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MULTIDIMENSIONAL SPECKLE NOISE, MODELLING AND FILTERING RELATED TO SAR DATA

by

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Appendix A

Calculation of $p_{v_1}(v_1)$ and $p_{v_2}(v_2)$

This appendix is devoted to derive the expressions of the probability density functions (pdf) $p_{v_1}(v_1)$ and $p_{v_2}(v_2)$, on the basis of

$$v_1 = \cos(\phi) \tag{A.1}$$
$$v_2 = \sin(\phi) \tag{A.2}$$

where ϕ represents the Hermitian product phase difference, whose pdf is given by Eq. (2.110) on page 31.

Let x be a random variable characterized by the pdf $p_x(x)$. Let $y = f(x)$ be a function with the following roots

$$y = f(x_1) = f(x_2) = \ldots = f(x_n) = \ldots. \tag{A.3}$$

The pdf of the variable y, denoted by $p_y(y)$, is obtained as given by Ref. [72] on page 95

$$p_y(y) = \sum_k \frac{p_x(x_k)}{|f'(x_k)|} \tag{A.4}$$

where x_k represents the solutions of $y = f(x)$ and $f'(x)$ is the derivative of $f(x)$.

The first pdf to obtain is $p_{v_1}(v_1)$, on the basis of Eq. (A.1). Since the Hermitian product phase difference ϕ for SAR imagery is only considered within the interval $[-\pi, \pi)$, Eq. (A.1) presents the two following solutions

$$\phi_1 = \arccos(v_1) \tag{A.5}$$
$$\phi_2 = -\arccos(v_1) \tag{A.6}$$

such that $\phi_1 \in [0, \pi)$ and $\phi_2 \in [-\pi, 0)$. Since the phase difference pdf $p_\phi(\phi)$ is symmetric about 0 rad [1], it follows

$$p_\phi(\phi_1) = p_\phi(\phi_2) = p_\phi(\arccos(v_1)) \qquad \phi \in [-\pi, \pi). \tag{A.7}$$

Concerning the evaluation of the term $|f'(x_k)|$ in Eq. (A.4), for Eq. (A.1)

$$\left|\frac{\partial v_1}{\partial \phi}\right|_{\phi=\phi_k} = |-\sin(\phi)|_{\phi=\phi_k} = |\sin(\phi)|_{\phi=\phi_k}. \tag{A.8}$$

[1] In this case the phase mode ϕ_x is considered equal to 0 rad since interest is on assuming the phase pdf as noise.

Evaluating Eq. (A.8) for $\phi_k = \phi_2$

$$|\sin(\phi_2)| = |\sin(-\phi_1)| = |-\sin(\phi_1)| = |\sin(\phi_1)|. \tag{A.9}$$

Consequently, only the solution $\phi_k = \phi_1$ has to be evaluated. Considering Eq. (A.5)

$$|\sin(\phi_1)| = |\sin(\arccos(v_1))| = \sqrt{1 - v_1^2} \tag{A.10}$$

where $\arccos(x) = \arcsin(\sqrt{1 - x^2})$ has been employed [213].

Introducing the results given by Eqs. (A.7) and (A.10) into (A.4)

$$p_{v_1}(v_1) = 2\frac{p_\phi(\arccos(v_1))}{\sqrt{1 - v_1^2}} \qquad v_1 \in [-1, 1). \tag{A.11}$$

Considering $p_\phi(\phi)$, where β takes the value

$$\beta = |\rho|\cos(\phi_1) = |\rho| v_1 \tag{A.12}$$

on the basis of Eq. (A.5), $p_{v_1}(v_1)$ has the expression

$$p_{v_1}(v_1) = \frac{1}{\pi\sqrt{1 - v_1^2}} \frac{\left(1 - |\rho|^2\right)\left[\left(1 - |\rho|^2 v_1^2\right)^{1/2} + |\rho|\, v_1\, (\pi - \arccos(|\rho|\, v_1))\right]}{\left(1 - |\rho|^2 v_1^2\right)^{3/2}} \qquad v_1 \in [-1, 1]. \tag{A.13}$$

The expression of $p_{v_2}(v_2)$ requires a slightly different analysis. In the phase interval $\phi \in [-\pi, \pi)$, Eq. (A.2) presents the two following solutions

$$\phi_1 = \arccos(v_2) \tag{A.14}$$

$$\phi_2 = \mathrm{sgn}(v_2)\pi - \arccos(v_2) \tag{A.15}$$

where $\mathrm{sgn}(\cdot)$ is the sign function, $\phi_1 \in [-\pi/2, \pi/2)$ and $\phi_2 \in [-\pi, -\pi/2) \cup [\pi/2, \pi)$.

