phi_{j,m_1}(x_1)\psi_{j,m_2}(x_2), \psi_{j,m_1}(x_1)\phi_{j,m_2}(x_2), \psi_{j,m_1}(x_1)\psi_{j,m_2}(x_2)\}_{m_1, m_2 \in \mathbb{Z}} \tag{3.68}$$

is a basis of the space $\mathbf{W}_j^2$. Since the space $L^2(\mathbb{R}^2)$ can be decomposed in orthogonal detail spaces as follows

$$L^2(\mathbb{R}^2) = \bigoplus_{j=-\infty}^{\infty} \mathbf{W}_j^2 \tag{3.69}$$

the basis

$$\{\phi_{j,m_1}(x_1)\psi_{j,m_2}(x_2), \psi_{j,m_1}(x_1)\phi_{j,m_2}(x_2), \psi_{j,m_1}(x_1)\psi_{j,m_2}(x_2)\}_{j, m_1, m_2 \in \mathbb{Z}} \tag{3.70}$$

is a basis of $L^2(\mathbb{R}^2)$.

As it has been shown, the space $\mathbf{V}_j^2$ can be decomposed into a coarse approximation space $\mathbf{V}_{j+1}^2$, whose orthonormal basis is defined by Eq. (3.64), and in a detail subspace $\mathbf{W}_{j+1}^2$. The detail space can be, at the same time, decomposed into the direct sum of three orthogonal subspaces $\mathbf{Q}_{j+1}^1$, $\mathbf{Q}_{j+1}^2$ and $\mathbf{Q}_{j+1}^3$ as it can be concluded from Eq. (3.67). The basis of these three subspaces are obtained in a separable way from the one-dimensional functions ϕ and ψ. As these functions are mutually orthogonal,

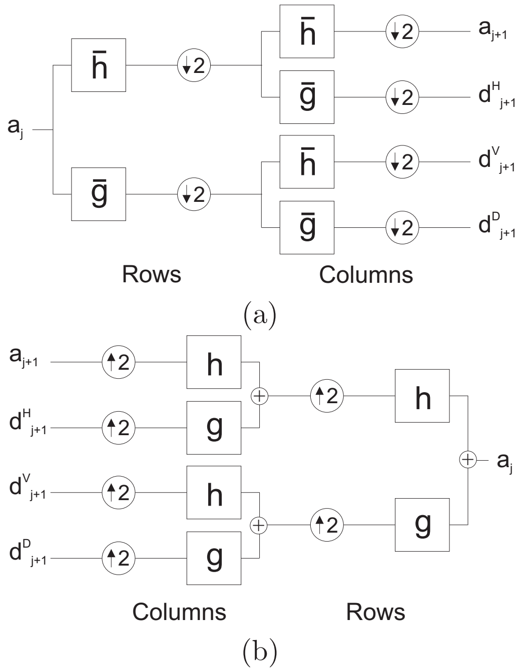

Figure 3.5: Separable two-dimensional filter bank which calculates the two-dimensional DWT for separable dimensions. (a) Fast two-dimensional DWT. (b) Fast two-dimensional IDWT. The coefficients a_j represent a coarse approximation of the original signal whereas d_j^k are the wavelet coefficients.

the product of them shall also lead to orthogonal functions. Then, the space $\mathbf{W}_j^2$ basis is composed by three sets of orthogonal functions or three wavelets

$$\psi_j^1(x_1, x_2) = \phi_j(x_1)\psi_j(x_2) \tag{3.71}$$
$$\psi_j^2(x_1, x_2) = \psi_j(x_1)\phi_j(x_2) \tag{3.72}$$
$$\psi_j^3(x_1, x_2) = \psi_j(x_1)\psi_j(x_2). \tag{3.73}$$

The Fourier transform of the function ϕ has a low-pass behavior, whereas the Fourier transform of ψ has a high-pass behavior. Hence, it can be concluded that the functions given by Eqs. (3.71), (3.72) and (3.73) are sensitive to different spatial frequencies. The function $\psi_j^1(x_1, x_2)$ is sensitive to horizontal details, hence, it is rewritten as $\psi_j^H(x_1, x_2)$. At the same time, $\psi_j^2(x_1, x_2)$ is sensitive to vertical details, being rewritten as $\psi_j^V(x_1, x_2)$. Finally, the function $\psi_j^3(x_1, x_2)$ is sensitive to diagonal details, therefore, it is denoted by $\psi_j^D(x_1, x_2)$.

