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Theta-duality in Abelian Varieties and the Bicanonical Map of Irregular Varieties

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CERTIFIQUEN

que la present memòria ha estat realitzada sota la seva direcció per Martí Lahoz Vilalta, i que constitueix la seva tesi per optar al grau de Doctor en Matemàtiques.

Barcelona, març de 2010.

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A l'Antònia, el Lluís i l'Erola

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Contents

Introduction

> Algebraic geometry is a mixture of the ideas of two Mediterranean cultures. It is the superposition of the Arab science of the lightning calculation of the solutions of equations over the Greek art of position and shape. This tapestry was originally woven on European soil and is still being refined under the influence of international fashion. Algebraic geometry studies the delicate balance between the geometrically plausible and the algebraically possible. Whenever one side of this mathematical teeter-totter outweighs the other, one immediately loses interest and runs off in search of a more exciting amusement.
>
> George R. Kempf. Algebraic Varieties.
> London Mathematical Society Lecture Notes Series **172**.

Abstract

Abelian varieties are some of the most studied higher dimensional algebraic varieties. In the complex case, they are constructed by quotienting $\mathbb{C}^g$ by an integral lattice, so they have a simple topological structure. However, on one hand there are many open problems concerning their geometry and, on the other hand, they are very useful to study other algebraic varieties.

Historically, in the context of algebraic geometry, the first example of higher dimensional abelian varieties are Jacobian varieties of non-

rational curves. Jacobian varieties have a natural principal polarization coming from the intersection product on the curve. Prym varieties are another classical example of abelian varieties already studied by Wirtinger in the late XIX century. They have also a natural principal polarization so they are principally polarized abelian varieties (ppav for short).

A principal polarization, i.e. a positive line bundle (modulo algebraic equivalence) with one section, gives rise to a duality theory on abelian varieties that resembles the classical duality theory on the projective space. Given a subscheme Y on a ppav, its theta-dual $T(Y)$ is the set parameterizing all the divisors representing the principal polarization that contain the given subscheme. A priori, this is just a set, but we can endow it with a natural scheme structure already introduced by Pareschi and Popa in [PP6].

Jacobian and Prym varieties are constructed from curves and they carry on some special subvarieties coming from the geometry of the curve. For example in the Jacobian case, the symmetric product of the curve $C^{(d)}$, for d less than the genus of the curve, maps birationally to a subvariety W_d in the Jacobian JC. These are the simplest Brill-Noether loci. The curve itself, is embedded in its Jacobian via the Abel-Jacobi map (case $d = 1$). Prym varieties are constructed from étale double covers of curves. When the base curve is non-hyperelliptic, the covering curve is embedded in the Prym variety via the Abel-Prym map. One of the goals of Chapter III is to compute the schematic theta-dual of these special geometrical subvarieties.

In the Jacobian case we obtain that $T(W_d) = W_{g-d-1}$ up to translation, where g is the genus of the curve. This result was already obtained by Pareschi and Popa in [PP6, §8.1] using cohomological classes to control the schematic structures. Our approach is based on the work of Polishchuk [Po2] and avoids using cohomological classes, controlling directly the sheaves involved in the definition of the schematic theta-dual.

In the Prym case, we compute the schematic theta-dual of the Abel-Prym curve. It turns out to be V^2, the second Prym-Brill-Noether locus as defined in [W3, (1.2)]. This locus has codimension 3 inside the Prym variety.

Why are these computations interesting? These special geometrical subvarieties inside the Jacobians and Prym varieties reflect geometric properties of the curves involved in their construction. Also, they have a special behavior as subvarieties of a ppav hopefully allowing to detect Jacobian and Prym varieties among all ppavs. To distinguish Jacobian (or Prym) varieties among all ppavs is known as the "geometric" Schottky problem. For example, the cohomological class of $[W_{g-d}] = [\Theta]^d/d!$ and a long standing conjecture states that the only ppavs (A, Θ) that have a subvariety representing this minimal cohomological class for some $d > 1$ are Jacobian varieties and one sporadic case, intermediate Jacobians of cubic threefolds. This conjecture is only completely solved in the case $d = \dim A - 1$ (this is the well-known Matsusaka-Ran criterion [Mat, Ra]). Pareschi and Popa have used the schematic theta-duality and the sheaves involved in its definition to propose in [PP6] a symmetric approach to this conjecture that allows to get some results for $d = 2$.

Observe that the case $d = \dim A - 1$ is not only a Schottky type result, but also a Torelli statement. That is, it allows us to recover the curve from its Jacobian and the principal polarization. However, in the Prym case, Donagi's construction [Do] shows that, even a Torelli result is not possible for Prym varieties in full generality. Indeed, different (in the sense of moduli) étale double coverings of curves give rise to isomorphic polarized Prym varieties. This is not the general behavior, but recently Izadi and Lange have shown that there are examples for arbitrarily high Clifford index [IL], disproving a conjecture by Donagi. However, we think that to know the schematic theta-dual of the Abel-Prym curve and its properties could be a good tool to understand the geometry of the Prym variety.

Going back to the Matsusaka-Ran criterion, we have said that it gives a Torelli statement. More precisely, it assures that an irreducible curve generating the Jacobian variety, with degree $g = \dim JC$ is an Abel-Jacobi curve, i.e. a translate of C embedded into its Jacobian. Moreover, g is the minimal degree of a non-degenerate irreducible curve in a ppav of dimension g. In the projective space, the non-degenerate curves of minimal degree are the rational normal curves. A classical result due to Castelnuovo, says that a finite collection of points in the projective space which is in linearly general position, but in special position with respect to quadrics, is contained in a unique rational normal curve.