Concerning the evaluation of the term $|f'(x_k)|$ in Eq. (A.4), for Eq. (A.2)

$$\left|\frac{\partial v_2}{\partial \phi}\right|_{\phi=\phi_k} = |\cos(\phi)|_{\phi=\phi_k}. \tag{A.16}$$

If Eq. (A.16) is evaluated for ϕ_2, Eq. (A.15), it follows

$$\begin{aligned} |\cos(\mathrm{sgn}(v_2)\pi - \arcsin(v_2))| &= |\cos(\mathrm{sgn}(v_2)\pi)\cos(\arcsin(v_2)) + \sin(\mathrm{sgn}(v_2)\pi)\sin(\arcsin(v_2))| \\ &= |-\cos(\arcsin(v_2))| = |\cos(\arcsin(v_2))|. \end{aligned} \tag{A.17}$$

Also in this case, only the first solution, Eq. (A.14), has to be evaluated for the derivative, Eq. (A.16),

$$|\cos(\phi_1)| = |\cos(\arcsin(v_2))| = \sqrt{1 - v_2^2} \tag{A.18}$$

where $\arcsin(x) = \arccos(\sqrt{1 - x^2})$ has been employed [213].

In this case, the terms $p_\phi(\phi_1)$ and $p_\phi(\phi_2)$ need separate evaluation. The parameter $\beta = |\rho|\cos(\phi_k)$ takes, for each solution, Eqs. (A.14) and (A.15), the expressions

$$\beta_{\phi_1} = |\rho|\cos(\phi_1) = |\rho|\cos(\arcsin(v_2)) = |\rho|\sqrt{1 - v_2^2} \tag{A.19}$$

$$\beta_{\phi_2} = |\rho|\cos(\phi_2) = |\rho|\cos(\mathrm{sgn}(v_2) - \arccos(v_2)) = -|\rho|\sqrt{1 - v_2^2}. \tag{A.20}$$

Considering Eq. (A.4), with the distribution $p_\phi(\phi)$ of Eq.(2.110) on page 31, where the results of Eqs. (A.18), (A.19) and (A.20) are introduced, the pdf $p_{v_2}(v_2)$ takes the expression

$$\begin{aligned} p_{v_2}(v_2) &= \frac{1}{\pi\sqrt{1 - v_2^2}} \frac{\left(1 - |\rho|^2\right)\left[\left(1 - |\rho|^2 (1 - v_2^2)\right)^{1/2} + |\rho|\sqrt{1 - v_2^2}\left(\frac{\pi}{2} - \arccos\left(|\rho|\sqrt{1 - v_2^2}\right)\right)\right]}{\left(1 - |\rho|^2 (1 - v_2^2)\right)^{3/2}} \\ & v_2 \in [-1, 1]. \end{aligned} \tag{A.21}$$

Appendix B

Calculation of N_c

The following appendix concerns the process to obtain the expression of the parameter N_c for an arbitrary number of looks N. As defined in Eq. (4.11), N_c is the mean value of the random process v_1. Consequently, this value can be found as

$$N_c = \int_{-1}^{1} v_1 p_{v_1}(v_1)\,\mathrm{d}v_1 = \int_{-\pi}^{\pi} \cos(\phi) p_\phi(\phi)\,\mathrm{d}\phi. \tag{B.1}$$