In brief, the previous paragraphs state that given a function $f(x_1, x_2) \in L^2(\mathbb{R}^2)$, it can be projected into the separable detail space $\mathbf{V}_j^2$. The projection into this space is given by the addition in four additional subspaces as follows

$$P_{\mathbf{V}_j^2} = P_{\mathbf{V}_{j+1}^2} + P_{\mathbf{Q}_{j+1}^H} + P_{\mathbf{Q}_{j+1}^V} + P_{\mathbf{Q}_{j+1}^D}. \tag{3.74}$$

As these projections are derived from a separable scheme in which the spatial dimensions are assumed independent, the two-dimensional fast wavelet transform is obtained by independently applying the Mallat algorithm to each one of the spatial dimensions, see Fig 3.5. The one-dimensional Mallat algorithm states that the transformed coefficients can be obtained by recursively iterating a basic cell, containing a low- and a high-pass filter followed by downsampling of a factor of 2, to the approximation coefficients. In the two-dimensional case, as the basic cell is iterated two times to derive the wavelet coefficients for each wavelet scale 2^j, four sets of coefficients are derived. One corresponding to the coarse approximation coefficients, denoted by $a_j[m_1, m_2]$, and three sets of detail or wavelet coefficients denoted by $d_j^H[m_1, m_2]$, $d_j^V[m_1, m_2]$ and $d_j^D[m_1, m_2]$. The analysis and reconstruction processes of the two-dimensional fast wavelet transform can be seen in Fig. 3.5.

3.3.6 Discrete Wavelet Packet Transform

The interest of analyzing a given function in a time-frequency framework of the spaces $L^2(\mathbb{R})$ or $L^2(\mathbb{R}^2)$ has lead to the construction of orthonormal bases for these spaces whose components are characterized for having a compact support, both in time or space and frequency. These ideas represent an extension of the Fourier analysis theory. This property allows to study particular function features at a given level of resolution. As it has been described by the two previous sections, the wavelet bases for the spaces $L^2(\mathbb{R})$ or $L^2(\mathbb{R}^2)$ perform a particular time-frequency plane tiling, in such a way that time resolution is higher for high frequencies than for low frequencies. What it would be desirable when analyzing a given function is the possibility to study a particular detail at a particular level of resolution. In terms of the time-frequency plane, it means the possibility to derive an arbitrary tiling for this plane, based on the signal properties, in order to obtain the maximum level of resolution, both in time and frequency at any desired point. The discrete wavelet transform has been obtained through a dyadic tiling of the time-frequency plane. This division of the time frequency plane has been shown to be connected with the multiresolution analysis and the perfect reconstruction filtering theory. An arbitrary tiling of the time-frequency can be derived by extending the ideas of wavelet and multiresolution analysis.

The multiresolution analysis, for the space $L^2(\mathbb{R})$ as well as for $L^2(\mathbb{R}^2)$, together with the idea of wavelet, showed the possibility to decompose any of the approximation spaces $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$ as the direct sum of a coarse approximation space $\mathbf{V}_{j+1}$ and a detail or wavelet space $\mathbf{W}_{j+1}$, see Eqs. (3.40) and (3.65). As a result, the union of the orthonormal bases of the spaces $\{\mathbf{W}_j\}_{j\in\mathbb{Z}}$ or $\{\mathbf{W}_j^2\}_{j\in\mathbb{Z}}$ are respectively orthonormal bases of the spaces $L^2(\mathbb{R})$ or $L^2(\mathbb{R}^2)$. The idea at the bottom of the wavelet packet transform, denoted by WPT, is to also split the detail spaces $\{\mathbf{W}_j\}_{j\in\mathbb{Z}}$ in the one-dimensional case, or the spaces $\{\mathbf{Q}_j^k\}_{j\in\mathbb{Z},k\in\{H,V,D\}}$ for the two-dimensional case. R. R. Coifman, Y. Meyer and M. V. Wickerhouser [156] proved that if $\{\theta_j(x-2^j m)\}_{m\in\mathbb{Z}}$ is an orthonormal basis of the space $\mathbf{U}_j$, given h and g as a pair of finite conjugate mirror filters such that

