Repeated Moral Hazard and Recursive Lagrangeans: Theory and Applications

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To my mother

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Abstract

This thesis elaborates a theoretical characterization of general dynamic agency problems based on recursive duality methods. With respect to current solution strategies, the main advantage of my approach is the possibility to analyze complicated models with many state variables, as it is the case in several macroeconomic situations. The first Chapter introduces the methodology and provides some numerical example. The second Chapter provides a characterization of optimal risk sharing contracts both in endowment and production economies, and shows how the approach is easy to apply to these problems. The third Chapter analyzes optimal unemployment insurance under different assumptions on access to financial markets and human capital trends.

Resumen

Esta tesis elabora una caracterización teórica de problemas de agencia dinámicos, basada en métodos recursivos duales. Respecto a las actuales estrategias de solución, la mayor ventaja de mi método es la posibilidad de analizar modelos complicados con muchas variables de estado, como es el caso en muchas situaciones macroeconómicas. El primer Capitulo introduce la metodología y presenta algunos ejemplos numéricos. El segundo Capitulo caracteriza contratos óptimos de participación al riesgo, en los dos casos de economía de dotación y de economía con producción, y demuestra que el método es muy simple en su aplicación a estos problemas. El tercer Capitulo analiza el seguro óptimo de desempleo bajo diferentes supuestos sobre el acceso al mercado financiero y la evolución del capital humano.

Foreword

This work develops a methodology to solve dynamic agency problems, and shows the main advantages of it by analyzing two economic situations that involve repeated moral hazard issues and that are extremely difficult to study with standard methods. It is composed of three chapters. The first introduces the dynamic principal-agent framework, and shows how it is possible to analyze it by means of a recursive Lagrangean approach along the lines of Marcet and Marimon (2009). It also develops a numerical algorithm that presents many advantages with respect to standard approaches, and in particular can be applied to problems with large state spaces without big difficulties [1].
The second chapter explores the issue of dynamic risk sharing in the presence of moral hazard, and in particular characterizes the differences between endowment economies and production economies. The third chapter is devoted to optimal unemployment insurance under various assumptions both on the access to financial markets and on human capital trends.
The dynamic agency framework has been useful in analyzing various economic situations in different fields: a non-exhaustive list includes unemployment insurance (Hopenhayn and Nicolini (1997), Shimer and Werning (forthcoming), Werning (2002), Pavoni (2007, forthcoming)), executive compensation (Clementi et al. (2008a,2008b), Clementi et al. (2006), Atkeson and Cole (2008)), entrepreneurship (Quadrini (2004), Paulson et al. (2006)), credit markets (Lehnert et al. (1999)), risk sharing (Zhao (2007), Friedman (1998)), and many more.
It turns out that these models are very complex, and it is very rare to derive closed form solutions. Therefore, the researcher must make use of numerical techniques. In late 80s, Abreu, Pearce and Stacchetti (1990) developed a recursive method (the "promised utilities approach") to solve dynamic agency problems, and their approach has been widely used by the profession. Unfortunately, their method poses serious numerical complexities already with just one state variable, and therefore many macroeconomic problems cannot be analyzed with this approach. In the first chapter, I show an alternative methodology to analyze the repeated moral hazard framework: I provide a theoretical justification for the use of Lagrangean techniques. I prove that the Lagrangean of the principal's optimization problem has a recursive structure along the lines of the work of Marcet and Marimon (2009). I then present a numerical algorithm that is based on the recursive Lagrangean, and show some examples of its application. The main advantage is the possibility to solve problems with several state variables without sensible complications, thus providing a valid alternative to the promised utilities approach also in cases in which the latter has demonstrated to be extremely complicated to implement.
In the second and third chapters, I use the methodology developed in Chapter 1 to

[1] The analysis of dynamic agency problems with large state spaces is very challenging, for a discussion see e.g. the 2008 Toulouse Lectures by Narayana Kocherlakota at http://www.econ.umn.edu/ nkocher/toulouse.html

analyze two economic problems that are characterized by repeated moral hazard. In Chapter 2, I analyze a model of risk sharing in an endowment economy. I then extend the model in two ways: first, I analyze an endowment economy where agents have non-monitorable access to financial markets. I then devote my attention to a production economy where each agent accumulates physical capital and uses it to produce the unique good in the economy. Moral hazard arises because unobservable effort from the agent can affect the distribution of productivity shocks. These models are important for the study of consumption-saving anomalies (see for an example Attanasio and Pavoni (2008)), international risk sharing (Mele (2008)), optimal fiscal pacts in federal constitutions and microcredit agreements.

I provide an analytical characterization of the optimal allocations, and then I show some numerical examples for the three different cases, in an economy with two agents. The main insight from computational exercise is that consumption inequality is very persistent under the optimal contract. Moreover, in the production economy, steady state capital tends to diverge: even if the two agents are identical in the first period in terms of capital endowment and preferences, on average in steady state an agent has more capital than the other.

Chapter 3 presents a model of unemployment insurance with human capital depreciation and hidden access to financial market, under the assumption that the worker can experience different spells of unemployment and employment, and compares the optimal provision of insurance obtained in this setup with the ones prescribed by other models.

Under the assumption of hidden access to financial market, there are two forces that go in opposite directions. The first is the standard *incentives-versus-insurance* effect, which recommend the planner to reduce consumption of the unemployed during unemployment spell, as in Hopenhayn and Nicolini (1997). The second is the self-insurance that the worker can get by saving during employment spells and that can potentially overthrown the provision of incentives by the planner: if the worker accumulates enough savings, he can just use them to prolong the unemployment period and exert suboptimal search effort.

Moreover, human capital depreciation implies that the longer a worker stays unemployed, the lower its wage when he will find a job: therefore, long-term unemployed have low incentives to look for a job. These three forces interact to determine the optimal scheme.

The possibility to apply recursive Lagrangean techniques to repeated moral hazard models opens the door to many applications. Models of repeated moral hazard with heterogeneous agents and endogenous states are largely unexplored territory. Given the numerical complexity to solve them, having a fast algorithm becomes crucial especially for quantitative exercises. The Lagrangean approach can also provide a useful tool for optimal taxation theory in economies with private information, models of entrepreneurial choice, DSGE models with financial frictions. One of the purposes of this thesis is to illustrate few applications in order to make clear the advantages of the approach in terms of computational tractability and speed. Future work will be devoted to other applications.

Contents

1 Repeated Moral Hazard and Recursive Lagrangeans

1.1 Introduction

In this paper, I show how to solve repeated moral hazard models with the use of recursive Lagrangean techniques. My approach allows the analysis of dynamic hidden-actions models with many state variables and many agents, which are instead untractable with commonly used solution strategies. Moreover, my methodology is simpler and numerically faster than the alternatives. I present the main idea in a simple model of dynamic agency, and then I show examples of economic models that are either very difficult to solve or even untractable under the traditional approach, but do not pose significant difficulties with my techniques.

There has been a lot of research, in last two decades, on dynamic versions of the principal-agent model[1]. Typically these models do not have closed form solution, therefore it is necessary to solve them numerically. The main technical difficulty is that the optimal allocation is history-dependent: the principal must keep track of the whole history of shock realizations, use it to extract information about the agent's unobservable behavior, and reward or punish the agent accordingly. As a consequence, it is not possible to derive a standard recursive representation of the principal's intertemporal maximization problem. The traditional way of dealing with this complication is based on the *promised utilities approach*: the model can be transformed in an auxiliary problem with the same solution, in which the principal optimally chooses allocations and agent's continuation value, taking as given the continuation value chosen in the previous period. The latter (also called *promised utility*) incorporates the whole history of the game, and hence continuation value becomes a new endogenous state variable to be chosen optimally. By using a standard argument, due to Abreu, Pearce and Stacchetti (1990) (APS henceforth) among others, it can be shown that the auxiliary problem has a recursive representation in a new state space that includes the continuation value and the state variables of the original problem. However, there is an additional complication: promised utilities must belong to a feasible set, which has to be characterized

[1]Recent contributions have focused both on the case in which agent's consumption is observable (see for example Rogerson (1985a), Spear and Srivastava (1987), Thomas and Worrall (1990), Phelan and Townsend (1991), Fernandes and Phelan (2000)) and more recently on the case in which agents can secretly save and borrow (Werning (2001), Abraham and Pavoni (2006, 2008, forthcoming)); other works have explored what happens with the presence of more than one agent (see e.g. Zhao (2007) and Friedman (1998)), while few researchers have extended the setup to production economies with capital (Clementi, Cooley and Di Giannatale (2008a,2008b)). Among applications, a non-exhaustive list includes unemployment insurance (Hopenhayn and Nicolini (1997), Shimer and Werning (forthcoming), Werning (2002), Pavoni (2007, forthcoming)), executive compensation (Clementi et al. (2008a,2008b), Clementi et al. (2006), Atkeson and Cole (2008)), entrepreneurship (Quadrini (2004), Paulson et al. (2006)), credit markets (Lehnert et al. (1999), and many more.

numerically before computation of the optimal allocation[2]. It is easy to characterize this set if there is just one exogenous shock, but it becomes complicated, if not computationally impossible, in models with several endogenous states. Therefore, with state-of-the-art approach, there is a huge class of models that are untractable even with numerical methods.

My paper provides a way to overcome the limits of the promised utilities approach, by extending the techniques of recursive Lagrangeans developed in Marcet and Marimon (2009) (MM henceforth) to the dynamic agency model[3]. With respect to the traditional approach, the main gain is in terms of tractability: under MM, I do not have to characterize any feasible set, since the recursive representation of the principal-agent problem is always well-defined with no need for additional constraints. Therefore, it is possible to find a solution also in presence of several endogenous state variables and many agents. I provide an algorithm based on the recursive Lagrangean which is much faster than the usual dynamic programming techniques and does not suffer from the same dimensionality issues.

To illustrate the method, I first apply it to the simplest version of the dynamic agency model as in Spear and Srivastava (1987). What is crucial to use the Lagrangean technique is the use of a first-order approach: I solve the agent's problem by taking first-order conditions with respect to effort, and use them as constraints in the principal's maximization problem[4]. I write down the Lagrangean, and using arguments similar to MM, I show that this saddle-point problem is recursive in an enlarged state space, which includes the stochastic output and an endogenously evolving Pareto-Negishi[5] weight attached to agent's utility. The latter has a natural interpretation: it summarizes the principal's promises, according to which the agent is rewarded or punished. If a "good" realization of the output is observed, the Pareto-Negishi weight increases, therefore the principal cares more about the utility of the agent and the agent gets more

[2]The feasible set is the fixed point of a set-operator (see Abreu, Pierce and Stacchetti (1990) for details). The standard numerical algorithm starts with an initial set large enough, and iteratively converges to the fixed point. Sleet and Yeltekin (2003) provide an efficient way of computing the value correspondence approximation.

[3]Sleet and Yeltekin (2008) apply recursive Lagrangean techniques to problem where there is private information about idyosyncratic stochastic preference shocks. In that framework, the structure of incentive compatibility constraints allows a direct application of MM techniques, while in this paper I need to transform and make some restrictive assumptions on the optimization problem in order to be able to write the Lagrangean. Moreover, Sleet and Yeltekin (2008) do not exploit the homogeneity of the value and policy functions, which is crucial in my proof strategy and in various numerical applications.

[4]In order to be sure that agent's first-order conditions are sufficient to get the optimal solution for the problem of the principal, I assume that Rogerson (1985b) conditions of monotone likelihood ratio and convex distribution function are satisfied.

[5]Lustig and Chien (2005) use the term "Pareto-Negishi weight" in a model of an endowment economy with limited enforcement, where agents face both aggregate and idyosyncratic shocks. In their work, the weight of each agent evolves stochastically in order to keep track of binding enforcement constraints.

consumption than in the previous period; analogously, if a "bad" outcome happens, the Pareto-Negishi weight decreases, hence the principal cares less about the utility of the agent and accordingly the agent gets less consumption than in the previous period. With the optimal choice of the Pareto-Negishi weight, the principal guarantees that the optimal allocation is incentive compatible.

Finally, I can obtain a solution from the Lagrangean first-order conditions[6]. This methodology is much simpler to implement, and less mathematically and computationally demanding than APS techniques.

Extending the basic approach to models with several state variables is straightforward. Imagine, to fix ideas, that we want to modify the baseline dynamic principal-agent model, by introducing observable capital accumulation. Assume that output is produced through a production technology that uses capital, and it is affected by a productivity shock, the distribution of which depend on agent's effort. It is easy to show that the Lagrangean associated with this extended model is recursive in a state space that includes the productivity shock, the Pareto-Negishi weight and the capital stock.

The numerical algorithm builds on the previous theoretical framework The basic idea is to find approximated policy functions by solving Lagrangean first-order conditions[7]. This algorithm is extremely fast in comparison with APS techniques. Computational speed depends in part on the fact that there is no need to characterize the feasible set for promised utilities. However, the main gain is obtained because solving a nonlinear system of equations is much faster than value function iteration.

After the detailed characterization of the optimal allocation in a model without endogenous states, I present few examples which are thought by the profession to be difficult to solve: I show how to use the recursive Lagrangean approach in a repeated moral hazard setup where the agent can accumulate assets without being monitored by the principal (as in Werning (2001) and Abraham and Pavoni (2006, forthcoming)), and in a dynamic risk-sharing problem with several agents where output depends on unobservable effort (as in Zhao (2007) and Friedman (1998)[8]). In both frameworks, I obtain

[6]Second-order conditions can be an issue in these models. The researcher can control for this problem by starting from different initial conditions and checking if the algorithm always converges to the same solution. All examples presented in my paper are robust to this check.

[7]The procedure is an application of the collocation method (see Judd (1998)): first, approximate the policy functions for allocations, the agent's continuation value and the value of the problem, over a set of grid nodes, with standard interpolation techniques (cubic splines or Chebichev polynomials); then, solve the Lagrangean first-order conditions with a nonlinear equation solver. Details are provided in the next sections.

[8]Friedman (1998) uses a very similar approach to the one presented in my paper. He analyzes a dynamic risk sharing problem in which there is a finite number of agents, and each of them exerts unobservable effort (this model is briefly presented in Section 1.5). As in my work, he characterizes the recursivity of the optimal contract by using Pareto-Negishi weights instead of continuation values, also if he does not directly apply the Lagrangean approach. His work is focused on theoretical results, though, and it does not provide any numerical example. Finally, he does not exploit the homogeneity properties of the value function to reduce the dimensionality of the state space, as I do here.

a recursive representation in the state space that contains the natural states and the endogenous Pareto-Negishi weight(s).
Finally, I present two examples of models that are untractable under the APS approach: a problem of optimal executive compensation scheme, and an international risk-sharing model with moral hazard and physical capital.
The paper is organized as follows: Section 1.2 introduces the basic framework of repeated moral hazard and explains how to obtain the recursive Lagrangean. Section 1.3 shows how to introduce endogenous observable states in the analysis by presenting an example of a production economy. Section 1.4 develops the treatment of cases with unobservable endogenous states, by analyzing a model with hidden asset accumulation. Section 1.5 presents a framework with several agents. Section 1.6 explains the details of the algorithm, and provides some numerical simulation for the models described in previous sections. Section 1.7 provides economic models that are untractable with APS techniques, and shows how to solve them with the Lagrangean approach. Section 1.8 concludes.

1.2 The basic model

In order to illustrate the Lagrangean approach, I start with a dynamic agency problem without endogenous states, where APS do not pose significant problems. In the next sections, I will extend the analysis to other setups in which the presence of endogenous state variables makes the use of APS techniques challenging for the researchers.
The economy is inhabited by a risk neutral principal and a risk averse agent. Time is discrete, and the state of the world follows an observable Markov process $\{s_t\}_{t=0}^{\infty}$, where $s_t \in S$, and $\#S = I$. The realizations of the process are public information. I will denote with subscripts the single realizations, and with superscripts the histories:

$$s^t \equiv \{s_0, ..., s_t\} \in S^{t+1}$$

At each period, the agent gets a state-contingent income flow $y(s_t)$, enjoys consumption $c_t(s^t)$, receives a transfer $\tau_t(s^t)$ from the principal, and exerts a costly unobservable action $a_t(s^t) \in A \subseteq \mathbb{R}_+$, $\{0\} \in A$, and A is bounded. I will refer to $a_t(s^t)$ as action or effort.
The costly action affects the future probability distribution of the state of the world. For simplicity, let $\widehat{s}_i$, $i = 1, 2, ..., I$ be the possible realizations of $\{s_t\}$ and let them be ordered such that $y(s_t = \widehat{s}_1) < y(s_t = \widehat{s}_2) < ... < y(s_t = \widehat{s}_I)$. Let $\pi(s_{t+1} = \widehat{s}_i \mid s_t, a_t(s^t))$ be the probability that state tomorrow is $\widehat{s}_i \in S$ conditional on past state and effort exerted by the agent at the beginning of the period[9], with $\pi(s_0 = \widehat{s}_I) = 1$. I assume $\pi(\cdot)$ is twice continuously differentiable in $a_t(s^t)$, and has *full support*: $\pi(s_{t+1} = \widehat{s}_i \mid s_t, a) > 0$

[9]Notice that I allow for persistence; in the numerical examples, I focus on i.i.d. shocks, but it should be clear that persistence does not create particular problems neither theoretically nor numerically.

$\forall i$, $\forall a$, $\forall s_t$. Let $\Pi\left(s^{t+1} \mid s_0, a^t\left(s^t\right)\right) = \prod_{j=0}^{t} \pi\left(s_{j+1} \mid s_j, a_j\left(s^j\right)\right)$ be the probability of history s^{t+1} induced by the history of unobserved actions $a^t\left(s^t\right) \equiv \left(a_0\left(s^0\right), a_1\left(s^1\right), ..., a_t\left(s^t\right)\right)$. The instantaneous utility of the agents is

$$u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right)$$

with $u(\cdot)$ strictly increasing, strictly concave and satisfying Inada conditions, while $\upsilon(\cdot)$ is strictly increasing and strictly convex; both are twice continuously differentiable. I also assume the instantaneous utility is uniformly bounded. The agent does not accumulate assets autonomously: the only source of insurance is the principal. Then, the budget constraint of the agent will be simply:

$$c_t\left(s^t\right) = y\left(s_t\right) + \tau_t\left(s^t\right) \quad \forall s^t, t \geq 0$$

Both principal and agent are fully committed once they sign the contract at time zero. A *contract* (or *allocation*) in this framework is a plan $\left(a^\infty, c^\infty, \tau^\infty\right) \equiv \{a_t\left(s^t\right), c_t\left(s^t\right), \tau_t\left(s^t\right)$ $\forall s^t \in S^{t+1}\}_{t=0}^{\infty}$ that belongs to the following set:

$$\begin{aligned}\Gamma^{MH} \equiv & \left\{\left(a^\infty, c^\infty, \tau^\infty\right) : a_t\left(s^t\right) \in A, \quad c_t\left(s^t\right) \geq 0, \right. \\ & \left. \tau_t\left(s^t\right) = c_t\left(s^t\right) - y\left(s_t\right) \quad \forall s^t \in S^{t+1}, t \geq 0\right\}\end{aligned}$$

Assume, for simplicity, that the discount factor of the agent and the principal is the same. The principal evaluates allocations according to the following

$$\begin{aligned}P\left(s_0; a^\infty, c^\infty, \tau^\infty\right) &= -\sum_{t=0}^{\infty}\sum_{s^t} \beta^t \tau_t\left(s^t\right) \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) \\ &= \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \left[y\left(s_t\right) - c_t\left(s^t\right)\right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) \qquad (1.1)\end{aligned}$$

therefore efficient contracts can be characterized by maximizing (1.1), subject to incentive compatibility and to the requirement of providing at least a minimum level of ex-ante utility V^{out} to the agent:

$$\begin{aligned}W\left(s_0\right) = & \max_{\{a_t(s^t), c_t(s^t)\}_{t=0}^{\infty} \in \Gamma^{MH}} \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \left[y\left(s_t\right) - c_t\left(s^t\right)\right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) \\ s.t. \quad & a^\infty \in \arg\max_{\{a_t(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \left[u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right)\right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)\end{aligned} \qquad (1.2)$$

$$\sum_{t=0}^{\infty}\sum_{s^t} \beta^t \left[u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right)\right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) \geq V^{out} \qquad (1.3)$$

I will call this *the original problem*. Notice that (1.2) is a very complicated object. In this work, I use the first order conditions of the agent's problem as a substitute for the constraint (1.2). In order to guarantee that this substitution leads to the actual solution of the original problem, I assume that Rogerson (1985b) conditions of monotone likelihood ratio and convexity of the distribution are satisfied:

Condition 1 (Monotone Likelihood-Ratio Condition (MLRC)) $\widehat{a} \leq \widehat{\widehat{a}} \Longrightarrow \frac{\pi(s_{t+1}=s_i|s_t,\widehat{a})}{\pi\left(s_{t+1}=s_i|s_t,\widehat{\widehat{a}}\right)}$ *is nonincreasing in* i.

The above property can be restated in a simpler way: if $\pi(\cdot)$ is differentiable, then MLRC is equivalent to $\frac{\pi_a(s_{t+1}=s_i|s_t,a)}{\pi(s_{t+1}=s_i|s_t,a)}$ being nondecreasing in i for any a, where $\pi_a(\cdot)$ is the derivative of $\pi(\cdot)$ with respect to a. An important consequence of the MLRC is the following: let $F(\cdot)$ be the cumulative distribution function of $\pi(\cdot)$; then MLRC implies that the density function $F'(s_{t+1} = s_i \mid s_t, a)$ is nonpositive for any i and every a. Therefore, more effort implies a first order stochastic dominance shift of the distribution (see Rogerson (1985b)). The second condition is:

Condition 2 (Convexity of the Distribution Function Condition (CDFC)) *The cumulative distribution function is convex:* $F''(s_{t+1} = s_i \mid s_t, a)$ *is nonnegative for any* i *and every* a.

This condition implies that the cumulative distribution function is convex. In more intuitive terms, MLRC asks for the state of nature to be "sufficiently informative" about the unobservable effort, while CDFC says that this informativeness has "decreasing returns to scale".

I now define the problem of the agent and I derive his first order conditions with respect to effort. The problem of the agent, given the principal's strategy profile $\tau^\infty \equiv \{\tau_t(s^t)\}_{t=0}^\infty$, is:

$$V(s_0; \tau^\infty) = \max_{\{c_t(s^t),a_t(s^t)\}_{t=0}^\infty \in \Gamma^{MH}} \left\{ \sum_{t=0}^\infty \sum_{s^t} \beta^t \left[u\left(c_t\left(s^t\right)\right) - \upsilon\left(\left(a_t\left(s^t\right)\right)\right)\right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) \right\}$$

The first order condition for effort is:

$$\begin{aligned} \upsilon'\left(a_t\left(s^t\right)\right) = \sum_{j=1}^\infty \beta^j \sum_{s^{t+j}|s^t} \pi_a\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right) \times \\ \times \left[u\left(c_{t+j}\left(s^{t+j}\right)\right) - \upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right] \Pi\left(s^{t+j} \mid s^{t+1}, a^{t+j}\left(s^{t+j} \mid s^{t+1}\right)\right) \end{aligned} \tag{1.4}$$

Intuitively, the marginal cost of effort today (LHS) has to be equal to future expected benefits (RHS) in terms of expected future utility. The use of (1.4) is key to my approach, since it allows me to write the Lagrangean of the principal's problem. In the following, for simplicity I will refer to (1.4) as the *incentive-compatibility constraint* (ICC).

We can write the Pareto problem of the principal as:

$$
\begin{aligned}
W(s_0) = & \max_{\{a_t(s^t),c_t(s^t)\}_{t=0}^{\infty}\in\Gamma^{MH}} \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left[y(s_t)-c_t(s^t)\right]\Pi\left(s^t \mid s_0, a^{t-1}(s^{t-1})\right) \\
s.t. \quad & \upsilon'\left(a_t(s^t)\right) = \sum_{j=1}^{\infty}\beta^j\sum_{s^{t+j}|s^t}\frac{\pi_a\left(s_{t+1}\mid s_t, a_t(s^t)\right)}{\pi\left(s_{t+1}\mid s_t, a_t(s^t)\right)}\times \qquad (1.5) \\
& \times\left[u\left(c_{t+j}(s^{t+j})\right)-\upsilon\left(a_{t+j}(s^{t+j})\right)\right]\Pi\left(s^{t+j}\mid s^t, a^{t+j-1}(s^{t+j-1}\mid s^t)\right) \\
& \forall s^t, t\geq 0 \\
& \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left[u\left(c_t(s^t)\right)-\upsilon\left(a_t(s^t)\right)\right]\Pi\left(s^t\mid s_0, a^{t-1}(s^{t-1})\right)\geq V^{out}
\end{aligned}
$$

a The Lagrangean approach

In order to write the Lagrangean of the Pareto problem, notice that (1.3) must be binding in the optimum: otherwise, the principal can increase her expected discounted utility by asking the agent to increase effort in period 0 by $\delta > 0$, provided that δ is small enough. Therefore (1.3) will be associated with a strictly positive Lagrange multiplier (say, γ), which will be a function of V^{out}: for every V^{out}, there will be a γ associated with (1.3). This Lagrange multiplier can be seen as a Pareto-Negishi weight on the agent's utility. Since each γ implies a unique V^{out}, I can fully characterize the Pareto frontier of this economy by solving the problem for different values of γ between zero and infinity. Hence, in the following, I am going to consider γ as a parameter, that represents the constraint (1.3). Moreover, notice that by fixing γ, V^{out} will appear in the Lagrangean only in the constant term γV^{out}, thus it will be irrelevant for the optimal allocation. Given these considerations, Problem (1.5) can be seen as the constrained maximization of a social welfare function, where the Pareto weight for the principal and the agent are, respectively, 1 and γ:

$$
\begin{aligned}
W^{SWF}(s_0) = & \max_{\{a_t(s^t),c_t(s^t)\}_{t=0}^{\infty}\in\Gamma^{MH}} \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left[y(s_t)-c_t(s^t)\right]\Pi\left(s^t\mid s_0, a^{t-1}(s^{t-1})\right)+ \\
& +\gamma\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left[u\left(c_t(s^t)\right)-\upsilon\left(a_t(s^t)\right)\right]\Pi\left(s^t\mid s_0, a^{t-1}(s^{t-1})\right) \\
s.t. \quad & \upsilon'\left(a_t(s^t)\right) = \sum_{j=1}^{\infty}\beta^j\sum_{s^{t+j}|s^t}\frac{\pi_a\left(s_{t+1}\mid s_t, a_t(s^t)\right)}{\pi\left(s_{t+1}\mid s_t, a_t(s^t)\right)}\times \\
& \times\left[u\left(c_{t+j}(s^{t+j})\right)-\upsilon\left(a_{t+j}(s^{t+j})\right)\right]\Pi\left(s^{t+j}\mid s^t, a^{t+j-1}(s^{t+j-1}\mid s^t)\right)
\end{aligned}
$$

Let $\beta^t \lambda_t(s^t)\Pi(s^t \mid s_0, a^{t-1}(s^{t-1}))$ be the Lagrange multiplier associated to each ICC. I can therefore write the Lagrangean as:

$$
\begin{aligned}
&L(s_0, \gamma, c^\infty, a^\infty, \lambda^\infty) = \\
&= \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left\{y(s_t) - c_t\left(s^t\right) + \gamma\left[u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right)\right]\right\}\Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) + \\
&\quad - \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\lambda_t\left(s^t\right)\left\{\upsilon'\left(a_t\left(s^t\right)\right) - \sum_{j=1}^{\infty}\beta^j\sum_{s^{t+j}|s^t}\frac{\pi_a\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)}{\pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)}\times\right. \\
&\quad \left.\times\left[u\left(c_{t+j}\left(s^{t+j}\right)\right) - \upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right]\Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1} \mid s^t\right)\right)\right\}\times \\
&\qquad \times\Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)
\end{aligned}
$$

The Lagrangean can be manipulated with simple algebra to get the following expression:

$$
\begin{aligned}
L(s_0, \gamma, c^\infty, a^\infty, \lambda^\infty) &= \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left\{y(s_t) - c_t\left(s^t\right) + \phi_t\left(s^t\right)\left[u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right)\right] + \right. \\
&\left. -\lambda_t\left(s^t\right)\upsilon'\left(a_t\left(s^t\right)\right)\right\}\Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)
\end{aligned}
$$

where

$$
\phi_t\left(s^{t-1}, s_t\right) = \gamma + \sum_{i=0}^{t-1}\lambda_i\left(s^i\right)\frac{\pi_a\left(s_{i+1} \mid s_i, a_i\left(s^i\right)\right)}{\pi\left(s_{i+1} \mid s_i, a_i\left(s^i\right)\right)}
$$

The intuition is simple. For any s^t, the expression $\lambda_t(s^t)\frac{\pi_a\left(s_{t+1}|s_t, a_t\left(s^t\right)\right)}{\pi\left(s_{t+1}|s_t, a_t\left(s^t\right)\right)}$ is a *planner's promise* about how much she will increase the weight of agent's welfare in the future, depending on which realization of state s_{t+1} is observed. By keeping track of all λ's and $\frac{\pi_a}{\pi}$'s realized in the past, $\phi_t(s^t)$ summarizes all the promises made by the planner in previous periods. In this framework, there is straightforward interpretation of $\phi_t(s^t)$: it is the *Pareto-Negishi weight* of the agent's lifetime utility, that evolves *endogenously* in order to track agent's effort with the following recursive law of motion:

$$
\phi_{t+1}\left(s^t, \widehat{s}\right) = \phi_t\left(s^t\right) + \lambda_t\left(s^t\right)\frac{\pi_a\left(s_{t+1} = \widehat{s} \mid s_t, a_t\left(s^t\right)\right)}{\pi\left(s_{t+1} = \widehat{s} \mid s_t, a_t\left(s^t\right)\right)} \quad \forall\widehat{s} \in S \tag{1.6}
$$

$$
\phi_0\left(s^0\right) = \gamma
$$

To better understand the role of $\phi_t(s^t)$, let us assume there are only two possible realizations of the state of nature: $s_t \in \{s_L, s_H\}$ At time 0, the weight is equal to γ. In period 1, given our assumption on the likelihood ratio, the Pareto-Negishi weight is higher than γ if the principal observes s_H, while it is lower than γ if she observes s_L (a formal proof of this fact is obtained in Lemma 1). Therefore the agent is rewarded by a higher weight in the social welfare function of the principal (i.e., the principal cares more about him) if a good state of nature is observed, while it is punished by a lower weight (i.e., the principal cares less about him) if a bad state of nature happens.

b Recursive formulation

By the duality theory (see for example Luenberger (1969)), we know that a solution of the original problem corresponds to a saddle point of the Lagrangean, i.e. the contract

$$\left(c^{\infty *}, a^{\infty *}, \tau^{\infty *}\right) = \left\{c_t^*\left(s^t\right), a_t^*\left(s^t\right), y\left(s_t\right) - c_t^*\left(s^t\right) \quad \forall s^t \in S^{t+1}\right\}_{t=0}^{\infty}$$

is a solution for the original problem if there exist a sequence $\left\{\lambda_t^*\left(s^t\right) \quad \forall s^t \in S^{t+1}\right\}_{t=0}^{\infty}$ of Lagrange multipliers such that $\left(c^{\infty *}, a^{\infty *}, \lambda^{\infty *}\right) = \left\{c_t^*\left(s^t\right), a_t^*\left(s^t\right), \lambda_t^*\left(s^t\right) \quad \forall s^t \in S^{t+1}\right\}_{t=0}^{\infty}$ satisfy:

$$L\left(s_0, \gamma, c^{\infty}, a^{\infty}, \lambda^{\infty *}\right) \leq L\left(s_0, \gamma, c^{\infty *}, a^{\infty *}, \lambda^{\infty *}\right) \leq L\left(s_0, \gamma, c^{\infty *}, a^{\infty *}, \lambda^{\infty}\right)$$

It is possible to recursively characterize the solutions of the Lagrangean. In particular, it is possible to show that value and policy functions depend on the state of the world s_t and the Pareto-Negishi weight $\phi_t\left(s^t\right)$. The reader not interested in the details can skip this section and jump directly to the characterization of the optimal allocation.
I follow the strategy of MM by showing that a generalized version of (1.5) is recursive in an enlarged state space. Let me define the following generalized version of (1.5):

$$\begin{aligned}
W_\theta^{SWF}\left(s_0\right) = & \max_{\left\{a_t\left(s^t\right), c_t\left(s^t\right)\right\}_{t=0}^{\infty} \in \Gamma^{MH}} \overline{\phi}^0 \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \left[y\left(s_t\right) - c_t\left(s^t\right)\right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) + \\
& + \gamma \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \left(u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right)\right) \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) \\
s.t. \quad & \upsilon'\left(a_t\left(s^t\right)\right) = \sum_{j=1}^{\infty} \beta^j \sum_{s^{t+j} \mid s^t} \frac{\pi_a\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)}{\pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)} \times \\
& \times \left[u\left(c_{t+j}\left(s^{t+j}\right)\right) - \upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right] \Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1} \mid s^t\right)\right) \\
& \forall s^t, t \geq 0
\end{aligned}$$

Notice that if $\overline{\phi}^0 = 1$, then we are back to (1.5). We can write down the Lagrangean of this problem by assigning a Lagrange multiplier $\beta^t \lambda_t\left(s^t\right)$ to each ICC constraint:

$$\begin{aligned}
& L_\theta\left(s_0, \gamma, c^{\infty}, a^{\infty}, \lambda^{\infty}\right) = \\
& = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \left\{\overline{\phi}^0 \left[y\left(s_t\right) - c_t\left(s^t\right)\right] + \gamma\left[u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right)\right]\right\} \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) + \\
& \quad - \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \lambda_t\left(s^t\right) \left\{\upsilon'\left(a_t\left(s^t\right)\right) - \sum_{j=1}^{\infty} \beta^j \sum_{s^{t+j} \mid s^t} \frac{\pi_a\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)}{\pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)} \times \right. \\
& \quad \left. \times \left[u\left(c_{t+j}\left(s^{t+j}\right)\right) - \upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right] \Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1} \mid s^t\right)\right)\right\} \times \\
& \quad \times \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)
\end{aligned}$$

Notice that $r\left(a, c, s\right) \equiv y\left(s\right) - c$ is uniformly bounded by natural debt limits, so there exists a lower bound $\underline{\kappa}$ such that $r\left(a, c, s\right) \geq \underline{\kappa}$. We can therefore define $\kappa < \frac{\underline{\kappa}}{1-\beta}$.

Define $\varphi(\phi,\lambda,a,s') \equiv \phi + \lambda\frac{\pi_a(s'|s,a)}{\pi(s'|s,a)}$, $h_0^P(a,c,s) \equiv r(a,c,s)$, $h_1^P(a,c,s) \equiv r(a,c,s) - \kappa$, $h_0^{ICC}(a,c,s) \equiv u(c) - \upsilon(a)$, $h_1^{ICC}(a,c,s) \equiv -\upsilon'(a)$, $\theta \equiv \begin{bmatrix}\phi^0 & \phi\end{bmatrix} \in \mathbb{R}^2$, $\chi \equiv \begin{bmatrix}\lambda^0 & \lambda\end{bmatrix}$ and

$$\begin{aligned} h(a,c,\theta,\chi,s) &\equiv \theta h_0(a,c,s) + \chi h_1(a,c,s) \\ &\equiv \begin{bmatrix}\phi^0 & \phi\end{bmatrix}\begin{bmatrix} h_0^P(a,c,s) \\ h_0^{ICC}(a,c,s)\end{bmatrix} + \begin{bmatrix}\lambda^0 & \lambda\end{bmatrix}\begin{bmatrix} h_1^P(a,c,s) \\ h_1^{ICC}(a,c,s)\end{bmatrix} \end{aligned}$$

which is homogenous of degree 1 in (θ,χ). The Lagrangean can be written as:

$$L_\theta(s_0,\gamma,c^\infty,a^\infty,\chi^\infty) = \sum_{t=0}^{\infty}\sum_{s^t}\beta^t h\left(a_t\left(s^t\right),c_t\left(s^t\right),\theta_t\left(s^t\right),\chi_t\left(s^t\right),s_t\right)\Pi\left(s^t \mid s_0,a^{t-1}\left(s^{t-1}\right)\right)$$

where

$$\begin{aligned} \theta_{t+1}\left(s^t,\widehat{s}\right) &= \varphi\left(\theta_t\left(s^t\right),\chi_t\left(s^t\right),a_t\left(s^t\right),\widehat{s}\right) \quad \forall \widehat{s}\in S \\ \theta_0\left(s^0\right) &= \begin{bmatrix}\overline{\phi}^0 & \gamma\end{bmatrix} \end{aligned}$$

Notice that the constraint defined by $h_1^P(a,c,s)$ is never binding by definition, therefore $\lambda_t^0(s^t) = 0$ and $\phi_t^0(s^t) = \overline{\phi}^0$ $\forall s^t, t \geq 0$, which implies that the only relevant state variable is $\phi_t(s^t)$. We can associate a saddle point functional equation to this Lagrangean

$$J(s,\theta) = \min_{\chi}\max_{a,c}\left\{h(a,c,\theta,\chi,s) + \beta\sum_{s'}\pi(s' \mid s,a)\,J(s',\theta'(s'))\right\} \tag{1.7}$$
$$s.t. \quad \theta'(s') = \theta + \chi\frac{\pi_a(s' \mid s,a)}{\pi(s' \mid s,a)} \quad \forall s'$$

In order to show that there is a unique value function $J(s,\theta)$ that solves (1.7), it is sufficient to prove that the operator on the right hand side of the functional equation is a contraction[10].

There are two technical differences with the original framework in MM. First, the endogenous evolution of the Pareto-Negishi weight is a deviation from MM, since in their paper the law of motion of the costate variable $\theta_t(s^t)$ only depends on $\chi_t(s^t)$, while here also depends on $a_t(s^t)$. Second, the probability distribution of the future states is endogenous and depends on the optimal effort $a_t(s^t)$. I show in Proposition 1 that the argument in MM works also here with some minor modifications.

[10]Messner and Pavoni (2004) show that, also if the value of the problem (1.7) is unique, the policy function associated with it can be suboptimal or even unfeasible. To avoid these issues, though, it is sufficient to impose that the policy function satisfies all the constraints of the original problem. Since I solve for the Lagrangean first-order conditions, I always impose all the constraints.

Proposition 1 *Fix an arbitrary constant $K > 0$ and let $K_\theta = \max\{K, K\|\theta\|\}$. The operator*

$$\begin{aligned}
(T_K f)(s,\theta) &\equiv \min_{\{\chi>0:\|\chi\|\le K_\theta\}} \max_{a,c} \left\{ h(a,c,\theta,\chi,s) + \beta \sum_{s'} \pi(s' \mid s,a) f(s', \theta'(s')) \right\} \\
s.t.\quad \theta'(s') &= \theta + \chi \frac{\pi_a(s' \mid s,a)}{\pi(s' \mid s,a)} \quad \forall s'
\end{aligned}$$

is a contraction.