The expression of N_c will be derived through the second integral of Eq. (B.1). In order to have the expression of N_c for an arbitrary number of looks, the multi-look pdf of ϕ is needed. The expression of this distribution can be found in [22]

$$\begin{aligned} p_{\phi,N}(\phi) &= \frac{(1-|\rho|^2)^N}{2\pi}\left[\frac{(2N-2)!}{[(N-1)!]^2 2^{2(N-1)}}\left(\frac{(2N-1)\beta}{(1-\beta^2)^{N+\frac{1}{2}}}\left(\frac{\pi}{2}+\arcsin(\beta)\right)+\frac{1}{(1-\beta^2)^N}\right)\right. \\ &+ \left.\frac{1}{2(N-1)}\sum_{r=0}^{N-2}\frac{\Gamma(N-\frac{1}{2})}{\Gamma(N-\frac{1}{2}-r)}\frac{\Gamma(N-1-r)}{\Gamma(N-1)}\frac{1+(2r+1)\beta^2}{(1-\beta^2)^{r+2}}\right] \end{aligned} \tag{B.2}$$

where $\beta = |\rho|\cos(\phi)$. The second integral of Eq. (B.1) can be calculated by using the distribution given in Eq. (B.2). This integral can be divided into four additional integrals:

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$$\begin{aligned} &\frac{(1-|\rho|^2)^N}{2\pi}\frac{(2N-2)!(2N-1)}{[(N-1)!]^2 2^{2(N-1)}}\frac{\pi}{2}|\rho|\int_{-\pi}^{\pi}\frac{\cos^2(\phi)}{(1-|\rho|^2\cos^2(\phi))^{N+\frac{1}{2}}}\,\mathrm{d}\phi \\ = &\frac{(1-|\rho|^2)^N}{2\pi}\frac{(2N-2)!(2N-1)}{[(N-1)!]^2 2^{2(N-1)}}\frac{\pi}{2}|\rho| B\left(\frac{1}{2},\frac{3}{2}\right)\,{}_2F_1\left(\frac{3}{2},N+\frac{1}{2};2;|\rho|^2\right). \end{aligned} \tag{B.3}$$

$B(x,y)$ represents the beta function and ${}_2F_1(a,b;c;z)$ represents the Gauss hypergeometric function, whose definition is

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{B.4}$$

where, $(x)_n = \Gamma(x+n)/\Gamma(x)$ represents the Pochhammer symbol. The integral in Eq. (B.3) has been solved by means of Eq. 3.682 on page 431 of Ref. [171].

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$$\frac{(1-|\rho|^2)^N}{2\pi}\frac{(2N-2)!(2N-1)}{[(N-1)!]^2 2^{2(N-1)}}|\rho|\int_{-\pi}^{\pi}\frac{\cos^2(\phi)\arcsin(|\rho|\cos(\phi))}{(1-|\rho|^2\cos^2(\phi))^{N+\frac{1}{2}}}\,\mathrm{d}\phi$$
$$= \frac{(1-|\rho|^2)^N}{2\pi}\frac{(2N-2)!(2N-1)}{[(N-1)!]^2 2^{2(N-1)}}|\rho|2\int_{0}^{\pi}\frac{\cos^2(\phi)\arcsin(|\rho|\cos(\phi))}{(1-|\rho|^2\cos^2(\phi))^{N+\frac{1}{2}}}\,\mathrm{d}\phi = 0 \tag{B.5}$$

since the function inside the integral is an odd function in the interval $\phi \in [0, \pi)$ with respect to $\phi = \pi/2$.

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$$\frac{(1-|\rho|^2)^N}{2\pi}\frac{(2N-2)!}{[(N-1)!]^2 2^{2(N-1)}}\int_{-\pi}^{\pi}\frac{\cos(\phi)}{(1-|\rho|^2\cos^2(\phi))^{N}}\,\mathrm{d}\phi$$
$$= \frac{(1-|\rho|^2)^N}{2\pi}\frac{(2N-2)!}{[(N-1)!]^2 2^{2(N-1)}}2\int_{0}^{\pi}\frac{\cos(\phi)}{(1-|\rho|^2\cos^2(\phi))^{N}}\,\mathrm{d}\phi = 0. \tag{B.6}$$

Using the same argument as in the previous point, one can deduce that the integral is also zero in this case.