$$\theta_{j+1}^0(x) = \sum_{m=-\infty}^{\infty} h[n]\theta_j(x-2^j m) \tag{3.75}$$

$$\theta_{j+1}^1(x) = \sum_{m=-\infty}^{\infty} g[n]\theta_j(x-2^j m). \tag{3.76}$$

The family $\{\theta_{j+1}^0(x-2^{j+1}m), \theta_{j+1}^1(x-2^{j+1}m)\}$ is an orthonormal basis of $\mathbf{U}_j$. Consequently, a wavelet packet transform is obtained by iterating the basic cell of filters defined for the fast wavelet transform, not only at the coarse approximation branch, but also at the detail branch. This issues means, that the Mallat algorithm can be also extended to calculate the fast wavelet packet transform [23]. However, it is not necessary to split every subspace at every scale. The splitting of a subspace can be done based on deterministic criteria but also depending on the signal properties, adapting thus, the time-frequency plane to the particularities of a given signal. This special procedure receives the name of best basis selection [157, 129], as the orthonormal basis in which a particular function is projected depends on the function itself.

The wavelet packet concept can be also straightforwardly extended to the $L^2(\mathbb{R}^2)$ case using, as in the one-dimensional case, the tensor product of spaces. In this case, the wavelet transform receives the name of wavelet packet quad-three.

3.4 Construction of Wavelets

As it is clear from what has been presented throughout this chapter, the definition of the wavelet theory, as well as its associated concepts, has been performed without the necessity to give a particular expression for the wavelet function ψ. The only restriction a wavelet function has to fulfill is the admissibility condition, Eq. (3.21), which ensures the existence of an inversion formula for the CWT. This condition is not too much restrictive respect to the choice for the wavelet function. As a consequence, additional constraints have to be imposed on the wavelet function in order to obtain transformation schemes with particular features. A clear example has been given in the case of orthonormal linear expansions of a function $f \in L^2(\mathbb{R})$. In this case, the wavelet function has to be orthogonal to its time and scale translations.

A wavelet function can be classified on the basis of its particular properties. These properties shall determine, for instance, the properties of the DWT, or the wavelet series, associated with a particular wavelet function. Hence, one can deduce that the wavelet function has to be selected depending on the particular function's feature.

3.4.1 Wavelet Function's Properties

Most applications based on the wavelet analysis theory try to exploit its capability to concentrate the energy of a function in a reduced set of wavelet coefficients. As it has been already mentioned, the ability to concentrate the function's energy depends on the properties of the wavelet function ψ, on the properties of the basis extracted out of ψ, $\{\psi_{j,m}\}_{j,m \in L^2(\mathbb{R})}$, but also, on the characteristics of the function under analysis. In the following, the main properties of the wavelet functions shall be analyzed in order to give some guidelines to select the suitable wavelet function for a particular goal.

Orthogonal Wavelets

Multiresolution analysis in $L^2(\mathbb{R})$ has demonstrated that wavelet series expansions are related with perfect reconstruction filtering theory. Under the proper selection of these filters, see Section 3.3.3, it is possible to derive a wavelet mother function ψ, from which, a family of orthonormal wavelets $\{\psi_{j,m}\}_{j,m \in \mathbb{Z}}$ can be obtained. Orthogonality (also orthonormality) is a convenient property in many signal processing applications, as it is possible to relate the $L^2(\mathbb{R})$-norm directly with the value of the discrete wavelet coefficients, i.e., the Parseval theorem, Eq. (3.4), holds. This is important as errors present in the original data will not grow under the transformations and stable calculations are possible. Moreover, orthogonal wavelet bases give rise to non-redundant representations of functions.