Proof. *Appendix A.* ■

Proposition 1 shows that the saddle point problem is recursive in the state space $(s,\theta) \in S \times \mathbb{R}^2$. All the other theorems in MM apply directly to my framework: the result of Proposition 1 is valid for any $K > 0$, and since whenever the Lagrangean has a solution the Lagrange multipliers are bounded, then a recursive solution of the Problem $(SPFE)$ is a solution of the Lagrangean, and more importantly it is a solution of the original problem. As a consequence, we can restrict the search of optimal contracts to the set of policy functions that are Markovian in the space $(s,\theta) \in S \times \mathbb{R}^2$. But remember that the first element of θ is constant for any t and the only actual endogenous state is $\phi_t(s^t)$; therefore, from this point of view, finding the optimal contract has the same numerical complexity as finding the optimal allocations in a standard stochastic neoclassical growth model.
Notice that, since in the Lagrangean formulation we eliminated the constant γV^{out}, the value of the original problem is:

$$W(s_0) = W^{SWF}(s_0) - \gamma V^{out} = J\left(s_0, \begin{bmatrix} 1 & \gamma \end{bmatrix}\right) - \gamma V^{out}$$

where $V^{out} = V(s_0; \tau^{\infty *})$ is the agent's lifetime utility implied by the optimal contract. Another important consequence of Proposition 1 is that the value function $J(s,\theta)$ is homogeneous of degree 1 (and consequently, policy functions for allocations are homogeneous of degree zero)[11]: this fact will be important in the last example in Section 1.5.

c Characterization of the optimal contract

In this section I show few properties of the optimal contract. Those properties are the analogous, in the Lagrangean approach, of well known results in the literature. Let us go back to the problem with $\overline{\phi}^0 = 1$. We can take the first order conditions of the Lagrangean:

$$c_t(s^t): \quad 0 = -1 + \phi_t(s^t) u_c(c_t(s^t)) \tag{1.8}$$

[11]This is made clear in the proof.

$$a_t\left(s^t\right): \quad 0=-\lambda_t\left(s^t\right)\upsilon''\left(a_t\left(s^t\right)\right)-\phi_t\left(s^t\right)\upsilon'\left(a_t\left(s^t\right)\right)+ \tag{1.9}$$
$$+\sum_{j=1}^{\infty}\beta^j\sum_{s^{t+j}|s^t}\frac{\pi_a\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}{\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}\left\{y\left(s_t\right)-c_t\left(s^t\right)-\lambda_{t+j}\left(s_{t+j}\right)\upsilon'\left(a_{t+j}\left(s^{t+j}\right)\right)-\right.$$
$$\left.+\phi_{t+j}\left(s^{t+j}\right)\left[u\left(c_{t+j}\left(s^{t+j}\right)\right)-\upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right]\right\}\Pi\left(s^{t+j}\mid s^t,a^{t+j-1}\left(s^{t+j-1}\right)\right)+$$
$$+\beta\lambda_t\left(s^t\right)\sum_{s^{t+1}|s^t}\frac{\partial\left(\frac{\pi_a(\cdot)}{\pi(\cdot)}\right)}{\partial a}\left[u\left(c_{t+1}\left(s^{t+1}\right)\right)-\upsilon\left(a_{t+1}\left(s^{t+1}\right)\right)\right]\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)$$

and

$$\lambda_t\left(s^t\right) \quad : \quad 0=-\upsilon'\left(a_t\left(s^t\right)\right)+\sum_{j=1}^{\infty}\sum_{s^{t+j}|s^t}\frac{\pi_a\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}{\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}\times \tag{1.10}$$
$$\times\left[\beta^j\left[u\left(c_{t+j}\left(s^{t+j}\right)\right)-\upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right]\Pi\left(s^{t+j}\mid s^t,a^{t+j-1}\left(s^{t+j-1}\mid s^t\right)\right)\right]$$

Lemma 1 makes clear how $\phi_t\left(s^t\right)$ incorporates the promises of the principal. From (1.8) we can see that $c_{t+1}\left(s^{t+1}\right)=u_c^{-1}\left(\frac{1}{\phi_{t+1}\left(s^{t+1}\right)}\right)$, then $c_{t+1}\left(s^{t+1}\right)$ is increasing in $\phi_{t+1}\left(s^{t+1}\right)$. Lemma 1 says that, tomorrow,the principal will reward a high income realization with higher consumption than today, and a low income realization with lower consumption than today[12].

Lemma 1 *In the optimal contract,* $\phi_{t+1}\left(s^t,\widehat{s}_1\right)<\phi_t\left(s^t\right)<\phi_{t+1}\left(s^t,\widehat{s}_I\right)$ *for any* t.

Proof. *Appendix A.* ■

The following Proposition characterizes the long run properties of the Pareto Negishi weight.

Proposition 2 $\phi_t\left(s^t\right)$ *is a martingale that converges to zero almost surely.*

Proof. *Appendix A.* ■

Proposition 2 is the well known result that $\frac{1}{u_c\left(c_t\left(s^t\right)\right)}$ evolves as a martingale (see Rogerson (1985a)). The a.s.-convergence to zero is the so called *immiseration property* that implies zero consumption almost surely as $t\to\infty$, which is a standard result in models with asymmetric information (see Thomas and Worrall (1990), for example). In this framework, the immiseration property has an intuitive interpretation: in order to keep strong incentives for the agent, the planner must ensure that the Pareto-Negishi weight goes to zero almost surely as $t\to\infty$ for any possible sequence of realizations of the income shock.

[12]Thomas and Worrall (1990) prove the same property with APS techniques.

The result in Proposition 2 is obtained by using the law of motion of $\phi_t(s^t)$ and (1.8), which yields

$$E_t^a\left[\frac{1}{u_c\left(c_{t+1}\left(s^{t+1}\right)\right)}\right]=\frac{1}{u_c\left(c_t\left(s^t\right)\right)}$$

We can use Jensen's inequality and the strict concavity of $u(\cdot)$ to get that $E_t^a\left[u_c\left(c_{t+1}\left(s^{t+1}\right)\right)\right] > u_c\left(c_t\left(s^t\right)\right)$: the profile of expected consumption is decreasing across time.

1.3 Repeated moral hazard with capital accumulation

The previous setup did not have any endogenous natural state variable. In this Section, I provide an example of a repeated moral hazard model with capital accumulation, in order to show how easy is to extend the basic framework with the Lagrangean approach. The setup is similar to Clementi, Cooley and Di Giannatale (2008a,b): it is a production economy where a good is produced with the use of capital. The production function is:

$$y_t\left(s^t\right)=A\left(s_t\right)f\left(k_{t-1}\left(s^{t-1}\right)\right)$$

where $A(s_t)$ is a productivity shock. The feasibility constraint is:

$$c_t\left(s^t\right)+i_t\left(s^t\right)\leq A\left(s_t\right)f\left(k_{t-1}\left(s^{t-1}\right)\right)$$

where $i_t(s^t)$ is investment in physical capital. The law of motion for capital is:

$$k_t\left(s^t\right)=i_t\left(s^t\right)+(1-\delta)\,k_{t-1}\left(s^{t-1}\right)\qquad k_{-1}\text{ given}$$

where δ is the depreciation rate of capital. Combining feasibility and the law of motion for capital, we get the following resource constraint:

$$c_t\left(s^t\right)+k_t\left(s^t\right)-(1-\delta)\,k_{t-1}\left(s^{t-1}\right)\leq A\left(s_t\right)f\left(k_{t-1}\left(s^{t-1}\right)\right)$$

$$k_{-1}\text{ given}$$

The instantaneous utility of the agent is

$$u\left(c_t\left(s^t\right)\right)-\upsilon\left(a_t\left(s^t\right)\right)$$

The set of feasible contracts is then

$$\begin{aligned}\Gamma^K\equiv\Big\{&\left(a^\infty,c^\infty,k^\infty\right):a_t\left(s^t\right)\in A,\quad c_t\left(s^t\right)\geq 0,\quad k_t\left(s^t\right)\in K\subseteq\mathbb{R}_+,\\&c_t\left(s^t\right)+k_t\left(s^t\right)-(1-\delta)\,k_{t-1}\left(s^{t-1}\right)\leq A\left(s_t\right)f\left(k_{t-1}\left(s^{t-1}\right)\right)\quad\forall s^t\in S^{t+1},t\geq 0\Big\}\end{aligned}$$

Notice that, with the same assumptions on primitives as in the previous Section, the incentive compatibility constraint is the same as (1.4). The Pareto-constrained allocation can be found by solving:

$$W^{SWF}(s_0, k_{-1}) = \max_{\{c_t(s^t), a_t(s^t), k_t(s^t)\}_{t=0}^{\infty} \in \Gamma^K} \left\{ \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \left[A(s_t) f\left(k_{t-1}\left(s^{t-1}\right)\right) - c_t\left(s^t\right) - \right.\right.$$
$$\left. -k_t\left(s^t\right) + (1-\delta) k_{t-1}\left(s^{t-1}\right) \right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) +$$
$$\left. + \gamma \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \left[u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right) \right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) \right\}$$

$$s.t. \quad \upsilon_a'\left(a_t\left(s^t\right)\right) = \sum_{j=1}^{\infty} \beta^j \sum_{s^{t+j}|s^t} \frac{\pi_a\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)}{\pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)} \times \tag{1.11}$$
$$\times \left[u\left(c_{t+j}\left(s^{t+j}\right)\right) - \upsilon\left(a_{t+j}\left(s^{t+j}\right)\right) \right] \Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1} \mid s^t\right)\right)$$

$$c_t\left(s^t\right) + k_t\left(s^t\right) - (1-\delta) k_{t-1}\left(s^{t-1}\right) \le A(s_t) f\left(k_{t-1}\left(s^{t-1}\right)\right)$$
$$k_{-1} \text{ given}$$

We associate a Lagrange multiplier $\beta^t \lambda_t\left(s^t\right)$ to any ICC constraint. We can now write down the Lagrangean of this problem and manipulate it to get:

$$L\left(s_0, \gamma, c^{\infty}, a^{\infty}, k^{\infty}, \lambda^{\infty}\right) =$$
$$= \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \left\{ A(s_t) f\left(k_{t-1}\left(s^{t-1}\right)\right) - c_t\left(s^t\right) - k_t\left(s^t\right) + (1-\delta) k_{t-1}\left(s^{t-1}\right) + \right.$$
$$\left. +\phi_t\left(s^t\right) \left[u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right) \right] - \lambda_t\left(s^t\right) \upsilon_a'\left(a_t\left(s^t\right)\right) \right\} \Pi\left(s^t \mid h_0, a^{t-1}\left(s^{t-1}\right)\right)$$

where

$$\phi_{t+1}\left(s^t, \widehat{s}\right) = \phi_t\left(s^t\right) + \lambda_t\left(s^t\right) \frac{\pi_a\left(s_{t+1} = \widehat{s} \mid s_t, a_t\left(s^t\right)\right)}{\pi\left(s_{t+1} = \widehat{s} \mid s_t, a_t\left(s^t\right)\right)} \quad \forall \widehat{s} \in S$$
$$\phi_0\left(s^0\right) = \gamma$$

As in the previous section, we can associate a saddle point functional equation to (a generalized version of) the Lagrangean, and show that the operator is a contraction (see Appendix B for the details). In this case, therefore, the solution will be a Markovian policy function that depends on capital and the Pareto weight.

1.4 Repeated moral hazard with hidden assets

Werning (2001, 2002) and Abraham and Pavoni (2006, 2008, forthcoming) (AP from here on) analyze a model with hidden effort and hidden assets: the agent can borrow or lend without being monitored by the principal. This problem generates a continuum of incentive constraints (for each possible income realization, there is a continuum of possible asset positions). Hence the feasible set of continuation values has infinite dimension and APS techniques cannot be used. In order to overcome this complication, they characterize the optimal contract by defining an auxiliary problem, where agent's first-order conditions over effort and bonds are used as constraints for the principal's problem. They show that the solution of their auxiliary problem is characterized by three state variables (income, promised utility and consumption marginal utility), and can be solved recursively by value function iteration. Abraham and Pavoni (2006, forthcoming) also provide a numerical procedure to verify if the first-order approach delivers the true incentive compatible allocation. Even if their work is big step ahead in the analysis of this class of models, the use of APS arguments makes their numerical algorithm too slow for calibration purposes and not easily adaptable to more complicated extensions. In this section, I show how the Lagrangean approach can easily deal with this framework.

Let $\{b_t(s^t)\}_{t=-1}^{\infty}$, b_{-1} given, be a sequence of one-period bond that the agent pays 1 today, getting R tomorrow. Assume that the principal cannot monitor the bond market, so that the asset accumulation is unobservable to her. Then agent's budget constraint becomes:

$$c_t\left(s^t\right) + b_t\left(s^t\right) = y\left(s_t\right) + \tau_t\left(s^t\right) + Rb_{t-1}\left(s^{t-1}\right)$$

while the instantaneous utility function for the agent is the same as in Section 1.2. We have to solve now the following agent's problem:

$$\begin{aligned}
&\widetilde{V}\left(s_0, b_{-1}; \tau^{\infty}\right) = \\
&= \max_{\{c_t(s^t), b_t(s^t), a_t(s^t)\}_{t=0}^{\infty} \in \Gamma^{HA}} \left\{ \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \left[u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right)\right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) \right\}
\end{aligned}$$

where

$$\begin{aligned}
\Gamma^{HA} \equiv & \left\{ \left(a^{\infty}, c^{\infty}, b^{\infty}, \tau^{\infty}\right) : a_t\left(s^t\right) \in A, \quad c_t\left(s^t\right) \geq 0, \right. \\
& \left. c_t\left(s^t\right) + b_t\left(s^t\right) = y\left(s_t\right) + \tau_t\left(s^t\right) + Rb_{t-1}\left(s^{t-1}\right) \quad \forall s^t \in S^{t+1}, t \geq 0 \right\}
\end{aligned}$$

Accordingly, agent's first order conditions with respect to the unobservable variables (i.e., effort and bond holdings) are (1.4) and the following Euler equation:

$$u'\left(c_t\left(s^t\right)\right) = \beta R \sum_{s_{t+1}} u'\left(c_{t+1}\left(s^t, s_{t+1}\right)\right) \pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right) \tag{1.12}$$

Assume $\beta R = 1$ to simplify algebra. The presence of hidden assets requires (1.12) to be included in the set of constraints for the principal's problem.

Let $\beta^t \eta_t(s^t)$ be the Lagrange multiplier for (1.12), and $\beta^t \lambda_t(s^t)$ the Lagrange multiplier for ICC. The Lagrangean can be manipulated to get:

$$L(s_0, \gamma, c^\infty, a^\infty, \lambda^\infty, \eta^\infty) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \left\{ y(s_t) - c_t(s^t) + \phi_t(s^t) \left[u(c_t(s^t)) - \upsilon(a_t(s^t)) \right] + \right.$$
$$\left. -\lambda_t(s^t) \upsilon'(a_t(s^t)) + \left[\eta_t(s^t) - \beta^{-1} \zeta_t(s^t) \right] u_c(c_t(s^t)) \right\} \Pi\left(s^t \mid s_0, a^{t-1}(s^{t-1})\right) \quad (1.13)$$

where

$$\phi_{t+1}(s^t, \widehat{s}) = \phi_t(s^t) + \lambda_t(s^t) \frac{\pi_a(s_{t+1} = \widehat{s} \mid s_t, a_t(s^t))}{\pi(s_{t+1} = \widehat{s} \mid s_t, a_t(s^t))} \quad \forall \widehat{s} \in S \quad \text{and} \quad \phi_0(s^0) = \gamma$$

$$\zeta_{t+1}(s^t, \widehat{s}) = \eta_t(s^t) \quad \forall \widehat{s} \in S \quad \text{and} \quad \zeta_0(s^0) = 0$$

This problem is characterized by two costate variables: the Pareto weight $\phi_t(s^t)$ and the new costate $\zeta_t(s^t)$, which keeps track of the Euler equation. Using the same arguments of Proposition 1, it is possible to show that the problem is recursive in the state space that includes (s, ϕ, ζ) as states variables (see Appendix C for details).

1.5 Multiple agents

The Lagrangean method can make a big difference also in models with many agents. In this section, I present a model of dynamic risk-sharing with a finite number of agents, where each of them exerts unobservable effort. The setup presented here is inspired, with minor differences, by Zhao (2007), who solves numerically the same model with APS techniques.

There are N agents indexed by $i = 1, ..., N$. Each agent is subject to an observable Markov state process $\{s_{it}\}_{t=0}^{\infty}$, where $s_{it} \in S_i$. where s_{i0} is known, and the process is common knowledge. The process is independent across agents. Let $S \equiv \underset{i=1}{\overset{N}{\times}} S_i$ and $s_t \equiv \{s_{1t}, ..., s_{Nt}\} \in S$ be the state of nature in the economy, let $s^t \equiv \{s_0, ..., s_t\}$ be the history of these realizations. In the following, let $x_t(s^t) \equiv (x_{1t}(s^t), ..., x_{Nt}(s^t))$ for any generic variable x.

Each agent exerts a costly action $a_{it}(s^t)$, which is unobservable to other players. This action affects next period distribution of states of nature: let $\pi(s_{i,t+1} \mid s_{it}, a_{it}(s^t))$ be the probability that state is $s_{i,t+1}$ conditional on past state and effort exerted by the agent in period t. Therefore, since the processes are independent across agents, we can define $\Pi(s^{t+1} \mid s_0, a^t(s^t)) = \prod_{i=1}^{N} \prod_{j=0}^{t} \pi(s_{i,j+1} \mid s_{ij}, a_{ij}(s^j))$ to be the cumulated probability of an history s^{t+1} given the whole history of unobserved actions $a^t(s^t) \equiv (a_0(s^0), a_1(s^1), ..., a_t(s^t))$. I assume $\pi(s_{i,t+1} \mid s_{it}, a_{it}(s^t))$ is differentiable in $a_{it}(s^t)$ as many time as necessary, and I denote its derivative with respect to $a_{it}(s^t)$ as $\pi_{a_i}(s_{i,t+1} \mid s_{it}, a_{it}(s^t))$.

The utility of each agent is

$$u\left(c_{it}\left(s^t\right)\right)-\upsilon\left(a_{it}\left(s^t\right)\right)$$

and the resource constraint of the economy is:

$$\sum_{i=1}^{N} c_{it}\left(s^t\right) \leq \sum_{i=1}^{N} y_{it}\left(s_{it}\right) \tag{1.14}$$

where $y_{it}\left(s_{it}\right)$ is the stochastic endowment of each agent.
A feasible contract is a sequence $\left(a^{\infty}, c^{\infty}\right) \equiv \left\{c_t\left(s^t\right), a_t\left(s^t\right)\right\}_{t=0}^{\infty}$ such that (1.14) is satisfied. Therefore

$$\begin{aligned}\Gamma^{MA} \;\; \equiv \;\; & \Big\{\left(a^{\infty}, c^{\infty}\right): a_t\left(s^t\right) \in A, \quad c_t\left(s^t\right) \geq 0, \\ & \left.\sum_{i=1}^{N} c_{it}\left(s^t\right) \leq \sum_{i=1}^{N} y_{it}\left(s_{it}\right) \quad \forall s^t \in S^{t+1}, t \geq 0\right\}\end{aligned}$$

Let $\omega \equiv \left\{\omega_i\right\}_{i=1}^{N}$ be a vector of weights, and assume MLRC and CDFC are satisfied in this economy. Since first-order condition with respect to effort for each agent is the same as in Section 1.2, the constrained efficient allocation is the solution of the following maximization problem:

$$P\left(s_0\right)=\max_{\left\{c_t\left(s^t\right), a_t\left(s^t\right)\right\}_{t=0}^{\infty}}\left\{\sum_{i=1}^{N} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \omega_i\left[u\left(c_{it}\left(s^t\right)\right)-\upsilon\left(a_{it}\left(s^t\right)\right)\right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)\right\}$$

$$\begin{aligned} s.t. \quad \upsilon'\left(a_{it}\left(s^t\right)\right) = & \sum_{j=1}^{\infty} \sum_{s^{t+j} \mid s^t} \beta^j \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}\left[u\left(c_{i,t+j}\left(s^{t+j}\right)\right)-\right. \\ & \left.-\upsilon\left(a_{i,t+j}\left(s^{t+j}\right)\right)\right] \Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1}\right)\right) \\ & \forall i=1, \ldots, N \end{aligned}$$

$$\sum_{i=1}^{N} c_{it}\left(s^t\right)=\sum_{i=1}^{N} y_i\left(s_{it}\right)$$

Let $\beta^t \lambda_{it}\left(s^t\right)$ be the Lagrange multiplier for the incentive-compatibility constraint of agent i. We can substitute for the resource constraint, and write the Lagrangean as:

$$\begin{aligned} & L\left(s_0, \omega, c^{\infty}, a^{\infty}, \lambda^{\infty}\right)= \\ & =\sum_{i=1}^{N} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t\left\{\phi_{it}\left(s^t\right)\left[u\left(c_{it}\left(s^t\right)\right)-\upsilon\left(a_{it}\left(s^t\right)\right)\right]-\right. \\ & \left.-\lambda_{it}\left(s^t\right) \upsilon'\left(a_{it}\left(s^t\right)\right)\right\} \Pi\left(s^t \mid h_0, a^{t-1}\left(s^{t-1}\right)\right) \end{aligned}$$

where

$$\begin{aligned} \phi_{i,t+1}\left(s^t, s_{t+1}\right) & = \phi_{it}\left(s^t\right)+\lambda_{it}\left(s^t\right) \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)} \\ \phi_{i0}\left(s_0\right) & = \omega_i \end{aligned}$$

The new variables $\phi_{it}(s^t)$, $i = 1, ..., N$, are endogenously evolving Pareto-Negishi weights which have the same interpretation as in previous Sections. Chapter 2 characterizes in detail the optimal allocation, therefore I refer the reader to it for an economic analysis of this model.
It is possible to prove the recursivity of this problem in the space (s_t, ϕ_t) of endogenous Pareto weights and state of nature (see Appendix D for details). Due to the homogeneity properties of the value function, the relevant state space can be reduced:

$$\frac{1}{\phi_1} J(s, \phi_1, ..., \phi_N) = J\left(s, 1, \frac{\phi_2}{\phi_1}, ..., \frac{\phi_N}{\phi_1}\right) \equiv \widetilde{J}\left(s, \frac{\phi_2}{\phi_1}, ..., \frac{\phi_N}{\phi_1}\right)$$

therefore we only need $N-1$ endogenous states.
In Section 1.6 I show a numerical example for the case of 2 agents, which is examined in detail by Zhao (2007) with APS techniques. With respect to the latter, my approach is much simpler, since the endogenous states are summarized by the ratio of the Pareto weights $\theta \equiv \frac{\phi_2}{\phi_1}$ (Appendix D shows how to rewrite Lagrangean first-order conditions in terms of the ratio θ). Therefore, in terms of numerical complexity, solving this model with my approach has the same difficulty of a stochastic neoclassical growth model.

1.6 Numerical simulations: a new algorithm

In this Section, I present some numerical examples of the models described in previous Sections. The main advantage with respect to APS is that I do not need to characterize the feasible set for continuation values, which allows me to solve models with a large number of state variables. All the examples presented here are not calibrated.

a The algorithm

For simplicity, I assume that the Markov process has only two possible realizations $(s^L < s^H)$. I also assume there is no persistence across time (i.e., the state is i.i.d.), and I use the simpler notation $\pi(a_t) = \pi\left(s_{t+1} = s^H \mid a_t\right)$. The numerical procedure is a collocation algorithm (see Judd (1998)) over the first-order conditions of the Lagrangean. From the recursive formulation we know that policy functions depend on the natural states of the problem and on the costates (i.e., Pareto weights) that come out from the Lagrangean approach. Let ς be the vector of allocations, χ be the vector of Lagrange multipliers, $x \in X$ be the vector of natural states, and $\theta \in \Theta$ be the vector of costates, and define $R(s, \varsigma, \chi, x, \theta)$ as the objective function in the saddle point functional equation, and $r(s, \varsigma, \chi, x, \theta)$ as the instantaneous utility function for the agent. We therefore proceed as follows:

1. Fix γ and define a discrete grid $G \subset S \times X \times \Theta$ for natural states and costates.

2. Approximate policy functions for allocations ς and Lagrange multipliers χ, the value function of the principal J and the continuation value of the agent U using cubic splines (or Chebychev polynomials, depending on the application), and set initial conditions for the approximation parameters[13]

3. For any $(s, x, \theta) \in G$, use a nonlinear solver[14] to solve for the Lagrangean first order conditions and the following two equations for the continuation value U and the value function J:

$$\begin{aligned} U(s, x, \theta) &= r(s, \varsigma, \chi, x, \theta) + \\ &\quad +\beta\left[\pi(a) U\left(s^H, x'^H, \theta'^H\right) + (1-\pi(a)) U\left(s^L, x'^L, \theta'^L\right)\right] \quad (1.15) \\ J(s, x, \theta) &= R(s, \varsigma, \chi, x, \theta) + \\ &\quad +\beta\left[\pi(a) J\left(s^H, x'^H, \theta'^H\right) + (1-\pi(a)) J\left(s^L, x'^L, \theta'^L\right)\right] \quad (1.16) \end{aligned}$$

I use the Miranda-Fackler Compecon toolbox for function approximation. I check the degree of approximation by calculating the residuals of the Lagrangean first order conditions (i.e., how much the numerically approximated first-order conditions are different from zero) on a grid $G^{test} \subseteq G$ with many gridpoints. In all applications, steps 1-3 are applied first to a grid with very few gridpoints, and then I increase the precision of the approximation by applying steps 1-3 to a finer grid. In general, a good approximation is obtained with few gridpoints. The algorithm is coded in Matlab.

Repeated moral hazard

In order to make the algorithm clear, I provide a detailed example of the procedure in the case of a standard repeated moral hazard setup. I simplify the notation by writing a generic variable as x_t instead of $x_t(s^t)$. I assume that the income process has two possible realizations ($y^L = y\left(s^L\right)$ and $y^H = y\left(s^H\right)$). I also assume there is no persistence across time (i.e., the state is i.i.d.).

The Lagrangean becomes:

$$L = E_0^a \sum_{t=0}^{\infty} \beta^t \left\{(y_t - c_t) + \phi_t\left[u(c_t) - \upsilon(a_t)\right] - \lambda_t \upsilon'(a_t)\right\}$$

[13] In the next subsection, I provide a clarification about why it is important to parametrize the value function and the continuation value, by means of an example.

[14] In all applications presented in this paper, I use a version of the Broyden algorithm coded by Michael Reiter.

with

$$\begin{aligned}
\phi_{t+1}^H &= \phi_t + \lambda_t \frac{\pi_a(a_t)}{\pi(a_t)} \\
\phi_{t+1}^L &= \phi_t - \lambda_t \frac{\pi_a(a_t)}{1-\pi(a_t)}
\end{aligned}$$

$$\phi_0\left(s^0\right) = \gamma$$

where E_t^a is the expectation operator over histories induced by the probability distribution $\pi(a_t)$. The first-order conditions can be rewritten as

$$c_t: \quad u'(c_t) = \frac{1}{\phi_t} \tag{1.17}$$

$$\begin{aligned}
a_t \quad : \quad & 0 = -\lambda_t \upsilon''(a_t) - \phi_t \upsilon'(a_t) + \\
& +\pi_a(a_t)\beta E_{t+1}^a \left\{ \sum_{j=1}^{\infty} \beta^{j-1} \left\{ (y_{t+j} - c_{t+j}) - \lambda_{t+j}\upsilon'(a_{t+j}) \right.\right. \\
& \left.\left. +\phi_{t+j}\left[u(c_{t+j}) - \upsilon(a_{t+j})\right]\right\} \mid y_{t+1} = y^H \right\} + \\
& -\pi_a(a_t)\beta E_{t+1}^a \left\{ \sum_{j=1}^{\infty} \beta^{j-1} \left\{ (y_{t+j} - c_{t+j}) - \lambda_{t+j}\upsilon'(a_{t+j}) \right.\right. \\
& \left.\left. +\phi_{t+j}\left[u(c_{t+j}) - \upsilon(a_{t+j})\right]\right\} \mid y_{t+1} = y^L \right\} + \\
& +\beta\lambda_t \pi(a_t) \frac{\partial\left(\frac{\pi_a(a_t)}{\pi(a_t)}\right)}{\partial a_t} \left[u(c_{t+1}) - \upsilon(a_{t+1}) \mid y_{t+1} = y^H\right] + \\
& +\beta\lambda_t (1-\pi(a_t)) \frac{\partial\left(\frac{-\pi_a(a_t)}{1-\pi(a_t)}\right)}{\partial a_t} \left[u(c_{t+1}) - \upsilon(a_{t+1}) \mid y_{t+1} = y^L\right]
\end{aligned} \tag{1.18}$$

and

$$\begin{aligned}
\lambda_t \quad : \quad & 0 = -\upsilon'(a_t) + \pi_a(a_t)\beta E_{t+1}^a \left\{ \sum_{j=1}^{\infty} \beta^{j-1} \left[u(c_{t+j}) - \upsilon(a_{t+j}) | y_{t+1} = y^H\right] \right\} + \\
& -\pi_a(a_t)\beta E_{t+1}^a \left\{ \sum_{j=1}^{\infty} \beta^{j-1} \left[u(c_{t+j}) - \upsilon(a_{t+j}) | y_{t+1} = y^L\right] \right\}
\end{aligned} \tag{1.19}$$

Notice that

$$\begin{aligned}
& J\left(y^i, \phi_{t+1}^i\right) = E_{t+1}^a \left\{ \sum_{j=1}^{\infty} \beta^{j-1} \left\{ (y_{t+j} - c_{t+j}) - \lambda_{t+j}\upsilon'(a_{t+j}) \right.\right. \\
& \left.\left. +\phi_{t+j}\left[u(c_{t+j}) - \upsilon(a_{t+j})\right]\right\} \mid y_{t+1} = y^i \right\} \\
& \qquad\qquad\qquad\qquad i = H, L
\end{aligned}$$

and

$$U\left(y^i,\phi_{t+1}^i\right)=E_{t+1}^a\left\{\sum_{j=1}^{\infty}\beta^{j-1}\left[u\left(c_{t+j}\right)-\upsilon\left(a_{t+j}\right)|y_{t+1}=y^i\right]\right\}$$
$$i=H,L$$

Therefore we can rewrite (1.18) and (1.19) as

$$\begin{aligned}
a_t \quad &: \quad 0=-\lambda_t\upsilon''\left(a_t\right)-\phi_t\upsilon'\left(a_t\right)+\beta\pi_a\left(a_t\right)\left[J\left(y^H,\phi_{t+1}\right)-J\left(y^L,\phi_{t+1}\right)\right]+ \qquad (1.20)\\
&+\beta\lambda_t\left\{\pi\left(a_t\right)\frac{\partial\left(\frac{\pi_a(a_t)}{\pi(a_t)}\right)}{\partial a_t}\left[u\left(c_{t+1}\right)-\upsilon\left(a_{t+1}\right)\mid y_{t+1}=y^H\right]+\right.\\
&\left.+\left(1-\pi\left(a_t\right)\right)\frac{\partial\left(\frac{-\pi_a(a_t)}{1-\pi(a_t)}\right)}{\partial a_t}\left[u\left(c_{t+1}\right)-\upsilon\left(a_{t+1}\right)\mid y_{t+1}=y^L\right]\right\}
\end{aligned}$$

$$\lambda_t: \quad 0=-\upsilon'\left(a_t\right)+\beta\pi_a\left(a_t\right)\left[U\left(y^H,\phi_{t+1}^H\right)-U\left(y^L,\phi_{t+1}^L\right)\right] \qquad (1.21)$$

I fix γ and I choose a discrete grid for ϕ_t that contains γ. I approximate with cubic splines a, λ, U and J on each grid node. I get consumption directly from ϕ by using (1.17): $c=u'^{-1}\left(\phi^{-1}\right)$. Finally I solve the system of nonlinear equations that includes (1.20), (1.21), (1.15) and (1.16).
I choose the following functional forms:

$$\begin{aligned}
u\left(c\right) &= \frac{c^{1-\sigma}}{1-\sigma}\\
\upsilon\left(a\right) &= \alpha a^{\varepsilon}\\
\pi\left(a\right) &= a^{\nu}, \quad a\in(0,1)
\end{aligned}$$

The baseline parameters are summarized in the table:

α	ε	ν	σ	y^L	y^H	β	γ
0.5	2	0.5	2	0	1	0.95	0.5955

The algorithm delivers a set of parameterized policy functions. The solution is obtained in around 5 seconds in a state-of-the-art laptop. Figure 1.1 shows consumption, effort, the next period Pareto weights and the ICC Lagrange multiplier as functions of the current state ϕ. As we already said, consumption is increasing in ϕ, while effort is decreasing in the Pareto weight. Notice also that the policy functions for the Pareto weights satisfy Lemma 1. The Lagrange multiplier, interestingly, is an increasing function of the current state: as long as ϕ increases (i.e., as long as the realizations of high income is preponderant), the shadow cost of enforcing an incentive compatible allocation decreases.

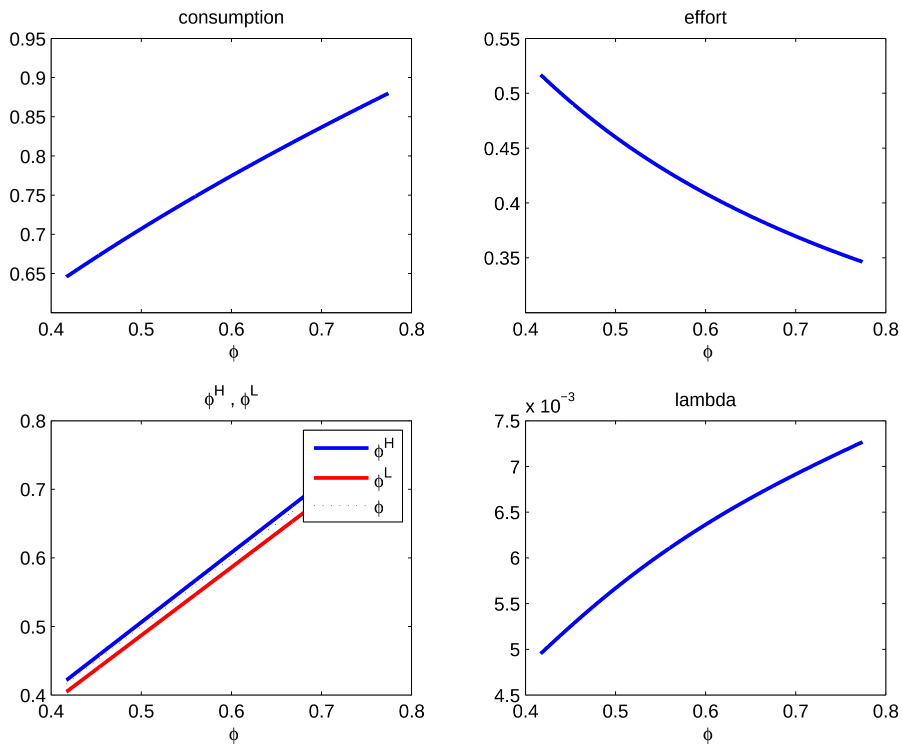

Figure 1.1: Pure moral hazard: policy functions

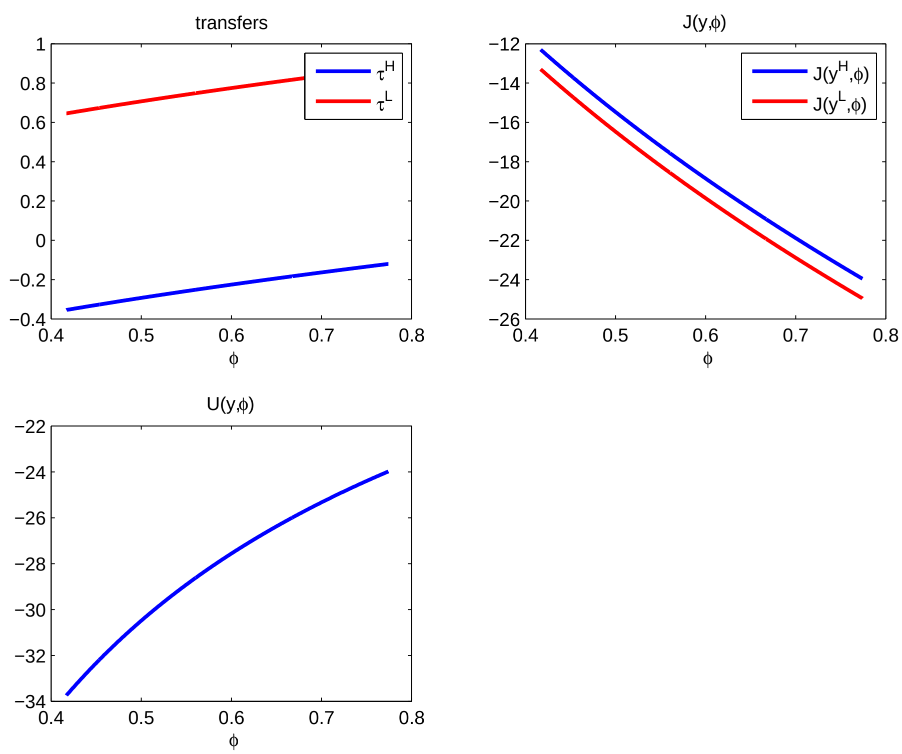

Figure 1.2: Pure moral hazard: policy functions (cont.)

Figure 1.2 represents the parameterized policy functions for transfers, continuation value of the agent and value function of the principal. Transfers are increasing in ϕ, as agent's lifetime utility; at the contrary, planner value is monotone decreasing in the Pareto weight.

Figure 1.3 and 1.4 show the average allocations across 50 thousands independent simulations for 200 periods, starting with $y_0 = y^H$. In general, these simulations are in line with previous studies[15]: average consumption decreases while effort, on the other hand, increases on average. As in Thomas and Worrall (1990), the average path for agent's lifetime utility is decreasing, while the Lagrange multiplier λ is reduced on average along the optimal path. Interestingly, ϕ does not show a monotone pattern. To understand the last plot of Figure 1.4, let us notice that it is possible to derive the asset holdings of the principal from optimal allocations (Appendix E shows the details): according to the simulations, average assets must decrease across time.