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$$\frac{(1-|\rho|^2)^N}{2\pi}\frac{1}{2(N-1)}\int_{-\pi}^{\pi}\cos(\phi)\sum_{r=0}^{N-2}\frac{\Gamma(N-\frac{1}{2})}{\Gamma(N-\frac{1}{2}-r)}\frac{\Gamma(N-1-r)}{\Gamma(N-1)}\frac{1+(2r+1)\beta^2}{(1-\beta^2)^{r+2}}\,\mathrm{d}\phi$$
$$= \frac{(1-|\rho|^2)^N}{2\pi}\frac{1}{2(N-1)}\sum_{r=0}^{N-2}\frac{\Gamma(N-\frac{1}{2})}{\Gamma(N-\frac{1}{2}-r)}\frac{\Gamma(N-1-r)}{\Gamma(N-1)}$$
$$\cdot\left[\int_{-\pi}^{\pi}\frac{\cos(\phi)}{(1-|\rho|^2\cos^2(\phi))^{r+2}}\,\mathrm{d}\phi + |\rho|^2\int_{-\pi}^{\pi}\frac{(2r+1)\cos^3(\phi)}{(1-|\rho|^2\cos^2(\phi))^{r+2}}\,\mathrm{d}\phi\right] = 0. \tag{B.7}$$

One can deduce that both integrals equal zero considering the same arguments employed in the previous two points.

Taking into account the values of the different integrals in the previous four points, the value of N_c, for a given number of looks N, is

$$N_c = (1-|\rho|^2)^N\frac{(2N-2)!(2N-1)|\rho|}{[(N-1)!]^2 2^{2(N-1)}}\frac{\pi}{4}\,{}_2F_1\left(\frac{3}{2}, N+\frac{1}{2}; 2; |\rho|^2\right). \tag{B.8}$$

Introducing the expression ${}_2F_1(a, b; c; x) = (1-x)^{c-b-a}\,{}_2F_1(c-a, c-b; c; x)$, the value of N_c for single-look imagery, i.e., $N = 1$, is

$$N_c = \frac{\pi}{4}|\rho|\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; |\rho|^2\right). \tag{B.9}$$

In order to calculate the expression of $\sigma^2_{v_1}$, the second moment of v_1 has to be calculated

$$E\{v_1^2\} = \int_{-1}^{1} v_1^2 p_{v_1}(v_1)\,\mathrm{d}v_1 = \int_{-\pi}^{\pi}\cos^2(\phi)p_\phi(\phi)\,\mathrm{d}\phi. \tag{B.10}$$

Eq. (B.2) presents the pdf of the phase ϕ for an arbitrary number of looks. The value of Eq. (B.10) can not be found in this case due to the complexity of the pdf given by Eq. (B.2). Instead, only the expression of $E\{v_1^2\}$ for one-look imagery is obtained. Considering $N = 1$, Eq. (B.2) simplifies to the expressions given by Eqs. (2.110) and (5.13). Hence, Eq. (B.10) is

$$E\{v_1^2\} = \frac{1-|\rho|^2}{2\pi}\int_{-\pi}^{\pi}\frac{\cos^2(\phi)\beta(\frac{1}{2}\pi+\arcsin(\beta))}{(1-\beta^2)^{\frac{3}{2}}}\,\mathrm{d}\phi + \frac{1-|\rho|^2}{2\pi}\int_{-\pi}^{\pi}\frac{\cos^2(\phi)}{1-\beta^2}\,\mathrm{d}\phi \tag{B.11}$$

where $\beta = |\rho|\cos(\phi)$. The first integral of Eq. (B.11) can be obtained as

$$\begin{aligned} &\frac{1-|\rho|^2}{2\pi}\int_{-\pi}^{\pi}\frac{\cos^2(\phi)\beta(\frac{1}{2}\pi+\arcsin(\beta))}{(1-|\rho|^2\cos^2(\phi))^{\frac{3}{2}}}\,\mathrm{d}\phi \\ =\ &\frac{|\rho|(1-|\rho|^2)}{2\pi}\int_{-\pi}^{\pi}\frac{\cos^3(\phi)\frac{1}{2}\pi}{(1-|\rho|^2\cos^2(\phi))^{\frac{3}{2}}}\,\mathrm{d}\phi+\frac{|\rho|(1-|\rho|^2)}{2\pi}\int_{-\pi}^{\pi}\frac{\cos^3(\phi)\arcsin(|\rho|\cos(\phi))}{(1-|\rho|^2\cos^2(\phi))^{\frac{3}{2}}}. \end{aligned} \quad \text{(B.12)}$$