The handicap an orthogonal wavelet presents is the non existence of compactly supported, linear phase (symmetric wavelet), orthogonal wavelets, except for the Haar wavelet [135]. This wavelet has some characteristics that make it not suitable to be applied with smooth functions. The importance of linear phase lies in the fact that, in the absence of it, the reconstruction process can induce distortions. Returning to the case of interest, the wavelet function ψ has been shown to be determined by the scaling function ϕ or the discrete sequence $\{h_m\}_{m \in \mathbb{Z}}$. Indeed, let ϕ be a real-valued scaling function, associated with a real-value, compactly supported sequence $\{h_m\}_{m \in \mathbb{Z}}$, ϕ is a linear phase function if the sequence $\{h_m\}_{m \in \mathbb{Z}}$ is symmetric, that is $h_{M-m} = h_m \ \forall m \in \mathbb{Z}$ and all the zeros of the associated polynomial

$$H(e^{j\omega}) = \sum_{m=0}^{M} h_n e^{j\omega n} \tag{3.77}$$

that lie on the unit circle have even multiplicities [135, 24]. Thus, the scaling function has linear phase if $\widehat{h}(\omega)$ has linear phase. By using Eq. (3.45), it can be concluded that ψ has linear phase if $\widehat{g}(\omega)$ has also linear phase.

The lack of linear phase for compactly supported wavelet function can be solved in two different ways. First, by allowing the wavelet and the scaling function to be complex functions [158, 159]. Second, orthogonality is a strong condition on the wavelet functions, therefore by relaxing it, compactly supported wavelet functions with linear phase can be obtained. An important class of compactly supported wavelets with linear phase are the biorthogonal wavelet functions.

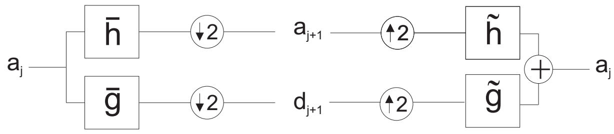

Figure 3.6: Two-branch filter bank which calculates the biorthogonal wavelet transform. The fast biorthogonal wavelet transform is calculated by the pair of filters (h, g). The fast inverse biorthogonal wavelet transform is calculated by the filters $(\widetilde{h}, \widetilde{g})$.

Biorthogonal Wavelets

Orthogonal compactly supported wavelet bases for $L^2(\mathbb{R})$, based on multiresolution analysis theory, can be obtained by iterating a two-branch filter bank with the same filters in the analysis and reconstruction steps. This condition can be relaxed by allowing different filters in the analysis and the reconstruction steps of the Mallat algorithm. When the discrete wavelet transform was introduced in Section 3.3.2, it was shown that linear expansions of functions in which the analysis and the reconstruction bases are different are possible within the frame theory. In this case, two dual bases were defined, leading to two dual linear expansions, see Eq. (3.25). Under proper orthogonality constrains over these bases, biorthogonal wavelet bases can be derived.

As in the case of multiresolution analysis, biorthogonal wavelet bases can be directly related with perfect reconstruction filter-bank schemes. M. Vetterli demonstrated that it is possible to obtain compactly supported, perfect reconstruction filters in those cases in which analysis and reconstruction filters are different [25, 160]. In such a case, the pair of analysis filters are denoted by (h, g) and the reconstruction filters are denoted by $(\widetilde{h}, \widetilde{g})$. Fig. 3.6 depicts a two-branch perfect reconstruction filter bank with biorthogonal filters.