Pareschi and Popa [PP4] have discovered that the, a priori naïf, analogy between projective spaces and ppavs (A, Θ), where the role played by the hyperplanes in the projective space is played by divisors algebraically equivalent to Θ, goes further. In fact, they have seen that divisors algebraically equivalent to 2Θ in a ppav and quadrics in the projective space, play the same role with respect to curves of minimal degree. The Castelnuovo result of Pareschi and Popa says that in a ppav of dimension g, the existence of a collection of $g+2$ distinct points in general position with respect to Θ, but special with respect to 2Θ, implies that A is the Jacobian of a curve C. Moreover, the $g+2$ points have to be contained in an Abel-Jacobi curve. Thus, Abel-Jacobi curves play the role of rational normal curves, and the analogue of Castelnuovo's result contains a Schottky statement. As a corollary, they also get a Torelli statement, recovering the curve as the intersection of all divisors algebraically equivalent to 2Θ that contain the given $g+2$ points.

The purpose of Chapter IV is to extend this Pareschi and Popa [PP4] result to possibly non-reduced subschemes as Eisenbud and Harris did in the projective case [EH]. We remark that, already the fact that a finite subscheme general with respect to Θ, but special with respect to 2Θ, is contained in a smooth curve (i.e. is curvilinear) is not obvious.

While in chapthers III and IV we have studied ppavs constructed from curves and coverings of curves and how can they be distinguished among all ppavs, in Chapter V we will move to the context of birational geometry and pluricanonical maps. There, the role played in our methods by the Abel-Jacobi map and the Jacobian variety will be substituted by its higher dimensional analogue, i.e. the Albanese map and the Albanese variety. In this context, the natural generalization of the non-rational curves are the irregular varieties, that is, varieties whose Albanese variety is not trivial. However, we must say that the Albanese variety, constructed in the complex case from the Hodge structure, and as the dual of the Picard variety in the general case, it is not principally polarized in general. Moreover, the Albanese map is not necessarily an embedding. Anyway, the study of the Albanese variety and the Albanese map gives a lot of geometrical information on the original variety X. For example, the dimension of a generic fiber of the Albanese map controls the vanishing of the higher cohomologies of a general paracanonical line bundle of X (see [GL1, GL2]). In particular, when the Albanese map is generically finite onto its image (we will say the X is of maximal Albanese dimension), the canonical Euler characteristic of X is positive or zero.

In the study of complex projective algebraic varieties, the natural maps that are provided by the differential forms defined in the variety, have a special importance. Observe that, the Albanese map can be constructed from the differential 1-forms in our original variety and maps X to a complex torus. Whereas, the differential n-forms (where n is the dimension of X) produce a map to a projective space, known as the canonical map. The multiples of the canonical linear series produce in this way the pluricanonical maps. So, these are the classical and *canonical* ways to represent general algebraic varieties into better behaved algebraic varieties.

An interesting problem is to study how the properties of the Albanese map are related with the properties of the pluricanonical maps. Since both are constructed from differential forms in our original variety X, there is certainly a relation between them, eventhough its geometrical consequences are not obvious.

Varieties of general type are those whose m-th pluricanonical map, for m big enough, induces a birational equivalence between X and its image (here m denotes the multiple of the canonical linear series). In this case, is common to say abusively that the m-th pluricanonical map is birational. Almost by definition, pluricanonical maps are an essential tool to study varieties of general type. There are two related main problems on this subject.

One problem is to give a bound on m depending only on the dimension of X for which the m-th pluricanonical map is birational. The existence of this bound in any dimension has been proved by Hacon-M^{c}Kernan [HM] and independently by Takayama [Ta] using the previous work of Kawamata, Siu and Tsuji in the ambient of the Minimal Model Program. Already in [HM], they pose the problem to find explicit (hopefully small) values of this minimal m for each dimension. In the case of surfaces, this had already been solved sharply by Bombieri [Bo] showing that the 5-th pluricanonical map is enough to ensure the birationality. In the case of threefolds, Chen and Chen [CC] have proved that $m = 73$ is enough.

The second problem is to classify or, at least, give geometrical or numerical restrictions for varieties whose m-th pluricanonical map is non-birational for low m. In the case of surfaces, this problem was also studied by Bombieri in [Bo], where he gave sharp numerical conditions for the birationality of the tetracanonical and tricanonical map. The numerical classification started by Bombieri of surfaces whose bicanonical map is non-birational, was not sharp. Since then, lots of efforts have been devoted to classify completely the minimal surfaces with non-birational bicanonical map (see for example [CCM, CFM, CM1, CM2, Xi]).

The study of the birationality of the m-th pluricanonical map for low m, in any dimension but restricted to irregular varieties was initiated by Chen and Hacon. For example, they show in [CH1] that, if X is a variety of general type and whose Albanese map is generically finite on its image, then $\chi(\omega_X) > 0$ implies that the tricanonical map is birational. A similar result was obtained later by Pareschi and Popa. They introduced a new invariant for irregular varieties, the generic vanishing index $\mathrm{gv}(\omega_X)$, and they proved that irregular varieties with $\mathrm{gv}(\omega_X) \geqslant 1$ have very ample tricanonical map away from the exceptional locus of the Albanese map (see [PP3]).

The main purpose of the second part of this Thesis is to study the bicanonical map of irregular varieties. First, let us recall what is known in the surface case. Assume that S is an irregular surface whose bicanonical map is non-birational. It is common to say that S presents the standard case if it admits a fibration by curves of genus 2. Indeed, suppose that S admits a fibration by curves of genus 2. Then, the bicanonical system of the surface restricts to a subsystem of the bicanonical map of the general fiber. Since the bicanonical map of a curve of genus 2 is non-birational, the bicanonical map of S cannot certainly be birational. Bombieri already showed that if S does not present the standard case, then the autointersection of the canonical divisor is small, i.e. $K_S^2 \leqslant 10$. Hence, surfaces with non-birational bicanonical map, but not presenting the standard case, are contained in a finite number of components of the moduli space of surfaces of general type. There are two examples of such surfaces:

(a) symmetric products of a curve of genus 3 (classical);
(b) double covers of principally polarized abelian surface branched at a divisor $D \in |2\Theta|$ (as it was observed by Catanese).