From Eq. (B.7), it can be proved that the first integral of Eq. (B.12) equals zero. The second integral of Eq. (B.12) can not be solved directly. Hence the Taylor series of $\arcsin(|\rho|\cos(\phi))$ is included to solve the integral. Thus,

$$\begin{aligned} &\frac{|\rho|(1-|\rho|^2)}{2\pi}\int_{-\pi}^{\pi}\frac{\cos^3(\phi)\arcsin(|\rho|\cos(\phi))}{(1-|\rho|^2\cos^2(\phi))^{\frac{3}{2}}} \\ =\ &\frac{|\rho|(1-|\rho|^2)}{2\pi}\int_{-\pi}^{\pi}\frac{\cos^3(\phi)\sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(2n+1)}(|\rho|\cos(\phi))^{2n+1}}{(1-|\rho|^2\cos^2(\phi))^{\frac{3}{2}}} \\ =\ &\frac{|\rho|(1-|\rho|^2)}{2\pi}\sum_{n=0}^{\infty}\frac{(2n)!|\rho|^{2n+1}}{4^n(n!)^2(2n+1)}\int_{-\pi}^{\pi}\frac{\cos^{2n+4}(\phi)}{(1-|\rho|^2\cos^2(\phi))^{\frac{3}{2}}}\,\mathrm{d}\phi \\ =\ &\frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(2n)!B\left(\frac{1}{2},n+\frac{5}{2}\right)}{4^{(n-1)}(n!)^2(2n+1)}\frac{|\rho|^{2n+2}}{(1-|\rho|^2)^{\frac{1}{2}}}\,{}_2F_1\left(\frac{1}{2},\frac{3}{2};n+3;\frac{|\rho|^2}{1-|\rho|^2}\right) \end{aligned} \quad \text{(B.13)}$$

where $B(x,y)$ represents the Beta function. The development of Eq. (B.13) is based on the equalities for the Hypergeometric function: ${}_2F_1(a,b;c;x) = (1-x)^{c-b-a}\,{}_2F_1(c-a,c-b;c;x)$ and ${}_2F_1(a,b;c;x) = {}_2F_1(b,a;c;x)$, and on the expression 3.682 on page 431 of Ref. [171].

The second integral of Eq. (B.11) equals

$$\frac{1-|\rho|^2}{2\pi}\int_{-\pi}^{\pi}\frac{\cos^2(\phi)}{1-|\rho|^2\cos^2(\phi)}\,\mathrm{d}\phi=\frac{\sqrt{1-|\rho|^2}}{1+\sqrt{1-|\rho|^2}}. \quad \text{(B.14)}$$

Consequently, considering the results given by Eqs. (B.12) and (B.13), the expression of $E\{v_1^2\}$ is

$$E\{v_1^2\}=\frac{\sqrt{1-|\rho|^2}}{1+\sqrt{1-|\rho|^2}}+\frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(2n)!B\left(\frac{1}{2},n+\frac{5}{2}\right)}{4^{(n-1)}(n!)^2(2n+1)}\frac{|\rho|^{2n+2}}{(1-|\rho|^2)^{\frac{1}{2}}}\,{}_2F_1\left(\frac{1}{2},\frac{3}{2};n+3;\frac{|\rho|^2}{1-|\rho|^2}\right). \quad \text{(B.15)}$$

The expression of the variance of the noise term v_1, for single-look SAR imagery, is found as $\sigma_{v_1}^2 = E\{v_1^2\} - N_c^2$, where Eqs. (B.9) and (B.15) are considered. As one observed, the expression of $\sigma_{v_1}^2$ is not very useful since the dependence on the coherence $|\rho|$ is not clear.