M. Vetterli gave the conditions under which, the filter bank presented in Fig. 3.6 performs a perfect reconstruction [160]

$$\widehat{h}^*(\omega)\widehat{\widetilde{h}}(\omega) + \widehat{h}^*(\omega+\pi)\widehat{\widetilde{h}}(\omega+\pi) = 2 \tag{3.78}$$

$$\widehat{g}^*(\omega)\widehat{\widetilde{g}}(\omega) + \widehat{g}^*(\omega+\pi)\widehat{\widetilde{g}}(\omega+\pi) = 2 \tag{3.79}$$

$$\widehat{g}^*(\omega)\widehat{\widetilde{h}}(\omega) + \widehat{g}^*(\omega+\pi)\widehat{\widetilde{h}}(\omega+\pi) = 0 \tag{3.80}$$

$$\widehat{h}^*(\omega)\widehat{\widetilde{g}}(\omega) + \widehat{h}^*(\omega+\pi)\widehat{\widetilde{g}}(\omega+\pi) = 0. \tag{3.81}$$

Multiresolution analysis showed that perfect reconstruction filter banks can be understood as expansions in $\ell^2(\mathbb{Z})$. As a consequence, the pairs of filters (h, g) and $(\widetilde{h}, \widetilde{g})$ are associated with two different MRAs of $L^2(\mathbb{R})$ which are defined by the pairs of functions (ϕ, ψ) and $(\widetilde{\phi}, \widetilde{\psi})$. The functions $(\widetilde{\phi}, \widetilde{\psi})$ are called the dual scaling function and the dual wavelet function, respectively. Consequently, it is possible to define two wavelet families $\{\psi_{j,m}\}_{j,m\in\mathbb{Z}}$ and $\{\widetilde{\psi}_{j,m}\}_{j,m\in\mathbb{Z}}$ [161, 135]. These two bases are biorthogonal bases of $L^2(\mathbb{R})$ if the following biorthogonality conditions are satisfied

$$\left\langle \phi(x), \widetilde{\phi}(x-m) \right\rangle = \delta[m] \tag{3.82}$$

$$\left\langle \psi_{j,m}, \widetilde{\psi}_{j',m'} \right\rangle = \delta[m-m']\delta[j-j']. \tag{3.83}$$

These biorthogonality conditions can be also written in terms of the associated spaces with the MRA's. Therefore, for $\mathbf{V}_j = \mathbf{V}_{j+1} \oplus \mathbf{W}_{j+1}$ and $\widetilde{\mathbf{V}}_j = \widetilde{\mathbf{V}}_{j+1} \oplus \widetilde{\mathbf{W}}_{j+1}$, the biorthogonal conditions given in Eqs. (3.82) and (3.83) lead to [23]

$$\mathbf{W}_j \perp \widetilde{\mathbf{V}}_j\, , \; \widetilde{\mathbf{W}}_j \perp \mathbf{V}_j\, , \; \widetilde{\mathbf{W}}_j \perp \mathbf{W}_j \tag{3.84}$$

Finally, the pair of filters (h, g) and $(\widetilde{h}, \widetilde{g})$ can be changed in the analysis and reconstruction steps of the Mallat algorithm, leading to two different function's expansions

$$f = \sum_{j,m\in\mathbb{Z}} \langle f, \psi_{j,m}\rangle \widetilde{\psi}_{j,m} = \sum_{j,m\in\mathbb{Z}} \langle f, \widetilde{\psi}_{j,m}\rangle \psi_{j,m}. \tag{3.85}$$

Compactly Supported Wavelets

If the scaling function and the wavelet function are compactly supported functions, the pair of filters (h, g) associated to the corresponding MRA will have a finite number of non-zero coefficients. From a point of view of the discrete wavelet transform, only a finite number of operations are needed in order to calculate it. For those cases in which the filters have very long responses, or even are infinite impulse responses, a fast decay is desired as this response can be reasonably approximated by a finite impulse response.

The ability of the wavelet transform to produce high amplitude coefficients is associated with the effective filter's length. If a given function has a singularity at a position x_0 inside the support of the wavelet $\psi_{j,m}$, it may be possible that $\langle f, \psi_{j,m}\rangle$ has a large amplitude. As a consequence, if the wavelet functions has a support of length K, at each scale 2^j, there are K wavelets $\psi_{j,m}$ whose support includes x_0. Hence, the number of high amplitude coefficients is reduced with short support wavelets.