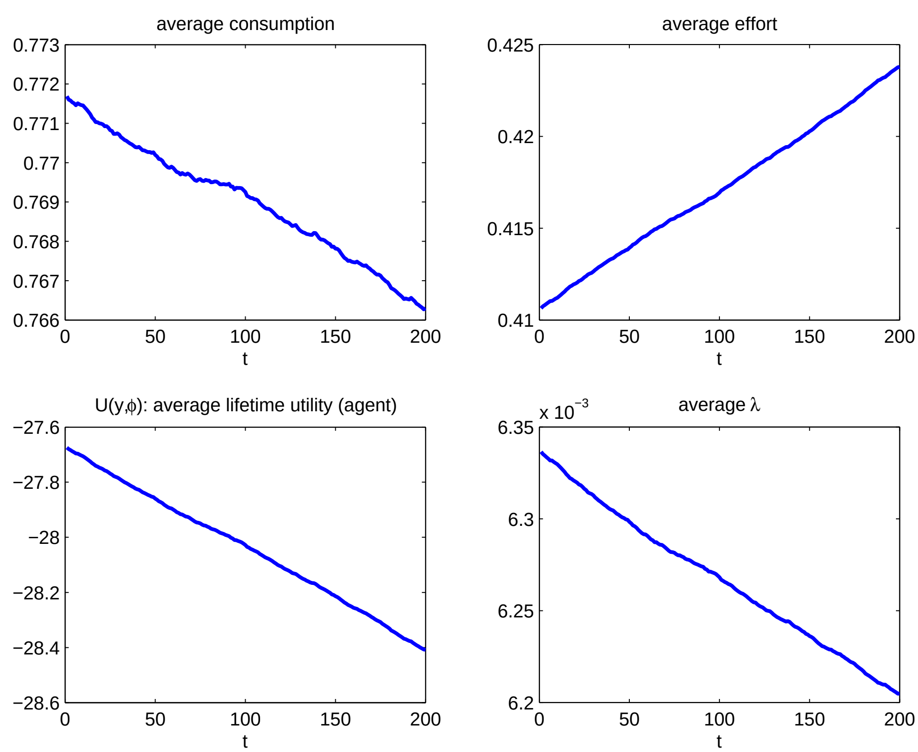

Figure 1.3: Pure moral hazard, average over 50000 independent simulations

Finally, Figure 1.5 shows the Pareto frontier: it is decreasing and strictly concave.

[15]The fact that each simulations starts with the same initial value for the shock explains the jump in period 1 for many series.

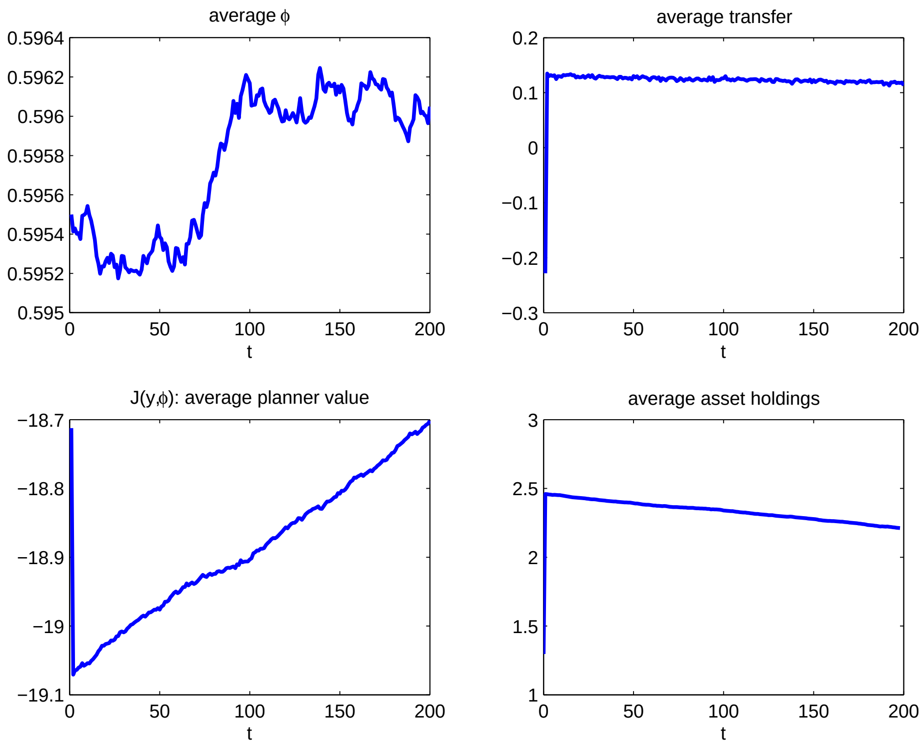

Figure 1.4: Pure moral hazard, average over 50000 independent simulations (cont.)

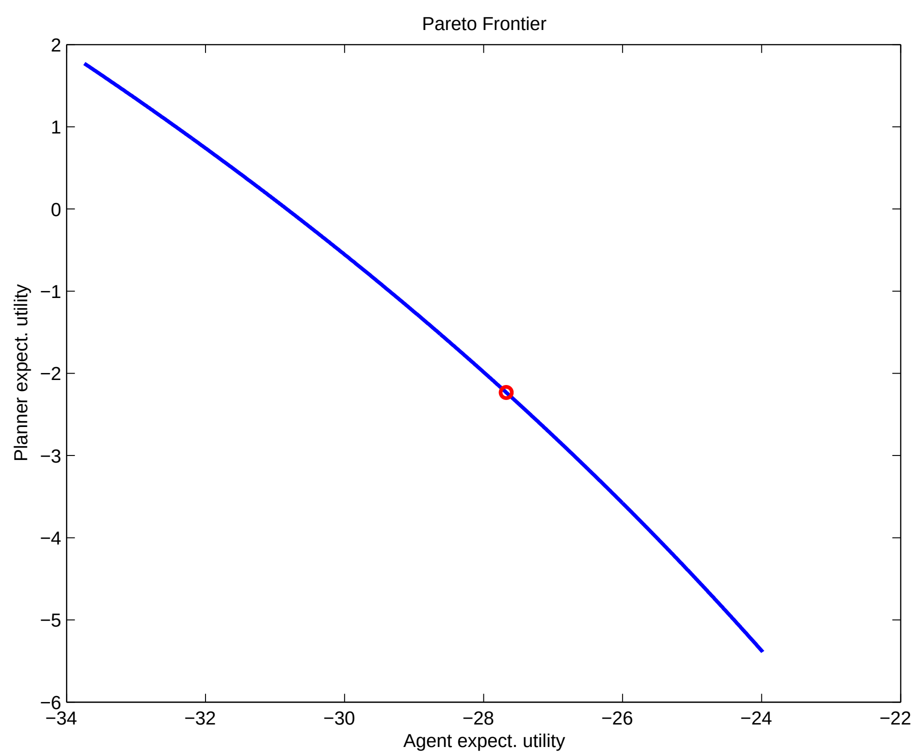

Figure 1.5: Pure moral hazard: Pareto frontier

An example with hidden assets

I maintain the same functional forms and parameters as in the previous example. The solution is obtained in around 15 seconds. Policy functions for consumption, agent lifetime utility and λ depicted in Figure 1.6 and 1.7 are strictly increasing and concave in both costates, while effort is strictly decreasing and convex.

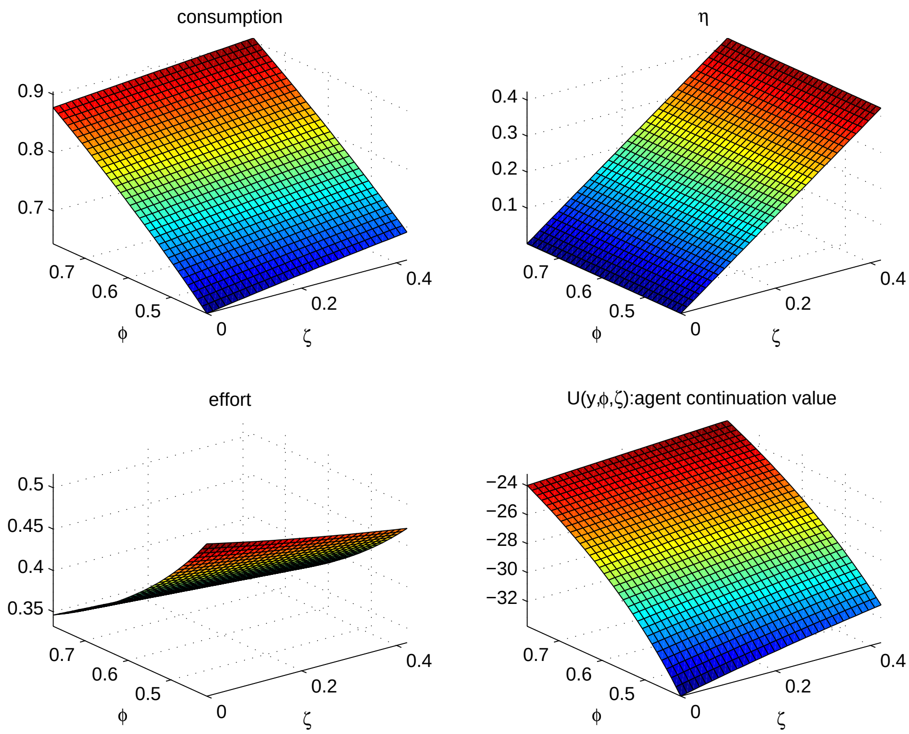

Figure 1.6: Moral hazard with hidden assets, policy functions

The simulated series in Figure 1.8 and 1.9 confirm the results in Abraham and Pavoni: on average, consumption and lifetime utility increase across time, while effort decreases. Asset holdings (see Appendix E to see how they are calculated) also increase on average.

Finally, Figure 1.10 shows the Pareto frontier for different ζ_0 (the natural one is zero): it is decreasing and strictly concave. An application of the verification procedure described in the Appendix C shows that the first-order approach is also correct.

An example of risk sharing with moral hazard

I assume that there are two identical agents and they have the same weight in the social welfare function, and I maintain the same functional forms and parameters, except for income realizations:

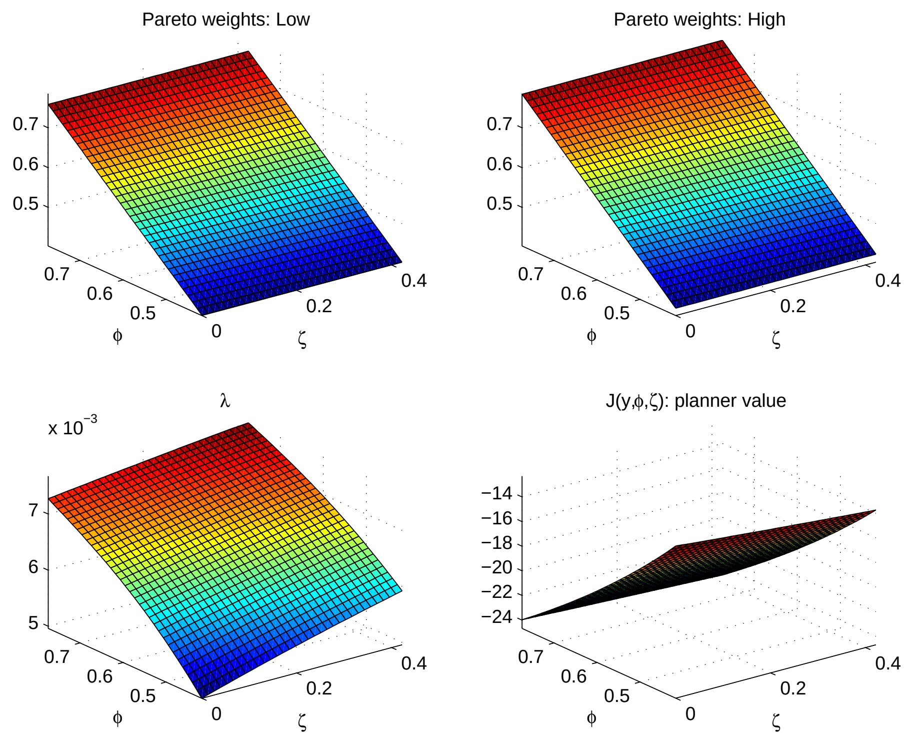

Figure 1.7: Moral hazard with hidden assets, policy functions (cont.)

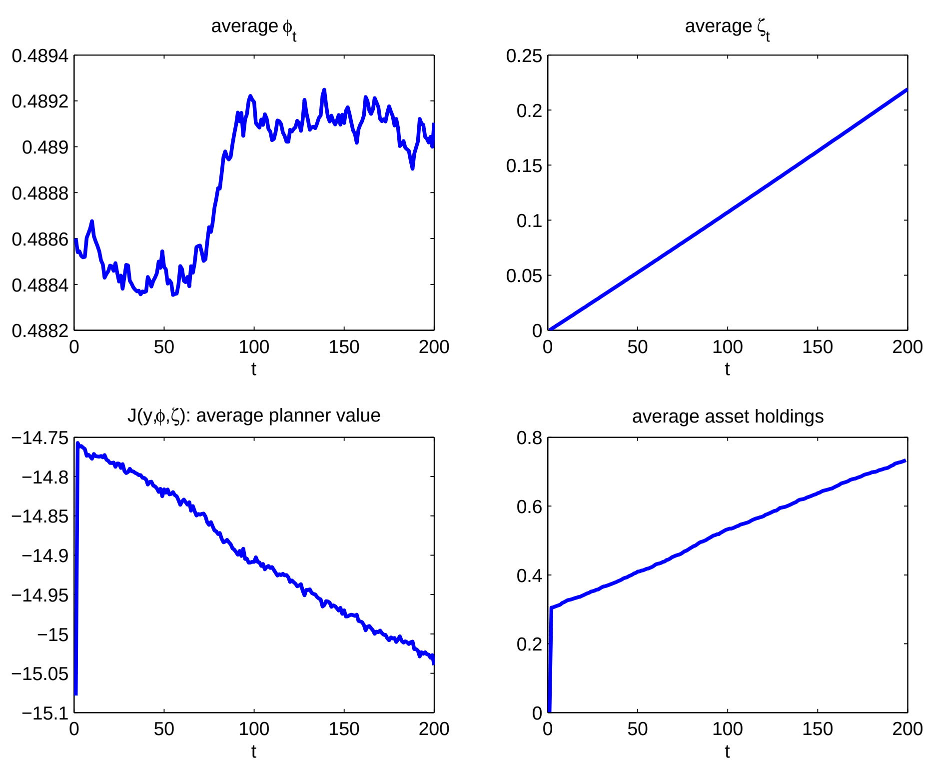

Figure 1.8: Moral hazard with hidden assets, average over 50000 independent simulations

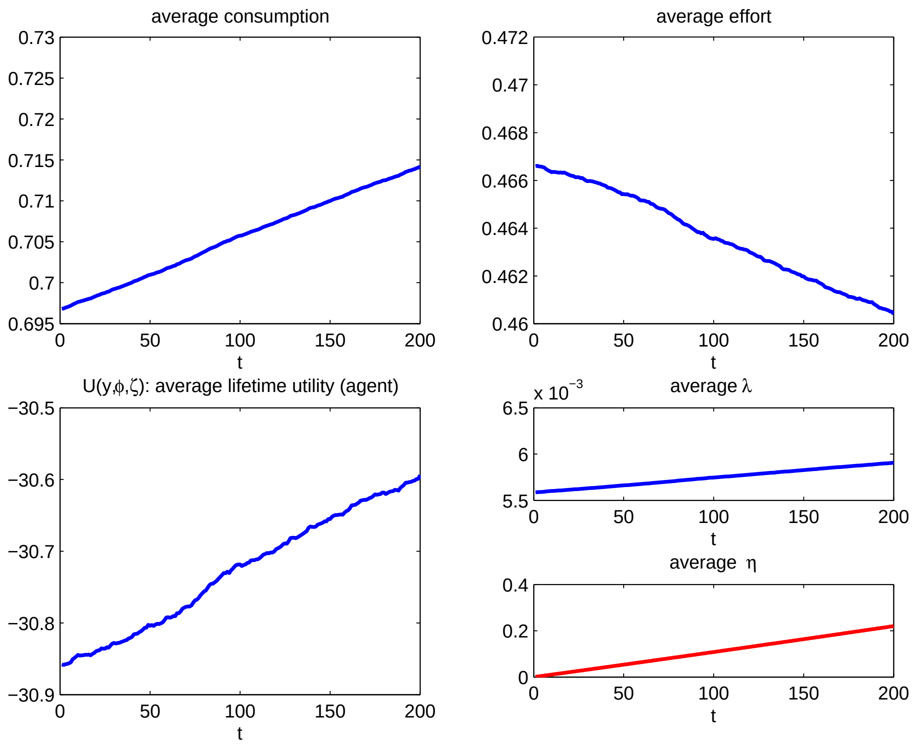

Figure 1.9: Moral hazard with hidden assets, average over 50000 independent simulations (cont.)

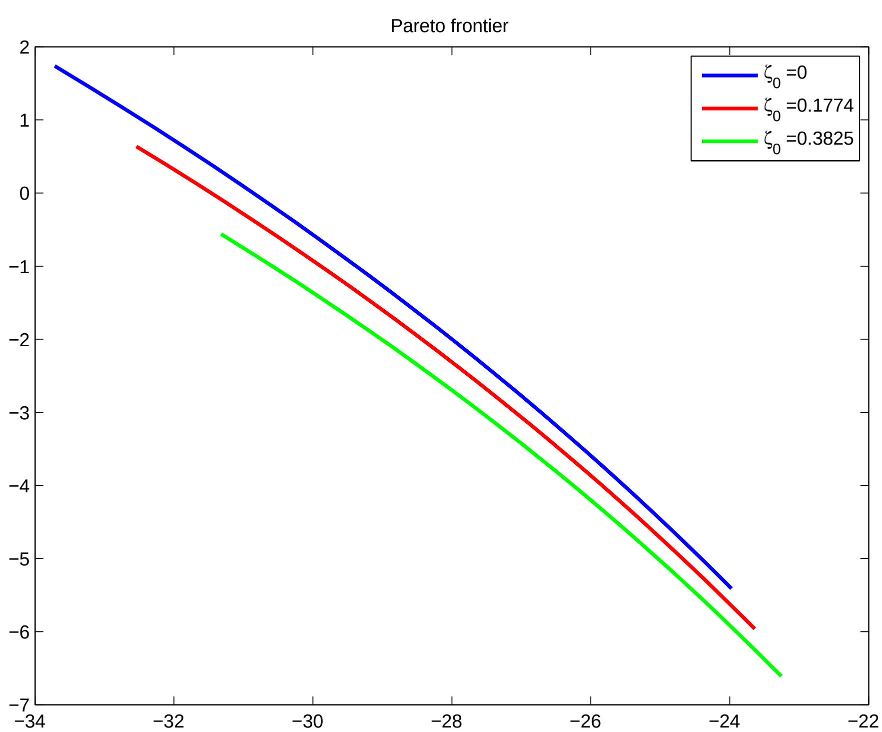

Figure 1.10: Pure moral hazard: Pareto frontier

α_i	ε_i	ν_i	σ_i	y_i^L	y_i^H	β	ω_i
0.5	2	0.5	2	.4	.6	0.95	0.5

Also in this case, my results are in line with the theoretical and numerical findings of Zhao (2007) and Friedman (1998). Remember from Section 1.5 that the relevant state is the ratio of endogenous Pareto weights $\theta \equiv \frac{\phi_2}{\phi_1}$. In this case, I get a solution in around 13 seconds. Figures 1.11 and 1.12 show that agent 1 consumption and lifetime utility are decreasing in θ for any possible state of the world while effort is increasing in θ, while the contrary happens to agent 2.

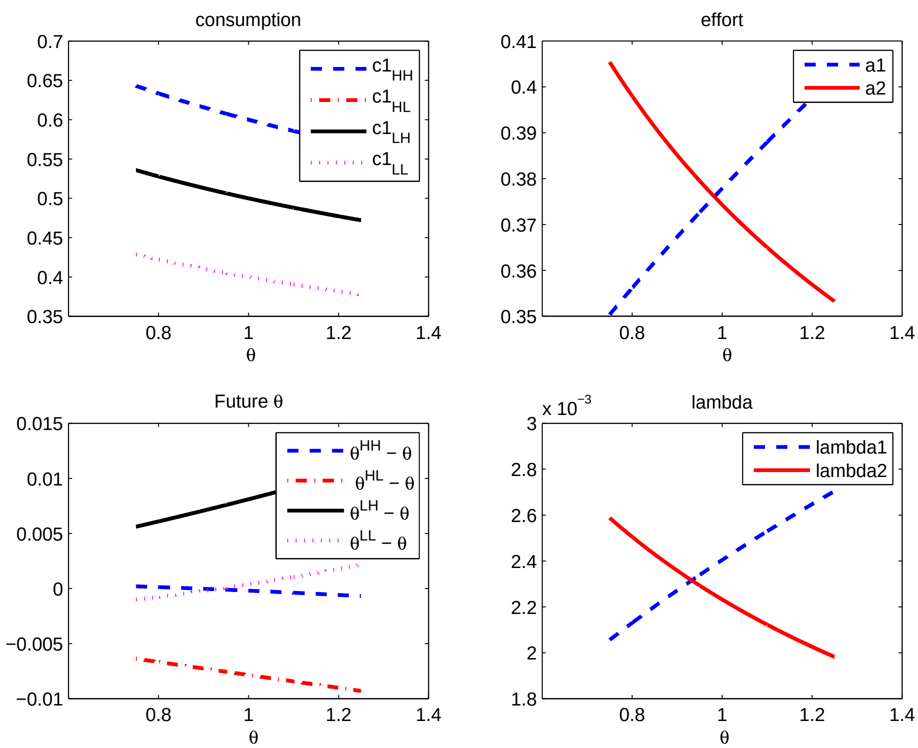

Figure 1.11: Risk sharing with moral hazard, policy functions (2 agents)

Given that $\theta = \frac{u'(c_1)}{u'(c_2)}$ (see first-order conditions in Appendix D), we can use it as a measure of consumption inequality: Figure 1.13 and 1.14 show a sample path of 200 periods.

Notice that θ is very persistent, confirming the theoretical result that θ evolves as a submartingale (see Chapter 2 for characterization of this property with the Lagrangean approach). Finally, Figure 1.15 shows a decreasing, strictly concave Pareto frontier.

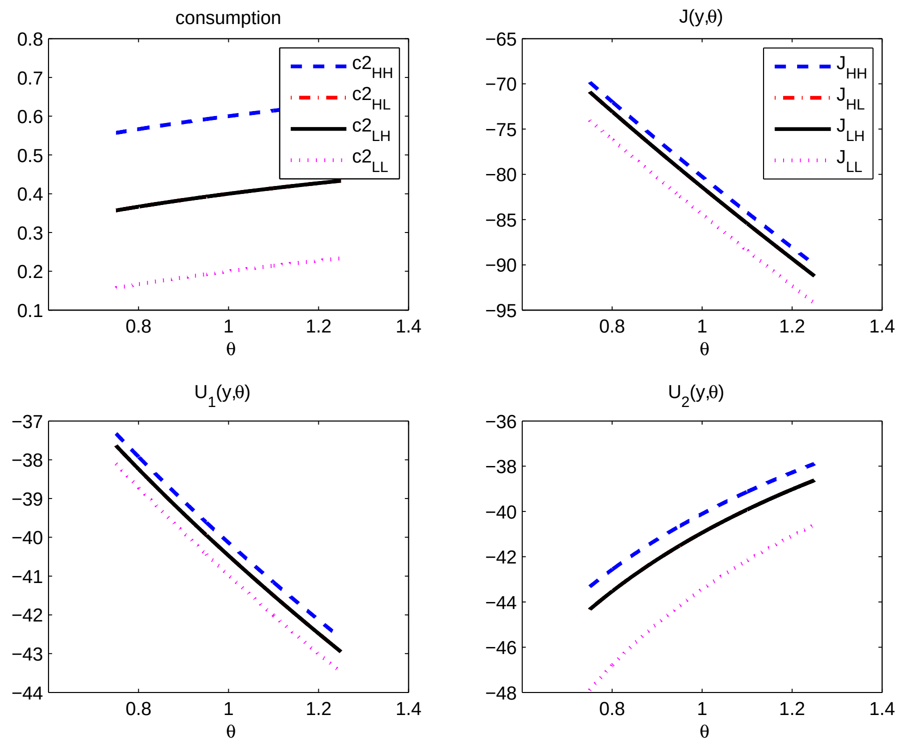

Figure 1.12: Risk sharing with moral hazard, policy functions (2 agents) (cont.)

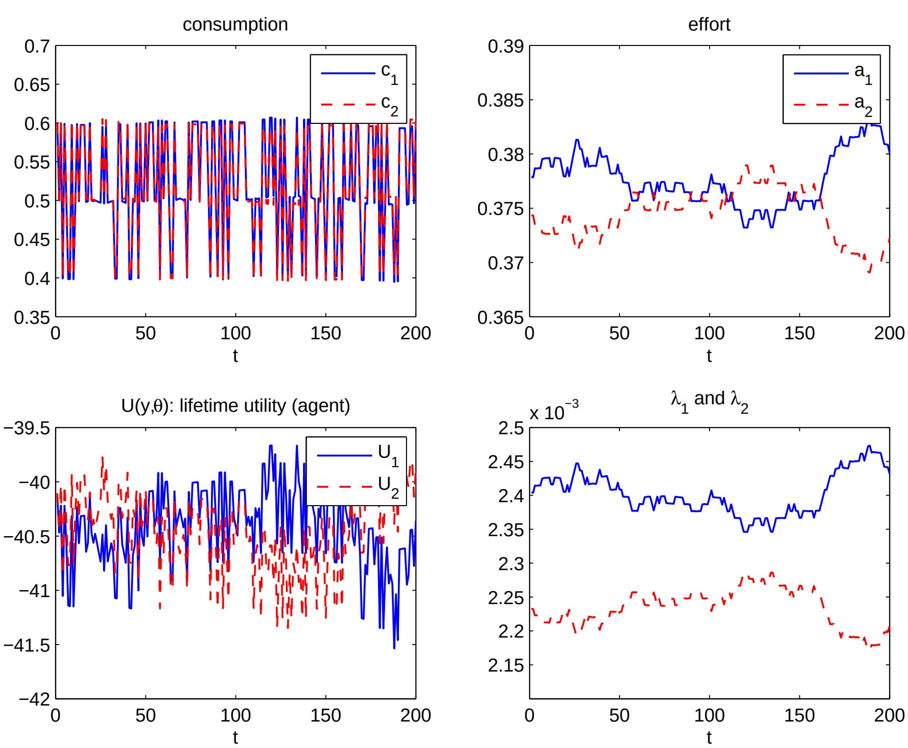

Figure 1.13: Risk sharing with moral hazard, sample path (2 agents)

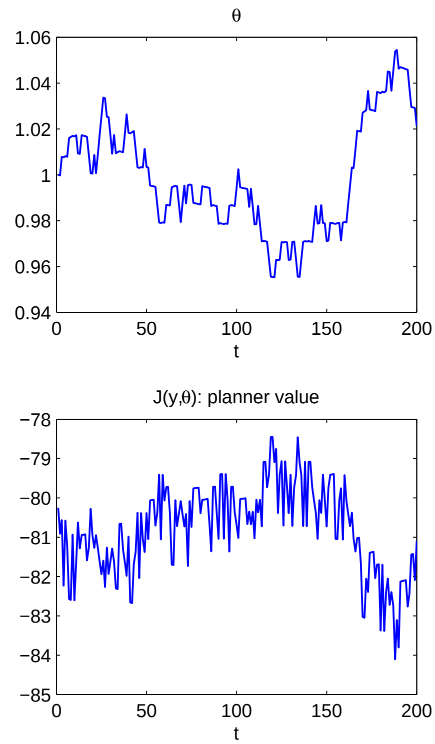

Figure 1.14: Risk sharing with moral hazard, sample path (2 agents) (cont.)

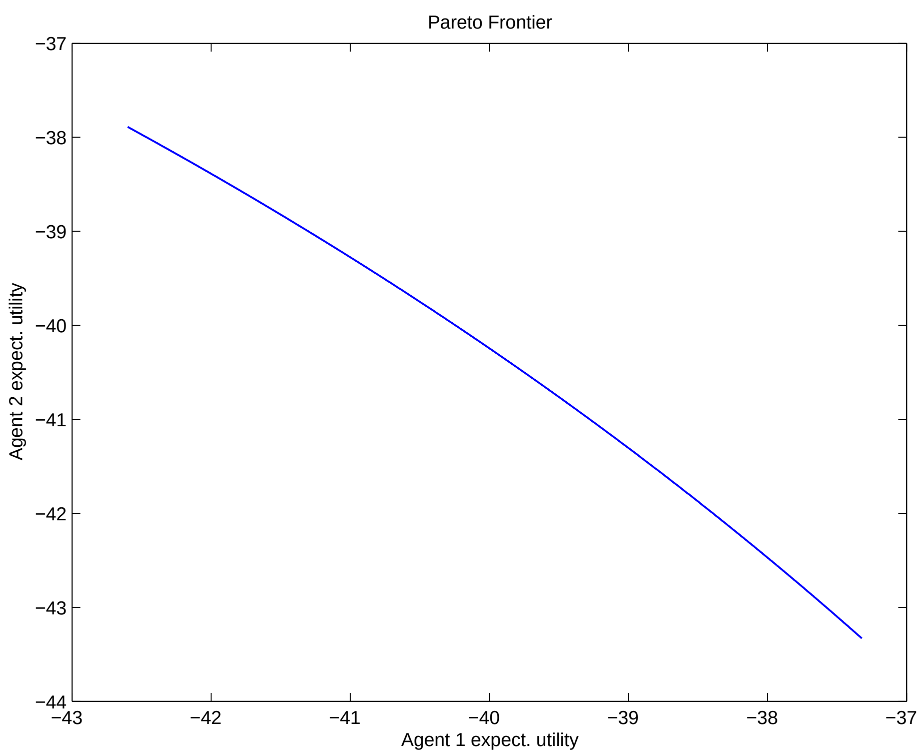

Figure 1.15: Risk sharing with moral hazard, Pareto frontier (2 agents)

1.7 Models that are untractable under APS techniques

In this Section, I provide two examples of models that are untractable with APS techniques.

a International risk sharing with moral hazard

Assume we live in a world with N countries. In each country, there is a representative entrepreneur endowed with an initial amount of capital $k_{i,-1}$, $i = 1, ..., N$ and with a production technology

$$y_{it}\left(s^{t}\right) = A_{i}\left(s_{i}^{t}\right) f\left(k_{i,t-1}\left(s^{t-1}\right)\right)$$

where $A_{i}\left(s_{i}^{t}\right)$ is a productivity shock and s_{i}^{t} is the history of states of the world in country i. Each entrepreneur enjoys consumption, and exerts unobservable effort that affects the productivity shock in the next period through the probability function $\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^{t}\right)\right)$. Assume the state of the world in each country is iid across time and across countries. Define $s_{t} = \left(s_{1t}, s_{2t}, ..., s_{Nt}\right)$. The budget contraint of entrepreneur of country i is:

$$c_{it}\left(s^{t}\right) + i_{it}\left(s^{t}\right) + \sum_{j\neq i}^{N} nx_{it}^{j}\left(s^{t}\right) \leq A_{i}\left(s^{t}\right) f\left(k_{i,t-1}\left(s^{t-1}\right)\right)$$

where $i_{it}\left(s^{t}\right)$ is investment in physical capital, and $nx_{it}^{j}\left(s^{t}\right)$ is net exports of country i with country j. The law of motion of capital is:

$$k_{it}\left(s^{t}\right) = i_{it}\left(s^{t}\right) + \left(1-\delta_{i}\right) k_{i,t-1}\left(s^{t-1}\right)$$

where δ_{i} is the depreciation rate of capital in country i. The world resource constraint becomes

$$\sum_{i=1}^{N} c_{it}\left(s^{t}\right) + \sum_{i=1}^{N} k_{it}\left(s^{t}\right) - \sum_{i=1}^{N}\left(1-\delta_{i}\right) k_{i,t-1}\left(s^{t-1}\right) \leq \sum_{i=1}^{N} A_{i}\left(s^{t}\right) f\left(k_{i,t-1}\left(s^{t-1}\right)\right)$$

$$k_{i,-1} \text{ given } \forall i = 1, ..., N$$

Notice that this model is an extension of the one presented in Section 1.5. The Pareto-constrained allocation can be found by solving:

$$P\left(s_0, \{k_{i,-1}\}_{i=1}^N\right) = \max_{\left\{\{c_{it}(s^t), a_{it}(s^t), k_{it}(s^t)\}_{i=1}^N\right\}_{t=0}^\infty} \left\{\sum_{i=1}^N \sum_{t=0}^\infty \beta^t \sum_{s^t} \omega_i \left[u\left(c_{it}\left(s^t\right)\right) - \right.\right.$$
$$\left.\left. -\upsilon\left(a_{it}\left(s^t\right)\right)\right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)\right\}$$

$$s.t. \quad \upsilon'\left(a_{it}\left(s^t\right)\right) = \sum_{j=1}^\infty \sum_{s^{t+j}|s^t} \beta^j \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)} \times \tag{1.22}$$
$$\left[u\left(c_{i,t+j}\left(s^{t+j}\right)\right) - \upsilon\left(a_{i,t+j}\left(s^{t+j}\right)\right)\right] \Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1}\right)\right)$$
$$\forall i = 1, ..., N$$

$$\sum_{i=1}^N c_{it}\left(s^t\right) + \sum_{i=1}^N k_{it}\left(s^t\right) - \sum_{i=1}^N \left(1-\delta_i\right) k_{i,t-1}\left(s^{t-1}\right) \leq$$
$$\leq \sum_{i=1}^N A_i\left(s^t\right) f\left(k_{i,t-1}\left(s^{t-1}\right)\right)$$
$$k_{i,-1} \text{ given } \forall i = 1, ..., N$$

We can now write down the Lagrangean of this problem:

$$L\left(\varsigma^\infty, \nu^\infty, k^\infty, \phi^\infty\right) =$$
$$= \sum_{i=1}^N \sum_{t=0}^\infty \sum_{s^t} \beta^t \left\{\phi_{it}\left(s^t\right)\left[u\left(c_{it}\left(s^t\right)\right) - \upsilon\left(a_{it}\left(s^t\right)\right)\right] - \right.$$
$$\left. -\lambda_{it}\left(s^t\right) \upsilon'\left(a_{it}\left(s^t\right)\right)\right\} \Pi\left(s^t \mid h_0, a^{t-1}\left(s^{t-1}\right)\right)$$

where

$$\begin{aligned} \phi_{i,t+1}\left(s^t, s_{t+1}\right) &= \phi_{it}\left(s^t\right) + \lambda_{it}\left(s^t\right) \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)} \\ \phi_{i0}\left(s_0\right) &= \omega_i \end{aligned}$$

A derivation of the properties of this model and some numerical examples are included in Chapter 2.

b CEO compensation

There is a growing literature on executive compensation schemes, which builds on the general principal-agent model. For example, Clementi et al (2006) shows in a two-period model how stock compensation to executives can be thought as a commitment device for the firm: since the firm cannot credibly commit to pay a severance package, then stock grants are used to give deferred compensation to the manager (because it is more difficult to default on stockholders' rights). Clementi et al. (2008a, 2008b) build a fully dynamic model with capital, and analyze some interesting properties of the model in line with data, but they do not include stocks as part of the compensation and they do not look at the no commitment case. The main difficulty in analyzing more realistic models is technical: the APS technique imposes a limit on the number of state variables that can be included in the setup.