Appendix C

Combination of Variance Expressions

In Chapter 4 and Chapter 5, the family of curves $1/2\,(1-|\rho|^2)^\alpha$, Eq. (4.12), is employed to approximate some expressions concerning variances, where $|\rho|$ represents the coherence between a pair of SAR images and α controls the curve's shape. In the case of the interferometric phasor noise model, this family is employed to approximate $\sigma^2_{v_1}$, Eq. (4.14), and $\sigma^2_{v_2}$ Eq. (4.18), whereas in the case of the Hermitian product speckle noise model it is employed to approximate $\sigma^2_{n_{a1}}$, Eq. (5.56), and $\sigma^2_{n_{a2}}$, Eq. (5.71). The noise terms v_1 and v_2, in the case of the interferometric phasor noise model, are combined to form the noise terms v_c and v_s, Eqs. (4.23) and (4.24) respectively. In a similar way, in the case of the Hermitian product speckle noise model, the noise terms n_{ar} and n_{ai}, Eqs. (5.85) and (5.86), result form the combination of n_{a1} and n_{a2}. Considering the expressions of the corresponding variance values $\sigma^2_{v_c}$, $\sigma^2_{v_s}$, $\sigma^2_{n_{ar}}$ and $\sigma^2_{n_{ai}}$ given respectively by Eqs. (4.27), (4.28), (5.89) and (5.90), all of them have expressions of the type

$$\frac{1}{2}(1-|\rho|^2)^{\alpha_1}\cos^2(\phi)+\frac{1}{2}(1-|\rho|^2)^{\alpha_2}\sin^2(\phi) \tag{C.1}$$

where ϕ is a phase measurement. In order to maintain the simplicity of the variance expressions, it is

wanted that the curves given by Eq. (C.1) belong also to the family $1/2\,(1-|\rho|^2)^\alpha$. This is reduced to find an exponent α_3, such that the curve

$$\frac{1}{2}(1-|\rho|^2)^{\alpha_3} \tag{C.2}$$

equals Eq. (C.1). Hence, the solution is

$$\alpha_3=\frac{\ln\left(\frac{1}{2}(1-|\rho|^2)^{\alpha_1}\cos^2(\phi)+\frac{1}{2}(1-|\rho|^2)^{\alpha_2}\sin^2(\phi)\right)}{\ln(1-|\rho|^2)}. \tag{C.3}$$

Fig. C.1 presents a plot of α_3 as a function of $|\rho|$ and ϕ, for the particular case in which $\alpha_1=1$ and $\alpha_2=1.5$. As it can be observed, the value of α_3 ranges between α_1 and α_2 as a function of the phase ϕ, whereas it is practically constant as a function of the coherence value $|\rho|$. In order to eliminate the dependence on ϕ, α_3 is considered

$$\alpha_3=\frac{\alpha_1+\alpha_2}{2} \tag{C.4}$$

which corresponds to the average value of α_3 in the space given by $|\rho|$ and ϕ. By considering α_3 equal to Eq. (C.4), instead to Eq. (C.3), α_3 is a constant value. The resulting curve with the exponent α_3 can be considered as the average curve of those defined by the exponents α_1 and α_2, since $\cos^2(\phi)$ and $\sin^2(\phi)$ have values in the range $[0, 1]$.

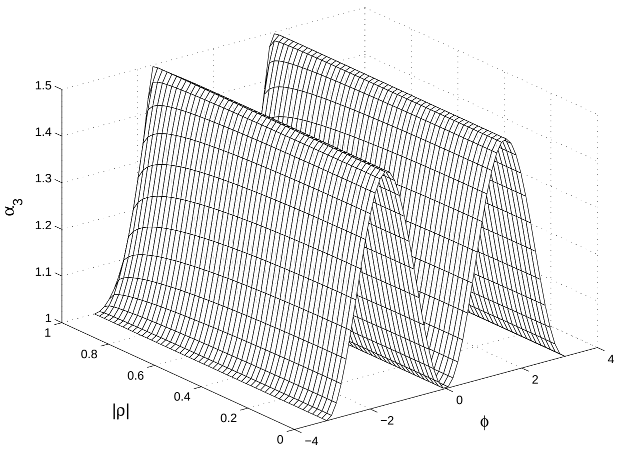

Figure C.1: Exponent α_3 as a function of $|\rho|$ and ϕ. In this case, $\alpha_1 = 1$ and $\alpha_2 = 1.5$.

Appendix D

Linear Least Squares Regression Analysis

Linear least squares regression analysis has been employed in Chapters 4 and 5 to measure quantitatively the validity of the proposed noise models with real data. This comparison consist on the analysis of first and second order statistical moments, in such a way that the regression lines measure the agreement between the theoretical expressions of these statistical moments, given by the proposed noise models, and the same values estimated from real data. This appendix concerns only how the regression analysis results have to be interpreted to measure the noise model's validity. The reader is directed to comprehensive references on this topic for a detailed analysis [214, 178].