Suppose that S has non-birational bicanonical map and does not present the standard case. Then, in a sequel of articles (see mainly [CCM, CM1, CM2]), it is shown that the Euler characteristic of S is $\chi(\omega_S) = 1$. Moreover, if the irregularity $q(S) \geqslant 3$, then S has to be birationally equivalent to one of the surfaces in case *(a)*. If the irregularity $q(S) = 2$, then S

must be birationally equivalent to one of the surfaces in case *(b)*.

Observe that the symmetric product of a curve of genus 3 can be seen as the theta-divisor of the curve of genus 3. In this sense, examples *(a)* and *(b)* generalize to arbitrary dimension, giving the following irregular varieties whose bicanonical map is non-birational:

(a') theta-divisors of an indecomposable ppav, and
(b') double covers of a ppav (A, Θ) branched along a reduced divisor of the linear series $|2\Theta|$.

The first part of Chapter V will be devoted to give a numerical criterion to ensure the birationality of the bicanonical map of irregular varieties. Indeed, we get an analogue result of that of Pareschi and Popa for the tricanonical map, that is, we see that irregular varieties with $\mathrm{gv}(\omega_X) \geqslant 2$ have birational bicanonical map. Thus, if X is an irregular variety whose bicanonical map is non-birational, then $\mathrm{gv}(\omega_X) \leqslant 1$. Roughly speaking, the standard case, *(a')* and *(b')*, can be seen as boundary ($\mathrm{gv}(\omega_X) = 1$) or sub-boundary ($\mathrm{gv}(\omega_X) = 0$) examples of this criterion.

As we have already pointed out, in the case of surfaces, examples *(a)* and *(b)* are the only ones not presenting the standard case and $q(S) \geqslant 2$. Moreover, the symmetric product S of a curve of genus 3 has $\mathrm{gv}(\omega_S) = 1$ and the double cover of principally polarized abelian surface branched along a reduced divisor in $|2\Theta|$ has $\mathrm{gv}(\omega_S) = 0$. In the second part of Chapter V, we will try to see whether this behavior extend to higher dimensions. More precisely, we will try to see whether examples *(a')* and *(b')* are the only cases without irregular fibrations, whose bicanonical map is non-birational and $q(X) \geqslant \dim X$. To make precise the assumption that an irregular variety has no irregular fibrations, we use the notion of primitive varieties introduced by Catanese [Ca]. In fact, we show that primitive varieties admit only very special fibrations to other irregular varieties.

On one hand, we prove that if X is a primitive variety whose bicanonical map is non-birational and $q(X) > \dim X$, then X is birationally equivalent to a theta-divisor of an indecomposable ppav (see [BLNP]). This characterize the primitive boundary cases of the numerical criterion for the bicanonical map, that is, primitive varieties with non-birational bicanonical map and $\mathrm{gv}(\omega_X) = 1$.

On the other hand, we start the study of primitive varieties of general type whose bicanonical map is non-birational and $q(X) = \dim X$. Under additional hypotheses we show that the only possibility is that X is birationally equivalent to a double cover of a ppav (A, Θ) branched along a reduced divisor of $|2\Theta|$. This corresponds to primitive varieties of general type with non-birational bicanonical map and $\mathrm{gv}(\omega_X) = 0$, which is the sub-boundary case with respect to the general criterion.

Thus, eventough Albanese varieties are not necessarily principally polarized, we see that the "atomic" cases of varieties whose bicanonical map is non-birational are constructed from principally polarized Albanese varieties.

Finally, we would like to emphasize that the techniques used throughout this Thesis are based on the Fourier-Mukai transform.

Summary

We want to give a concise, but precise overview of the Thesis. So, in what follows, the highlighted propositions and theorems are the main results of this Thesis. Their numbering corresponds to the order in which they will appear in the sequel (chapter in roman, section and numbering). To state them properly and make this short summary self-contained, let us introduce the Fourier-Mukai transform. We would also like to take this opportunity to show the main technical tools that will be used in this Thesis.

The Fourier-Mukai transform

The Fourier-Mukai transform appears in the early 80's in the work of Mukai [M2] to study the moduli of deformations of Picard sheaves. It is constructed using the Poincaré line bundle in $A \times \operatorname{Pic}^0 A$ and the projections $p : A \times \operatorname{Pic}^0 A \to A$ and $q : A \times \operatorname{Pic}^0 A \to \operatorname{Pic}^0 A$. We use the following notation,

$$\mathrm{R}\Phi_{\mathcal{P}} : \mathrm{D}^b(A) \to \mathrm{D}^b(\operatorname{Pic}^0 A), \qquad \text{where } \mathrm{R}\Phi_{\mathcal{P}}(\,\cdot\,) = \mathrm{R}q_*(p^*(\,\cdot\,)\underline{\otimes}\mathcal{P}),$$

and $\mathrm{D}^b(X)$ is the bounded derived category of coherent sheaves on X. Mukai proves in [M2, Thm. 2.2] that it gives an equivalence of categories between $\mathrm{D}^b(A)$ and $\mathrm{D}^b(\operatorname{Pic}^0 A)$, although A and $\operatorname{Pic}^0 A$ are, in general, non-isomorphic. The proof of this equivalence is based on the properties of the Poincaré line bundle $\mathcal{P}$.