Vanishing Moments

An important characteristic of the wavelet functions are the number of vanishing moments. A wavelet function ψ is said to have p vanishing moments if

$$\int_{-\infty}^{\infty} x^k \psi(x)\,\mathrm{d}x = 0 \quad 0 \leq k < p. \tag{3.86}$$

The first consequence that can be extracted form the definition of a vanishing moment is that the wavelet function is orthogonal to polynomials of degree $p-1$. Consequently, if a function can be locally approximated by a Taylor polynomial of degree k, if $k < p$, the discrete wavelet transform will produce small value coefficients [23]. The definition of vanishing moment can be also obtained in the Fourier domain. In this case let ψ to have p vanishing moments, as a result the Fourier transform $\widehat{\psi}(\omega)$ and its $p-1$ derivatives are zero at $\omega = 0$ [23,24]. In the same way, the filter $\widehat{h}(\omega)$ and its $p-1$ derivatives are zero at $\omega = \pi$. Vanishing moments are specially important in singularity detection and characterization by means of the continuous wavelet transform [162].

Despite there is not a clear relation between the number of vanishing moments and the support of the wavelet function, Daubechies, under the orthogonality constraint, demonstrated that a wavelet function with p vanishing moments has a minimum support of length $2p-1$ [149,135]. Daubechies wavelets are optimal wavelets as they have the minimum support for a given number of vanishing moments. As a result, it exist a trade-off between the number of vanishing moments and the wavelet support. For a smooth function, it would be desirable a wavelet function with a high number of vanishing moments in order to increase the number of small value coefficients. But for irregular functions, it is desirable to reduce the wavelet support in order to minimize the number of large amplitude coefficients.

3.4.2 Examples of Wavelets

As one can see from what has been presented up to this moment, there exist a large number of functions that fulfill the conditions to be a wavelet function. Depending on the constraints imposed on the wavelet function, as for instance orthogonality, this number can be reduced. But, even in this case, the number of possibilities is large. In the following, some important families of wavelet functions are presented.

Haar Wavelet

The Haar wavelet function is obtained with a MRA of piecewise constant functions. The scaling function is defined as the box function in time, i.e., $\phi(x) = \mathbf{1}_{[0,1]}$. The filter $\{h_m\}_{m\in\mathbb{Z}}$, determined by the corresponding multiresolution analysis is

$$h_m = \begin{cases} \frac{1}{\sqrt{2}} & m = 0, 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.87}$$

Therefore, the wavelet function has the expression $\psi(x) = \mathbf{1}_{[0,1/2]} - \mathbf{1}_{[1/2,1]}$. The Haar wavelet is the only compactly supported, symmetric, orthogonal wavelet. The main drawback of this wavelet is that it is not suitable to approximate smooth functions as it is a irregular function with only one vanishing moment.

Shannon Wavelet

The Shannon wavelet represents a dual respect to the Haar wavelet. The Shannon wavelet corresponds to a scaling function which is a box in frequency, i.e., $\widehat{\phi}(\omega) = \mathbf{1}_{[-\pi,\pi)}$. The low pass filter is found to be $\widehat{h}(\omega) = \sqrt{2}\,\mathbf{1}_{[-\pi/2,\pi/2)}$. Using Eq. (3.47) it can be seen, that the wavelet function has the expression

$$\psi(x) = \frac{\sin 2\pi(x-1/2)}{2\pi(x-1/2)} - \frac{\pi(x-1/2)}{\pi(x-1/2)}. \tag{3.88}$$

It can be demonstrated that the Shannon wavelet has an infinite number of vanishing moments. The main problem of the Shannon wavelet is that it presents a very low decay which tends to increase the number of high amplitude coefficients.