Let X and Y to be two variables such that X contains the theoretical values of a given parameter θ and Y contains the estimated values of θ from real data. The least squares regression analysis measures the least squares regression line

$$Y = a_0 + a_1 X \tag{D.1}$$

where a_0 is the constant coefficient and a_1 the regression line slope. Hence, if the estimated value Y of the parameter θ equals the theoretical value given by X, it implies $a_0 = 0$ and $a_1 = 1$.

Two additional quantities are also provided in order to measure quantitatively the agreement between the estimated regression line and the real measurements. The first measure is the *estimate standard error*, denoted by s. The parameter s measures the scatter about the regression line of Y on X. that is, it is an absolute measure of how good the estimated line fits the means of the variable Y. Its value is given by

$$s = \sqrt{\frac{\sum (Y - \widehat{Y})^2}{N}} \tag{D.2}$$

where $\widehat{Y}$ represents the value of Y given by Eq. (D.1), Y is the actual value of the parameter under analysis and N denotes the number of samples. Hence, the lower the value of s, the better the fit of the estimated line to the real relation between X and Y.

The second parameter is the *correlation coefficient*, denoted by r, whose expression is

$$r = \sqrt{\frac{\sum (\widehat{Y} - \overline{Y})^2}{\sum (Y - \overline{Y})^2}} \tag{D.3}$$

where Y is the value of the parameter θ estimated from data, $\widehat{Y}$ is the value given by Eq. (D.1) and $\overline{Y}$ is the mean value of Y. The value of r is in the interval $[0, 1]$. The parameter r is a relative measure of the degree of linear association between X and Y. A value of r equal to 1 indicates that Eq. (D.1) is able to explain all the variance present in the observations. On the contrary, the lower the value of r, the higher the amount of the observation's variance Eq. (D.1) can not be explained.

The validity of the proposed noise models is quantitatively measured by the parameters a_0, a_1, s and r for the mean and variance statistical moments. First, the closer the parameters a_0 and a_1 to 0 and 1, respectively, the larger the agreement between theoretical expressions and values derived from real data. This agreement is measured by s and r. Therefore, the lower the value of s the better the fit between theoretical and experimental values. In this case, the parameter r can be considered as a quantity measuring how much close the theoretical expressions of the mean and variance are to the corresponding values derived from real data. A value equal to 1 indicates that the degree of linear association between the theoretical values and the experimental ones is complete. Considering that $a_0 = 0$ and $a_1 = 1$, it can be concluded that the theory completely describes data. On the contrary, the lower the value of r, the less the capacity of the theoretical expressions to estimate data values.

Finally, since the theoretical mean and variance expressions are directly derived from the different terms of the noise models, if these expressions fit with data, it can be concluded that the noise models are able to fully explain data behavior.

Appendix E

Calculation of $E\{z^2\cos(\phi)\}$, $E\{z_c z_2\}$ and $E\{z_1 z_2\}$

This first part of this appendix focus on calculating the expression of the term $E\{z^2\cos(\phi)\}$

$$E\{z^2\cos(\phi)\} = \int_0^\infty \int_{-\pi}^{\pi} z^2\cos(\phi) p_{z,\phi}(z,\phi)\,\mathrm{d}\phi\,\mathrm{d}z. \tag{E.1}$$

The expression of $p_{z,\phi}(x,\phi)$ is given in Eq. (5.9) on page 111. Hence, considering $\phi_x = 0$

$$\begin{aligned} E\{z^2\cos(\phi)\} &= \int_0^\infty \int_{-\pi}^{\pi} z^2\cos(\phi) p_{z,\phi}(x,\phi)\,\mathrm{d}\phi\,\mathrm{d}z \\ &= \int_0^\infty \frac{2z^3}{\pi\psi^2(1-|\rho|^2)} K_0\left(\frac{2z}{\psi(1-|\rho|^2)}\right) \int_{-\pi}^{\pi} \cos(\phi)\exp\left(\frac{2|\rho| z\cos(\phi)}{\psi(1-|\rho|^2)}\right)\mathrm{d}\phi\,\mathrm{d}z \\ &= \int_0^\infty \frac{2z^3}{\pi\psi^2(1-|\rho|^2)} K_0\left(\frac{2z}{\psi(1-|\rho|^2)}\right) I_0\left(\frac{2|\rho| z}{\psi(1-|\rho|^2)}\right)\mathrm{d}\phi \qquad \text{(E.2)} \\ &= \psi^2|\rho|(1-|\rho|^2)^3\frac{9\pi}{16}\,{}_2F_1\left(\frac{5}{2},-\frac{1}{2};2;|\rho|^2\right) \\ &= \psi^2|\rho|\frac{9\pi}{16}\,{}_2F_1\left(-\frac{1}{2},-\frac{1}{2};2;|\rho|^2\right). \qquad \text{(E.3)} \end{aligned}$$