In the following example, I describe a model of executive compensation with base salary, dividends and stocks for a firm that produces a good with capital, and where the manager has the option to leave each period. A continuum of risk-neutral investors own a firm which produces a good using capital

$$y_t\left(s^t\right) = A\left(s_t\right) f\left(k_{t-1}\left(s^{t-1}\right)\right)$$

where $A\left(s_t\right)$ is a productivity shock. The risk-averse manager of this firm can be compensated with base salary $w_t\left(s^t\right)$, and with stocks $\sigma_t\left(s^t\right) \leq 1$, where 1 is the total number of stocks. Manager's effort is unobservable, and affects the probability distribution of productivity shock $A\left(s_t\right)$. The feasibility constraint for the firm is:

$$w_t\left(s^t\right) + d_t\left(s^t\right) + i_t\left(s^t\right) \leq A\left(s_t\right) f\left(k_{t-1}\left(s^{t-1}\right)\right), \qquad k_{-1} \text{ given}$$

where $i_t\left(s^t\right)$ is investment in physical capital and $d_t\left(s^t\right)$ are distributed dividends. The law of motion for capital is:

$$k_t\left(s^t\right) = i_t\left(s^t\right) + (1-\delta) k_{t-1}\left(s^{t-1}\right)$$

where δ is the depreciation rate of capital. Combining feasibility and the law of motion for capital, we get the following resource constraint:

$$w_t\left(s^t\right) + d_t\left(s^t\right) + k_t\left(s^t\right) - (1-\delta) k_{t-1}\left(s^{t-1}\right) \leq A\left(s_t\right) f\left(k_{t-1}\left(s^{t-1}\right)\right)$$

I assume that the firm does not issue new stocks, and that manager's consumption is perfectly monitorable. The latter assumption is equivalent to assume that the manager receives income only from the firm and cannot save or invest in other stocks or bonds. Therefore manager's consumption $c_t\left(s^t\right)$ is given by:

$$c_t\left(s^t\right) = w_t\left(s^t\right) + \sigma_{t-1}\left(s^{t-1}\right) d_t\left(s^t\right) + p_t\left(s^t\right)\left(\sigma_t\left(s^t\right) - \sigma_{t-1}\left(s^{t-1}\right)\right)$$

where $p_t\left(s^t\right)$ is the price of stocks. Since each investor is risk-neutral, and owns $1-\sigma_t\left(s^t\right)$ stocks, then their budget constraint is

$$c_t^I\left(s^t\right) - p_t\left(s^t\right)\left(\sigma_t\left(s^t\right) - \sigma_{t-1}\left(s^{t-1}\right)\right) = d_t\left(s^t\right)\left(1 - \sigma_{t-1}\left(s^{t-1}\right)\right)$$

and therefore, by imposing a no-bubble condition:

$$\begin{aligned} p_t\left(s^t\right) &= \beta \sum_{s^{t+1}|s^t} \left[p_{t+1}\left(s^{t+1}\right) + d_{t+1}\left(s^{t+1}\right)\right] \pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right) \\ &= \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^t} \beta^j d_{t+j}\left(s^{t+j}\right) \Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1} \mid s^t\right)\right) \end{aligned}$$

The instantaneous utility of the manager is

$$u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right)$$

The set of feasible contracts is then

$$\begin{aligned} \Gamma^{CEO} \equiv \Big\{ & \left(a^\infty, c^\infty, w^\infty, i^\infty, d^\infty, \sigma^\infty, k^\infty\right) : a_t\left(s^t\right) \in A, \quad c_t\left(s^t\right) \geq 0, \quad k_t\left(s^t\right) \in K \subseteq \mathbb{R}_+, \\ & p_t\left(s^t\right)\left(\sigma_t\left(s^t\right) - \sigma_{t-1}\left(s^{t-1}\right)\right) \geq 0, \quad \sigma_t\left(s^t\right) \in [0,1], \\ & c_t\left(s^t\right) + k_t\left(s^t\right) - (1-\delta) k_{t-1}\left(s^{t-1}\right) \leq A\left(s_t\right) f\left(k_{t-1}\left(s^{t-1}\right)\right) \quad \forall s^t \in S^{t+1}, t \geq 0 \Big\} \end{aligned}$$

The problem of the manager is

$$\begin{aligned} \max_{\{c_t(s^t), a_t(s^t)\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \left[u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right)\right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) \\ s.t. \quad & c_t\left(s^t\right) \leq w_t\left(s^t\right) + \sigma_{t-1}\left(s^{t-1}\right) d_t\left(s^t\right) + p_t\left(s^t\right)\left(\sigma_t\left(s^t\right) - \sigma_{t-1}\left(s^{t-1}\right)\right) \end{aligned}$$

From this maximization, we obtain the following first-order condition for effort:

$$\begin{aligned} \upsilon_a'\left(a_t\left(s^t\right)\right) = \sum_{j=1}^{\infty} \beta^j \sum_{s^{t+j}|s^t} & \frac{\pi_a\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)}{\pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)} \times \\ & \times \left[u\left(c_{t+j}\left(s^{t+j}\right)\right) - \upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right] \Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1} \mid s^t\right)\right) \end{aligned}$$

I assume that in each period the manager has the option to quit; if he quits, he immediately finds another job in another firm which is of the same size (i.e., has the same capital) of the first one. The investor sets a contract that makes quitting undesirable for the manager, i.e. I add a participation constraint to the investor's problem. The Pareto-constrained allocation can be found by solving:

$$\begin{aligned} & W\left(s_0, k_{-1}, \sigma_{-1}\right) = \\ & \max_{\{w_t(s^t), a_t(s^t), d_t(s^t), \sigma_t(s^t), k_t(s^t)\}_{t=0}^{\infty} \in \Gamma^{CEO}} \Bigg\{ \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \Big[A\left(s_t\right) f\left(k_{t-1}\left(s^{t-1}\right)\right) - w_t\left(s^t\right) - \\ & \qquad -k_t\left(s^t\right) + (1-\delta) k_{t-1}\left(s^{t-1}\right) - \sigma_{t-1}\left(s^{t-1}\right) d_t\left(s^t\right) - \\ & \qquad -p_t\left(s^t\right)\left(\sigma_t\left(s^t\right) - \sigma_{t-1}\left(s^{t-1}\right)\right)\Big] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) \Bigg\} \end{aligned}$$

$$\begin{aligned} s.t. \quad \upsilon_a'\left(a_t\left(s^t\right)\right) = \sum_{j=1}^{\infty} \beta^j \sum_{s^{t+j}|s^t} & \frac{\pi_a\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)}{\pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)} \times \\ & \times \left[u\left(c_{t+j}\left(s^{t+j}\right)\right) - \upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right] \Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1} \mid s^t\right)\right) \end{aligned} \tag{1.23}$$

plus the following constraints

$$p_t\left(s^t\right)=\sum_{j=1}^{\infty}\sum_{s^{t+j}|s^t}\beta^j d_{t+j}\left(s^{t+j}\right)\Pi\left(s^{t+j}\mid s^t,a^{t+j-1}\left(s^{t+j-1}\mid s^t\right)\right) \tag{1.24}$$

$$w_t\left(s^t\right)+d_t\left(s^t\right)+k_t\left(s^t\right)-(1-\delta)\,k_{t-1}\left(s^{t-1}\right)\leq A\left(s_t\right)f\left(k_{t-1}\left(s^{t-1}\right)\right)$$
$$k_{-1}\text{ given}$$

$$c_t\left(s^t\right)\leq w_t\left(s^t\right)+\sigma_{t-1}\left(s^{t-1}\right)d_t\left(s^t\right)+p_t\left(s^t\right)\left(\sigma_t\left(s^t\right)-\sigma_{t-1}\left(s^{t-1}\right)\right)$$
$$\sigma_{-1}=0$$

$$\begin{aligned}&\sum_{j=0}^{\infty}\sum_{s^{t+j}|s^t}\beta^j\left[u\left(c_{t+j}\left(s^{t+j}\right)\right)-\upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right]\times\\&\qquad\times\Pi\left(s^{t+j}\mid s^t,a^{t+j-1}\left(s^{t+j-1}\mid s^t\right)\right)\geq V^{OUT}\left(s^t,k_{t-1}\left(s^{t-1}\right),0\right)\end{aligned} \tag{1.25}$$
$$\forall t\qquad\forall s^t$$

where $V^{OUT}\left(s^t,k_{t-1}\left(s^{t-1}\right),0\right)$ is the value for the manager of working in a new firm with the same capital as the one he leaves.

We associate a Lagrange multiplier $\beta^t\lambda_t\left(s^t\right)$ to any ICC constraint (1.23), $\beta^t\gamma_t\left(s^t\right)$ to the participation constraint (1.25), and $\beta^t\mu_t\left(s^t\right)$ to the price constraint (1.24). We can now write down the Lagrangean:

$$\begin{aligned}&L\left(s_0,w^\infty,a^\infty,d^\infty,\sigma^\infty,k^\infty,\lambda^\infty,\gamma^\infty,\mu^\infty\right)=\\&=\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left\{A\left(s_t\right)f\left(k_{t-1}\left(s^{t-1}\right)\right)-w_t\left(s^t\right)-k_t\left(s^t\right)+(1-\delta)\,k_{t-1}\left(s^{t-1}\right)+\right.\\&\qquad-\sigma_{t-1}\left(s^{t-1}\right)d_t\left(s^t\right)-p_t\left(s^t\right)\left(\sigma_t\left(s^t\right)-\sigma_{t-1}\left(s^{t-1}\right)\right)+\mu_t\left(s^t\right)p_t\left(s^t\right)-\zeta_t\left(s^t\right)d_t\left(s^t\right)+\\&\qquad+\left(\phi_t\left(s^t\right)+\gamma_t\left(s^t\right)\right)\left[u\left(c_t\left(s^t\right)\right)-\upsilon\left(a_t\left(s^t\right)\right)\right]+\\&\qquad\left.-\lambda_t\left(s^t\right)\upsilon_a'\left(a_t\left(s^t\right)\right)-\gamma_t\left(s^t\right)V^{OUT}\left(s^t,k_{t-1}\left(s^{t-1}\right),0\right)\right\}\Pi\left(s^t\mid h_0,a^{t-1}\left(s^{t-1}\right)\right)\end{aligned}$$

where

$$\begin{aligned}\phi_{t+1}\left(s^t,\widehat{s}\right)&=\phi_t\left(s^t\right)+\lambda_t\left(s^t\right)\frac{\pi_a\left(s_{t+1}=\widehat{s}\mid s_t,a_t\left(s^t\right)\right)}{\pi\left(s_{t+1}=\widehat{s}\mid s_t,a_t\left(s^t\right)\right)}+\gamma_t\left(s^t\right)\quad\forall\widehat{s}\in S\\\zeta_{t+1}\left(s^t,\widehat{s}\right)&=\zeta_t\left(s^t\right)+\mu_t\left(s^t\right)\quad\forall\widehat{s}\in S\end{aligned}$$

$$\zeta_0\left(s^0\right)=\phi_0\left(s^0\right)=0$$

As in the previous section, we can associate a saddle point functional equation to (a generalized version of) the Lagrangean, and show that the operator is a contraction (proof available upon request). In this case, therefore, the solution will be a Markovian policy function that depends on productivity shock, capital, stock holdings, the Pareto weight $\phi_t(s^t)$ and the costate $\zeta_t(s^t)$.

1.8 Conclusions

I have presented a Lagrangean approach to repeated moral hazard problems, and an algorithm which is much faster than the traditional APS techniques. My methodology allows the researcher to deal with models with many states, and to calibrate the simulated series to real data in a reasonable amount of time. A huge class of models which are untractable under standard techniques can be easily addressed with my approach.
This method has many possible applications. Given the speed, the algorithm can be useful (as a time-saving technique) also for those models that are tractable with traditional techniques, but computationally burdensome. Dynamic agency problems with hidden effort and hidden assets are a good example: while we have a good qualitative idea of the main predictions of this model, to the best of my knowledge we still do not have a quantitative assessment in a calibrated economy. Mainly this is due to numerical difficulties. The Lagrangean approach offers a chance to overcome these limits: we can easily calibrate models and match data, in order to better understand various issues as e.g. consumption-saving anomalies, optimal unemployment insurance with assets accumulation, DSGE models with financial frictions.
However, the main gain of the Lagrangean method can be seen in more complicated setups, which are untractable with current state-of-the-art algorithms. Models of repeated moral hazard with heterogeneous agents and endogenous states are a good example: they require to solve the problem of each agent and aggregate the resulting individual optimal choices, then iterating until a general equilibrium is found. APS techniques are unmanageable even with just two endogenous states, while for my approach it would be a simple computational task.
Other issues in which the Lagrangean approach has an advantage in terms of complexity of the framework are optimal taxation theory in economies with private information, models of entrepreneurial choice, and models of banking and credit markets.
There is a price to pay, though: either restricting the class of models we analyze, by imposing some mild assumptions on primitives (e.g., Condition 1 and 2 in this work), or in some cases by verifying numerically the optimality of the solution. In any case, these seem small costs compared to the benefit of analyzing issues that are at present unmanageable.

2 Dynamic Risk Sharing with Moral Hazard

2.1 Introduction

In this Chapter, I study optimal dynamic risk sharing between a pool of agents that exert hidden effort. I first analyze this problem in endowment economies, and then I extend the setup to a production economy with capital accumulation.

In a full information environment, an optimal contractual arrangement prescribes that agents pool their income and share it. However, in the case of multi-sided moral hazard, this would imply low incentives to exert effort. An optimal arrangement must then consider the trade-off between providing insurance and incentives.

Zhao (2007) analyzes the case of an endowment economy with two-sided moral hazard, and shows how to solve the model with recursive techniques based on the work of Abreu, Pearce and Stacchetti (1990). This approach is not easy to apply to production economies or even to endowment economies where the financial market is not monitorable by the planner. Mainly this is due to the complexities discussed in Chapter 1: as the number of state variables and agents increases, the APS approach becomes extremely burdensome for numerical simulations. Therefore, I will use the Lagrangean approach to study these models.

The characterization of the optimal allocation can be obtained by means of endogenously evolving Pareto-Negishi weights. We can see the planner problem as a social welfare maximization in which the planner observes the realization of income shocks (or productivity shocks in the production economy) and increases (respectively, decreases) the Pareto weight of each agent if the realization of the shock is "good" ("bad"). By optimally choosing these weights for each history, the planner makes sure to provide enough incentives to each agent.

I show that the problem has a recursive structure: Pareto weights are new state variables that keep track of the history of shocks' realizations. Therefore, policy functions depend on these Pareto weight, a fact which is exploited in the numerical examples.

First, I characterize the optimal contract in a simple endowment economy. I show that the ratio of Pareto weights for each couple of agents evolves as a submartingale. The main implication is that also the ratio of marginal utilities of consumption for each couple of agents evolves as a submartingale. Therefore, consumption inequality is very persistent in this framework [1].

Once I have characterized the optimal allocations in the simple endowment economy, I extend the setup to allow for hidden access to financial markets. Each agent can trade in a risk-free bond at market price, and these trades are not observable by the planner[2].

[1]While the long run behavior of inequality and therefore the characterization of the limit behavior of the submartingale is interesting *per se*, it is beyond the scope of this work and therefore it is left for future research.

[2]In a model with private information over labor productivity shocks, Golosov and Tsivinski (2007)

In this case, marginal rates of substitution between consumption today and tomorrow must be equalized across agents. This implies that the previous submartingale result for the ratio of marginal utility of consumption is not valid anymore. Indeed, the ratio of Pareto weights is still a submartingale, but this ratio is not equal to the consumption marginal utilities ratio anymore. Instead, it is possible to show that the planner wants to put a wedge between the ratio of Pareto weights and the consumption marginal utilities in order to distort the consumption/saving decisions of the agents[3].

Finally, I consider a production economy. In this case, the distribution of productivity shocks is affected by hidden effort. The submartingale result here applies to the ratio of marginal utilities of consumption multiplied by a wedge that depends on the return of capital. This property has effects on capital accumulation. In particular, numerical simulations in a 2-agents economy show that steady state capital diverge on average: one agent has higher capital than the other, also if they are identical at time zero in terms of preferences and capital endowment.

This work is very related to Friedman (1998), who studies the same model of Zhao (2007), and shows that the planner problem has a recursive structure in the space of Pareto weights, also if he does not directly apply the Lagrangean approach. His work is mainly theoretical, though, and it does not provide any numerical example. Finally, he does not exploit the homogeneity properties of the value function to reduce the dimensionality of the state space, as I do here.

The Chapter is organized as follows: Section 2.2 presents a model of risk sharing with repeated moral hazard in an endowment economy. Section 2.3 extends the basic setup to allow for hidden trades in the financial market. Section 2.4 analyze production economy with capital accumulation. Section 2.5 presents few numerical examples of simulated 2-agents economies, and Section 2.6 concludes.

2.2 An endowment economy

There are N agents indexed by $i = 1, ..., N$.[4]. Each agent receive a stochastic endowment, governed by an observable Markov state process $\{s_{it}\}_{t=0}^{\infty}$, where $s_{it} \in S_i$. I assume s_{i0} is known, and the process is common knowledge. I will denote with subscripts the single realizations, and with superscripts the whole histories of states:

$$s_i^t \equiv \{s_{i0}, ..., s_{it}\}$$

show, with numerical simulations, that private insurance markets can provide almost efficient levels of insurance without public intervention. A quantitative exploration of the efficiency of private insurance markets in this framework with hidden effort is left for future research

[3]In a decentralization of the constrained optimal allocation, this wedge will translate in a saving tax or subsidy

[4]In particular, in the numerical computations I work with $N = 2$ for simplicity.

I also assume that the processes are independent across agents. Let $s_t \equiv \{s_{1t}, ..., s_{Nt}\}$ be the state of nature in the economy, let $s^t \equiv \{s_0, ..., s_t\}$ be the history of their realizations.
The agent exerts a costly action $a_{it}(s^t)$, which is unobservable to other players. This action affects next period distribution of states of nature: let $\pi(s_{i,t+1} \mid s_{it}, a_{it}(s^t))$ be the probability that state is $s_{i,t+1}$ conditional on past state and effort exerted by the government in period t. Therefore, since the processes are independent across agents, we can define $\Pi(s^{t+1} \mid s_0, a^t(s^t)) = \prod_{i=1}^N \prod_{j=0}^t \pi(s_{i,j+1} \mid s_{ij}, a_{ij}(s^j))$ to be the cumulated probability of an history s^{t+1} given the whole history of unobserved actions $a^t(s^t) \equiv (a_0(s^0), a_1(s^1), ..., a_t(s^t))$. I assume $\pi(s_{i,t+1} \mid s_{it}, a_{it}(s^t))$ is differentiable in $a_{it}(s^t)$ as many time as necessary, and I denote its derivative with respect to $a_{it}(s^t)$ as $\pi_{a_i}(s_{i,t+1} \mid s_{it}, a_{it}(s^t))$. I also assume that $\pi(s_{i,t+1} \mid s_{it}, a_{it}(s^t))$ is increasing in $a_{it}(s^t)$ for "good" states, and decreasing for bad states[5].
The utility of the agent is

$$u\left(c_{it}\left(s^t\right)\right) - \upsilon\left(a_{it}\left(s^t\right)\right)$$

for which we assume as usual $u_c > 0$, $u_{cc} < 0$, $u_l > 0$, $u_{ll} > 0$ and Inada conditions, and $\upsilon' > 0$, $\upsilon'' \geq 0$. Aggregate resource constraint is:

$$\sum_{i=1}^N c_{it}\left(s^t\right) \leq \sum_{i=1}^N y_{it}\left(s^t\right) \tag{2.1}$$

where $y_i(s_t)$ is the endowment of agent i in period t. A contract is a pair of sequences $\{c_{it}(s^t), a_{it}(s^t)\}_{t=0}^\infty$ for each agent. I assume there is perfect commitment from all parts when they enter in the contract. Let us start with some definitions:

Definition 1 *A contract* $\{c_{it}(s^t), a_{it}(s^t)\}_{t=0}^\infty$ *is incentive compatible if* $\forall i$

$$a_i^\infty \in \arg\max \sum_{t=0}^\infty \sum_{s^t} \beta^t \left[u\left(c_{it}\left(s^t\right)\right) - \upsilon\left(a_{it}\left(s^t\right)\right)\right] \Pi\left(s^{t+1} \mid s^t, a^{t-1}\left(s^{t-1}\right)\right)$$

It is obviously difficult to deal with the incentive compatible constraint as defined above. In order to have a simpler problem, I apply a first order approach: I use the first order condition with respect to $a_{it}(s^t)$ of the government problem to characterize the optimal contract:

Definition 2 *A contract* $\{c_{it}(s^t), a_{it}(s^t)\}_{t=0}^\infty$ *is first order incentive compatible if*

$$\begin{aligned} \upsilon'\left(a_{it}\left(s^t\right)\right) &= \sum_{j=1}^\infty \sum_{s^{t+j}|s^t} \beta^j \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)} \left[u\left(c_{i,t+j}\left(s^{t+j}\right)\right) - \right. \\ &\left. -\upsilon\left(a_{i,t+j}\left(s^{t+j}\right)\right)\right] \Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1}\right)\right) \end{aligned} \tag{2.2}$$

[5]For example, imagine s_{it} can be s^H or s^L. If s^H is the good state, then $\pi_{a_i}\left(s^H \mid s_{it}, a_{it}(s^t)\right) > 0$, while $\pi_{a_i}\left(s^L \mid s_{it}, a_{it}(s^t)\right) < 0$.

In order to be sure that that first-order incentive compatibility is equivalent to incentive compatibility, we assume that Rogerson (1985b) conditions are satisfied.
Let $\{\omega_i\}_{i=1}^N$ be a given vector of initial Pareto weights. Therefore the constrained efficient allocation is the solution of the following maximization problem:

$$P(s_0) = \max_{\left\{\{c_{it}(s^t),a_{it}(s^t)\}_{i=1}^N\right\}_{t=0}^\infty} \left\{ \sum_{i=1}^N \omega_i \sum_{t=0}^\infty \beta^t \sum_{s^t} \left[u\left(c_{it}\left(s^t\right)\right) - \right.\right.$$
$$\left.\left. -\upsilon\left(a_{it}\left(s^t\right)\right)\right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)\right\}$$

$$s.t. \quad \upsilon'\left(a_{it}\left(s^t\right)\right) = \sum_{j=1}^\infty \sum_{s^{t+j}|s^t} \beta^j \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)} \left[u\left(c_{i,t+j}\left(s^{t+j}\right)\right) - \right.$$
$$\left. -\upsilon\left(a_{i,t+j}\left(s^{t+j}\right)\right)\right] \Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1}\right)\right)$$
$$\forall i = 1, ..., N$$

$$\sum_{i=1}^N c_{it}\left(s^t\right) = \sum_{i=1}^N y_{it}\left(s^t\right)$$

We can now write down the Lagrangean of this problem. Let $\lambda_{it}\left(s^t\right)$ be the Lagrange multiplier of the (first-order) incentive compatibility constraint for agent i, and let $\phi \equiv \{\phi_i\}_{i=1}^N$, $\varsigma \equiv \{c_i, a_i\}_{i=1}^N$, $\nu \equiv \{\lambda_i\}_{i=1}^N$. By applying the methodology of Marcet and Marimon (2009) and some tedious algebra, we get:

$$L\left(\varsigma^\infty, \nu^\infty, \phi^\infty\right) =$$
$$= \sum_{i=1}^N \sum_{t=0}^\infty \sum_{s^t} \beta^t \left\{\phi_{it}\left(s^t\right) \left[u\left(c_{it}\left(s^t\right)\right) - \upsilon\left(a_{it}\left(s^t\right)\right)\right] - \right.$$
$$\left. -\lambda_{it}\left(s^t\right) \upsilon'\left(a_{it}\left(s^t\right)\right)\right\} \Pi\left(s^t \mid h_0, a^{t-1}\left(s^{t-1}\right)\right)$$

where

$$\begin{aligned} \phi_{i,t+1}\left(s^t, s_{t+1}\right) &= \phi_{it}\left(s^t\right) + \lambda_{it}\left(s^t\right) \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)} \\ \phi_{i0}\left(s_0\right) &= \omega_i \end{aligned}$$

The new variable $\phi_{it}\left(s^t\right)$ is an endogenously evolving Pareto-Negishi weight which keeps track of the incentive compatibility constraint: $\lambda_{it}\left(s^t\right) > 0$ and then $\phi_{i,t+1}\left(s^t, s_{t+1}\right) < \phi_{it}\left(s^t\right)$ for "good" states of nature, and $\phi_{i,t+1}\left(s^t, s_{t+1}\right) > \phi_{it}\left(s^t\right)$ for "bad" states of nature.
The following Proposition shows that this problem is recursive:

Proposition 3 *The constrained efficient allocation solves the following functional equation*

$$
\begin{aligned}
W(s,\phi) &= \min_{\nu} \max_{\varsigma} \left\{ r(\varsigma,\nu,s,\phi) + \beta \sum_{s'} \pi(s' \mid s,a) W(s',\phi'(s')) \right\} \\
s.t. \quad & \phi_i'(s') = \phi_i + \lambda_i \frac{\pi_{a_i}(s' \mid s,a)}{\pi(s' \mid s,a)} \qquad i=1,...,N
\end{aligned}
$$

where

$$
r(\varsigma,\nu,s,\phi) \equiv \sum_{i=1}^{N} \{\phi_i [u(c_i) - \upsilon(a_i)] - \lambda_i \upsilon'(a_i)\}
$$

Moreover, the operator

$$
\begin{aligned}
(TW)(s,\phi) &= \min_{\nu} \max_{\varsigma} \left\{ r(\varsigma,\nu,s,\phi) + \beta \sum_{s'} \pi(s' \mid s,a) W(s',\phi'(s')) \right\} \\
s.t. \quad & \phi_i'(s') = \phi_i + \lambda_i \frac{\pi_{a_i}(s' \mid s,a)}{\pi(s' \mid s,a)} \qquad i=1,...,N
\end{aligned}
$$

is a contraction, and $W(\cdot,\phi)$ is homogenous of degree 1, while the policy correspondences $\varsigma(\cdot,\phi)$ and $\nu(\cdot,\phi)$ are homogeneous of degree zero.

Proof. *See Appendix D, Proposition 13.* ■

It is also possible to simplify the problem, in the case with two agents. In that case, we can show that the problem is equivalent to an auxiliary problem with only one endogenous state variable that corresponds to the ratio of the costate variables. In fact, $\frac{1}{\phi_1} W(s,\phi_1,\phi_2) = W\left(s,1,\frac{\phi_2}{\phi_1}\right) \equiv \widetilde{W}\left(s,\frac{\phi_2}{\phi_1}\right)$ for any s by the homogeneity property of the value function, and clearly $\varsigma(s,\phi_1,\phi_2) = \varsigma\left(s,1,\frac{\phi_2}{\phi_1}\right) \equiv \widetilde{\varsigma}\left(s,\frac{\phi_2}{\phi_1}\right)$ for any s and $\nu(s,\phi_1,\phi_2) = \nu\left(s,1,\frac{\phi_2}{\phi_1}\right) \equiv \widetilde{\nu}\left(s,\frac{\phi_2}{\phi_1}\right)$ by the homogeneity properties of the policy correspondences.

a Characterization of the contract

The following proposition is related to the result in Rogerson (1985a). It states that the optimal allocation satisfies an inverted Euler equation. Moreover, the ratio of marginal utilities depends on the ratio of the endogenous Pareto weights.

Proposition 4 *A constrained-efficient allocation $\left\{c_{it}(s^t), a_{it}(s^t)_{i=1}^{N}\right\}_{t=0}^{\infty}$ satisfies the following conditions:*

$$
\frac{\phi_{jt}(s^t)}{\phi_{it}(s^t)} = \frac{u_c(c_{it}(s^t))}{u_c(c_{jt}(s^t))} \tag{2.3}
$$

$$E_{i,t}\left\{\left[u_c\left(c_{i,t+1}\left(s^t,s_{t+1}\right)\right)\right]^{-1}\right\}=\frac{1}{u_c\left(c_{it}\left(s^t\right)\right)}\qquad \forall t\geq 0 \tag{2.4}$$

where $E_{i,t}$ is the expectation operator under the probabilities defined by the optimal action of agent i.

Proof. *First order condition for consumption of the Lagrangean imply directly first equation. For (2.4), rewrite first order condition for consumption at time $t+1$*

$$\phi_{i,t+1}\left(s^t,s_{t+1}\right)=\frac{1}{u_c\left(c_{i,t+1}\left(s^t,s_{t+1}\right)\right)}$$

Multiply it by $\pi\left(s_{i,t+1}\mid s_{i,t},a_{i,t}\left(s^t\right)\right)$ and sum over $s_{i,t+1}$ to get

$$\sum_{s_{i,t+1}}\phi_{i,t+1}\left(s^t,s_{t+1}\right)\pi\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)=$$
$$=\sum_{s_{i,t+1}}\frac{1}{u_c\left(c_{i,t+1}\left(s^t,s_{t+1}\right)\right)}\pi\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)$$

Now use the definition of $\phi_{i,t+1}\left(s^t,s_{t+1}\right)$ to substitute for it

$$\sum_{s_{i,t+1}}\left(\phi_{it}\left(s^t\right)+\lambda_{it}\left(s^t\right)\frac{\pi_{a_i}\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)}\right)\pi\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)=$$
$$=\sum_{s_{i,t+1}}\frac{1}{u_c\left(c_{i,t+1}\left(s^t,s_{t+1}\right)\right)}\pi\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)$$

and the first order condition for consumption at time t to obtain:

$$\frac{1}{u_c\left(c_{it}\left(s^t\right)\right)}+\sum_{s_{i,t+1}}\left(\lambda_{it}\left(s^t\right)\frac{\pi_{a_i}\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)}\right)\pi\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)=$$
$$=\sum_{s_{i,t+1}}\frac{1}{u_c\left(c_{i,t+1}\left(s^t,s_{t+1}\right)\right)}\pi\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)$$

It is now obvious that the second term in the LHS is zero, and we get (2.4) $\forall t\geq 0$. ■

Notice that by equation (2.4), $\frac{1}{u_c(c_{it}(s^t))}$ is a martingale with respect to the agent-i probability distribution generated by his own optimal effort. The following Proposition derives the implications of this result:

Proposition 5 *$\frac{u_c\left(c_{jt}\left(s^t\right)\right)}{u_c\left(c_{it}\left(s^t\right)\right)}$ is a submartingale.*

Proof. *Notice that*

$$\frac{u_c\left(c_{jt}\left(s^t\right)\right)}{u_c\left(c_{it}\left(s^t\right)\right)}=\frac{1}{u_c\left(c_{it}\left(s^t\right)\right)}\left[\frac{1}{u_c\left(c_{jt}\left(s^t\right)\right)}\right]^{-1}=$$
$$=\sum_{s_{i,t+1}}\frac{\pi\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)}{\sum_{s_{j,t+1}}\frac{u_c\left(c_{i,t+1}\left(s^t,s_{t+1}\right)\right)}{u_c\left(c_{j,t+1}\left(s^t,s_{t+1}\right)\right)}\pi\left(s_{j,t+1}\mid s_{jt},a_{jt}\left(s^t\right)\right)}$$

and therefore by Jensen's inequality

$$\begin{aligned}\frac{u_c\left(c_{jt}\left(s^t\right)\right)}{u_c\left(c_{it}\left(s^t\right)\right)} &\leq \sum_{s_{i,t+1}}\sum_{s_{j,t+1}}\frac{u_c\left(c_{j,t+1}\left(s^t,s_{t+1}\right)\right)}{u_c\left(c_{i,t+1}\left(s^t,s_{t+1}\right)\right)}\pi\left(s_{j,t+1}\mid s_{jt},a_{jt}\left(s^t\right)\right)\pi\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)= \\ &= \sum_{s_{t+1}}\frac{u_c\left(c_{j,t+1}\left(s^t,s_{t+1}\right)\right)}{u_c\left(c_{i,t+1}\left(s^t,s_{t+1}\right)\right)}\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)\end{aligned}$$

■

Proposition 5 determines the optimal consumption path. Notice that the submartingale result implies a persistent consumption inequality: once consumption of two agents start to diverge, this divergence will be long-lasting. Eventually, it can be reverted, but this will depend on the realization of idiosyncratic shocks.
We can also prove the following:

Proposition 6 $\Phi_t^{jk}\equiv\frac{\phi_{jt}\left(s^t\right)}{\phi_{kt}\left(s^t\right)}$ *is a submartingale.*

Proof. *Notice that we can always write:*

$$\begin{aligned}\frac{\phi_{jt}\left(s^t\right)}{\phi_{it}\left(s^t\right)} &= \frac{\sum_{s_{j,t+1}}\phi_{j,t+1}\left(s^t\right)\pi\left(s_{j,t+1}\mid s_{jt},a_{jt}\left(s^t\right)\right)}{\sum_{s_{k,t+1}}\phi_{k,t+1}\left(s^t\right)\pi\left(s_{k,t+1}\mid s_{kt},a_{kt}\left(s^t\right)\right)} \\ &= \frac{\sum_{s_{j,t+1}}\pi\left(s_{j,t+1}\mid s_{jt},a_{jt}\left(s^t\right)\right)}{\sum_{s_{k,t+1}}\frac{\phi_{k,t+1}\left(s^t\right)}{\phi_{j,t+1}\left(s^t\right)}\pi\left(s_{k,t+1}\mid s_{kt},a_{kt}\left(s^t\right)\right)} \\ &\leq \sum_{s_{j,t+1}}\sum_{s_{k,t+1}}\frac{\phi_{j,t+1}\left(s^t\right)}{\phi_{k,t+1}\left(s^t\right)}\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)\end{aligned}$$

where the last line comes from Jensen's inequality. ■

2.3 An endowment economy with unobservable bond markets

Take the same endowment economy but assume the planner cannot monitor the credit market. In this case, the agent of each country can trade one-period bond, buying it or selling it at the observable price $p_t\left(s^t\right)$. The agent has the following budget constraint:

$$c_{it}\left(s^t\right)+p_t\left(s^t\right)b_{it}\left(s^t\right)\leq y_{it}\left(s^t\right)+b_{i,t-1}\left(s^{t-1}\right)$$

The bond market must clear:

$$\sum_{i=1}^{N}b_{it}\left(s^t\right)=0$$

Following the first-order approach in Abraham and Pavoni (forthcoming), we now have to make sure that the effort and bond holding decisions of the agents are incentive compatible. We use the agent's first-order conditions with respect to effort and bond holding to characterize the optimal contract. The problem becomes:

$$P(s_0) = \max_{\left\{\{c_{it}(s^t), a_{it}(s^t)\}_{i=1}^N, p_t(s^t)\right\}_{t=0}^\infty} \left\{ \sum_{i=1}^N \sum_{t=0}^\infty \beta^t \sum_{s^t} \omega_i \left[u\left(c_{it}\left(s^t\right)\right) - \right.\right.$$
$$\left.\left. -\upsilon\left(a_{it}\left(s^t\right)\right)\right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)\right\}$$

$$s.t. \quad \upsilon'\left(a_{it}\left(s^t\right)\right) = \sum_{j=1}^\infty \sum_{s^{t+j}|s^t} \beta^j \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)} \left[u\left(c_{i,t+j}\left(s^{t+j}\right)\right) - \right.$$
$$\left. -\upsilon\left(a_{i,t+j}\left(s^{t+j}\right)\right)\right] \Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1}\right)\right)$$
$$\forall i = 1, ..., N$$

$$p_t\left(s^t\right) u'\left(c_{it}\left(s^t\right)\right) = \beta \sum_{s^{t+j}|s^t} u\left(c_{i,t+1}\left(s^{t+1}\right)\right) \pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)$$
$$\forall i = 1, ..., N$$

$$\sum_{i=1}^N c_{it}\left(s^t\right) = \sum_{i=1}^N y_{it}\left(s^t\right)$$

We can now write down the Lagrangean of this problem. Let $\phi \equiv \{\phi_i\}_{i=1}^N$,$\zeta \equiv \{\zeta_i\}_{i=1}^N$, $\varsigma \equiv \{c_i, a_i\}_{i=1}^N$, $\nu \equiv \{\lambda_i, \eta_i\}_{i=1}^N$:

$$L\left(\varsigma^\infty, \nu^\infty, \phi^\infty, \zeta^\infty\right) =$$
$$= \sum_{i=1}^N \sum_{t=0}^\infty \sum_{s^t} \beta^t \left\{ \phi_{it}\left(s^t\right) \left[u\left(c_{it}\left(s^t\right)\right) - \upsilon\left(a_{it}\left(s^t\right)\right)\right] - \lambda_{it}\left(s^t\right) \upsilon'\left(a_{it}\left(s^t\right)\right) + \right.$$
$$\left. + \left[p_t\left(s^t\right) \eta_{it}\left(s^t\right) - \zeta_{it}\left(s^t\right)\right] u'\left(c_{it}\left(s^t\right)\right)\right\} \Pi\left(s^t \mid h_0, a^{t-1}\left(s^{t-1}\right)\right)$$

where

$$\begin{aligned} \phi_{i,t+1}\left(s^t, s_{t+1}\right) &= \phi_{it}\left(s^t\right) + \lambda_{it}\left(s^t\right) \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)} \qquad \phi_{i0}\left(s_0\right) = \omega_i \\ \zeta_{i,t+1}\left(s^t, s_{t+1}\right) &= \eta_{it}\left(s^t\right) \qquad \zeta_{i0}\left(s_0\right) = 0 \end{aligned}$$

Notice that in this economy the marginal rate of substitution between today's consumption and tomorrow's consumption are equalized across agents. This is different from the previous simple endowment economy, where the marginal rate of substitution between today's consumption and tomorrow's consumption is governed by equation 2.4.
The following Proposition is the analog of Proposition 3:

Proposition 7 *The constrained efficient allocation solves the following functional equation*

$$\begin{aligned} W(s,\phi,\zeta) &= \min_{\nu} \max_{\varsigma, p} \left\{ r(\varsigma, p, \nu, s, \phi, \zeta) + \beta \sum_{s'} \pi(s' \mid s, a) W(s', \phi'(s'), \zeta') \right\} \\ s.t. &\qquad \phi_i'(s') = \phi_i + \lambda_i \frac{\pi_{a_i}(s' \mid s, a)}{\pi(s' \mid s, a)} \qquad i = 1, ..., N \\ &\qquad \zeta_i' = \eta_i \qquad i = 1, ..., N \end{aligned}$$

where

$$r(\varsigma, p, \nu, s, \phi, \zeta) \equiv \sum_{i=1}^{N} \{\phi_i [u(c_i) - \upsilon(a_i)] - \lambda_i \upsilon'(a_i) + (p\eta_i - \zeta_i) u'(c_i)\}$$

Moreover, the operator

$$\begin{aligned} (TW)(s,\phi,\zeta) &= \min_{\nu} \max_{\varsigma, p} \left\{ r(\varsigma, p, \nu, s, \phi, \zeta) + \beta \sum_{s'} \pi(s' \mid s, a) W(s', \phi'(s'), \zeta') \right\} \\ s.t. &\qquad \phi_i'(s') = \phi_i + \lambda_i \frac{\pi_{a_i}(s' \mid s, a)}{\pi(s' \mid s, a)} \\ &\qquad \zeta_i' = \eta_i \qquad i = 1, ..., N \end{aligned}$$

is a contraction, and $W(\cdot, \phi, \zeta)$ is homogenous of degree 1, while the policy correspondences $\varsigma(\cdot, \phi, \zeta)$ and $\nu(\cdot, \phi, \zeta)$ are homogeneous of degree zero.

The proof is a simple extension of the proof for Proposition 13 in Appendix D, and therefore it is omitted. Also in this case, it is possible to use homogeneity properties to reduce the state space.

a Characterization of the contract

Define $\Phi_t^{kj}(s^t) \equiv \frac{\phi_{kt}(s^t)}{\phi_{jt}(s^t)}$ and for any variable $x_{it}(s^t)$, let $\widetilde{x}_{it}(s^t) \equiv \frac{x_{it}(s^t)}{\phi_{it}(s^t)}$. Then the following Proposition defines the characteristics of the optimal contract.