Daubechies Wavelets

The Daubechies wavelets represent perhaps, the most known family of wavelet functions. These wavelets make possible to derive orthonormal bases for the space $L^2(\mathbb{R})$ with minimum length for a given number of vanishing moments [149, 135]. As shown by Eq. (3.30), compactly supported wavelets are obtained with a finite impulse response filter $\{h_m\}_{m\in\mathbb{Z}}$. As a consequence, $\widehat{h}(\omega)$ is a trigonometric polynomial function. Let ψ be a wavelet function with p vanishing moments, it has been shown that $\widehat{h}(\omega)$ must have a zero of order p at $\omega = \pi$, so, $\widetilde{h}(\omega)$ takes the form [135]

$$\widehat{h}(\omega) = \sqrt{2}\left(\frac{1+e^{-j\omega}}{2}\right)^p R(e^{-j\omega}). \tag{3.89}$$

For h having M non zero coefficients, $R(e^{-j\omega})$ is a polynomial of degree $m = M - 1 - p$. The problem here lies on finding the polynomial $R(e^{-j\omega})$ such $\widehat{h}(\omega)$ satisfies Eq. (3.37). By using the Bezout theorem for polynomials, I. Daubechies derived the expression for this type of wavelets.

Multiresolution analysis states that the continuous wavelet function can be obtained by the iteration of a two-band filter bank at the low-pass branch of the analysis step. From a practical point of view, this iteration is not given to the infinite, therefore it was suggested that it would be desirable that finitely iterated wavelet fulfills some regularity conditions, i.e., the iterated wavelet has to be continuous with several continuous derivatives. The regularity of the wavelet function can be translated to the filter h by introducing a flatness condition at the frequency $\omega = \pi$ as the number of zeros at this point. Daubechies wavelets give rise, therefore, to maximally flat filters.

The main disadvantage of the Daubechies wavelets is the lack of symmetry or antisymmetry, that is, the lack of linear phase [24]. The previous construction does not have a unique solution for a given length N. A different alternative is based on selecting the most symmetric wavelet which leads to wavelets with almost linear phase [163]. This type of wavelets are known as symlets.

Maximally Frequency Selective Wavelets

By introducing a flatness condition in the filter design of h as the number of zeros at $\omega = \pi$, maximally flat filters or orthogonal regular wavelets are derived. From a signal processing point of view, maximally flat filters have very poor frequency selectivity properties [164]. Maintaining the length of the filter in M coefficients, if the flatness constraint is reduced by reducing the number of zeros at $\omega = \pi$, it is possible to use these new degrees of freedom to impose frequency selectivity constrains. In [164], Rioul and Duhamel gave a Remez Exchange algorithm to derive orthonormal wavelets with improved frequency selectivity properties, increasing as a consequence the number of orthogonal wavelet families.

Spline Biorthogonal Wavelets

By removing orthonormality condition in the design of the wavelet basis for $L^2(\mathbb{R})$, it is possible to obtain biorthogonal families of wavelets. The main advantage of this type of construction is the increase of freedom in order to design the associated filters to the multiresolution analysis. In the orthogonal case, the filter design is reduced to the design of the low-pass filter coefficients $h[m]$, as the rest of the filters are derived from it. In the biorthogonal case, as different filters are designed for the analysis and reconstruction steps, there is more freedom to fix criteria as the filter support or the number of vanishing moments.

Let $\widehat{h}(\omega)$ be

$$\widehat{h}(\omega) = \sqrt{2}\exp\left(\frac{-j\epsilon\omega}{2}\right)\left(\cos\frac{\omega}{2}\right)^{p} \tag{3.90}$$

with $\epsilon = 0$ for p even and $\epsilon = 1$ for p odd, the scaling function is a box spline of degree $p-1$. As the wavelet function is a combination of spline boxes, it is also a compactly supported function.

The number of vanishing moments of ψ, $\widetilde{p}$ is a free parameter with the same parity as p. From Eq. (3.90), A. Cohen, I. Daubechies and Feauveau demonstrated that the dual low-pass filter has the following frequency response [151]

$$\widehat{\widetilde{h}}(\omega) = \sqrt{2}\exp\left(\frac{-j\epsilon\omega}{2}\right)\left(\cos\frac{\omega}{2}\right)^{\widetilde{p}}\sum_{k=0}^{q-1}\begin{pmatrix} q-1+k \\ k \end{pmatrix}\left(\sin\frac{\omega}{2}\right)^{2k}. \tag{3.91}$$