The expression given by Eq. (E.2) has been derived on the basis of the formula 8.431 on page 968 of Ref. [171]. Eq. (E.3) is derived on the basis of the equality ${}_2F_1(a,b;c;x) = (1-x)^{c-b-a}\,{}_2F_1(c-a,c-b;c;x)$.

The second part of this appendix focus on calculating the expression of the term $E\{z_c z_2\}$. Considering the expressions for z_c and z_2, respectively given by Eqs. (5.27) and (5.60), it follows

$$E\{z_c z_2\} = N_c E\{z^2\sin(\phi)\} \int_0^\infty \int_{-\pi}^{\pi} z^2\sin(\phi) p_{z,\phi}(z,\phi)\,\mathrm{d}\phi\,\mathrm{d}z. \tag{E.4}$$

Similarly, as performed in the case of Eq. (E.3)

$$\begin{aligned} N_c E\{z^2\sin(\phi)\} &= N_c \int_0^\infty \int_{-\pi}^{\pi} z^2\sin(\phi) p_{z,\phi}(x,\phi)\,\mathrm{d}\phi\,\mathrm{d}z \\ &= N_c \int_0^\infty \frac{2z^3}{\pi\psi^2(1-|\rho|^2)} K_0\left(\frac{2z}{\psi(1-|\rho|^2)}\right) \int_{-\pi}^{\pi} \sin(\phi)\exp\left(\frac{2|\rho| z\cos(\phi)}{\psi(1-|\rho|^2)}\right)\mathrm{d}\phi\,\mathrm{d}z \\ &= 0. \qquad \text{(E.5)} \end{aligned}$$

The phase integral in Eq. (E.5) is zero since the function to integrate consist on an odd function evaluated in the interval $[-\pi, \pi)$.

The last part of this appendix concerns the value of $E\{z_1 z_2\}$. If the expressions of z_1 and z_2, Eqs. (5.42) and (5.60) respectively, together with Eq. (E.5), it follows

$$\begin{aligned} E\{z_1 z_2\} &= E\{z^2 \sin(\phi)(\cos(\phi) - N_c)\} = E\{z^2 \cos(\phi)\sin(\phi)\} - N_c E\{z^2 \sin(\phi)\} \\ &= \int_0^\infty \int_{-\pi}^{\pi} z^2 \cos(\phi)\sin(\phi) p_{z,\phi}(z,\phi)\,\mathrm{d}\phi\,\mathrm{d}z. \end{aligned} \tag{E.6}$$

The value of Eq. (E.6) is

$$\begin{aligned} & N_c E\{z^2 \cos(\phi)\sin(\phi)\} \\ = \; & N_c \int_0^\infty \int_{-\pi}^{\pi} z^2 \cos(\phi)\sin(\phi) p_{z,\phi}(x,\phi)\,\mathrm{d}\phi\,\mathrm{d}z \\ = \; & N_c \int_0^\infty \frac{2z^3}{\pi\psi^2(1-|\rho|^2)} K_0\left(\frac{2z}{\psi(1-|\rho|^2)}\right) \int_{-\pi}^{\pi} \cos(\phi)\sin(\phi)\exp\left(\frac{2|\rho| z\cos(\phi)}{\psi(1-|\rho|^2)}\right)\mathrm{d}\phi\,\mathrm{d}z = 0. \end{aligned} \tag{E.7}$$

As in the case of Eq. (E.5), Eq. (E.7) is zero because the phase integral consist on the integration of an odd function on the phase interval $[-\pi, \pi)$.