More generally, given two varieties X and Y, we can substitute the Poincaré line bundle by an object in the derived category of the product $X \times Y$, and we can consider the generalized Fourier-Mukai transform between the derived categories of X and Y. In particular, given a variety that has a morphism to an abelian variety $a : X \to A$, Pareschi and Popa studied in a sequence of articles (see [PP8, PP3, PP7]) the generalized Fourier-Mukai transform from the derived category of X to the derived category of $\operatorname{Pic}^0 A$, i.e.

$$\mathrm{R}\Phi_{P_a} : \mathrm{D}^b(X) \to \mathrm{D}^b(\operatorname{Pic}^0 A), \qquad \text{where } \mathrm{R}\Phi_{P_a}(\,\cdot\,) = \mathrm{R}q_*(p^*(\,\cdot\,)\underline{\otimes}P_a)$$

and $P_a = (a \times \mathrm{id})^*\mathcal{P}$. When X is an abelian variety we recover the original Fourier-Mukai transform. Pareschi and Popa have also introduced the following invariant

Definition I.1.9 ([PP7, Def. 3.1]). Given a coherent sheaf $\mathcal{F}$ on X, the *generic vanishing index* of $\mathcal{F}$ (with respect to $a : X \to A$) is

$$\mathrm{gv}_a(\mathcal{F}) := \min_{i>0} \left\{ \operatorname{codim}_{\operatorname{Pic}^0 A}(\operatorname{supp} R^i\Phi_{P_a}\mathcal{F}) - i \right\}.$$

When a is the Albanese map, we omit it in the notation.

The name of this invariant comes from the Generic Vanishing Theorem of Green and Lazarsfeld [GL1, Thm. 1] that shows that,

$$\left.\begin{array}{l}\text{if } X \text{ is a compact, connected, Kähler manifold and it admits a map}\\ a\colon X\to A \text{ to a complex torus } A, \text{ whose generic fiber has dimension } k,\end{array}\right\}$$
$$\text{then } \operatorname{gv}_a(\omega_X)\geqslant -k.$$

Pareschi and Popa have shown, using commutative algebra, that the generic vanishing index of $\mathcal{F}$ controls some (local) properties of the Fourier-Mukai transform of the dual of $\mathcal{F}$, broadly extending a Green-Lazarsfeld's conjecture for the canonical sheaf [GL2, Prob. 6.2].

- On one hand, they have used the Auslander-Buchsbaum Theorem [Ma, Thm. 19.1] to show that given a coherent sheaf $\mathcal{F}$ on X we have the following equivalence (see [PP8, Thm. A])

  $$\operatorname{gv}_a(\mathcal{F}) \geqslant 0 \quad \text{if, and only if,} \quad R^i\Phi_{P_a}\,\mathrm{R}\Delta\,\mathcal{F} = 0, \text{ for all } i \neq \dim X,$$

  where $\mathrm{R}\Delta\,\mathcal{F} = \mathrm{R}\mathcal{H}om(\mathcal{F}, \omega_X)$ is the Grothendieck dual of $\mathcal{F}$.

- On the other hand, based on the Evans-Griffith Syzygy Theorem [EG, Cor. 1.7], they have found a dictionary between the non-negative values of the generic vanishing index and the local properties of the Fourier-Mukai transform of the dual sheaf. More precisely, for every $m \geqslant 0$

  $$\operatorname{gv}_a(\mathcal{F}) \geqslant m \quad \text{if, and only if,} \quad R^d\Phi_{P_a}\,\mathrm{R}\Delta\,\mathcal{F} \text{ is a } m\text{-th syzygy sheaf,}$$

  where $d = \dim X$ (see [PP7, Cor. 3.2]).

These two theorems are either essential for the proofs of the main results of this Thesis or as a motivation for some calculations. Therefore, they will often be mentioned. However, although they provide a very interesting duality dictionary, they have a mild geometrical content. As Pareschi and Popa have already done in their articles, we will use geometric inputs to obtain geometrical results. Roughly speaking, in the context of abelian varieties we will use the original Mukai Equivalence Theorem [M2, Thm 2.2] and, in the context of irregular varieties we will use the Generic Vanishing Theorems of Green and Lazarsfeld [GL1, GL2].

Principally polarized abelian varieties

Going back to the original setting of the Fourier-Mukai transform, when (A, Θ) is a principally polarized abelian variety (ppav), the polarization provides an isomorphism ϕ_Θ between the abelian variety and its dual. Then, the Fourier-Mukai transform can be seen as an autoequivalence,

$$\mathrm{R}\mathcal{S} : \mathrm{D}^b(A) \to \mathrm{D}^b(A), \qquad \text{where } \mathrm{R}\mathcal{S} = \phi_\Theta^* \circ \mathrm{R}\Phi_{\mathcal{P}}\,.$$

A first step to understand the Fourier-Mukai transform of a sheaf $\mathcal{F}$ is to study the set-theoretical support $R^i\mathcal{S}\,\mathcal{F}$. By the Base Change Theorem, it is contained in the sets

$$V^i(\mathcal{F}) = \big\{a \in A \,\big|\, h^i(\mathcal{F} \otimes \phi_\Theta(a)) > 0\big\}$$

known as the cohomological support loci of $\mathcal{F}$, i.e. $\operatorname{supp} R^i\mathcal{S}\,\mathcal{F} \subseteq V^i(\mathcal{F})$.

Pareschi and Popa have shown in [PP6] that, if an arbitrary curve C on a ppav (A,Θ) fulfils the condition $\mathrm{gv}(\mathcal{I}_{C/A}(\Theta)) \geqslant 0$, then it must be an Abel-Jacobi curve inside its Jacobian. Thus, they give a Torelli and a “geometric” Schottky result based on the generic vanishing index of the ideal sheaf of the curve twisted by Θ. In the Prym case, given an étale double cover $\pi : \widetilde{C} \to C$, when the curve C is non-hyperelliptic, we also have an Abel-Prym embedding $\widetilde{C} \hookrightarrow P$. However, Donagi’s construction shows that in some cases is not possible to recover $\widetilde{C}$ from P and its principal polarization. In order to study what happens in the Prym case and in collaboration with Sebastian Casalaina-Martin and Filippo Viviani, we have computed in [CLV] the cohomological support loci for ideal sheaf of an Abel-Prym curve twisted by Ξ and by 2Ξ.

Proposition III.2.7 and III.2.12 ([CLV, Thm. 3.1 and 4.2])**.** *Let (P,Ξ) be the Prym variety associated to the étale double cover $\pi : \widetilde{C} \to C$, where C is non-hyperelliptic. Consider $\widetilde{C} \hookrightarrow P$ the Abel-Prym embedding. Then,*

(a) $V^1(\mathcal{I}_{\widetilde{C}}(\Xi)) = V^0(\mathcal{I}_{\widetilde{C}}(\Xi))$ *with* $\dim V^0(\mathcal{I}_{\widetilde{C}}(\Xi)) = \dim P - 3$.