Proposition 8 *A constrained-efficient allocation $\left\{ \{c_{it}(s^t), a_{it}(s^t)\}_{i=1}^{N} \right\}_{t=0}^{\infty}$ satisfies the following*

$$\Phi_t^{jk}(s^t) = \left[\frac{u'(c_{jt}(s^t))}{u'(c_{kt}(s^t))} \right]^{-1} \cdot \frac{[p_t(s^t)\eta_{jt}(s^t) - \zeta_{jt}(s^t)] u''(c_{jt}(s^t))}{[p_t(s^t)\eta_{kt}(s^t) - \zeta_{kt}(s^t)] u''(c_{kt}(s^t))} \quad \forall k, j \tag{2.5}$$

$$\sum_{j=1}^{N} \eta_{jt}(s^t) u'(c_{jt}(s^t)) = 0 \tag{2.6}$$

Proof. *Take first order conditions with respect to consumption and bond price and rearrange.* ■

The differences with Section 2.2 are clear. First, in Section 2.2 we have found that $\Phi_t^{jk}\left(s^t\right) = \left[\frac{u'\left(c_{jt}\left(s^t\right)\right)}{u'\left(c_{kt}\left(s^t\right)\right)}\right]^{-1}$, while here we have a wedge between the ratio of Pareto weights and the inverse of the ratio of marginal utilities of consumption. This wedge depends on the price of bonds, the Lagrange multipliers associated to individual Euler equations, the new costates $\zeta_{jt}\left(s^t\right)$ with $j = 1, ..., N$ and the relative degree of concavity of the utility function (the ratio of second derivatives for consumption utility).
In the case of CRRA utility function for consumption, we can simplify the above result of equation (2.5):

Corollary 1 *Assume that* $u\left(c_i\right) \equiv \frac{c_i^{1-\sigma_i}}{1-\sigma_i}$. *Therefore*

$$\frac{c_{jt}}{c_{kt}} = \frac{\sigma_j}{\sigma_k} \cdot \frac{p_t\left(s^t\right)\widetilde{\eta}_{jt}\left(s^t\right) - \widetilde{\zeta}_{jt}\left(s^t\right)}{p_t\left(s^t\right)\widetilde{\eta}_{kt}\left(s^t\right) - \widetilde{\zeta}_{kt}\left(s^t\right)} \quad \forall k, j \tag{2.7}$$

Equation (2.7) shows that consumption inequality in this framework depends not only on the endogenous Pareto weights, but also on the shadow cost of the Euler equation. To fix ideas, imagine that $\sigma_j = \sigma_k$ and Pareto weights for agents j and k are identical at period t. Then,

$$\frac{c_{jt}}{c_{kt}} = \frac{p_t\left(s^t\right)\eta_{jt}\left(s^t\right) - \zeta_{jt}\left(s^t\right)}{p_t\left(s^t\right)\eta_{kt}\left(s^t\right) - \zeta_{kt}\left(s^t\right)} \tag{2.8}$$

which implies there can still be inequality due to different asset accumulation paths and different histories. Clearly, there are two reasons for inequality in this economy: one is induced by the incentive structure (governed by Pareto weights), where inequality is used as an incentive for increasing effort. The second source of inequality is wealth accumulation. The wedge in equation (2.5) reflects the second source of inequality.
Proposition 6 also holds in this economy. However, the implications are different from Section 2.2. Now it is the LHS of equation (2.5) that behaves as a submartingale: the ratio of marginal utilities of consumption multiplied by a wedge that depends on bond price, the Lagrange multiplier of the Euler equations and the new costate variables associated with them.
Moreover, equation (2.6) has the following consequence:

Corollary 2 *Assume* $N = 2$. *Define*

$$B_t\left(s^t\right) \equiv \frac{\sigma_1}{\sigma_2} \cdot \frac{p_t\left(s^t\right)\widetilde{\eta}_{1t}\left(s^t\right) - \widetilde{\zeta}_{1t}\left(s^t\right)}{p_t\left(s^t\right)\widetilde{\eta}_{2t}\left(s^t\right) - \widetilde{\zeta}_{2t}\left(s^t\right)}$$

Therefore

$$\frac{\eta_{1t}\left(s^t\right)}{\eta_{2t}\left(s^t\right)}\frac{u'\left(c_{1t}\left(s^t\right)\right)}{u'\left(c_{2t}\left(s^t\right)\right)} = -1 \tag{2.9}$$

$$c_{1t}\left(s^t\right) = \left[y_{1t}\left(s^t\right) + y_{2t}\left(s^t\right)\right]\frac{B_t\left(s^t\right)}{1 + B_t\left(s^t\right)} \tag{2.10}$$

The focus on a 2-agents economy helps understand the logic. Marginal utilities of consumption ratio is governed by the behavior of the Lagrange multipliers associated with individual Euler equations. The sign of $\eta_{it}\left(s^t\right)$ defines if an agent is the borrower or the lender in the economy. Remember that $\eta_{it}\left(s^t\right)$ is the Lagrange multiplier of Euler equation for agent i. Therefore, in the optimal contract, a positive sign of $\eta_{it}\left(s^t\right)$ implies that agent i will be willing to decrease her consumption today and increase it tomorrow, i.e. agent i wants to save. Viceversa, a negative sign of $\eta_{it}\left(s^t\right)$ indicates that agent i wants to borrow. The difference between $p_t\left(s^t\right)\eta_{it}\left(s^t\right)$ and $\zeta_{it}\left(s^t\right)$ drives the trades in the hidden assets market. What Corollary 2 says is that $\eta_{1t}\left(s^t\right)$ and $\eta_{2t}\left(s^t\right)$ have opposite sign. This implies that (except for period 0) also $\zeta_{1t}\left(s^t\right)$ and $\zeta_{2t}\left(s^t\right)$ have opposite signs. Therefore, in a 2-agents economy, as it is obvious, there will always be one agent indebted with the other. Which one will depend on the history of shocks that drive the value of $B_t\left(s^t\right)$.

2.4 A production economy

The previous endowment economies have interesting properties, and we are interested to see if these properties survive to the introduction of observable capital accumulation. Therefore, let us assume that each country has its own production function, which we assume is the same for both

$$y_{it}\left(s^t\right) = A_i\left(s^t\right) f\left(k_{i,t-1}\left(s^{t-1}\right)\right)$$

where $A_i\left(s^t\right)$ is a productivity parameter, and therefore the feasibility constraint becomes

$$\sum_{i=1}^{N} c_{it}\left(s^t\right) + \sum_{i=1}^{N} k_{it}\left(s^t\right) - \sum_{i=1}^{N}\left(1-\delta_i\right) k_{i,t-1}\left(s^{t-1}\right) \leq \sum_{i=1}^{N} A_i\left(s^t\right) f\left(k_{i,t-1}\left(s^{t-1}\right)\right)$$

$$k_{i,-1} \text{ given } \forall i = 1, ..., N$$

where δ_i is the depreciation rate of capital in country i.

The Pareto-constrained allocation can be found by solving:

$$P\left(s_0,\{k_{i,-1}\}_{i=1}^{N}\right)=\max_{\left\{\{c_{it}(s^t),a_{it}(s^t),k_{it}(s^t)\}_{i=1}^{N}\right\}_{t=0}^{\infty}}\left\{\sum_{i=1}^{N}\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\omega_i\left[u\left(c_{it}\left(s^t\right)\right)\ -\right.\right.$$
$$\left.\left.-\upsilon\left(a_{it}\left(s^t\right)\right)\right]\Pi\left(s^t\mid s_0,a^{t-1}\left(s^{t-1}\right)\right)\right\}$$

$$s.t.\quad \upsilon'\left(a_{it}\left(s^t\right)\right)=\sum_{j=1}^{\infty}\sum_{s^{t+j}|s^t}\beta^j\frac{\pi_{a_i}\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)}\times \tag{2.11}$$
$$\left[u\left(c_{i,t+j}\left(s^{t+j}\right)\right)-\upsilon\left(a_{i,t+j}\left(s^{t+j}\right)\right)\right]\Pi\left(s^{t+j}\mid s^t,a^{t+j-1}\left(s^{t+j-1}\right)\right)$$
$$\forall i=1,...,N$$

$$\sum_{i=1}^{N}c_{it}\left(s^t\right)+\sum_{i=1}^{N}k_{it}\left(s^t\right)-\sum_{i=1}^{N}\left(1-\delta_i\right)k_{i,t-1}\left(s^{t-1}\right)\leq$$
$$\leq\sum_{i=1}^{N}A_i\left(s^t\right)f\left(k_{i,t-1}\left(s^{t-1}\right)\right)$$
$$k_{i,-1}\text{ given }\forall i=1,...,N$$

We can now write down the Lagrangean of this problem:

$$L\left(\varsigma^\infty,\nu^\infty,k^\infty,\phi^\infty\right)=$$
$$=\sum_{i=1}^{N}\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left\{\phi_{it}\left(s^t\right)\left[u\left(c_{it}\left(s^t\right)\right)-\upsilon\left(a_{it}\left(s^t\right)\right)\right]-\right.$$
$$\left.-\lambda_{it}\left(s^t\right)\upsilon'\left(a_{it}\left(s^t\right)\right)\right\}\Pi\left(s^t\mid h_0,a^{t-1}\left(s^{t-1}\right)\right)$$

where

$$\begin{aligned}\phi_{i,t+1}\left(s^t,s_{t+1}\right)&=\phi_{it}\left(s^t\right)+\lambda_{it}\left(s^t\right)\frac{\pi_{a_i}\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1}\mid s_{it},a_{it}\left(s^t\right)\right)}\\ \phi_{i0}\left(s_0\right)&=\omega_i\end{aligned}$$

The following Proposition shows that this problem is recursive:

Proposition 9 *The constrained efficient allocation solves the following functional equation*

$$\begin{aligned}W\left(s,k,\phi\right)&=\min_{\nu}\max_{\varsigma,k'}\left\{r\left(\varsigma,k',\nu,s,k,\phi\right)+\beta\sum_{s'}\pi\left(s'\mid s,a\right)W\left(s',k',\phi'\left(s'\right)\right)\right\}\\ s.t.\qquad&\phi_i'\left(s'\right)=\phi_i+\lambda_i\frac{\pi_{a_i}\left(s'\mid s,a\right)}{\pi\left(s'\mid s,a\right)}\qquad i=1,...,N\end{aligned}$$

where

$$r(\varsigma, k', \nu, s, k, \phi) \equiv \sum_{i=1}^{N} \{\phi_i [u(c_i) - \upsilon(a_i)] - \lambda_i \upsilon'(a_i)\}$$

Moreover, the operator

$$\begin{aligned}(TW)(s,k,\phi) &= \min_{\nu} \max_{\varsigma, k'} \left\{ r(\varsigma, k', \nu, s, k, \phi) + \beta \sum_{s'} \pi(s' \mid s, a) W(s', k', \phi'(s')) \right\} \\ s.t. \qquad & \phi_i'(s') = \phi_i + \lambda_i \frac{\pi_{a_i}(s' \mid s, a)}{\pi(s' \mid s, a)} \qquad i = 1, ..., N\end{aligned}$$

is a contraction, and $W(s,k,\phi)$ is homogenous of degree 1 in ϕ, while the policy correspondences $\varsigma(s,k,\phi)$, $k'(s,k,\phi)$ and $\nu(s,k,\phi)$ are homogeneous of degree zero.

In the case of $N = 2$, it is possible to show that the state space can be reduced thanks to the homogeneity properties of the value and policy functions: the model is recursive in $(\theta_t(s^t), k_{1t-1}(s^{t-1}), k_{2t-1}(s^{t-1}))$, where $\theta_t(s^t) \equiv \frac{\phi_{2t}(s^t)}{\phi_{1t}(s^t)}$.

a Characterization of the contract

It should be obvious at this point that the submartingale result in Proposition 6 survives to the introduction of capital accumulation (the law of motion is the same as in the endowment economies). To characterize the optimal contract in more detail, define $R_{it}^K(s^t) \equiv A_i(s^t) f_k'(k_{i,t-1}(s^{t-1})) + (1-\delta_i)$. Therefore we can state the following:

Proposition 10 *A constrained-efficient allocation $\left\{ \{c_{it}(s^t), a_{it}(s^t), k_{it}(s^t)\}_{i=1}^N \right\}_{t=0}^{\infty}$ satisfies the following condition:*

$$\Phi_t^{ij}(s^t) = \left[\frac{u'(c_{it}(s^t))}{u'(c_{jt}(s^t))} \right]^{-1} \frac{\sum_{s^{t+1}|s^t} \phi_{it+1}(s^{t+1}) u'(c_{i,t+1}(s^{t+1})) R_{i,t+1}^K(s^{t+1}) \pi(s_{t+1} \mid s_t, a_t(s^t))}{\sum_{s^{t+1}|s^t} \phi_{jt+1}(s^{t+1}) u'(c_{j,t+1}(s^{t+1})) R_{j,t+1}^K(s^{t+1}) \pi(s_{t+1} \mid s_t, a_t(s^t))}$$

Proof. *Take Lagrangean first order conditions with respect to $k_{it}(s^t)$ and $k_{jt}(s^t)$ and divide the first for the second.* ■

Also in this framework, there is a wedge between the ratio of marginal utilities of consumption and the ratio of Pareto weights, that depends on next period's return of capital for each agent.

2.5 Numerical examples

The case of an endowment economy has been studied in Chapter 1, and therefore I focus my attention on the case with hidden savings and the case with production. I will assume for simplicity that in each setup there are only two agents.

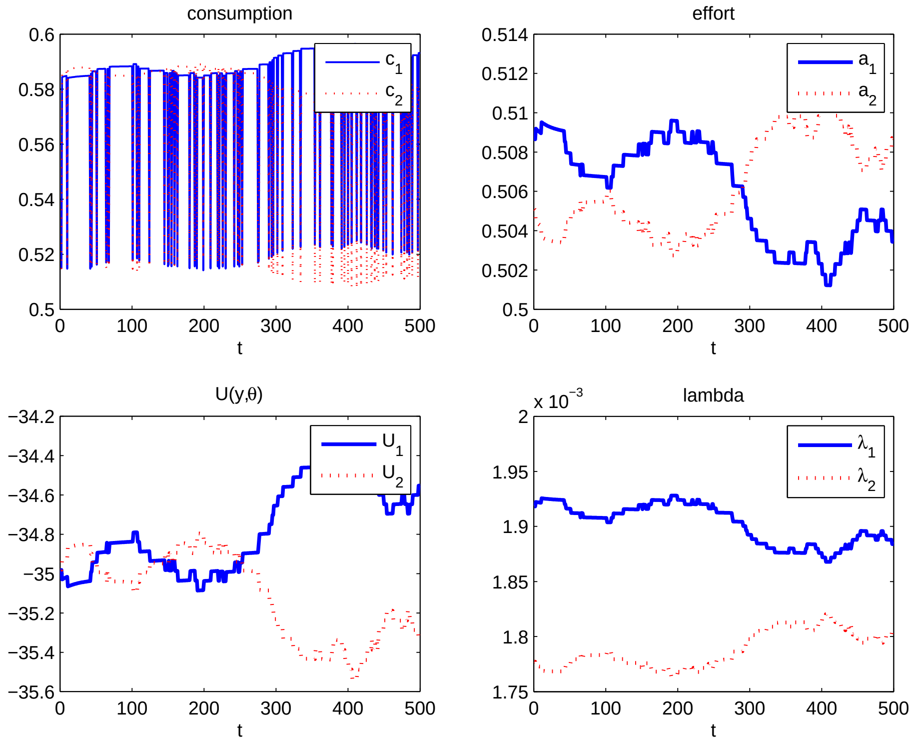

Figure 2.1: Production economy: sample path

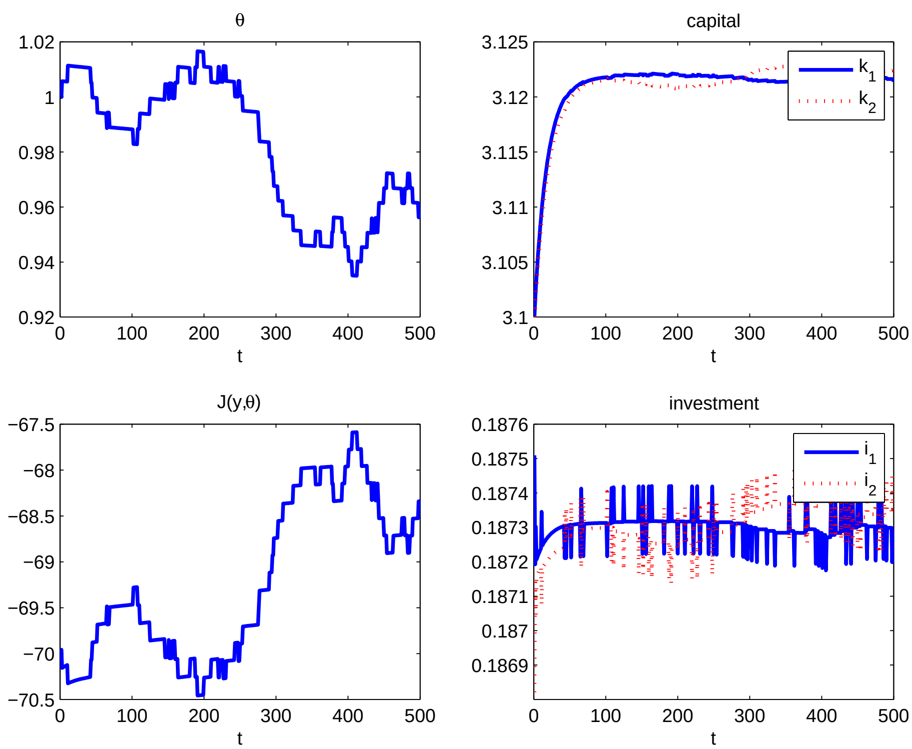

Figure 2.2: Production economy: sample path (cont.)

a Production economy

I keep the same functional forms of Section 1.6 in Chapter 1, and I choose the following production function for both agents:

$$f(k) = k^{\rho}$$

The baseline parameters are summarized in the table:

α_i	ε_i	ν_i	σ_i	A^L	A^H	β	ω_i	δ_i	ρ_i	k_0^i
0.05	2	0.1	2	0.45	0.55	0.95	0.5	0.06	0.3	3.1

The model is solved in around one minute. We therefore start with two identical agents. Figures 2.1 and 2.2 show a sample simulated path for this setup. Both consumption and investment are very volatile. Notice also that consumption inequality is very persistent, and this is reflected in the path of expected discounted utilities of each agent. Capital tends to slowly oscillate, and the agent who owns more capital changes aldo if very slowly. In Figures 2.3 and 2.4 I present a simulated sample path for the case in which agents are weighted differently at time zero. In particular I assume $\omega_1 = 0.4$ and $\omega_2 = 0.6$. It turns out that different initial weights imply a large transfer at time zero, which reallocate capital endowments from the country with high weight to the country with low weight. Consumption continues to be extremely volatile also if levels are different for the two agents. The agent that consumes less is the one that invest more and consequently accumulates more capital. Discounted utilities of the two agents tend to diverge in this sample path.
The average allocations based on 50000 simulations with a horizon of 500 periods are presented in Figure 2.5 and 2.6. The main result is the divergence of capital in the long run. This is due to the history dependence of investment: in each period, it is better to invest a little more in the production technology that has a better history of shocks. The ratio of Pareto weights θ keeps track of the history, and we see that it is different from one in the long run (remember that this is an average). The agent that accumulates more capital is also the agent that exerts more effort.
The last simulation I present is the average of 50000 simluations obtained by starting with different initial weights. In particular, I assume as before $\omega_1 = 0.4$ and $\omega_2 = 0.6$, while all the other parameters are the same as in the benchmark. The optimal allocations are shown in Figures 2.7 and 2.8. This results confirms the main hints of the sample path in Figures 2.3 and 2.4.

b Endowment economy with hidden wealth

The baseline parameterization is:

α_i	ε_i	ν_i	σ_i	β	ω_i
5	2	0.5	2	0.95	0.5

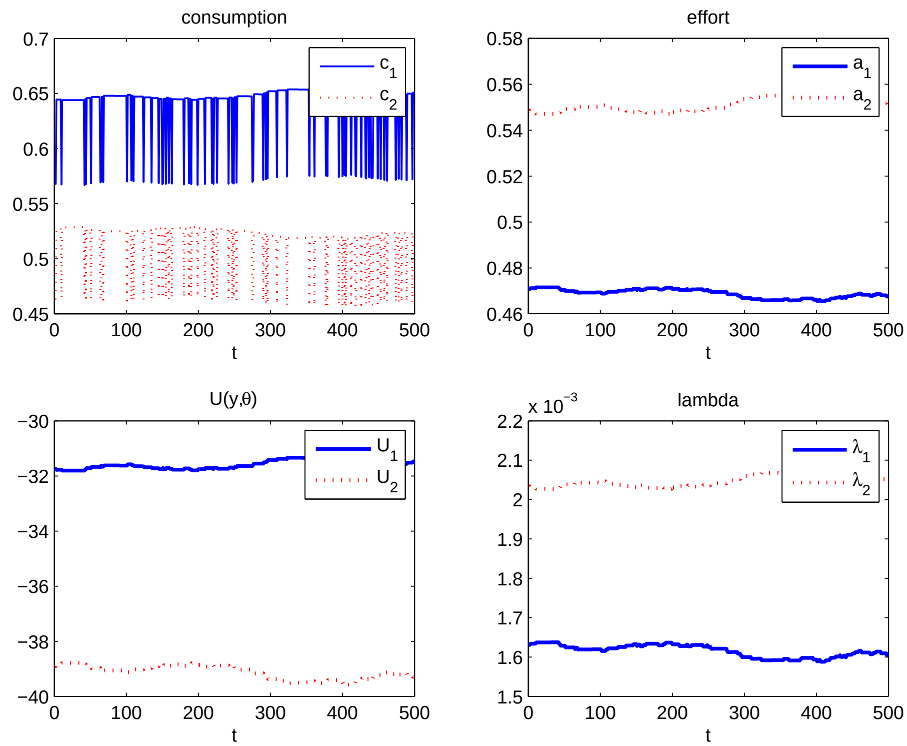

Figure 2.3: Production economy: sample path, different initial weights

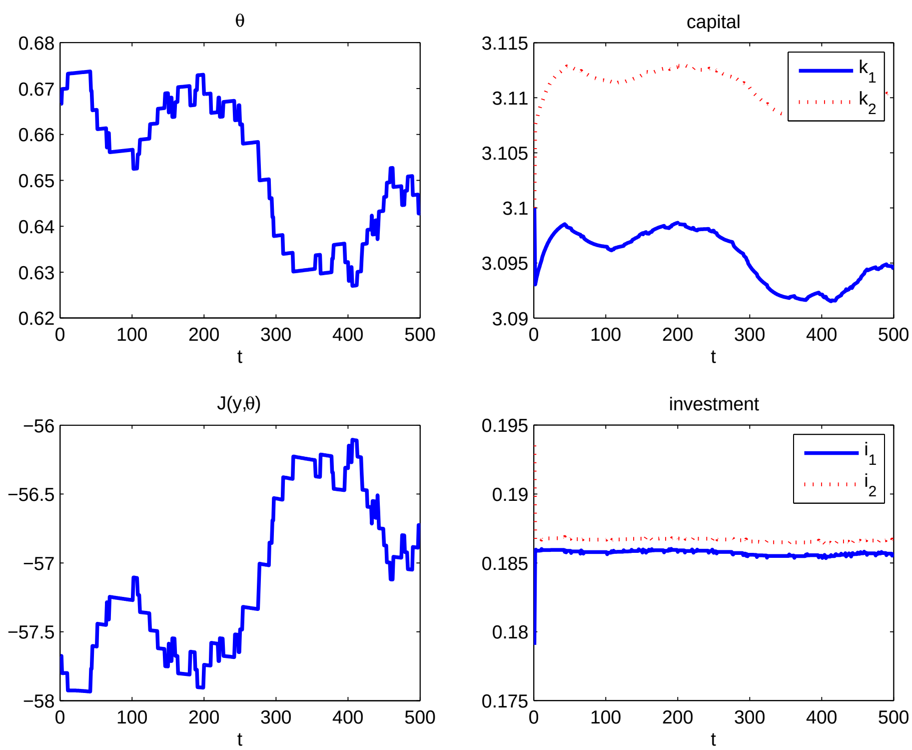

Figure 2.4: Production economy: sample path, different initial weights (cont.)

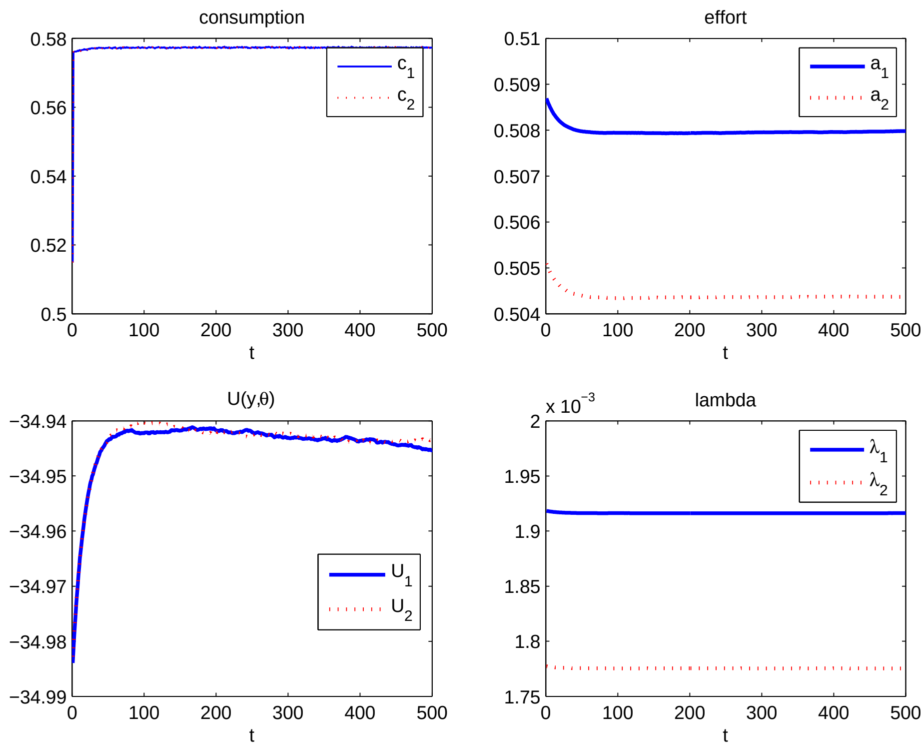

Figure 2.5: Production economy: average over 50000 simulations

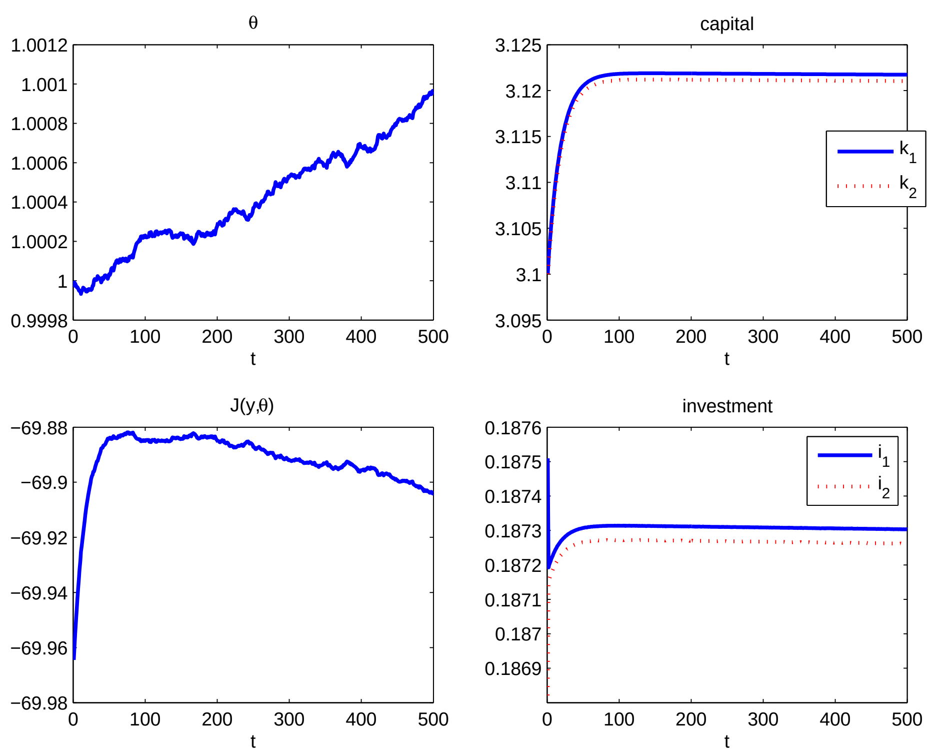

Figure 2.6: Production economy: average over 50000 simulations (cont.)

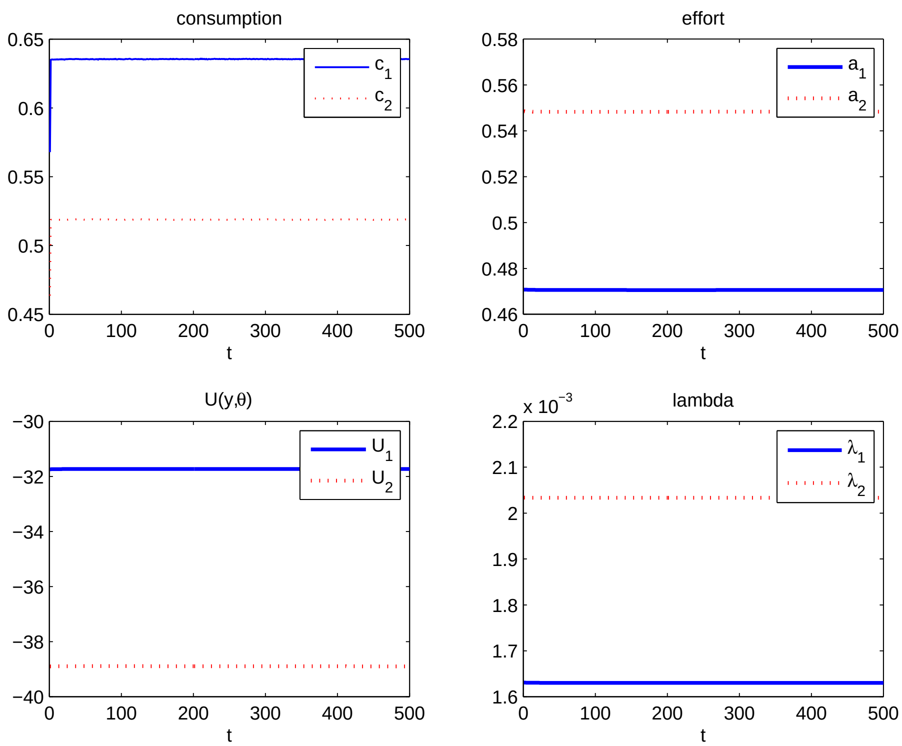

Figure 2.7: Production economy: average over 50000 simulations, different initial weights

The solution is found in less than one minute. Figures 2.9 and 2.10 present optimal allocations for a sample path of realized shocks.
In particular, in Figure 2.11 we can observe that asset positions become extreme quite soon.

Figures 2.12 and 2.13 present average allocations for 50000 simulations. One striking conclusion is that effort seems quite flat on average, also if it is decreasing for one agent and increasing for the other. From Figure 2.14 we can notice that it is not clear on average if there is one agent that tends to be always net debtor or net borrower.
I also show simulations for the case in which agents have different initial weights. Figures 2.15 and 2.16 show sample path allocations for $\omega_1 = 0.45$ and $\omega_2 = 0.55$. Notice that the initial consumption inequality is very important in determining future inequality, due to strong persistence. Asset positions in Figure 2.17 become much more extreme than in previous case with same initial weight. Therefore, different initial weights imply big wealth inequality in the long run. Finally averages over 50000 simulations in which agents have different initial weights are presented in Figures 2.18 and 2.19. Effort for high-weight agent tends to be lower, and this agent becomes permanently and hugely indebted with the other. Asset positions in Figure 2.20 become much more extreme than in previous case with same initial weight, confirming the behavior in the previous sample path.

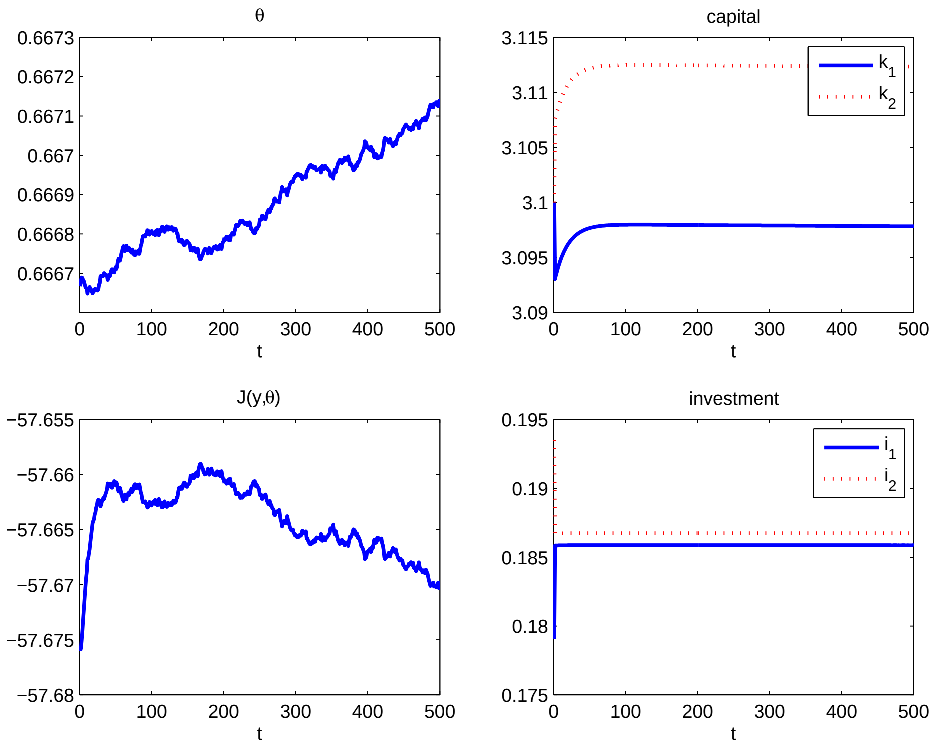

Figure 2.8: Production economy: average over 50000 simulations, different initial weights (cont.)

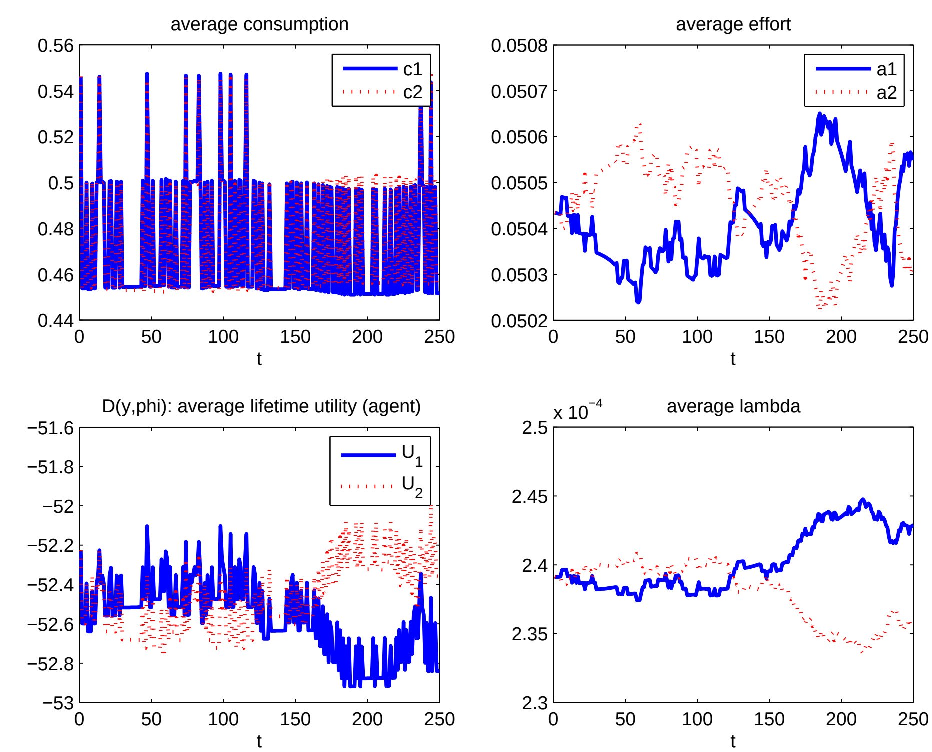

Figure 2.9: Endowment economy with hidden assets: sample path

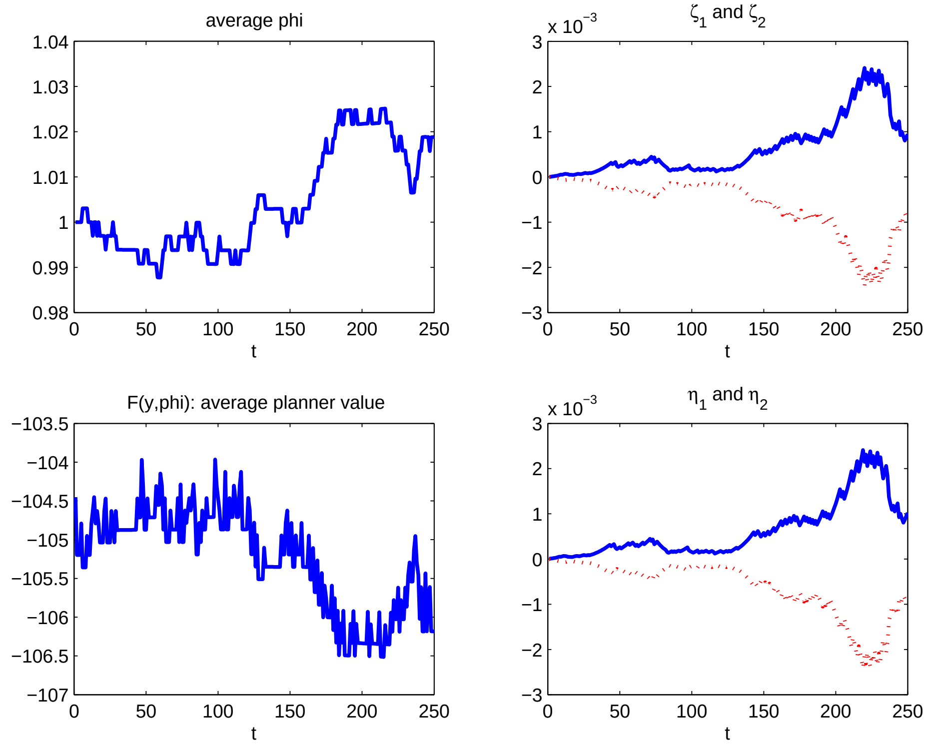

Figure 2.10: Endowment economy with hidden assets: sample path (cont.)