(b) $V^2(\mathcal{I}_{\widetilde{C}}(\Xi)) = P^-$.

$$\textit{Hence, } \mathrm{gv}(\mathcal{I}_{\widetilde{C}}(\Xi)) = -2.$$

(c) $V^0(\mathcal{I}_{\widetilde{C}}(2\Xi)) = P$ *if* $g \geqslant 4$ *and* $V^0(\mathcal{I}_{\widetilde{C}}(2\Xi))$ *is a point* q_0 *if* $g = 3$.

(d) $V^1(\mathcal{I}_{\widetilde{C}}(2\Xi)) = V^2(\mathcal{I}_{\widetilde{C}}(2\Xi)) = \{q_0\}$.

$$\textit{Hence, } \mathrm{gv}(\mathcal{I}_{\widetilde{C}}(2\Xi)) = \dim P - 2.$$

To prove *(d)* we have used an argument of Beauville as stated in [IvS, Lemma 2.4].

Theta-duality

In the previous proposition, the set $V^0(\mathcal{I}_{\widetilde{C}}(\Xi))$ has a special meaning. More generally, given a subvariety $Y \subset A$ of a ppav (A,Θ), $V^0(\mathcal{I}_Y(\Theta))$ can be seen as the divisors algebraically equivalent to Θ that contain Y. This is provided by the isomorphism given by the principal polarization. Thus, we have in (A,Θ) a duality theory similar to the duality theory of the projective space, where divisors algebraically equivalent to Θ play the role of hyperplanes in the projective space. The set $V^0(\mathcal{I}_Y(\Theta))$ is known as the set-theoretical theta-dual of Y. This classical point of view of the "theta-duality" mainly takes into account the set-theoretical structure of the theta-dual and the original variety Y. Our point of view is to consider the following natural scheme-theoretic structure on the theta-dual provided by the Fourier-Mukai transform.

Definition II.2.1 ([PP6, §4])**.** Let $Y \subset A$ be an arbitrary subscheme. The *schematic theta-dual of* Y is,

$$T(Y) = \operatorname{supp}\left((-1_A)^* \, R^g\mathcal{S}(\mathrm{R}\Delta(\mathcal{I}_Y(\Theta)))\right),$$

where we denote by supp the Fitting support (e.g. [Ei]).

This definition was proposed by Pareschi and Popa in [PP6, §4] and will be one of the leitmotif of Chapters III and IV.

In Chapter III we compute the theta-dual of the simplest Brill-Nother loci $W_d \subset JC$ (Proposition III.1.9)

$$T(W_d) = W_{g-d-1} \qquad \text{up to translation in } JC.$$

This result was already obtained by Pareschi and Popa in [PP6, §8.1] using the cohomological classes of the Brill-Nother loci to control the schematic structure of $T(W_d)$. Our approach is based on the work of Polishchuk [Po2] and we only work with the ideal sheaves and their Fourier-Mukai transform.

In the case of Prym varieties, we compute the theta-dual of the Abel-Prym curve.

Theorem III.2.9. *Let (P, Ξ) be the Prym variety associated to the étale double cover $\pi : \widetilde{C} \to C$, where C is non-hyperelliptic. Consider $\widetilde{C} \hookrightarrow P$ the Abel-Prym embedding. Then, we have the schematic equality (up to translation in P)*

$$T(\widetilde{C}) = V^2,$$

where V^2 is the second Prym-Brill-Noether locus as defined in [W3, (1.2)].

A geometric Schottky problem

As we pointed out in any g-dimensional ppav (A, Θ) we have a duality theory as in the projective space. Pareschi and Popa discovered that the analogy between divisors algebraically equivalent to Θ and hyperplanes extends also to divisors algebraically equivalent to 2Θ and quadrics in the projective space in the following sense. They define in [PP4] that a collection of $g + 2$ distinct points are in theta-general position, if there exist a theta-translate containing g of them, and avoiding the remaining ones. Then, they show that, if $g + 2$ distinct points in theta-general position on A impose less that $g + 2$ conditions on general 2Θ-translates, then A is the Jacobian of a curve C, and the $g+2$ points are contained in an Abel-Jacobi curve.

This resembles the classical Castelnuovo result in projective geometry, that says that a finite collection of points in $\mathbb{P}^r$ which is in linearly general position, but in special position with respect to quadrics, is contained in a unique rational curve. This classical result was extended by Eisenbud and Harris in [EH] to possibly non-reduced subschemes.

In the same spirit as Eisenbud and Harris, in collaboration with M. Gulbrandsen [GL], we extended this Pareschi and Popa result to possibly non-reduced subschemes. We say that a finite subscheme Γ is in theta-general position if, for every pair $\Gamma'' \subset \Gamma'$ of subschemes of Γ satisfying $\deg \Gamma' - 1 = \deg \Gamma'' \leq g$, there exists a theta-translate containing Γ'' but not Γ'. This is an analogous definition for linearly general position in the projective space. Then, the main result in [GL] is the following theorem.

Theorem IV.6.1. *Let $\Gamma \subset A$ be a theta-general finite subscheme of length $g+2$, imposing less than $g+2$ conditions on general 2Θ-translates. Then the following holds:*

(a) ***Schottky:*** *The principally polarized abelian variety (A, Θ) is isomorphic to a Jacobian $J(C)$ of a curve C, with its canonical polarization.*
(b) ***Castelnuovo:*** *The subscheme Γ is contained in an Abel-Jacobi curve, i.e. the image of an Abel-Jacobi map $C \to J(C)$.*
(c) ***Torelli:*** *The curve C equals the scheme theoretic intersection of all 2Θ-translates containing Γ.*

We remark that, already the fact that a subscheme Γ as in the theorem is contained in a non-singular curve (i.e. Γ is curvilinear) is not obvious. On the other hand, the converse to the theorem is easier, since a finite degree $g+2$ subscheme Γ of a curve C imposes less than $g+2$ conditions on general 2Θ-translates in the Jacobian.