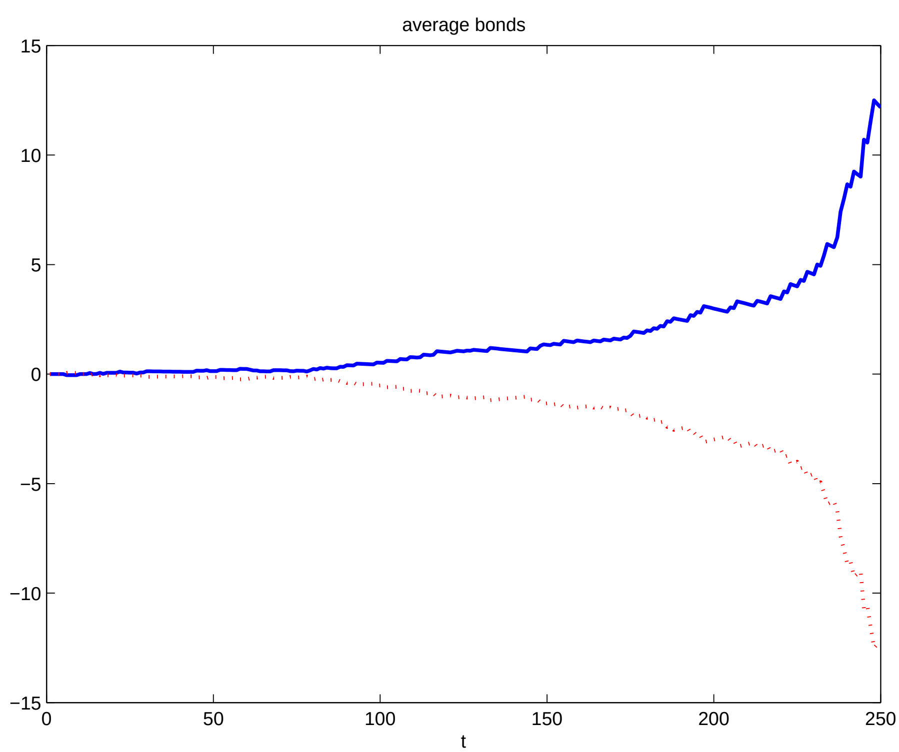

Figure 2.11: Endowment economy with hidden assets: sample path, bond positions

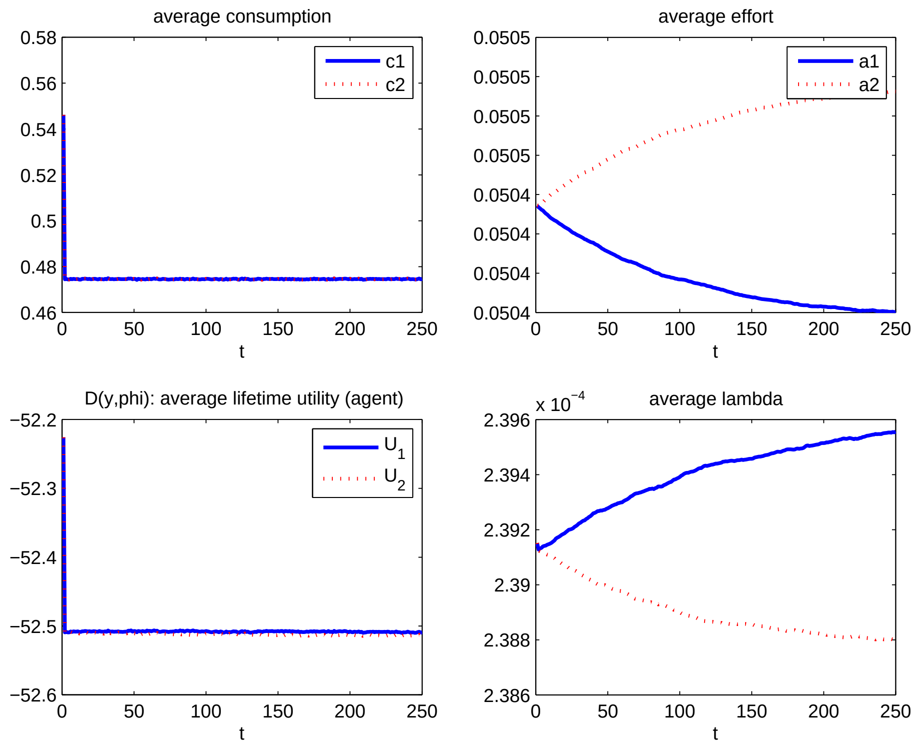

Figure 2.12: Endowment economy with hidden assets: average over 50000 simulations

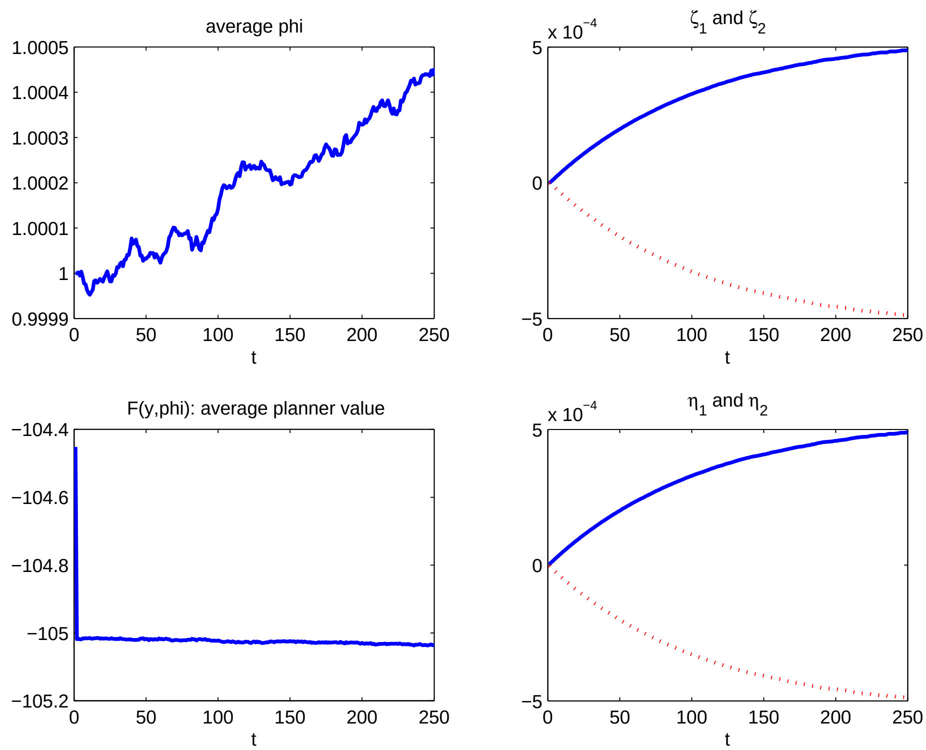

Figure 2.13: Endowment economy with hidden assets: average over 50000 simulations (cont.)

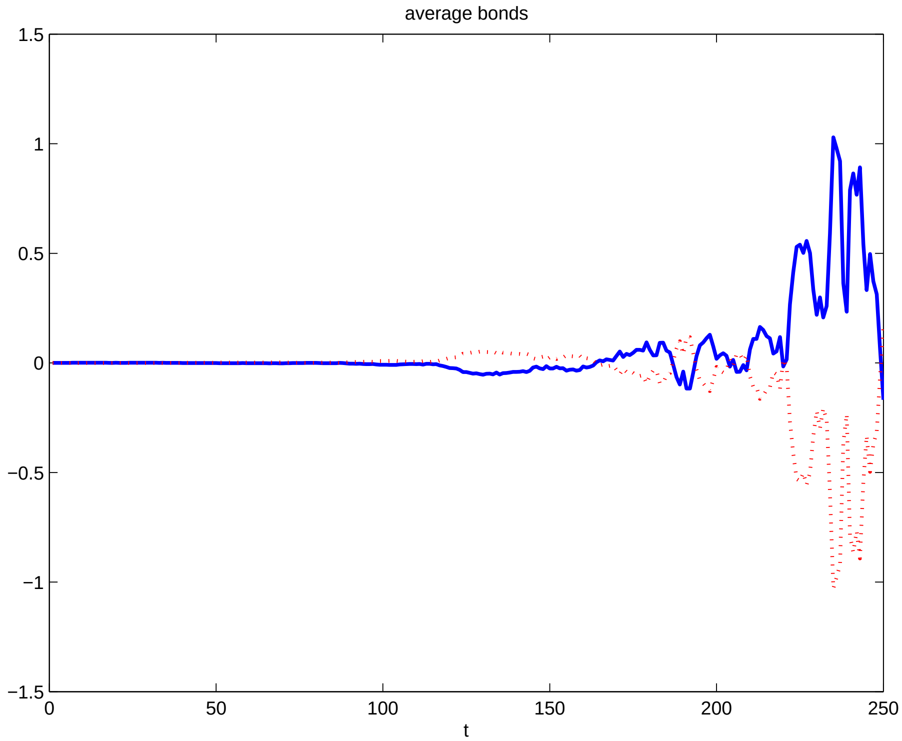

Figure 2.14: Endowment economy with hidden assets: average over 50000 simulations, bond positions

2.6 Conclusions

I have presented three different economies in which agents share their idiosyncratic risk under repeated multi-sided moral hazard. All three setups were analyzed by means of the Lagrangean approach presented in Chapter 1.

I have provided a characterization of the optimal arrangement for a simple endowment economy, for an endowment economy with hidden access to financial markets, and for a production economy. In all three cases, the efficient contract can be characterized by a submartingale result related to the ratio of marginal utilities of consumption. In particular, in the simple endowment economy this quantity behaves as a submartingale. In an endowment economy with non-monitorable access to financial markets, the submartingale behavior is associated to the ratio of marginal utilities of consumption multiplied by a wedge that depends on the bonds price and the shadow cost of the consumption-saving decision of each agent. In a production economy, we also have a wedge between the ratio of marginal utilities of consumption that depends on future returns on physical capital.

All numerical examples presented are not calibrated. Future work should be devoted to quantify welfare properties of the different setups with a realistic parametrization.

An important assumption maintained in the present work is full commitment. It will

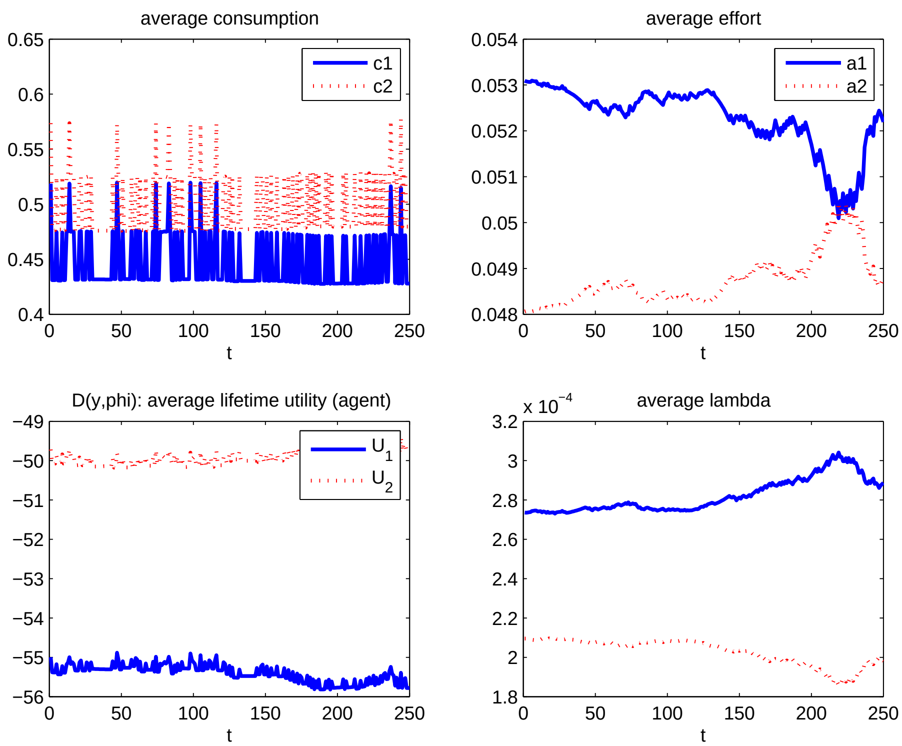

Figure 2.15: Endowment economy with hidden assets: sample path with different initial weights

be an interesting extension to analyze the case without commitment, in particular it is important to quantify the different welfare effects of private information versus enforcement frictions.

Finally, the possibility of decentralizing the optimal contract in a competitive equilibrium (perhaps with some frictions or policy instrument) is also an important issue which is left for future research.

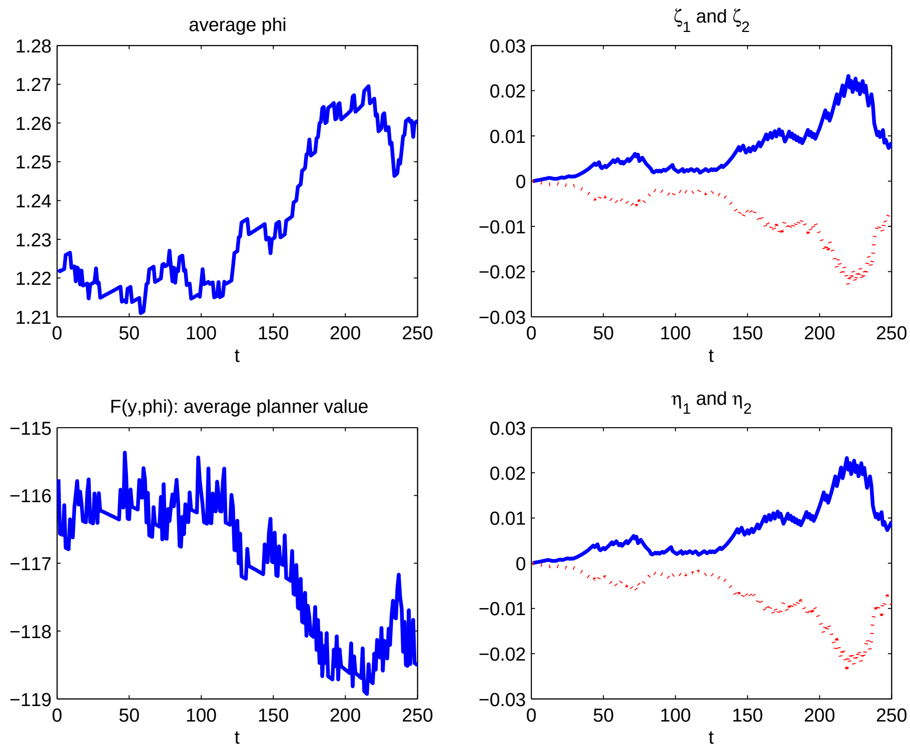

Figure 2.16: Endowment economy with hidden assets: sample path with different initial weights (cont.)

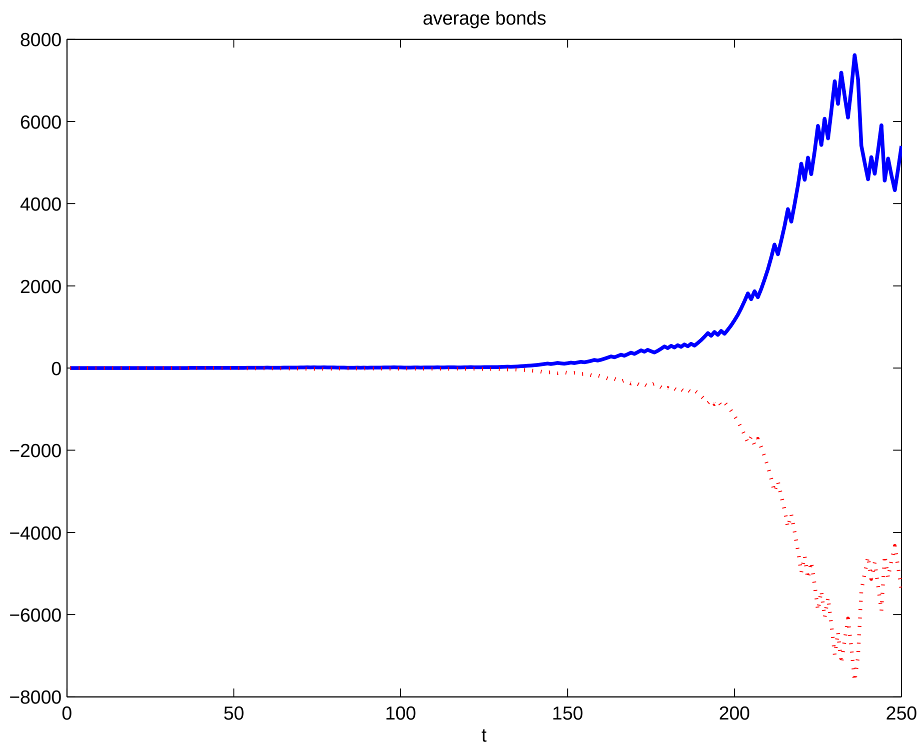

Figure 2.17: Endowment economy with hidden assets: sample path with different initial weights, bond positions

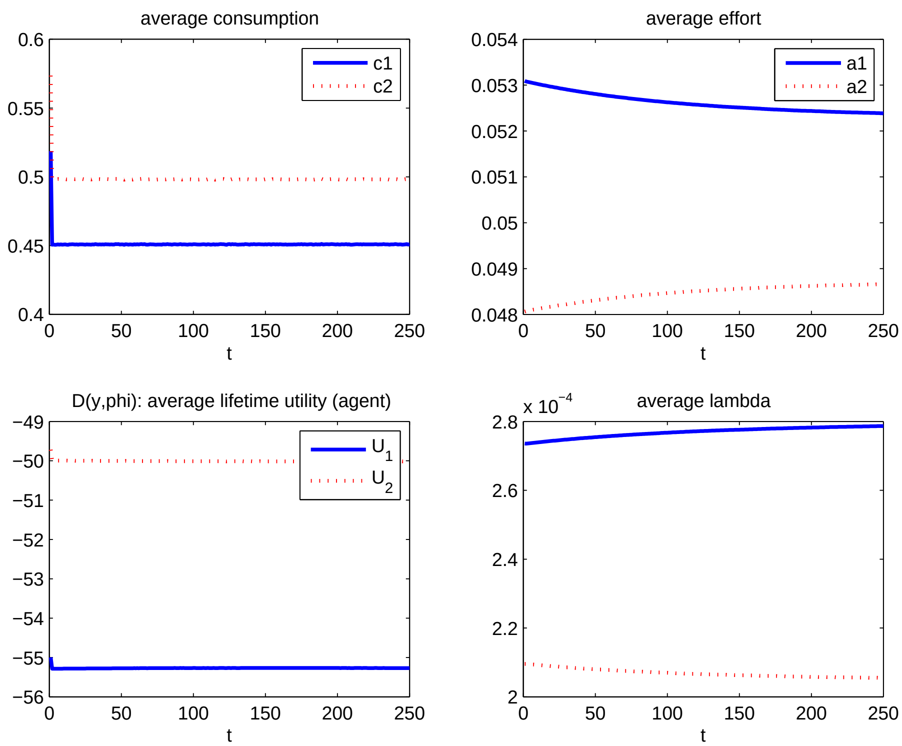

Figure 2.18: Endowment economy with hidden assets: average over 50000 simulations with different initial weights

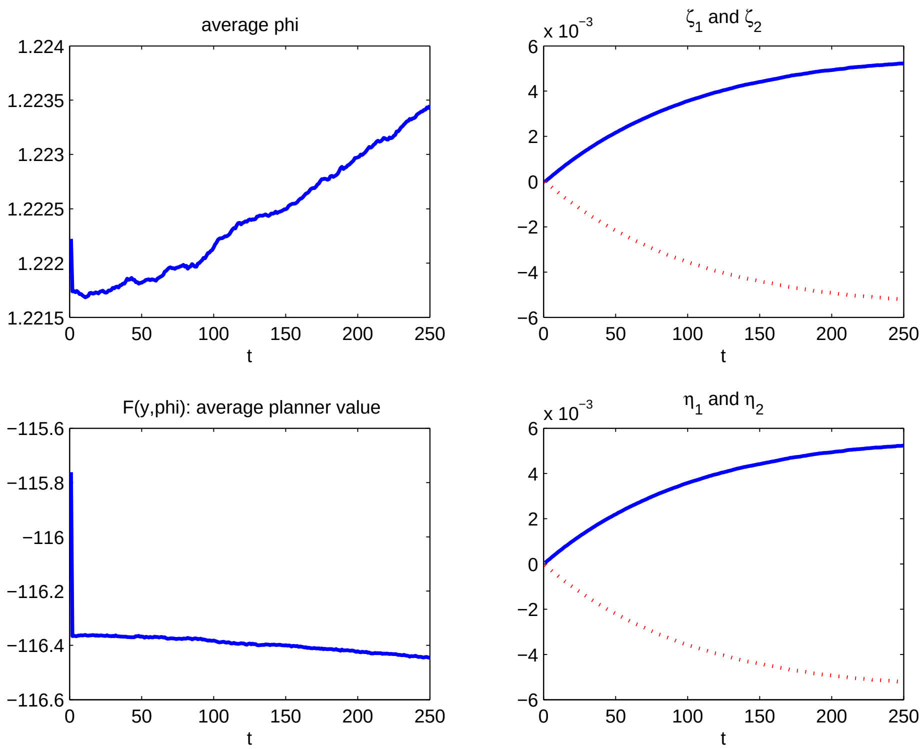

Figure 2.19: Endowment economy with hidden assets: average over 50000 simulations with different initial weights (cont.)

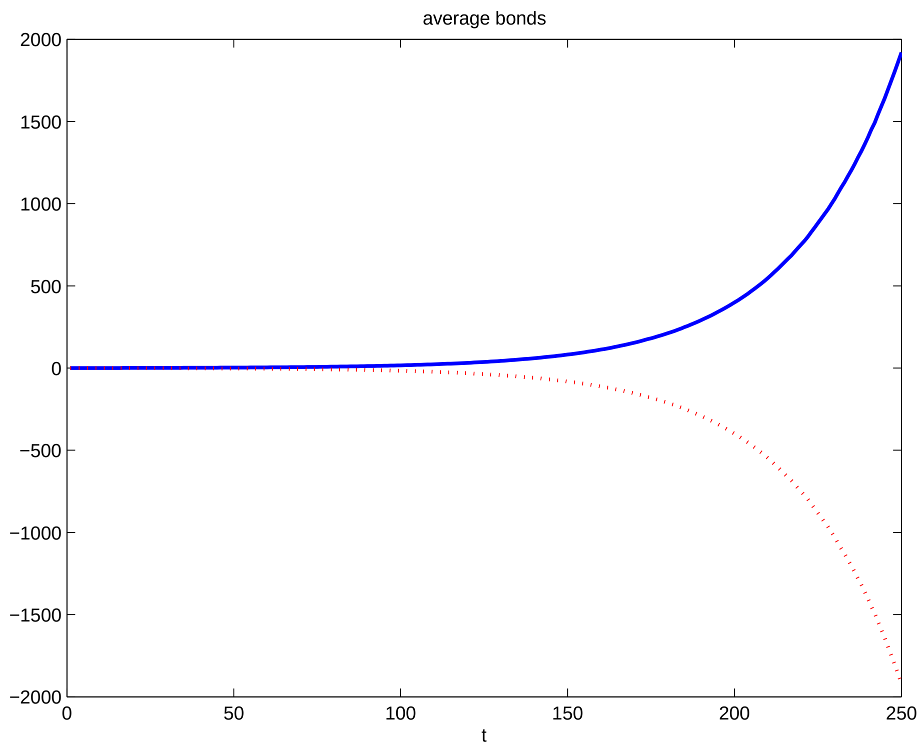

Figure 2.20: Endowment economy with hidden assets: average over 50000 simulations with different initial weights, bond positions

3 Unemployment Insurance, Human Capital and Financial Markets

3.1 Introduction

In this Chapter, I provide a first attempt at analyzing optimal unemployment insurance with human capital depreciation and hidden access to financial markets.
The seminal work of Hopenhayn and Nicolini (1997) suggests that an incentive compatible system of unemployment insurance must have decreasing unemployment benefit during unemployment spell. Two assumptions are crucial: first, there is no human capital accumulation or depreciation in their setup, and second, the worker has no access to financial markets.
However, it is a well documented fact that human capital depreciates during unemployment spells while increases during employment tenure. For example, Keane and Wolpin (1997) find an annual human capital depreciation rate for white US males during unemployment of 9.6% for blue collars and 36.5% for white collars. Moreover, there is also evidence that transition probabilities depend on the length of unemployment spell: van den Berg and van Ours (1994,1996) and Bover, Arellano and Bentolila (2002) find negative duration dependence in the unemployment hazard rate.
Starting from this evidence, Pavoni (forthcoming) shows that, in a model of UI with human capital depreciation and only two possible levels of effort, the result of Hopenhayn and Nicolini (1997) of a decreasing profile for unemployment benefits survives, but there is a point at which benefits stop decreasing and are kept constant by the principal. This is due to depreciation of human capital during unemployment spells: for very low levels of human capital, the principal has no interest in inducing the worker to find a new job, since both the probability of finding a job and the wage he will be receiving are too low.
On the other hand, the assumption of no access to financial markets seems extreme. Abraham and Pavoni (2006, forthcoming) characterize optimal allocation in a model with repeated moral hazard and hidden access to financial markets. In their setup, there is no human capital. The agent can save or borrow at a constant exogenous interest rate, but the principal cannot observe asset trades in the market. Under these assumptions, optimal unemployment benefits must be increasing during unemployment[1].
When both human capital depreciation and hidden access to financial markets are present, it is not clear which effect would prevail and therefore it is an open question if the optimal scheme must be decreasing or increasing. In this Chapter, I develop a model of optimal unemployment insurance with human capital depreciation and hidden asset markets, building on the work of Pavoni (forthcoming) and Abraham and Pavoni

[1]This is the case in my numerical examples: optimal transfers during unemployment are slightly increasing, passing from an initial replacement rate of 55.08% to replacement rate of 55.15% after one year of unemployment.

(2006, forthcoming). In my model, the worker can be employed or unemployed. I do not assume employment is an absorbing state as in Hopenhayn and Nicolini (1997): there can be alternate spells of unemployment and employment. The transition probability depends on unobservable effort[2] and on the level of human capital of the worker (as in Pavoni (forthcoming)). Moreover the worker has access to a hidden asset market as in Abraham and Pavoni (2006, forthcoming) with constant exogenous interest rate.

It turns out that the optimal insurance scheme is extremely generous with the worker: replacement rate is higher than 100% for more than two years, under my parametrization[3]. The profile of unemployment benefits is decreasing as in the standard Hopenhayn and Nicolini (1997) work. This very generous system is financed with a slightly increasing payroll tax.

The intuition behind this result is simple: the optimal insurance scheme takes into account that, on average, human capital (and hence wage) decreases over time. The optimal scheme therefore must give strong incentives to the worker to self insure himself. During unemployment, since the worker has no income, the optimal unemployment benefit must provide enough money for consumption and saving[4], also if this will affect negatively effort incentives: the wage will be lower forever after just one period of unemployment.

During employment, the optimal tax is almost constant, while in a model *à la* Hopenhayn and Nicolini (1997) with alternate spells of unemployment, the tax is decreasing during employment spells. The level of payroll tax is much lower than in the case without access to financial markets, since a large part of insurance is obtained through personal savings.

The counterintuitive result of a replacement rate larger than 100% must be interpreted with caution. The main problem is the assumption of a non-increasing human capital level. The evidence clearly shows increasing levels of human capital during employment, also if it is difficult to quantify the effective growth rate. It is likely that an increasing profile of human capital during employment spells could reverse the quantitative results. The analysis of this extension is beyond the scope of the present work, and therefore left for future research.

The Chapter is organized as follows. Section 3.2 presents the problem in presence of human capital depreciation without access to financial markets. Section 3.3 extends the setup to include hidden access to financial markets. Section 3.5 presents numerical examples and Section 3.6 concludes.

[2]Effort during unemployment is the classical search effort. During employment spells, it can be interpreted as job retention effort.

[3]My simulations are not calibrated to real data, but it should be noticed that human capital depreciation is assumed very low with respect to standard estimates.

[4]I conjecture that the initial replacement rate will be lower than 100% if I allow human capital to rise during employment spells. Current work-in-progress is devoted to understand the implications of a human capital trend in line with empirical data.

3.2 Unemployment insurance with human capital

Here I present the basic framework with human capital depreciation. This work departs from Pavoni (forthcoming) by assuming continuous effort choice as in the previous Chapters. I also depart from Hopenhayn and Nicolini (1997): they assume that employment is an absorbing state, while I allow for multiple spells of employment and unemployment, as in Zhao (2001) and in Wang and Williamson (1996, 2002). The worker is endowed with human capital, and his labor wage depends on it. Initial human capital is given and equal to h_{-1}. There are two possible states of the world: $S \equiv \{U, E\}$ where U means unemployed and E means employed. Human capital is constant during employment spells[5], while it depreciates during unemployment periods, according to the following process:

$$h_t\left(s^t\right) = \begin{cases} h_{t-1}\left(s^{t-1}\right) & s_t = E \\ \\ (1-\delta)\, h_{t-1}\left(s^{t-1}\right) & s_t = U \end{cases}$$

The wage is function of the human capital of the worker

$$y_t\left(s_t, h_{t-1}\left(s^{t-1}\right)\right) = \begin{cases} F\left(h_{t-1}\left(s^{t-1}\right)\right) & s_t = E \\ \\ 0 & s_t = U \end{cases}$$

I also maintain the assumption in Pavoni (forthcoming) about the dependence of transition probabilities on the human capital level of the worker: $\pi\left(s_{t+1} \mid s_t, h_{t-1}\left(s^{t-1}\right), a_t\left(s^t\right)\right)$. In this setup, the worker exerts effort when employed to increase the probability to keep the job; when unemployed, effort increases the probability of finding a job, similarly to Wang and Williamson (1996). The agent's problem is then:

$$V\left(s_0, h_{-1}; \tau^{\infty}\right) =$$

$$= \max_{\{c_t(s^t), a_t(s^t)\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \left[u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right)\right] \Pi\left(s^t \mid s_0, h^{t-2}\left(s^{t-2}\right), a^{t-1}\left(s^{t-1}\right)\right) \right\}$$

$$s.t. \qquad h_t\left(s^t\right) = \begin{cases} (1+\rho)\, h_{t-1}\left(s^{t-1}\right) & s_t = E \\ \\ (1-\delta)\, h_{t-1}\left(s^{t-1}\right) & s_t = U \end{cases}$$

$$h_t\left(s^t\right) \geq 0 \qquad h_{-1} \quad given$$

Accordingly, agent's first order conditions with respect to effort are

$$\upsilon'\left(a_t\left(s^t\right)\right) = \sum_{j=1}^{\infty} \beta^j \sum_{s^{t+j}|s^t} \frac{\pi_a\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)}{\pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)} \times \tag{3.1}$$

$$\times \left[u\left(c_{t+j}\left(s^{t+j}\right)\right) - \upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right] \Pi\left(s^{t+j} \mid s^t, h^{t+j-1}\left(s^{t+j-1} \mid s^{t-1}\right), a^{t+j}\left(s^{t+j} \mid s^t\right)\right)$$

[5] Even if it simplifies numerical exercises, it is not realistic to assume that human capital remains constant during employment spells. Current work-in-progress is devoted to explore the case in which human capital increases.

Let $\beta^t \lambda_t(s^t)$ the Lagrange multiplier for (3.2). I obtain the Lagrangean:

$$L(s_0, \gamma, c^\infty, a^\infty, h^\infty, \lambda^\infty) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \left\{ y(s_t) - c_t(s^t) + \phi_t(s^t) \left[u(c_t(s^t)) - v(a_t(s^t)) \right] + \right.$$
$$\left. - \lambda_t(s^t) v'(a_t(s^t)) \right\} \Pi\left(s^t \mid s_0, h^{t-2}(s^{t-2}), a^{t-1}(s^{t-1})\right)$$

where

$$\phi_{t+1}(s^t, \widehat{s}) = \phi_t(s^t) + \lambda_t(s^t) \frac{\pi_a(s_{t+1} = \widehat{s} \mid s_t, h_{t-1}(s^{t-1}), a_t(s^t))}{\pi(s_{t+1} = \widehat{s} \mid s_t, h_{t-1}(s^{t-1}), a_t(s^t))} \quad \forall \widehat{s} \in S$$
$$\phi_0(s^0) = \gamma$$

$$h_t(s^t) = \begin{cases} h_{t-1}(s^{t-1}) & s_t = E \\ (1-\delta) h_{t-1}(s^{t-1}) & s_t = U \end{cases}, \qquad h_{-1} \quad \text{given}$$

Using the same arguments of Proposition 1, it is possible to show that the problem is recursively characterized by the Pareto weight $\phi_t(s^t)$ and human capital $h_t(s^t)$. Notice that the characterization of the optimal contract is obtained by solving the same Lagrangean first-order conditions (1.8)-(1.9), with the only difference that policy functions depend both on the Pareto weight and human capital stock. Moreover, if $\delta = 0$, we are back to the standard unemployment insurance model with alternate unemployment spells *à la* Wang and Williamson (1996).

3.3 Unemployment insurance with human capital and hidden savings

It is possible to account for hidden savings by extending the previous setup. To this purpose, consider the previous model and assume the worker is allowed to save or borrow, but the credit market is not observable by the principal.

The agent's problem is then:

$$V\left(s_0, h_{-1}, b_{-1}; \tau^\infty\right) =$$
$$= \max_{\left\{c_t(s^t), a_t(s^t), b_t(s^t)\right\}_{t=0}^\infty} \left\{ \sum_{t=0}^\infty \sum_{s^t} \beta^t \left[u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right)\right] \times \right.$$
$$\left. \times \Pi\left(s^t \mid s_0, h^{t-2}\left(s^{t-2}\right), a^{t-1}\left(s^{t-1}\right)\right) \right\}$$

$$s.t. \qquad c_t\left(s^t\right) + b_t\left(s^t\right) = y_t\left(s_t, h_{t-1}\left(s^{t-1}\right)\right) + \tau_t\left(s^t\right) + Rb_{t-1}\left(s^{t-1}\right)$$

$$h_t\left(s^t\right) = \begin{cases} h_{t-1}\left(s^{t-1}\right) & s_t = E \\ \left(1-\delta\right) h_{t-1}\left(s^{t-1}\right) & s_t = U \end{cases}$$

$$h_t\left(s^t\right) \geq 0 \qquad h_{-1}, b_{-1} \quad given$$

We interpret $\tau_t\left(s^t\right)$ in the following way: when the worker is unemployed, $\tau_t\left(s^t\right)$ is the unemployment benefit; when the worker is employed, $\tau_t\left(s^t\right)$ is a tax or a transfer from the principal. We can derive agent's first order conditions with respect to the effort:

$$\upsilon'\left(a_t\left(s^t\right)\right) = \sum_{j=1}^\infty \beta^j \sum_{s^{t+j}|s^t} \frac{\pi_a\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)}{\pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)} \times \tag{3.2}$$
$$\times \left[u\left(c_{t+j}\left(s^{t+j}\right)\right) - \upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right] \Pi\left(s^{t+j} \mid s^t, h^{t+j-1}\left(s^{t+j-1} \mid s^{t-1}\right), a^{t+j}\left(s^{t+j} \mid s^t\right)\right)$$

and the first order condition with respect to bonds:

$$u'\left(c_t\left(s^t\right)\right) = \beta R \sum_{s^{t+1}|s^t} u'\left(c_{t+1}\left(s^{t+1}\right)\right) \pi\left(s_{t+1} \mid s_t, h_{t-1}\left(s^{t-1}\right), a_t\left(s^t\right)\right) \tag{3.3}$$

Let $\beta^t \lambda_t\left(s^t\right)$ the Lagrange multiplier for (3.2), and $\beta^t \eta_t\left(s^t\right)$ the Lagrange multiplier for (3.3). I obtain the following Lagrangean:

$$L\left(s_0, \gamma, c^\infty, a^\infty, h^\infty, \lambda^\infty\right)$$
$$= \sum_{t=0}^\infty \sum_{s^t} \beta^t \left\{ y\left(s_t\right) - c_t\left(s^t\right) + \phi_t\left(s^t\right) \left[u\left(c_t\left(s^t\right), l_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right)\right] + \right.$$
$$\left. - \lambda_t\left(s^t\right) \upsilon'\left(a_t\left(s^t\right)\right) + \left[\eta_t\left(s^t\right) - \beta^{-1}\zeta_t\left(s^t\right)\right] u_c\left(c_t\left(s^t\right)\right) \right\} \times$$
$$\times \Pi\left(s^t \mid s_0, h^{t-2}\left(s^{t-2}\right), a^{t-1}\left(s^{t-1}\right)\right)$$

where

$$\phi_{t+1}\left(s^t, \widehat{s}\right) = \phi_t\left(s^t\right) + \lambda_t\left(s^t\right) \frac{\pi_a\left(s_{t+1} = \widehat{s} \mid s_t, h_{t-1}\left(s^{t-1}\right), a_t\left(s^t\right)\right)}{\pi\left(s_{t+1} = \widehat{s} \mid s_t, h_{t-1}\left(s^{t-1}\right), a_t\left(s^t\right)\right)} \quad \forall \widehat{s} \in S$$

$$\phi_0\left(s^0\right) = \gamma$$

$$\zeta_{t+1}\left(s^t, \widehat{s}\right) = \eta_t\left(s^t\right) \quad \forall \widehat{s} \in S \quad \text{and} \quad \zeta_0\left(s^0\right) = 0$$

$$h_t\left(s^t\right) = \begin{cases} h_{t-1}\left(s^{t-1}\right) & s_t = E \\ \left(1-\delta\right) h_{t-1}\left(s^{t-1}\right) & s_t = U \end{cases}, \qquad h_{-1} \quad \text{given}$$

At this point it should be clear that the problem is recursively characterized by the Pareto weight $\phi_t\left(s^t\right)$, human capital $h_t\left(s^t\right)$ and costate variable $\zeta_t\left(s^t\right)$. Notice that, if $\delta = 0$, this is the model of repeated moral hazard with hidden savings presented in Section 1.4.

3.4 Characterization of the optimal allocations

Taking first order conditions of the Lagrangean for the problem without financial markets, and using the law of motion for the Pareto weight, we can obtain the following:

$$\frac{1}{u'\left(c_t\right)} = \beta R \left[\frac{1}{u'\left(c_{t+1}^E\right)} \pi^E\left(h_{t-1}, a_t\right) + \frac{1}{u'\left(c_{t+1}^U\right)}\left(1 - \pi^E\left(h_{t-1}, a_t\right)\right)\right]$$

which implies (by Jensen's inequality):

$$u'\left(c_t\right) < \beta R \left[u'\left(c_{t+1}^E\right) \pi^E\left(h_{t-1}, a_t\right) + u'\left(c_{t+1}^U\right)\left(1 - \pi^E\left(h_{t-1}, a_t\right)\right)\right]$$

This is the well known inverted Euler equation by Rogerson (1985a). There is a difference here, which depends on the fact that transition probabilities are function of the level of human capital. In particular, notice that if $\pi^E\left(0, a_t\right) = 0$ for any a_t, then there will be a period T such that for any $t \geq T$ consumption must be constant[6]. More generally, if the transition probability is monotone increasing in human capital, for low levels of human capital the unemployment benefit scheme will tend to be flatter.
The characterization is very different in the case of hidden access to credit markets. The standard Euler equation is at work here:

$$u'\left(c_t\right) = \beta R \left[u'\left(c_{t+1}^E\right) \pi^E\left(h_{t-1}, a_t\right) + u'\left(c_{t+1}^U\right)\left(1 - \pi^E\left(h_{t-1}, a_t\right)\right)\right]$$

[6] This is the case for the functional form used in the simulations. In the examples, I limit the horizon to 5 years, and T is not reached before the end of the simulation.

However, notice that exactly the same considerations are valid here: there will be a period T such that for any $t \geq T$ consumption must be constant, and if the transition probability is monotone increasing in human capital, we will observe a flatter benefit slope for low human capital. The main difference is that, on average, when the worker has no access to financial markets his consumption will be decreasing, while this is not true if he has access to financial markets.