The Schottky part of the theorem depends on the characterization of Jacobians by (possibly degenerated) trisecants to the Kummer variety (see [W2]). Furthermore, our approach differs from that of Pareschi and Popa by our systematic use of the Fourier-Mukai transform.

Irregular varieties

We say that a variety X is of general type if for some $m > 0$ the rational map associated to the linear system $|mK_X|$

$$\varphi_m = \varphi_{|mK_X|} : X \dashrightarrow \mathbb{P}^N = |mK_X|^\vee$$

gives a birational equivalence between X and its image through φ_m. In this situation, we will abusively say that φ_m is birational. The maps φ_m are called the pluricanonical maps of X and they are essential for the study of general type varieties, almost by definition. We will focus in the problem of classifying or, at least, giving numerical restrictions for varieties whose m-th pluricanonical map is non-birational for low m.

In the case of surfaces, this problem was almost completely solved by Bombieri in [Bo]. He proves that, if S is minimal surface of general type and $p_g(S) \geqslant 1$,

- for $m \geqslant 5$, φ_m is birational;
- if φ_4 is non-birational, then $K_S^2 = 1$ and $p_g(S) = 2$;
- if φ_3 is non-birational, then either $K_S^2 = 2$ and $p_g(S) = 3$ or $K_S^2 = 1$ and $p_g(S) = 2$.

For φ_2, the numerical classification started by Bombieri of surfaces whose bicanonical map is non-birational, was not sharp. Since then, lots of efforts due to Catanese, Ciliberto, Francia, Mendes Lopes, Pardini, Xiao Gang and others, have been devoted to classify completely the minimal surfaces with non-birational bicanonical map.

Suppose that S is a minimal surface of general type whose bicanonical map is non-birational. The paradigmatic case (known as the standard case) is when S admits a fibration to a curve such that the general fiber is a curve of genus 2. Since the bicanonical system of S restricts to the bicanonical system of the general fiber and $\varphi_{|2K_C|}$ is non-birational for a curve of genus 2, then $\varphi_{|2K_S|}$ cannot be birational. This kind of fibrations were studied carefully by Xiao Gang in [Xi]. When we suppose that S does not present the standard case, we have two different behaviors depending on $q(S) = h^1(S, \mathcal{O}_S)$,

(a) if $q(S) = 0$ and $p_g \geqslant 4$, Du Val had already proposed a list in [Du] that has been checked with modern techniques by C. Ciliberto, P. Francia and M. Mendes Lopes in [CFM];

(b) if $q(S) > 0$, we have two subcases,

  (b1) if $p_g(S) \geqslant 3$, then S is birationally equivalent to the symmetric product of a curve of genus 3 [CCM, Thm A].

  (b2) if $p_g(S) = 2$, then S is birationally equivalent to a double cover of a principally polarized abelian surface (A, Θ) branched along a smooth divisor in $|2\Theta|$ (see [CM2, Thm 1.1]).

The modern techniques to attack this problem are Reider's method, Bombieri-Francia's restriction method and various techniques involving the Albanese map and continuous systems of curves on a surface.

Going back to arbitrary dimension, we will extend, in some sense, the techniques involving the Albanese map and continuous systems of canonical divisors to higher dimensional varieties. Then, it is natural to restrict ourselves to irregular varieties, i.e. varieties with $q(X) = h^1(X, \mathcal{O}_X) > 0$. For these varieties, the Albanese variety is non-trivial (it has dimension $q(X)$) and the generalized Fourier-Mukai transform $\mathrm{R}\Phi_{P_a}$ plays an important role.

Let X be a smooth projective complex variety with $\mathrm{gv}(\omega_X) \geqslant 1$. Pareschi and Popa have shown[1] in [PP3, Thm. 6.1, Rem. 6.5] that $\omega_X^3 \otimes \alpha$ is very ample away from the exceptional locus of the Albanese map for every $\alpha \in \mathrm{Pic}^0 X$. This result is sharp, in the sense that there are varieties of general type and maximal Albanese dimension, such that $\mathrm{gv}(\omega_X) \geqslant 1$ and whose bicanonical map is not birational. Let us show the three paradigmatic constructions.

The first one is a generalization to higher dimensional varieties of case *(b1)* for surfaces.

Example A. *Let (A, Θ) be an indecomposable ppav, and let $X \to \Theta$ be a desingularization of Θ. The bicanonical map of X has degree 2 and it is not birational. Observe that, $\chi(\omega_X) = \mathrm{gv}(\omega_X) = 1$ and X is clearly of maximal Albanese dimension and general type.*

The second one is a generalization to higher dimensional varieties of case *(b2)* for surfaces .

Example B. *Let (A, Θ) be an indecomposable ppav. Without loss of generality, we can assume as above that Θ is symmetric, i.e. $\Theta = (-1)^*\Theta$. Consider a covering $h\colon \tilde{X} \to A$ (finite and surjective morphism) of A branched along a divisor in $|2\Theta|$. The bicanonical map of X has degree 2 and it is not birational. Observe that, $\chi(\omega_X) = 1$ and $\mathrm{gv}(\omega_X) = 0$*

The third example shows how can we construct other examples from the previous ones and it is a generalization to higher dimensional varieties of the standard case for surfaces.

Example C. *Let $f : X \to Y$ be a fibration, i.e. a surjective morphism with connected fibers, and suppose that the general fiber F has non-birational bicanonical map. Since the bicanonical map of X restricts to a subsystem of the bicanonical*

[1] As we have already mentioned a similar result was obtained previously by Chen and Hacon in [CH1, Thm. 4.4], saying that if X is a variety of general type and maximal Albanese dimension, then $\chi(\omega_X) > 0$ implies that the rational map associated to ω_X^3 is birational.

map of F, the bicanonical map of X cannot be birational.
Therefore, if a variety X has a fibration whose general fiber has non-birational bicanonical map, then X has non-birational bicanonical map.