3.5 Numerical examples

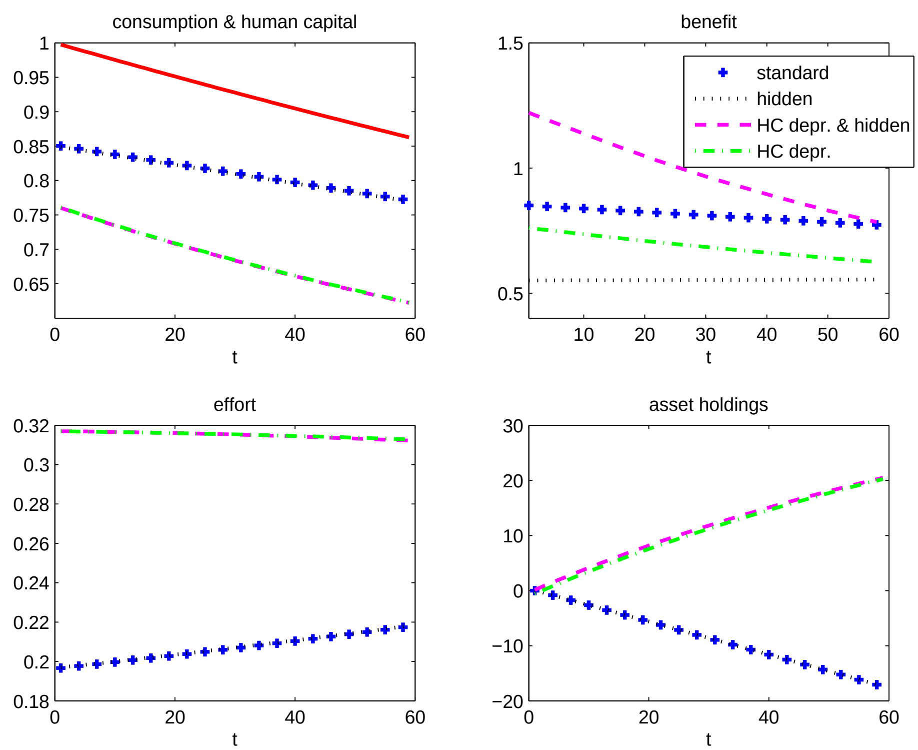

Figure 3.1: Consumption, replacement rate, effort and asset holdings for unemployed. Note: red solid line in the NW panel is human capital

In this section, I compare 4 different models. The first is the standard unemployment insurance model, with alternating unemployment spells as in Wang and Williamson (1996). The second is the model with human capital depreciation. The third is Abraham and Pavoni (forthcoming) model with hidden borrowing and lending. Finally, the fourth model accounts for hidden assets and human capital depreciation. I assume that one period is equivalent to one month, I use CRRA utility function for consumption, and quadratic disutility of effort as in the previous Chapters. The depreciation parameter δ is set such that yearly depreciation rate is 3%, as in Pavoni (forthcoming). The probability function is chosen to be

$$\pi(a) = a^{\nu} h^{1-\nu}, \quad a \in (0,1)$$

and notice that if $h = 1$ and $\delta = 0$, we are back to the case with no human capital analyzed in previous sections. All four models are based on the following parameterization:

α	ε	ν	σ	h^{MAX}	δ	β
0.5	2	0.1	2	1	0.0025	0.995

The value for initial Pareto weight of the agent is chosen so that the principal has zero expected cost, i.e.

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ y\left(s_t\right) - c_t\left(s^t\right) \right\} = 0$$

and I also assume $\beta R = 1$. In each simulation, the worker starts with a human capital level of 1 and zero asset holdings.
It is important to understand that this is not a calibrated version. The objective of the numerical exercise is to show some qualitative properties of the optimal allocation and policies. Future work will be devoted to a more careful quantitative assessment of the optimal unemployment compensation.
In Figure 3.1 it is possible to see the main differences between the four models. First, notice that the standard model (i.e. the model without access to credit market and without human capital depreciation) shows a decreasing replacement rate and increasing effort as in the previous literature. Adding human capital depreciation does not change the decreasing path of unemployment benefits, also if the initial replacement rate is much lower. This is what we expected: incentives to find a job must be stronger, since each month spent as unemployed implies a lower future wage, therefore the planner wants the worker to find a job quickly.
The big surprise comes from the case in which we have both human capital depreciation and hidden access to credit market: in this case, initial replacement rate is higher than 1. The worker already obtains a lot of insurance by saving. Moreover, anticipating that he will spend some periods unemployed, and therefore his wage will be lower in the future, he also wants to save during unemployment. Due to this reason, the optimal thing to do for the planner is to provide a very high unemployment benefit: part of it is consumed, but part of it is saved. The main point is that, even if the benefit is very generous, the consumption profile is decreasing during unemployment, therefore the agent has an incentive to look for a job. The benefit is decreasing during unemployment spell, but notice that it is larger than one for a very long period (around two years). This mainly depends on the parametrization chosen. For lower human capital depreciation rates, the replacement rate is less than one (see Figure 3.2). I conjecture that lower interest rates and a positive trend for human capital during employment periods can overturn this strange result.
Figure 3.1 also shows that consumption is much lower if we have human capital depreciation (around 10% lower than in the cases with constant human capital). Effort is

much higher, instead, also if it has a decreasing trend. Effort is lower and increasing for the models without human capital depreciation.

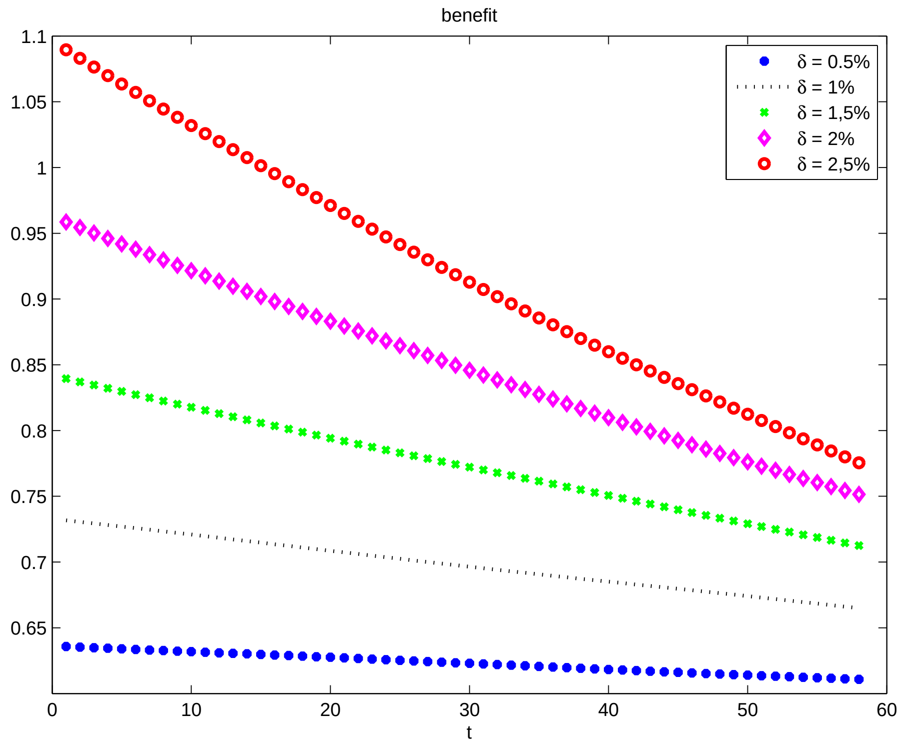

Figure 3.2: Unemployment benefits for different annual depreciation rates

It is also interesting to analyze what happens when the worker is employed (Figure 3.3). In this case, we can see that consumption is increasing in all cases. Notice that, when the worker has no access to financial market, the wage tax is decreasing during employment spell, while if he can secretly save or borrow the wage tax is almost constant. Moreover, if the agent has access to financial markets, the tax with human capital depreciation is around 50% higher than when human capital is constant.

Figure 3.4 gives a sample path for a generic worker. The fast and big accumulation of savings for self-insurance is evident in this figure, in comparison to what happens in a standard model and in a model with hidden assets accumulation but constant human capital. It is also worth noticing that unemployment spells are short (one or two periods).

Finally, I present the average allocation for 50000 simulations in Figure 3.5. On average, with human capital depreciation the worker rapidly accumulates a big stock of savings. This is not true if there is no human capital depreciation: in the standard model, the worker behaves as if he is getting indebted, while in the model with hidden assets and constant human capital he saves but in small amounts (see Figure 3.6).

3.6 Conclusions

In this Chapter, I have analyzed a model of optimal unemployment insurance with human capital depreciation and hidden access to financial markets. These two forces tend to drive unemployment benefits in opposite directions: depreciation of human capital makes the benefits decreasing during unemployment spell, while hidden assets accumulation does the opposite. Which effect dominates is a quantitative issue. In the parametrization used here, the first effect dominates. Benefits are decreasing during unemployment spell, but they are very generous for long time: for more than two years, the replacement rate is higher than 100%.

As mentioned, this is a first attempt to analyze this topic. In particular, simulations are not based on a serious calibration of the economy. An crucial assumption is the non-increasing trend for human capital. This is highly counterfactual. Current work-in-progress is exploring this issue, by using data on lifetime human capital trends to calibrate the model. Moreover, transition probabilities from unemployment and to unemployment are treated symmetrically, i.e. the same amount of effort and human capital generate the same transition probability to and from unemployment. A careful analysis should instead calibrate transition probabilities to data of actual flows.

Another important drawback of this analysis is the use of a partial equilibrium framework. However, endogenizing financial markets is important if we want to have a quantitative assessment of the importance of self-insurance versus public insurance. A general equilibrium treatment is a much more complicated task and beyond the scope of this work, and therefore will be the focus of future research.

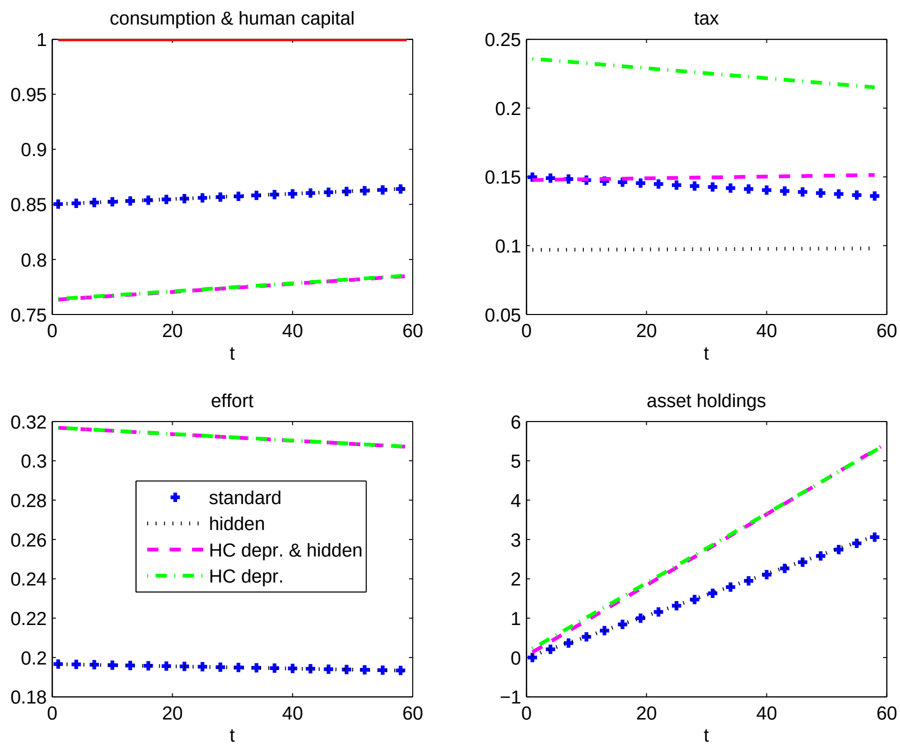

Figure 3.3: Consumption, replacement rate, effort and asset holdings for employed. Note: red solid line in the NW panel is human capital

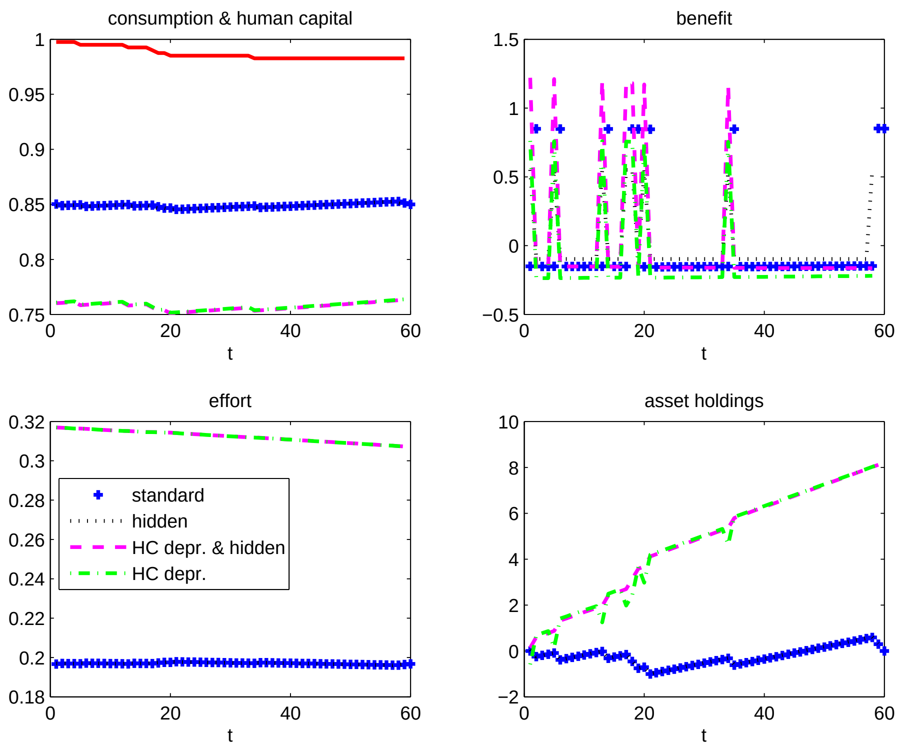

Figure 3.4: Consumption, replacement rate, effort and asset holdings, sample path. Note: red solid line in the NW panel is human capital

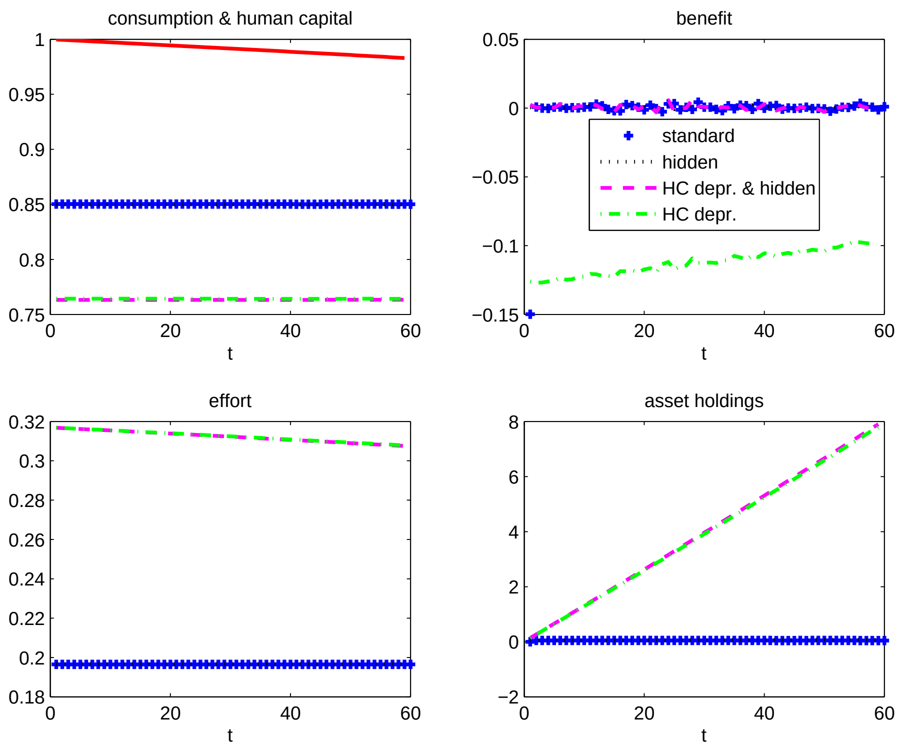

Figure 3.5: Consumption, replacement rate, effort and asset holdings, average for 50000 simulations. Note: red solid line in the NW panel is human capital

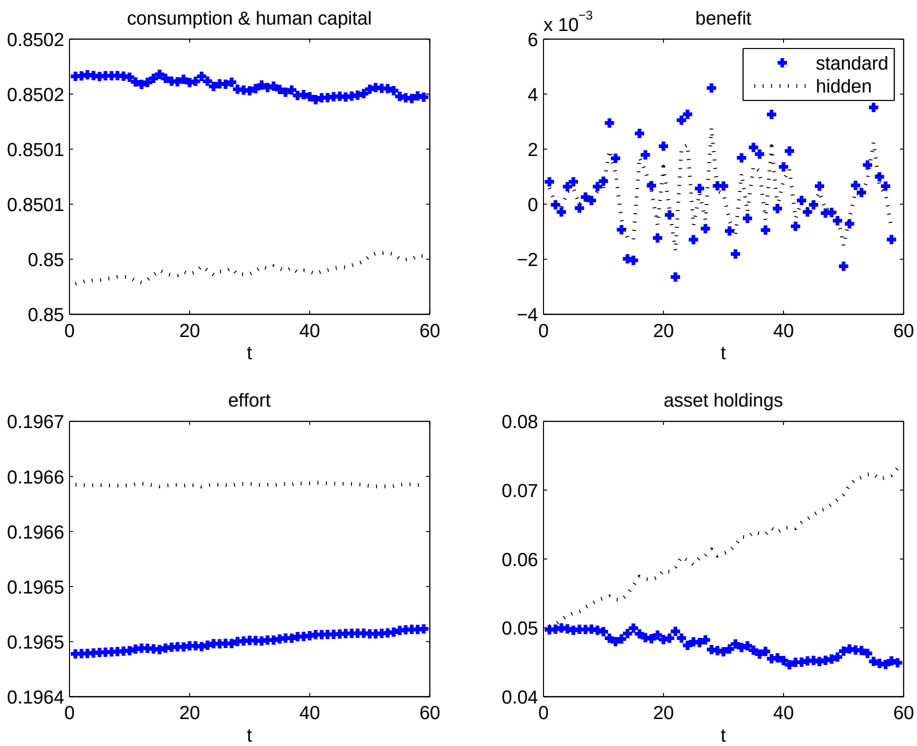

Figure 3.6: Consumption, replacement rate, effort and asset holdings, average for 50000 simulations: standard and hidden assets models. Note: red solid line in the NW panel is human capital

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A Proofs of Chapter 1

In this Appendix A, I collect all the proofs of Chapter 1
Proposition 1 Fix an arbitrary constant $K > 0$ and let $K_\theta = \max\{K, K\|\theta\|\}$. The operator

$$\begin{aligned}(T_K f)(s,\theta) &\equiv \min_{\{\chi>0:\|\chi\|\leq K_\theta\}} \max_{a,c} \left\{ h(a,c,\theta,\chi,s) + \beta \sum_{s'} \pi(s' \mid s,a) f(s',\theta'(s')) \right\} \\ s.t. \quad \theta'(s') &= \theta + \chi \frac{\pi_a(s' \mid s,a)}{\pi(s' \mid s,a)} \quad \forall s'\end{aligned}$$

is a contraction.
Proof. The space

$$\begin{aligned} M = \{ f : S \times \mathbb{R}^2 \longrightarrow \mathbb{R} \quad s.t. & \\ a) \qquad & \forall \alpha > 0 \quad f(\cdot, \alpha\theta) = \alpha f(\cdot,\theta) \\ b) \qquad & f(s,\cdot) \text{ is continuous and bounded} \quad \} \end{aligned}$$

will be our candidate, with norm

$$\|f\| = \sup\{|f(s,\theta)| : \|\theta\| \leq 1, s \in S\}$$

Marcet and Marimon (2009) show that M is a nonempty complete metric space. Now, fix a positive constant K and let $K_\theta = \max\{K, K\|\theta\|\}$. Define the auxiliary operator

$$\begin{aligned}(T_K f)(s,\theta) &\equiv \min_{\{\chi>0:\|\chi\|\leq K_\theta\}} \max_{a,c} \left\{ h(a,c,\theta,\chi,s) + \beta \sum_{s'} \pi(s' \mid s,a) f(s',\theta'(s')) \right\} \\ s.t. \quad \theta'(s') &= \theta + \chi \frac{\pi_a(s' \mid s,a)}{\pi(s' \mid s,a)} \quad \forall s'\end{aligned}$$

I have to show that $T_K : M \longrightarrow M$. Notice that

$$(T_K f)(s,\theta) = \theta h_0(a^*,c^*,s) + \chi^* h_1(a^*,c^*,s) + \beta \sum_{s'} \pi(s' \mid s,a^*) f\left(s', \theta^{*\prime}(s')\right)$$

hence by Schwartz's inequality

$$\begin{aligned}\|(T_K f)(s,\theta)\| \leq\; & \|\theta\| \|h_0(a^*,c^*,s)\| + \max\{K, K\|\theta\|\} \|h_1(a^*,c^*,s)\| \\ & + \beta \left(\max\{K, K\|\theta\|\} \left\| \frac{\pi_a(s' \mid s,a^*)}{\pi(s' \mid s,a^*)} \right\| + \|\theta\| \right) \left\| f\left(s', \frac{\theta^{*\prime}(s')}{\|\theta^{*\prime}(s')\|}\right) \right\|\end{aligned}$$

and therefore $(T_K f)(s,\phi)$ is bounded. A generalized Maximum Principle argument gives continuity of $(T_K f)(s,\phi)$. To check for homogeneity properties, let (a^*, c^*, χ^*) be such that

$$(T_K f)(s,\theta) = h(a^*,c^*,\theta,\chi^*,s) + \beta \sum_{s'} \pi(s' \mid s,a^*) f\left(s', \theta^{*\prime}(s')\right)$$

Then for any $\alpha > 0$ we get

$$\alpha \left(T_K f\right)(s,\theta) = \alpha \left[h\left(a^*, c^*, \theta, \chi^*, s\right) + \beta \sum_{s'} \pi\left(s' \mid s, a^*\right) f\left(s', \theta^{*\prime}\left(s'\right)\right)\right]$$

Therefore

$$\begin{aligned} & h\left(a^*, c^*, \alpha\theta, \alpha\chi^*, s\right) + \beta \sum_{s'} \pi\left(s' \mid s, a^*\right) f\left(s', \alpha\theta^{*\prime}\left(s'\right)\right) \\ = \;& \alpha \left[h\left(a^*, c^*, \theta, \chi^*, s\right) + \beta \sum_{s'} \pi\left(s' \mid s, a^*\right) f\left(s', \theta^{*\prime}\left(s'\right)\right)\right] \end{aligned}$$

Now take a generic χ, then define $\theta'_{a^*}\left(s'\right) = \varphi\left(\alpha\theta, \chi, a^*, s'\right)$ and $\theta'_a\left(s'\right) = \varphi\left(\theta, \chi^*, a, s'\right)$ for a feasible a. We can write:

$$\begin{aligned} & h\left(a^*, c^*, \alpha\theta, \chi, s\right) + \beta \sum_{s'} \pi\left(s' \mid s, a^*\right) f\left(s', \theta'_{a^*}\left(s'\right)\right) \\ = \;& \alpha \left[h\left(a^*, c^*, \theta, \frac{\chi}{\alpha}, s\right) + \beta \sum_{s'} \pi\left(s' \mid s, a^*\right) f\left(s', \frac{\theta'_{a^*}\left(s'\right)}{\alpha}\right)\right] \\ \geq \;& \alpha \left[h\left(a^*, c^*, \theta, \chi^*, s\right) + \beta \sum_{s'} \pi\left(s' \mid s, a^*\right) f\left(s', \theta^{*\prime}\left(s'\right)\right)\right] \\ \geq \;& \alpha \left[h\left(a, c, \theta, \chi^*, s\right) + \beta \sum_{s'} \pi\left(s' \mid s, a\right) f\left(s', \theta'_a\left(s'\right)\right)\right] \end{aligned}$$

and therefore

$$\begin{aligned} \left(T_K f\right)(s, \alpha\theta) &= h\left(a^*, c^*, \alpha\theta, \alpha\chi^*, s\right) + \beta \sum_{s'} \pi\left(s' \mid s, a^*\right) f\left(s', \alpha\theta^{*\prime}\left(s'\right)\right) \\ &= \alpha \left(T_K f\right)(s, \theta) \end{aligned}$$

and therefore the operator preserves the homogeneity properties. To see monotonicity, let $g, u \in M$ such that $g \leq h$. Therefore

$$\begin{aligned} & \max_{a,c} \left\{h\left(a, c, \theta, \chi, s\right) + \beta \sum_{s'} \pi\left(s' \mid s, a\right) g\left(s', \theta'\left(s'\right)\right)\right\} \\ \leq \;& \max_{a,c} \left\{h\left(a, c, \theta, \chi, s\right) + \beta \sum_{s'} \pi\left(s' \mid s, a\right) u\left(s', \theta'\left(s'\right)\right)\right\} \end{aligned}$$

and then

$$\begin{aligned} & \min_{\{\chi \geq 0 : \|\chi\| \leq K_\theta\}} \max_{a,c} \left\{h\left(a, c, \theta, \chi, s\right) + \beta \sum_{s'} \pi\left(s' \mid s, a\right) g\left(s', \theta'\left(s'\right)\right)\right\} \\ \leq \;& \min_{\{\chi \geq 0 : \|\chi\| \leq K_\theta\}} \max_{a,c} \left\{h\left(a, c, \theta, \chi, s\right) + \beta \sum_{s'} \pi\left(s' \mid s, a\right) u\left(s', \theta'\left(s'\right)\right)\right\} \end{aligned}$$

which implies $(T_K g)(s,\theta) \leq (T_K u)(s,\theta)$. To see discounting, let $k \in \mathbb{R}_+$, and define $f+k \in M$ as $(f+k)(s,\theta) = f(s,\theta)+k$. Therefore:

$$\begin{aligned} & \max_{a,c}\left\{h(a,c,\theta,\chi,s)+\beta\sum_{s'}\pi(s'\mid s,a)(g+k)(s',\theta'(s'))\right\} \\ = & \max_{a,c}\left\{h(a,c,\theta,\chi,s)+\beta\sum_{s'}\pi(s'\mid s,a)\,g(s',\theta'(s'))+\beta k\right\} \\ = & \max_{a,c}\left\{h(a,c,\theta,\chi,s)+\beta\sum_{s'}\pi(s'\mid s,a)\,g(s',\theta'(s'))\right\}+\beta k \end{aligned}$$

Hence we get

$$\begin{aligned} T_K(f+k)(s,\theta) &= \\ &= \min_{\{\chi\geq 0:\|\chi\|\leq K_\theta\}}\max_{a,c}\left\{h(a,c,\theta,\chi,s)+\beta\sum_{s'}\pi(s'\mid s,a)(f+k)(s',\theta'(s'))\right\} \\ &= \min_{\{\chi\geq 0:\|\chi\|\leq K_\theta\}}\max_{a,c}\left\{h(a,c,\theta,\chi,s)+\beta\sum_{s'}\pi(s'\mid s,a)\,f(s',\theta'(s'))\right\}+\beta k \\ &= (T_K f)(s,\theta)+\beta k \end{aligned}$$

and then $T_K(f+k) \leq T_K f + \beta k$. Now it is possible to use the above properties to show the contraction property for the operator T_K. In order to see this, let $f, g \in M$. By homogeneity, we get

$$\begin{aligned} f(s,\phi) &= g(s,\theta)+f(s,\theta)-g(s,\theta) \\ &\leq g(s,\theta)+|f(s,\theta)-g(s,\theta)| \end{aligned}$$

and then

$$f(s,\theta) \leq g(s,\theta)+\|f(s,\theta)-g(s,\theta)\|$$

Now applying the operator T_K and using monotonicity and discounting we get:

$$\begin{aligned} (T_K f)(s,\theta) &\leq T_K(g+\|f-g\|)(s,\theta) \\ &\leq (T_K g)(s,\theta)+\beta\|f-g\| \end{aligned}$$

which implies finally

$$\|T_K f - T_K g\| \leq \beta\|f-g\|$$

and given $\beta \in (0,1)$ this concludes the proof that the operator T_K is a contraction. ■

Lemma 1 *In the optimal contract,* $\phi_{t+1}(s^t,\widehat{s}_1) < \phi_t(s^t) < \phi_{t+1}(s^t,\widehat{s}_I)$ *for any* t.
Proof. Notice first that, for any t, $\exists i,j : \pi_a(\widehat{s}_i \mid s_t, a_t^*(s^t)) > 0$ and $\pi_a(\widehat{s}_j \mid s_t, a_t(s^t)) < 0$. Suppose not: then the only possibility is that $\pi_a(\widehat{s}_i \mid s_t, a_t(s^t)) = 0$ for any i

(otherwise, $\sum_{\widehat{s}_i} \pi_a(\widehat{s}_i \mid s_t, a_t(s^t)) \neq 0$, which is impossible). This implies, by (1.10), $0 = v'(a_t(s^t))$ which is a contradiction since $v(\cdot)$ is strictly increasing. Adding the full support assumption and the fact that $\lambda_t(s^t) > 0$, we get that $\exists i, j : \phi_{t+1}(s^t, \widehat{s}_j) < \phi_t(s^t) < \phi_{t+1}(s^t, \widehat{s}_i)$. By MLRC, $\phi_{t+1}(s^t, \widehat{s}_1) \leq \phi_{t+1}(s^t, \widehat{s}_j)$ for any j and $\phi_{t+1}(s^t, \widehat{s}_i) \leq \phi_{t+1}(s^t, \widehat{s}_I)$ for any i, which proves the statement. ■

Proposition 2 $\phi_t(s^t)$ *is a martingale that converges to zero.*
Proof. Use the law of motion of $\phi_t(s^t)$ and take expectations on both sides:

$$\sum_{s_{t+1}} \phi_{t+1}\left(s^t, s_{t+1}\right) \pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right) =$$

$$= \phi_t\left(s^t\right) + \lambda_t\left(s^t\right) \sum_{s_{t+1}} \frac{\pi_a(s_{t+1} \mid s_t, a_t(s^t))}{\pi(s_{t+1} \mid s_t, a_t(s^t))} \pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)$$

Notice that $\lambda_t(s^t) \sum_{s_{t+1}} \frac{\pi_a\left(s_{t+1}|s_t, a_t\left(s^t\right)\right)}{\pi(s_{t+1}|s_t, a_t(s^t))} \pi(s_{t+1} \mid s_t, a_t(s^t)) = 0$, which implies

$$E_t^a\left[\phi_{t+1} \mid s^t\right] = \phi_t\left(s^t\right) \tag{A.1}$$

where $E_t^a[\cdot]$ is the expectation operator induced by $a_t(s^t)$. Therefore $\phi_t(s^t)$ is a martingale. To see that it converges to zero, rewrite (A.1) by using (1.8):

$$E_t^a\left[\frac{1}{u_c(c_{t+1}(s^{t+1}))}\right] = \frac{1}{u_c(c_t(s^t))}$$

By Inada conditions, $\frac{1}{u_c(c_t(s^t))}$ is bounded above zero and below infinity. Therefore $\phi_t(s^t)$ is a nonnegative martingale, and by Doob's theorem it converges almost surely to a random variable (call it X). To see that $X = 0$ almost surely, I follow the proof strategy of Thomas and Worrall (1990), to which I refer for details. Suppose not, and take a path $\{s^t\}_{t=0}^{\infty}$ such that $\lim_{t\to\infty} \phi_t(s^t) = \overline{\phi} > 0$ and state $\widehat{s}_I$ happens infinitely many times. I claim that this sequence cannot exist. Take a subsequence $\left\{s^{t(k)}\right\}_{k=1}^{\infty}$ of $\{s^t\}_{t=0}^{\infty}$ such that $s_{t(k)} = \widehat{s}_I\ \forall k$. This subsequence has to converge to some limit $\overline{\overline{\phi}} > 0$, since at some point will be in a ϵ-neighborhood of $\overline{\phi}$ for some $\epsilon > 0$. Call $f(\phi_t(s^t), \widehat{s}_i) = \phi_{t+1}(s^t, \widehat{s}_i)$ and notice that $f(\cdot)$ is continuous, hence $\lim_{k\to\infty} f\left(\phi_{t(k)}\left(s^{t(k)}\right), \widehat{s}_I\right) = f\left(\overline{\overline{\phi}}, \widehat{s}_I\right)$. By definition, $f\left(\phi_{t(k)}\left(s^{t(k)}\right), \widehat{s}_I\right) = \phi_{t(k)+1}(s^t, \widehat{s}_I)$, then $\lim_{k\to\infty} \phi_{t(k)+1}\left(s^{t(k)}, \widehat{s}_I\right) = f\left(\overline{\overline{\phi}}, \widehat{s}_I\right)$. However, notice that it must be $\lim_{k\to\infty} \phi_{t(k)}\left(s^{t(k)}\right) = \overline{\phi}$ and $\lim_{k\to\infty} \phi_{t(k)+1}\left(s^{t(k)}, \widehat{s}_I\right) = \overline{\phi}$. But by Lemma 1, $\phi_{t(k)}\left(s^{t(k)}\right) < \phi_{t(k)+1}\left(s^{t(k)}, \widehat{s}_I\right)$ for any k. Therefore, this is a contradiction and this sequence cannot exist. Since paths where state $\widehat{s}_I$ occurs only a finite number of times have probability zero, this implies that

$$\Pr\left\{\lim_{t\to\infty} \phi_t\left(s^t\right) > 0\right\} = 0$$

which implies $X = 0$ almost surely. ■

B Capital accumulation

Recursivity

Following the same steps as in Section 1.2, define the following generalized version of the problem:

$$\begin{aligned}
W_{\Phi}^{SWF}\left(s_0,k_{-1}\right) = \max_{\{a_t(s^t),c_t(s^t),k_t(s^t)\}_{t=0}^{\infty}\in\Gamma^K} \overline{\phi}^0 \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left[A\left(s^t\right)f\left(k_{t-1}\left(s^{t-1}\right)\right)-c_t\left(s^t\right)-\right.\\
\left.-k_t\left(s^t\right)+\left(1-\delta\right)k_{t-1}\left(s^{t-1}\right)\right]\Pi\left(s^t\mid s_0,a^{t-1}\left(s^{t-1}\right)\right)+\\
+\gamma\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left[u\left(c_t\left(s^t\right)\right)-\upsilon\left(a_t\left(s^t\right)\right)\right]\Pi\left(s^t\mid s_0,a^{t-1}\left(s^{t-1}\right)\right)\\
s.t.\quad \upsilon_a'\left(a_t\left(s^t\right)\right)=\sum_{j=1}^{\infty}\beta^j\sum_{s^{t+j}|s^t}\frac{\pi_a\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}{\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}\times\\
\times\left[u\left(c_{t+j}\left(s^{t+j}\right)\right)-\upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right]\Pi\left(s^{t+j}\mid s^t,a^{t+j-1}\left(s^{t+j-1}\mid s^t\right)\right)\\
\forall s^t,t\geq 0\\
c_t\left(s^t\right)+k_t\left(s^t\right)-\left(1-\delta\right)k_{t-1}\left(s^{t-1}\right)\leq A\left(s_t\right)f\left(k_{t-1}\left(s^{t-1}\right)\right)
\end{aligned}$$

The Lagrangean is:

$$\begin{aligned}
L_{\Phi}\left(s_0,\gamma,c^{\infty},a^{\infty},\lambda^{\infty}\right)=\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left\{\overline{\phi}^0\left[A\left(s^t\right)f\left(k_{t-1}\left(s^{t-1}\right)\right)-c_t\left(s^t\right)-k_t\left(s^t\right)+\right.\right.\\
\left.\left.+\left(1-\delta\right)k_{t-1}\left(s^{t-1}\right)\right]+\gamma\left[u\left(c_t\left(s^t\right)\right)-\upsilon\left(a_t\left(s^t\right)\right)\right]\right\}\Pi\left(s^t\mid s_0,a^{t-1}\left(s^{t-1}\right)\right)+\\
-\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\lambda_t\left(s^t\right)\left\{\upsilon'\left(a_t\left(s^t\right)\right)-\sum_{j=1}^{\infty}\beta^j\sum_{s^{t+j}|s^t}\frac{\pi_a\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}{\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}\times\right.\\
\times\left[u\left(c_{t+j}\left(s^{t+j}\right)\right)-\upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right]\times\\
\left.\times\Pi\left(s^{t+j}\mid s^t,a^{t+j-1}\left(s^{t+j-1}\mid s^t\right)\right)\right\}\Pi\left(s^t\mid s_0,a^{t-1}\left(s^{t-1}\right)\right)
\end{aligned}$$

Notice that $r\left(a,c,k',k,s\right)\equiv A\left(s\right)f\left(k\right)-c-k'+\left(1-\delta\right)k$ is uniformly bounded, so there exists a lower bound $\underline{\kappa}$ such that $r\left(a,c,k',k,s\right)\geq\underline{\kappa}$. We can therefore define $\kappa<\frac{\underline{\kappa}}{1-\beta}$. Define $\varphi\left(\phi,\lambda,s'\right)\equiv\phi+\lambda\frac{\pi_a(s'|s,a)}{\pi(s'|s,a)}$, $h_0^P\left(a,c,k',k,s\right)\equiv r\left(a,c,k',k,s\right)$, $h_1^P\left(a,c,k',k,s\right)\equiv r\left(a,c,k',k,s\right)-\kappa$, $h_0^{ICC}\left(a,c,k',k,s\right)\equiv u\left(c\right)-\upsilon\left(a\right)$, $h_1^{ICC}\left(a,c,k',k,s\right)\equiv-\upsilon'\left(a\right)$, $\Phi\equiv\begin{bmatrix}\phi^0 & \phi\end{bmatrix}\in\mathbb{R}^2$, $\Lambda\equiv\begin{bmatrix}\lambda^0 & \lambda\end{bmatrix}$ and

$$\begin{aligned}
h\left(a,c,k',\Phi,\Lambda,k,s\right) &\equiv \Phi h_0\left(a,c,k',k,s\right)+\Lambda h_1\left(a,c,k',k,s\right)\\
&\equiv \begin{bmatrix}\phi^0 & \phi\end{bmatrix}\begin{bmatrix}h_0^P\left(a,c,k',k,s\right)\\ h_0^{ICC}\left(a,c,k',k,s\right)\end{bmatrix}+\begin{bmatrix}\lambda^0 & \lambda\end{bmatrix}\begin{bmatrix}h_1^P\left(a,c,k',k,s\right)\\ h_1^{ICC}\left(a,c,k',k,s\right)\end{bmatrix}
\end{aligned}$$

which is homogenous of degree 1 in (Φ, Λ). The Lagrangean can be written as:

$$
\begin{aligned}
& L_{\Phi}\left(s_0, k_{-1}, \gamma, c^{\infty}, a^{\infty}, k^{\infty}, \Lambda^{\infty}\right)= \\
& \quad =\sum_{t=0}^{\infty} \sum_{s^t} \beta^t h\left(a_t\left(s^t\right), c_t\left(s^t\right), k_t\left(s^t\right), \Phi_t\left(s^t\right), \Lambda_t\left(s^t\right), k_{t-1}\left(s^{t-1}\right), s_t\right) \times \\
& \quad \times \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_{t+1}\left(s^t, \widehat{s}\right) & = \varphi\left(\Phi_t\left(s^t\right), \Lambda_t\left(s^t\right), \widehat{s}\right) \quad \forall \widehat{s} \in S \\
\Phi_0\left(s^0\right) & = \begin{bmatrix} \overline{\phi}^0 & \gamma \end{bmatrix}
\end{aligned}
$$

We can associate a saddle point functional equation to this Lagrangean

$$
\begin{gathered}
J(s, k, \Phi)=\min_{\Lambda} \max_{a, c, k'}\left\{h\left(a, c, k', \Phi, \Lambda, k, s\right)+\beta \sum_{s'} \pi\left(s' \mid s, a\right) J\left(s', k', \Phi'\left(s'\right)\right)\right\} \\
s.t. \quad \Phi'\left(s'\right)=\Phi+\Lambda \frac{\pi_a\left(s' \mid s, a\right)}{\pi\left(s' \mid s, a\right)} \quad \forall s'
\end{gathered}
$$

The following Proposition shows that the RHS operator is a contraction mapping.