In the case of surfaces, the previous outlined results (see [CM1, CFM, CM2]) show that when S has non-birational bicanonical map and $q(S) \geqslant 2$, it falls in one of the previous Examples A, B or C.

Thus, the result of Pareschi and Popa regarding the tricanonical map, the behavior of the bicanonical map in the case of surfaces, and the three shown examples justify our aim to,

- give an analogous numerical restriction to that of Pareschi and Popa for varieties such that the rational map associated to $\omega_X^2 \otimes \alpha$ is not birational for some $\alpha \in \operatorname{Pic}^0 X$;
- classify varieties (of maximal Albanese dimension) such that the bicanonical is non-birational.

For the first question we obtain the following answer.

Theorem V.5.1. *Let X be a smooth projective complex variety. If $\operatorname{gv}(\omega_X) \geqslant 2$, then the rational map associated to $\omega_X^2 \otimes \alpha$ is birational onto its image for every $\alpha \in \operatorname{Pic}^0 X$.*

This numerical criterion, based on the generic vanishing index, implies also the following Corollary

Corollary V.5.2. *Let X be a smooth projective complex variety of maximal Albanese dimension such that the bicanonical map is not birational. Then $0 \leqslant \operatorname{gv}(\omega_X) \leqslant 1$. Moreover, it admits a fibration onto a normal projective variety Y with $0 \leqslant \dim Y < \dim X$, any smooth model $\tilde{Y}$ of Y is of maximal Albanese dimension, and*

1. *either, the general fibers map onto divisors in a fixed abelian variety*
2. *or the general fibers map onto a fixed abelian variety.*

In any case,

$$q(X) - \dim X \leqslant q(\tilde{Y}) - \dim Y + \operatorname{gv}(\omega_X).$$

Observe that we are allowing $\dim Y = 0$, so we can distinguish between to cases:

(i) When $\dim Y > 0$ we have an actual fibration. For surfaces, this corresponds to the standard case and the general fibers are curves of genus 2 that

   1. either they are embedded into a fixed abelian surface via the Abel-Jacobi map,
   2. or they are coverings branched in two points of a fixed elliptic curve.

In arbitrary dimension, it is expected that, when $\dim Y > 0$, the bicanonical map of the general fiber is non-birational.

(ii) When $\dim Y = 0$, we have that

1. either X maps to a divisor in its Albanese variety,
2. or X maps onto its Albanese variety.

To show that this situation is analogous to what happens in dimension 2, i.e. that Examples A and B provide the only possible constructions, it remains to show the following properties.

In case 1, the Albanese map is birational (it follows easily that $\mathrm{Alb}\, X$ is a ppav and X is birationally equivalent to a theta-divisor.)

In case 2, the Albanese map is generically $(2:1)$, the $\mathrm{Alb}\, X$ is a ppav and the corresponding covering is branched along a reduced divisor in $|2\Theta|$.

Before focussing in case *(ii)* to see the progresses made in this Thesis, let us outline the main ingredients on the proof of the previous theorem.

The proof of Theorem V.5.1 is based in the following birationality criterion proved in collaboration with M.A. Barja, J.C. Naranjo and G. Pareschi in [BLNP].

Theorem V.4.9 ([BLNP, Thm. 4.13])**.** *Let X be an irregular variety with a map to an abelian variety $a : X \to A$, such that $\mathrm{gv}_a(\omega_X) \geqslant 1$. Let U_0 be the complement of $V^1_a(\omega_X)$ in $\mathrm{Pic}^0 A$ and*

$$\mathcal{B}_a(p) = \{\alpha \in U_0 \,|\, p \in \mathrm{Bs}(\omega_X \otimes a^*\alpha)\}\,.$$

Suppose that for general p in X, $\mathrm{codim}_{\mathrm{Pic}^0 A}\, \mathcal{B}_a(p) \geqslant 2$, then $\omega_X^2 \otimes a^*\alpha$ is birational for all $\alpha \in \mathrm{Pic}^0 A$. In particular, ω_X^2 is birational.

The proof of this theorem is based on the generic vanishing theorems of Green and Lazarsfeld ([GL1, GL2]), some results of Kollár concerning the higher direct images of the canonical sheaf ([K1, K2]) and the continuously globally generation introduced by Pareschi and Popa ([PP1, PP3]). We also use broadly the dictionary established by Pareschi and Popa between the non-negative values of the generic vanishing properties of the canonical sheaf and the properties of the generalized Fourier-Mukai transform of the structural sheaf. Moreover a new geometrical tool comes in to substitute Mukai Equivalence Theorem.

Proposition V.3.2 ([BLNP, Prop. 6.1])**.** *Let X be a smooth variety of dimension d, equipped with a non-trivial morphism to an abelian variety $a\colon X \to A$ (over any algebraically closed field k) such that the map $a^*\colon \mathrm{Pic}^0 A \to \mathrm{Pic}^0 X$ is an embedding. Then*

$$R^d\Phi_{P_a}(\omega_X) \cong k(\hat{0}),$$

where $P_a = (a \times \mathrm{id})^*\mathcal{P}$ *and* $\mathcal{P}$ *is the Poincaré line bundle on* $A \times \operatorname{Pic}^0 A$.

When X itself is an abelian variety (or a complex torus), we recover a well-known result proved by Mumford in [Mu2, pg. 128] that is crucial in the proof of the Mukai Equivalence Theorem. As a first consequence we obtain a characterization of abelian varieties (see Propostion V.3.9 or [BLNP, Prop. 4.10]).

Primitive varieties

We would like to classify varieties of maximal Albanese dimension and non-birational bicanonical map that do not admit "irregular fibrations" (so they must fall in case *(ii)* in the previous discussion). First, let us formalize which kind of varieties are we talking about.