Proposition 11 *Fix an arbitrary constant $K>0$ and let $K_{\Phi}=\max\{K, K\|\Phi\|\}$. The operator*

$$
\begin{aligned}
\left(T_K f\right)(s, k, \Phi) & \equiv \min_{\{\Lambda>0:\|\Lambda\| \leq K_{\Phi}\}} \max_{a, c, k'}\left\{h\left(a, c, k', \Phi, \Lambda, k, s\right)+\beta \sum_{s'} \pi\left(s' \mid s, a\right) f\left(s', k', \Phi'\left(s'\right)\right)\right\} \\
s.t. \quad \Phi'\left(s'\right) & = \Phi+\Lambda \frac{\pi_a\left(s' \mid s, a\right)}{\pi\left(s' \mid s, a\right)} \quad \forall s'
\end{aligned}
$$

is a contraction.

Proof. *Straightforward by repeating the steps to prove Proposition 1 in the following space of functions:*

$$
\begin{aligned}
M=\{f: S \times K \times \mathbb{R}^2 \longrightarrow \mathbb{R} \quad & s.t. \\
& a) \quad \forall \alpha>0 \quad f(\cdot, \cdot, \alpha \Phi)=\alpha f(\cdot, \cdot, \Phi) \\
& b) \quad f(s, \cdot, \cdot) \text{ is continuous and bounded } \}
\end{aligned}
$$

with norm

$$
\|f\|=\sup\{|f(s, k, \Phi)|:\|\Phi\| \leq 1, s \in S, k \in K\}
$$

■

First-order conditions

From the Lagrangean

$$
\begin{aligned}
&L\left(\varsigma^{\infty},\nu^{\infty},k^{\infty},\phi^{\infty}\right)= \\
&\quad=\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left\{A\left(s_t\right)f\left(k_{t-1}\left(s^{t-1}\right)\right)-c_t\left(s^t\right)-k_t\left(s^t\right)+\left(1-\delta\right)k_{t-1}\left(s^{t-1}\right)+\right.\\
&\qquad\left.+\phi_t\left(s^t\right)\left[u\left(c_t\left(s^t\right)\right)-\upsilon\left(a_t\left(s^t\right)\right)\right]-\lambda_t\left(s^t\right)\upsilon'\left(a_t\left(s^t\right)\right)\right\}\Pi\left(s^t\mid h_0,a^{t-1}\left(s^{t-1}\right)\right)
\end{aligned}
$$

we get:

$$
c_t\left(s^t\right):\quad 0=-1+\phi_t\left(s^t\right)u_c\left(c_t\left(s^t\right)\right)
$$

$$
\begin{aligned}
a_t\left(s^t\right)\quad:\quad & 0=-\lambda_t\left(s^t\right)\upsilon''\left(a_t\left(s^t\right)\right)-\phi_t\left(s^t\right)\upsilon'\left(a_t\left(s^t\right)\right)+\\
&+\sum_{j=1}^{\infty}\beta^j\sum_{s^{t+j}|s^t}\frac{\pi_a\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}{\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}\left\{A\left(s_{t+j}\right)f\left(k_{t+j-1}\left(s^{t+j-1}\right)\right)-\right.\\
&-c_{t+j}\left(s^{t+j}\right)-k_{t+j}\left(s^{t+j}\right)+\\
&+\left(1-\delta\right)k_{t+j-1}\left(s^{t+j-1}\right)-\lambda_{t+j}\left(s^{t+j}\right)\upsilon'\left(a_{t+j}\left(s^{t+j}\right)\right)+\\
&\left.+\phi_{t+j}\left(s^{t+j}\right)\left[u\left(c_{t+j}\left(s^{t+j}\right)\right)-\upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right]\right\}\Pi\left(s^{t+j}\mid s^t,a^{t+j-1}\left(s^{t+j-1}\right)\right)+\\
&+\beta\lambda_t\left(s^t\right)\sum_{s^{t+1}|s^t}\frac{\partial\left(\frac{\pi_a\left(s_{t+1}|s_t,a_t\left(s^t\right)\right)}{\pi\left(s_{t+1}|s_t,a_t\left(s^t\right)\right)}\right)}{\partial a}\times\\
&\qquad\times\left[u\left(c_{t+1}\left(s^{t+1}\right)\right)-\upsilon\left(a_{t+1}\left(s^{t+1}\right)\right)\right]\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_t\left(s^t\right)\quad:\quad & 0=-\upsilon_a'\left(a_t\left(s^t\right)\right)+\sum_{j=1}^{\infty}\beta^j\sum_{s^{t+j}|s^t}\frac{\pi_a\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}{\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}\times\\
&\times\left[u\left(c_{t+j}\left(s^{t+j}\right)\right)-\upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right]\Pi\left(s^{t+j}\mid s^t,a^{t+j-1}\left(s^{t+j-1}\mid s^t\right)\right)
\end{aligned}
$$

$$
k_t\left(s^t\right):\quad 0=1-\beta\sum_{s^{t+1}|s^t}\left[A\left(s_{t+1}\right)f'\left(k_t\left(s^t\right)\right)+\left(1-\delta\right)\right]\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)
$$

C Hidden assets

Recursivity

Define the following generalized version of the problem:

$$W_\theta^{SWF}(s_0) = \max_{\{a_t(s^t),c_t(s^t)\}_{t=0}^\infty \in \Gamma^{HA}} \overline{\phi}^0 \sum_{t=0}^\infty \sum_{s^t} \beta^t \left[y(s_t) - c_t(s^t)\right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) +$$

$$+\gamma \sum_{t=0}^\infty \sum_{s^t} \beta^t \left(u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right)\right) \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)$$

$$s.t. \quad \upsilon'\left(a_t\left(s^t\right)\right) = \sum_{j=1}^\infty \beta^j \sum_{s^{t+j}|s^t} \frac{\pi_a\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)}{\pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)} \times$$

$$\times \left[u\left(c_{t+j}\left(s^{t+j}\right)\right) - \upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right] \Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1} \mid s^t\right)\right)$$

$$\forall s^t, t \geq 0$$

$$u'\left(c_t\left(s^t\right)\right) = \beta R \sum_{s_{t+1}} u'\left(c_{t+1}\left(s^t, s_{t+1}\right)\right) \pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)$$

The Lagrangean is:

$$L_\theta\left(s_0, \gamma, c^\infty, a^\infty, \lambda^\infty, \eta^\infty\right) = \sum_{t=0}^\infty \sum_{s^t} \beta^t \left\{\overline{\phi}^0 \left[y\left(s_t\right) - c_t\left(s^t\right)\right] +\right.$$

$$\left.+\gamma\left[u\left(c_t\left(s^t\right)\right) - \upsilon\left(a_t\left(s^t\right)\right)\right]\right\} \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) +$$

$$-\sum_{t=0}^\infty \sum_{s^t} \beta^t \lambda_t\left(s^t\right) \left\{\upsilon'\left(a_t\left(s^t\right)\right) - \sum_{j=1}^\infty \beta^j \sum_{s^{t+j}|s^t} \frac{\pi_a\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)}{\pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)} \times\right.$$

$$\left.\times\left[u\left(c_{t+j}\left(s^{t+j}\right)\right) - \upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right] \Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1} \mid s^t\right)\right)\right\} \times$$

$$\times \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) +$$

$$+\sum_{t=0}^\infty \sum_{s^t} \beta^t \eta_t\left(s^t\right) \left[u_c\left(c_t\left(s^t\right)\right) - \sum_{s^t+1} u_c\left(c_{t+1}\left(s^{t+1}\right)\right) \pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)\right] \times$$

$$\times \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)$$

Notice that $r(a,c,s) \equiv y(s) - c$ is uniformly bounded by debt limits, therefore there exists a lower bound $\underline{\kappa}$ such that $r(a,c,s) \geq \underline{\kappa}$. As before, we can define $\kappa < \frac{\underline{\kappa}}{1-\beta}$, $\varphi^1(\phi,\lambda,s') \equiv \phi + \lambda \frac{\pi_a(s'|s,a)}{\pi(s'|s,a)}$, $\varphi^2(\zeta,\eta,s') \equiv \eta$, $\Psi(\phi,\zeta,\lambda,\eta,s') \equiv \begin{bmatrix} \varphi^1(\phi,\lambda,s') \\ \varphi^2(\zeta,\eta,s') \end{bmatrix}$, $h_0^P(a,c,s) \equiv r(a,c,s)$, $h_1^P(a,c,s) \equiv r(a,c,s) - \kappa$, $h_0^{ICC}(a,c,s) \equiv u(c) - \upsilon(a)$, $h_1^{ICC}(a,c,s) \equiv -\upsilon'(a)$, $h_0^{EE}(a,c,s) \equiv -\beta^{-1} u_c'(c)$, $h_1^{EE}(a,c,s) \equiv u_c'(c)$, $\theta \equiv \begin{bmatrix} \phi^0 & \phi & \zeta \end{bmatrix} \in \mathbb{R}^3$, $\chi \equiv$

$\begin{bmatrix}\lambda^0 & \lambda & \eta\end{bmatrix}$ and

$$\begin{aligned}
h(a,c,\theta,\chi,s) &\equiv \theta h_0(a,c,s) + \chi h_1(a,c,s) \\
&\equiv \begin{bmatrix}\phi^0 & \phi & \zeta\end{bmatrix}\begin{bmatrix} h_0^P(a,c,s) \\ h_0^{ICC}(a,c,s) \\ h_0^{EE}(a,c,s)\end{bmatrix} + \begin{bmatrix}\lambda^0 & \lambda & \eta\end{bmatrix}\begin{bmatrix} h_1^P(a,c,s) \\ h_1^{ICC}(a,c,s) \\ h_1^{EE}(a,c,s)\end{bmatrix}
\end{aligned}$$

which is homogenous of degree 1 in (θ, χ). The Lagrangean can be written as:

$$L_\theta(s_0,\gamma,c^\infty,a^\infty,\chi^\infty) =$$
$$= \sum_{t=0}^{\infty}\sum_{s^t}\beta^t h\left(a_t\left(s^t\right), c_t\left(s^t\right), \theta_t\left(s^t\right), \chi_t\left(s^t\right), s_t\right)\Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)$$

where

$$\begin{aligned}
\theta_{t+1}\left(s^t,\widehat{s}\right) &= \Psi\left(\theta_t\left(s^t\right), \chi_t\left(s^t\right), \widehat{s}\right) \quad \forall\widehat{s}\in S \\
\theta_0\left(s^0\right) &= \begin{bmatrix}\overline{\phi}^0 & \gamma & 0\end{bmatrix}
\end{aligned}$$

We can associate a saddle point functional equation to this Lagrangean

$$J(s,\theta) = \min_{\chi}\max_{a,c}\left\{h(a,c,\theta,\chi,s) + \beta\sum_{s'}\pi\left(s' \mid s,a\right)J\left(s',\theta'\left(s'\right)\right)\right\} \qquad \text{(C.1)}$$
$$s.t. \quad \theta'\left(s'\right) = \Psi\left(\theta,\chi,s'\right) \quad \forall s'$$

The following Proposition shows that the RHS operator is a contraction mapping.

Proposition 12 *Fix an arbitrary constant $K > 0$ and let $K_\theta = \max\{K, K\|\theta\|\}$. The operator*

$$(T_K f)(s,\theta) \equiv \min_{\{\chi>0:\|\chi\|\le K_\theta\}}\max_{a,c}\left\{h(a,c,\theta,\chi,s) + \beta\sum_{s'}\pi\left(s' \mid s,a\right)f\left(s',\theta'\left(s'\right)\right)\right\}$$
$$s.t. \quad \theta'\left(s'\right) = \Psi\left(\theta,\chi,s'\right) \quad \forall s'$$

is a contraction.

Proof. *Straightforward by repeating the steps to prove Proposition 1 in the following space of functions:*

$$\begin{aligned}
M = \{f : S\times\mathbb{R}^3 \longrightarrow \mathbb{R} \quad & s.t. \\
& a) \quad \forall\alpha > 0 \quad f(\cdot,\alpha\theta) = \alpha f(\cdot,\theta) \\
& b) \quad f(s,\cdot) \text{ is continuous and bounded} \quad \}
\end{aligned}$$

with norm

$$\|f\| = \sup\{|f(s,\theta)| : \|\theta\| \le 1, s\in S\}$$

■

First-order conditions for the hidden asset model

$$L\left(s_0,\gamma,c^{\infty},a^{\infty},\lambda^{\infty},\eta^{\infty}\right)=\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left\{y\left(s_t\right)-c_t\left(s^t\right)+\phi_t\left(s^t\right)\left[u\left(c_t\left(s^t\right)\right)-\upsilon\left(a_t\left(s^t\right)\right)\right]+\right.$$
$$\left.-\lambda_t\left(s^t\right)\upsilon'\left(a_t\left(s^t\right)\right)+\left[\eta_t\left(s^t\right)-\beta^{-1}\zeta_t\left(s^t\right)\right]u_c\left(c_t\left(s^t\right)\right)\right\}\Pi\left(s^t\mid s_0,a^{t-1}\left(s^{t-1}\right)\right)$$

$$c_t\left(s^t\right):\quad 0=-1+\phi_t\left(s^t\right)u_c\left(c_t\left(s^t\right)\right)+\left[\eta_t\left(s^t\right)-\beta^{-1}\zeta_t\left(s^t\right)\right]u_{cc}\left(c_t\left(s^t\right)\right)$$

$$\begin{aligned}
a_t\left(s^t\right)\quad:\quad & 0=-\lambda_t\left(s^t\right)\upsilon''\left(a_t\left(s^t\right)\right)-\phi_t\left(s^t\right)\upsilon'\left(a_t\left(s^t\right)\right)+\\
&+\sum_{j=1}^{\infty}\beta^j\sum_{s^{t+j}|s^t}\frac{\pi_a\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}{\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}\left\{y\left(s_t\right)-c_t\left(s^t\right)-\right.\\
&-\lambda_{t+j}\left(s_{t+j}\right)\upsilon'\left(a_{t+j}\left(s^{t+j}\right)\right)-+\phi_{t+j}\left(s^{t+j}\right)\left[u\left(c_{t+j}\left(s^{t+j}\right)\right)-\upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right]+\\
&\left.+\left[\eta_{t+j}\left(s^{t+j}\right)-\beta^{-1}\zeta_{t+j}\left(s^{t+j}\right)\right]u_c\left(c_{t+j}\left(s^{t+j}\right)\right)\right\}\times\\
&\qquad\times\Pi\left(s^{t+j}\mid s^t,a^{t+j-1}\left(s^{t+j-1}\right)\right)+\\
&+\beta\lambda_t\left(s^t\right)\sum_{s^{t+1}|s^t}\frac{\partial\left(\frac{\pi_a\left(s_{t+1}|s_t,a_t\left(s^t\right)\right)}{\pi\left(s_{t+1}|s_t,a_t\left(s^t\right)\right)}\right)}{\partial a}\times\\
&\qquad\times\left[u\left(c_{t+1}\left(s^{t+1}\right)\right)-\upsilon\left(a_{t+1}\left(s^{t+1}\right)\right)\right]\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
\lambda_t\left(s^t\right)\quad:\quad & 0=-\upsilon'\left(a_t\left(s^t\right)\right)+\sum_{j=1}^{\infty}\sum_{s^{t+j}|s^t}\frac{\pi_a\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}{\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)}\times\\
&\times\left[\beta^j\left[u\left(c_{t+j}\left(s^{t+j}\right)\right)-\upsilon\left(a_{t+j}\left(s^{t+j}\right)\right)\right]\Pi\left(s^{t+j}\mid s^t,a^{t+j-1}\left(s^{t+j-1}\mid s^t\right)\right)\right]
\end{aligned}$$

$$\eta_t\left(s^t\right):\quad 0=u'\left(c_t\left(s^t\right)\right)-\sum_{s^{t+1}|s^t}u'\left(c_{t+1}\left(s^t,s_{t+1}\right)\right)\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)$$

The verification procedure

There are not known conditions under which the first-order approach is valid in the framework with hidden effort and hidden assets. Therefore, we cannot be sure that a first-order approach delivers the correct optimal allocation: it is possible that the solution obtained does not satisfies the true incentive compatibility constraint of the original problem. However we can verify it by a simple numerical procedure similar to the one proposed by Abraham and Pavoni (forthcoming): we remaximize the lifetime utility of the agent, by taking as given the optimal transfer scheme implied by the solution of the Pareto problem; if remaximization delivers a welfare gain to the agent, the solution obtained with first-order approach does not satisfy incentive compatibility. Instead, if no gain is possible, then the first-order approach is valid.
We solve the following problem:

$$V\left(s_0, b_{-1}, \gamma, 0\right) =$$
$$= \max_{\left\{c_t^V(s^t), a_t^V(s^t), b_t^V(s^t)\right\}_{t=0}^{\infty} \in \Gamma} \left\{ \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \left[u\left(c_t^V\left(s^t\right)\right) - v\left(a_t^V\left(s^t\right)\right)\right] \Pi\left(s^t \mid s_0, a^{V,t-1}\left(s^{t-1}\right)\right) \right\}$$

$$s.t. \quad c_t^V\left(s^t\right) + b_t^V\left(s^t\right) = y\left(s_t\right) + T(s_t, \phi_t\left(s^t\right), \zeta_t\left(s^t\right)) + Rb_{t-1}^V\left(s^{t-1}\right)$$
$$b_{-1} \quad \text{given}$$

$$\phi_{t+1}\left(s^t, \widehat{s}\right)\right) = \varphi^1(\widehat{s}, \phi_t\left(s^t\right), \zeta_t\left(s^t\right)) \quad \forall \widehat{s} \in S \quad \text{and} \qquad \phi_0\left(s^0\right) = \gamma$$

$$\zeta_{t+1}\left(s^t, \widehat{s}\right) = \varphi^2(\widehat{s}, \phi_t\left(s^t\right), \zeta_t\left(s^t\right)) \quad \forall \widehat{s} \in S \quad \text{and} \qquad \zeta_0\left(s^0\right) = 0$$

where $T(\cdot)$, $\varphi^1(\cdot)$ and $\varphi^2(\cdot)$ are the policy functions derived from Lagrangean (1.13), and are exogenous from the point of view of the agent (they define the transfer policy of the principal). It is obvious that this problem is recursive in the state space (s, ϕ, ζ, b), but notice that ϕ and ζ are exogenous states. Once we get the policy functions and the value function, we can calculate the welfare gain from reoptimization with respect to the optimal allocation obtained with first-order approach: if this difference is zero (in numerical terms), then the Lagrangean first-order method delivers the solution of the original problem.

D Multiple agents

Recursivity

First of all, notice that our problem is already in the form of a SWF maximizations, therefore we can apply directly the saddle-point functional equation to it. In this case, let $r(a_i, c_i, s) \equiv u(c_i) - \upsilon(a_i)$, $\varphi(\phi_i, \lambda_i, s') \equiv \phi_i + \lambda_i \frac{\pi_{a_i}\left(s_i' | s_i, a_i\right)}{\pi\left(s_i' | s_i, a_i\right)}$, $h_0^i(a, c, s) \equiv r(a_i, c_i, s)$, $h_1^i(a, c, s) \equiv -\upsilon'(a_i)$, and

$$h(a, c, \phi, \lambda, s) \equiv \phi h_0(a, c, s) + \lambda h_1(a, c, s)$$

which is homogenous of degree 1 in (ϕ, λ). The Lagrangean can be written as:

$$L(s_0, \omega, c^\infty, a^\infty, \lambda^\infty) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t h\left(a_t\left(s^t\right), c_t\left(s^t\right), \phi_t\left(s^t\right), \lambda_t\left(s^t\right), s_t\right) \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)$$

where

$$\begin{aligned} \phi_{t+1}\left(s^t, \widehat{s}\right) &= \varphi\left(\phi_t\left(s^t\right), \lambda_t\left(s^t\right), \widehat{s}\right) \quad \forall \widehat{s} \in S \\ \phi_0\left(s^0\right) &= \omega \end{aligned}$$

where of course $\phi \in \mathbb{R}^N$. We can associate a saddle point functional equation to this Lagrangean

$$J(s, \phi) = \min_{\lambda} \max_{a,c} \left\{ h(a, c, \phi, \lambda, s) + \beta \sum_{s'} \pi(s' \mid s, a) J(s', \phi'(s')) \right\} \tag{D.1}$$
$$s.t. \quad \phi'(s') = \phi + \lambda \frac{\pi_a(s' \mid s, a)}{\pi(s' \mid s, a)} \quad \forall s'$$

Again, we are left to prove the following:

Proposition 13 *Fix an arbitrary constant $K > 0$ and let $K_\theta = \max\{K, K\|\phi\|\}$. The operator*

$$\begin{aligned} (T_K f)(s, \phi) &\equiv \min_{\{\lambda > 0: \|\lambda\| \leq K_\theta\}} \max_{a,c} \left\{ h(a, c, \phi, \lambda, s) + \beta \sum_{s'} \pi(s' \mid s, a) f(s', \phi'(s')) \right\} \\ s.t. \quad \phi'(s') &= \phi + \lambda \frac{\pi_a(s' \mid s, a)}{\pi(s' \mid s, a)} \quad \forall s' \end{aligned}$$

is a contraction.

Proof. *Straightforward by repeating the steps to prove Proposition 1 in the following space of functions:*

$$
\begin{aligned}
M = \{ f : S \times \mathbb{R}^N \longrightarrow \mathbb{R} \quad s.t. & \\
& a) \qquad \forall \alpha > 0 \quad f(\cdot, \alpha\phi) = \alpha f(\cdot, \phi) \\
& b) \qquad f(s, \cdot) \ \text{ is continuous and bounded} \quad \}
\end{aligned}
$$

with norm

$$
\|f\| = \sup \{ |f(s, \phi)| : \|\phi\| \leq 1, s \in S \}
$$

■

The next Subsection shows first-order conditions for the case where $N = 2$.

First-order conditions (N=2)

The Lagrangean is

$$
\begin{aligned}
& L(s_0, \omega, c^{\infty}, a^{\infty}, \lambda^{\infty}) = \\
& = \sum_{i=1}^{2} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \left\{ \phi_{it}(s^t) \left[u\left(c_{it}(s^t)\right) - \upsilon\left(a_{it}(s^t)\right) \right] - \right. \\
& \qquad \left. - \lambda_{it}(s^t)\, \upsilon'\left(a_{it}(s^t)\right) \right\} \Pi\left(s^t \mid h_0, a^{t-1}\left(s^{t-1}\right)\right)
\end{aligned}
$$

$$
/c_{1t}(s^t) : \qquad \phi_{1t}(s^t)\, u'\left(c_{1t}(s^t)\right) = \phi_{2t}(s^t)\, u'\left(c_{2t}(s^t)\right)
$$

$$
\begin{aligned}
/a_{it}(s^t) \quad : \quad & \phi_{it}(s^t)\, \upsilon'\left(a_{it}(s^t)\right) + \lambda_{it}(s^t)\, \upsilon''\left(a_{it}(s^t)\right) \\
& = \sum_{i=1}^{2} \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^t} \beta^j \left\{ \phi_{i,t+j}(s^t) \left[u\left(c_{i,t+j}\left(s^{t+j}\right)\right) - \right. \right. \\
& \left. - \upsilon\left(a_{i,t+j}\left(s^{t+j}\right)\right) \right] - \lambda_{i,t+j}\left(s^{t+j}\right) \upsilon'\left(a_{i,t+j}\left(s^{t+j}\right)\right) \Big\} \times \\
& \times \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}(s^t)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}(s^t)\right)} \Pi\left(s^{t+j} \mid s_t, a^{t+j-1}\left(s^{t+j-1}\right)\right) + \\
& + \beta \lambda_{it}(s^t) \sum_{s^{t+1}|s^t} \frac{\partial \left(\frac{\pi_{a_i}\left(s_{i,t+1} | s_{it}, a_{it}(s^t)\right)}{\pi\left(s_{i,t+1} | s_{it}, a_{it}(s^t)\right)} \right)}{\partial a_i} \times \\
& \times \left[u\left(c_{i,t+1}\left(s^{t+1}\right)\right) - \upsilon\left(a_{i,t+1}\left(s^{t+1}\right)\right) \right] \pi\left(s_{t+1} \mid s_t, a_t(s^t)\right)
\end{aligned}
$$

$$/\lambda_{it}\left(s^t\right) \quad : \qquad -\upsilon'\left(a_{it}\left(s^t\right)\right) + \sum_{j=1}^{\infty}\sum_{s^{t+j}|s^t}\beta^j \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}\left[u\left(c_{i,t+j}\left(s^{t+j}\right)\right) - \right.$$

$$\left. -\upsilon\left(a_{i,t+j}\left(s^{t+j}\right)\right)\right]\Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1}\right)\right) = 0$$

We can restate all the above equations in only one endogenous state variable $\theta_t\left(s^t\right) \equiv \frac{\phi_{2t}\left(s^t\right)}{\phi_{1t}\left(s^t\right)}$, by using the homogeneity properties of policy functions and value function. For a generic variable $x_{it}\left(s^t\right)$, define $\widetilde{x}_{it}\left(s^t\right) = \frac{x_{it}\left(s^t\right)}{\phi_{it}\left(s^t\right)}$, thus:

$$/c_{1t}\left(s^t\right) : \qquad \frac{u'\left(c_{1t}\left(s^t\right)\right)}{u'\left(c_{2t}\left(s^t\right)\right)} = \theta_t\left(s^t\right)$$

$$\begin{aligned}
/a_{1t}\left(s^t\right) \quad : \qquad & \upsilon'\left(a_{1t}\left(s^t\right)\right) + \widetilde{\lambda}_{1t}\left(s^t\right)\upsilon''\left(a_{1t}\left(s^t\right)\right) = \\
& = \beta\sum_{s^{t+1}|s^t}\left(1 + \widetilde{\lambda}_{1t}\left(s^t\right)\frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}\right) \times \\
& \times \pi_{a_1}\left(s_{1,t+1} \mid s_{1t}, a_{1t}\left(s^t\right)\right)\pi\left(s_{2,t+1} \mid s_{2t}, a_{2t}\left(s^t\right)\right)\widetilde{J}\left(s_t, \theta_t\left(s^t\right)\right) + \\
& + \beta\widetilde{\lambda}_{1t}\left(s^t\right)\sum_{s^{t+1}|s^t}\frac{\partial\left(\frac{\pi_{a_1}\left(s_{1,t+1}|s_{1t},a_{1t}\left(s^t\right)\right)}{\pi\left(s_{1,t+1}|s_{1t},a_{1t}\left(s^t\right)\right)}\right)}{\partial a_i} \times \\
& \times\left[u\left(c_{1,t+1}\left(s^{t+1}\right)\right) - \upsilon\left(a_{1,t+1}\left(s^{t+1}\right)\right)\right]\pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)
\end{aligned}$$

$$\begin{aligned}
/a_{2t}\left(s^t\right) \quad : \qquad & \theta_t\left(s^t\right)\upsilon'\left(a_{2t}\left(s^t\right)\right) + \theta_t\left(s^t\right)\widetilde{\lambda}_{2t}\left(s^t\right)\upsilon''\left(a_{2t}\left(s^t\right)\right) = \\
& = \beta\sum_{s^{t+1}|s^t}\left(1 + \widetilde{\lambda}_{1t}\left(s^t\right)\frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}\right) \times \\
& \times \pi\left(s_{1,t+1} \mid s_{1t}, a_{1t}\left(s^t\right)\right)\pi_{a_2}\left(s_{2,t+1} \mid s_{2t}, a_{2t}\left(s^t\right)\right)\widetilde{J}\left(s_t, \theta_t\left(s^t\right)\right) + \\
& + \beta\theta_t\left(s^t\right)\widetilde{\lambda}_{2t}\left(s^t\right)\sum_{s^{t+1}|s^t}\frac{\partial\left(\frac{\pi_{a_2}\left(s_{2,t+1}|s_{2t},a_{2t}\left(s^t\right)\right)}{\pi\left(s_{2,t+1}|s_{2t},a_{2t}\left(s^t\right)\right)}\right)}{\partial a_2} \times \\
& \times\left[u\left(c_{2,t+1}\left(s^{t+1}\right)\right) - \upsilon\left(a_{2,t+1}\left(s^{t+1}\right)\right)\right]\pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)
\end{aligned}$$

$$/\lambda_{it}\left(s^t\right) \quad : \qquad -\upsilon'\left(a_{it}\left(s^t\right)\right) + \sum_{j=1}^{\infty}\sum_{s^{t+j}|s^t}\beta^j \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}\left[u\left(c_{i,t+j}\left(s^{t+j}\right)\right) - \right.$$

$$\left. -\upsilon\left(a_{i,t+j}\left(s^{t+j}\right)\right)\right]\Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1}\right)\right) = 0$$

E Bond holdings

Repeated moral hazard

We can define bond holdings recursively as:

$$\begin{aligned}
b_t\left(s^t\right) &= -E_t^a\sum_{j=1}^{\infty}\beta^j\left(y_{t+j}-c_{t+j}\right)= \\
&= -E_t^a\sum_{j=1}^{\infty}\beta^j\left\{\left(y_{t+j}-c_{t+j}\right)+\phi_{t+j}\left[u\left(c_{t+j}\right)-\upsilon\left(a_{t+j}\right)\right]-\lambda_{t+j}\upsilon'\left(a_{t+j}\right)\right\}+ \\
&\quad +E_t^a\sum_{j=1}^{\infty}\beta^j\left\{\phi_{t+j}\left[u\left(c_{t+j}\right)-\upsilon\left(a_{t+j}\right)\right]-\lambda_{t+j}\upsilon'\left(a_{t+j}\right)\right\} \\
&= -\beta E_t^a J\left(y_{t+1},\phi_{t+1}\right)+E_t^a\sum_{j=1}^{\infty}\beta^j\left\{\phi_{t+j}\left[u\left(c_{t+j}\right)-\upsilon\left(a_{t+j}\right)\right]-\lambda_{t+j}\upsilon'\left(a_{t+j}\right)\right\} \\
&= -\beta E_t^a J\left(y_{t+1},\phi_{t+1}\right)+E_t^a\sum_{j=1}^{\infty}\beta^j\left\{\phi_{t+j}\left[u\left(c_{t+j}\right)-\upsilon\left(a_{t+j}\right)\right]\right\} \\
&\quad -E_t^a\sum_{j=1}^{\infty}\beta^j\lambda_{t+j}E_{t+1}^a\sum_{k=0}^{\infty}\beta^k\frac{\pi_a\left(a_{t+j+1}\right)}{\pi\left(a_{t+j+1}\right)}\left[u\left(c_{t+j+k+1}\right)-\upsilon\left(a_{t+j+k+1}\right)\right] \\
&= -\beta E_t^a J\left(y_{t+1},\phi_{t+1}\right)+E_t^a\sum_{j=1}^{\infty}\beta^j\left\{\phi_{t+j}\left[u\left(c_{t+j}\right)-\upsilon\left(a_{t+j}\right)\right]\right\} \\
&\quad -E_t^a\sum_{j=1}^{\infty}\beta^j\lambda_{t+j}\frac{\pi_a\left(a_{t+j}\right)}{\pi\left(a_{t+j}\right)}U\left(s_{t+j+1},\phi_{t+j+1}\right)
\end{aligned}$$

and notice that

$$\begin{aligned}
\phi_t^*U\left(s_t,\phi_t^*\right) &= \phi_t^*\left[u\left(c_t^*\right)-\upsilon\left(a_t^*\right)\right]+\phi_t^*\beta E_t^a U\left(s_{t+1},\phi_{t+1}^*\right) \\
&= \phi_t^*\left[u\left(c_t^*\right)-\upsilon\left(a_t^*\right)\right]+\beta E_t^a\phi_t^*\frac{\phi_{t+1}^*}{\phi_{t+1}^*}U\left(s_{t+1},\phi_{t+1}^*\right) \\
&= \phi_t^*\left[u\left(c_t^*\right)-\upsilon\left(a_t^*\right)\right]+\beta E_t^a\frac{\phi_t^*}{\phi_{t+1}^*}\phi_{t+1}^*U\left(s_{t+1},\phi_{t+1}^*\right) \\
&= \phi_t^*\left[u\left(c_t^*\right)-\upsilon\left(a_t^*\right)\right]+\beta E_t^a\phi_{t+1}^*U\left(s_{t+1},\phi_{t+1}^*\right)= \\
&= \phi_t^*\left[u\left(c_t^*\right)-\upsilon\left(a_t^*\right)\right]+E_t^a\sum_{j=1}^{\infty}\beta^j\phi_{t+j}^*\left[u\left(c_{t+j}^*\right)-\upsilon\left(a_{t+j}^*\right)\right]
\end{aligned}$$

due to homogeneity of degree zero of the policy functions and of $U(s,\cdot)$. Therefore

$$\begin{aligned}
b_t\left(s^t\right) &= -\beta E_t^a J\left(y_{t+1},\phi_{t+1}\right)+\beta E_t^a \phi_{t+1} U\left(y_{t+1},\phi_{t+1}\right) \\
&\quad -E_t^a \sum_{j=1}^{\infty} \beta^j \lambda_{t+j} \frac{\pi_a\left(a_{t+j}\right)}{\pi\left(a_{t+j}\right)} U\left(s_{t+j+1},\phi_{t+j+1}\right) \\
\text{by Abel's formula} &= -\beta E_t^a J\left(y_{t+1},\phi_{t+1}\right)+\beta E_t^a \phi_{t+1} U\left(y_{t+1},\phi_{t+1}\right) \\
&\quad -E_t^a \sum_{j=1}^{\infty} \beta^j \left(\phi_{t+j+1}-\phi_{t+1}\right)\left[u\left(c_{t+j+1}^*\right)-\upsilon\left(a_{t+j+1}^*\right)\right] \\
&= -\beta E_t^a J\left(y_{t+1},\phi_{t+1}\right)+\beta E_t^a \phi_{t+1} U\left(y_{t+1},\phi_{t+1}\right) \\
&\quad -E_t^a \sum_{j=1}^{\infty} \beta^j \left(\phi_{t+j+1}\right)\left[u\left(c_{t+j+1}^*\right)-\upsilon\left(a_{t+j+1}^*\right)\right] \\
&\quad +E_t^a \sum_{j=1}^{\infty} \beta^j \phi_{t+1}\left[u\left(c_{t+j+1}^*\right)-\upsilon\left(a_{t+j+1}^*\right)\right] \\
&= -\beta E_t^a J\left(y_{t+1},\phi_{t+1}\right)+\beta E_t^a \phi_{t+1}\left[U\left(y_{t+1},\phi_{t+1}\right)+E_{t+1}^a U\left(y_{t+2},\phi_{t+2}\right)\right] \\
&\quad -\beta E_t^a \phi_{t+2} U\left(y_{t+2},\phi_{t+2}\right) \\
&= -\beta E_t^a J\left(y_{t+1},\phi_{t+1}\right)+\beta E_t^a \phi_{t+1}\left[U\left(y_{t+1},\phi_{t+1}\right)\right] \\
&\quad -E_t^a \lambda_{t+1} \frac{\pi_a\left(a_{t+1}\right)}{\pi\left(a_{t+1}\right)} \beta E_{t+1}^a U\left(y_{t+2},\phi_{t+2}\right)
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
b_t\left(s^t\right) &= -\beta E_t^a J\left(y_{t+1},\phi_{t+1}\right)+\beta E_t^a \phi_{t+1} U\left(y_{t+1},\phi_{t+1}\right) \\
&\quad -E_t^a \lambda_{t+1} \frac{\pi_a\left(a_{t+1}\right)}{\pi\left(a_{t+1}\right)}\left[U\left(y_{t+1},\phi_{t+1}\right)-u\left(c_{t+1}^*\right)+\upsilon\left(a_{t+1}^*\right)\right]
\end{aligned}$$

where the second line is due to the optimality of the contract.

Hidden assets

Starting from the previous result, in this case we can write

$$\begin{aligned}
b_t\left(s^t\right) &= -E_t^a \sum_{j=1}^{\infty} \beta^j \left(y_{t+j} - c_{t+j}\right) = \\
& -\beta E_t^a J\left(y_{t+1}, \phi_{t+1}\right) + \beta E_t^a \phi_{t+1} U\left(y_{t+1}, \phi_{t+1}\right) - \\
& -E_t^a \lambda_{t+1} \frac{\pi_a\left(a_{t+1}\right)}{\pi\left(a_{t+1}\right)} \left[U\left(y_{t+1}, \phi_{t+1}\right) - u\left(c_{t+1}^*\right) + \upsilon\left(a_{t+1}^*\right)\right] - \\
& -E_t^a \sum_{j=1}^{\infty} \beta^j \left[\eta_{t+j} - \beta^{-1} \zeta_{t+j}\right] u'\left(c_{t+j}\right) \\
&= -\beta E_t^a J\left(y_{t+1}, \phi_{t+1}\right) + \beta E_t^a \phi_{t+1} U\left(y_{t+1}, \phi_{t+1}\right) - \\
& -E_t^a \lambda_{t+1} \frac{\pi_a\left(a_{t+1}\right)}{\pi\left(a_{t+1}\right)} \left[U\left(y_{t+1}, \phi_{t+1}\right) - u\left(c_{t+1}^*\right) + \upsilon\left(a_{t+1}^*\right)\right] \\
& \underbrace{-E_t^a \sum_{j=1}^{\infty} \beta^j \left[\eta_{t+j} - \beta^{-1} \zeta_{t+j}\right] u'\left(c_{t+j}\right) - E_t^a \zeta_{t+1} u'\left(c_{t+1}\right)}_{=0 \text{ by definition}} + E_t^a \zeta_{t+1} u'\left(c_{t+1}\right) \\
&= -\beta E_t^a J\left(y_{t+1}, \phi_{t+1}\right) + \beta E_t^a \phi_{t+1} U\left(y_{t+1}, \phi_{t+1}\right) - \\
& -E_t^a \lambda_{t+1} \frac{\pi_a\left(a_{t+1}\right)}{\pi\left(a_{t+1}\right)} \left[U\left(y_{t+1}, \phi_{t+1}\right) - u\left(c_{t+1}^*\right) + \upsilon\left(a_{t+1}^*\right)\right] + \\
& +E_t^a \zeta_{t+1} u'\left(c_{t+1}\right)
\end{aligned}$$