Definition V.6.1 ([Ca, Def. 1.24])**.** An irregular Kähler manifold such that $\dim V^i(\omega_X) = 0$ for all $i > 0$ is called *primitive*.

As we have already claimed, the fibrations to maximal Albanese dimension varieties that primitive varieties admit are very special.

Proposition V.6.3. *Let* $f\colon X \to Y$ *be a fibration from a primitive variety* X *to a variety* Y *whose Albanese map is generically finite onto its image. Then*

(a) Y *is birational to an abelian variety.*

(b) Let F *be a general smooth fiber and* $\rho\colon \operatorname{Pic}^0 X \to \operatorname{Pic}^0 F$ *the restriction map. Then*

$$\ker \rho = f^* \operatorname{Pic}^0 Y.$$

When X is a primitive variety $\mathrm{gv}(\omega_X) = q(X) - \dim X$. If X is of maximal Albanese dimension $\mathrm{gv}(\omega_X) \geqslant 0$ by Green-Lazarsfeld's Theorem [GL1, Thm. 1]. Recall that, as in the case of surfaces, we expect that primitive varieties of general type with maximal Albanese dimension and non-birational bicanonical map have to be

- either birationally equivalent to a theta-divisor as in Example A (that has $\mathrm{gv}(\omega_X) = 1$)
- or birationally equivalent to a double cover of a ppav branched along a reduced divisor in $|2\Theta|$ like in Example B (that has $\mathrm{gv}(\omega_X) = 0$).

Hence it is natural to split the study in two cases, when $q(X) > \dim X$ we expect Example A and when $q(X) = \dim X$ we expect Example B.

Primitive varieties with $q(X) > \dim X$**.** In this case, in collaboration with M.A. Barja, J.C. Naranjo and G. Pareschi, we achieve the full classification of primitive varieties with $q(X) > \dim X$ and non-birational bicanonical map. Thus, we get the expected result:

Theorem V.6.7 ([BLNP, Thm. A]). *Let X be primitive smooth complex projective variety such that $\dim X < q(X)$. The following are equivalent*

(a) the bicanonical map of X is non-birational,

(b) X is birationally equivalent to a theta-divisor of an indecomposable principally polarized abelian variety.

Recall that, by our numerical criterion (see Theorem V.5.1) we already know that $\mathrm{gv}(\omega_X) = 1$ and X maps to a divisor in its Albanese variety. The key point in proving the previous result is to see that the classification of primitive varieties with $\mathrm{gv}(\omega_X) = 1$ and non-birational bicanonical map is equivalent to the classification of primitive varieties with $\chi(\omega_X) = \mathrm{gv}(\omega_X) = 1$ (these are the boundary examples with $\chi(\omega_X) = 1$ of the higher-dimensional Castelnuovo-de Franchis inequality proven by Pareschi and Popa in [PP7, Cor. 4.1]).

Then, we conclude the proof of the previous result by extending a cohomological characterization of theta-divisors due to Hacon and Pardini [HP1, Prop. 4.2] that has been proved independently, with a different proof, by Lazarsfeld and Popa [LP, Prop. 3.13].

Proposition V.6.4 ([BLNP, Prop. 3.1]). *Let X be a d-dimensional compact Kähler manifold such that:*

(a) X is primitive;

(b) $d < q = q(X)$;

(c) $\chi(\omega_X) = 1$.

Then $\mathrm{Alb}\, X$ *is a principally polarized abelian variety and the Albanese map* $\mathrm{alb}\colon X \to \mathrm{Alb}\, X$ *maps X birationally onto a theta-divisor.*

Primitive varieties with $q(X) = \dim X$. It is remarkable that in this situation, even in the case of surfaces, the equivalence between the classification of primitive varieties of general type with $\chi(\omega_X) = 1$ and primitive varieties of general type with non-birational bicanonical map, is no longer true. Indeed, there are primitive varieties with $\chi(\omega_X) = 1$, $\dim X = q(X)$ and birational bicanonical map. The first known example of this fact, was pointed out by Chen and Hacon in [CH2], where they construct a minimal surface S with $\chi(\omega_S) = 1$, $q(S) = 2$ and $K_S^2 = 5$. This surface is birational to a triple cover of an abelian surface A and its bicanonical map is birational.

This example shows that the problem of classifying primitive varieties of general type with non-birational bicanonical map and $q(X) = \dim X$ is more subtle.

We concentrate and work out the case of Galois abelian covers. That is, when X of maximal Albanese dimension and $q(X) = \dim X$, the Albanese map is generically finite and surjective. We suppose that $\mathrm{Alb}(X)$ is simple, that the Stein factorization of the Albanese map factors through a Galois abelian cover and it has rational singularities. In this situation, we show that the unique case where the bicanonical map of X is non-birational, occurs when X is birationally equivalent to a double cover over a ppav branched along a reduced divisor in the linear series $|2\Theta|$.

Theorem V.6.13. *Let X be a primitive smooth complex variety of general type, $q(X) = \dim X$ and suppose $\mathrm{Alb}\, X$ is simple. Then, the following are equivalent,*

- *(a) the bicanonical map of X is non-birational and the finite part of the Stein factorization of $\mathrm{alb}\colon X \to \mathrm{Alb}\, X$ is an abelian Galois cover with rational singularities,*
- *(b) X is birationally equivalent to a double cover of an indecomposable principally polarized abelian variety (A, Θ), branched along a reduced divisor in $|2\Theta|$.*

In fact, almost the same argument that we use in the previous theorem, proves the the following Proposition.

Proposition V.6.16. *Let X be a primitive smooth complex variety of general type and $q(X) = \dim X$. Then, the following are equivalent,*

- *(a) the finite part of the Stein factorization of $\mathrm{alb}\colon X \to \mathrm{Alb}\, X$ is an abelian Galois cover, has rational singularities and $\chi(\omega_X) = 1$,*
- *(b) X is birationally equivalent to a double cover of an indecomposable principally polarized abelian variety (A, Θ), branched along a reduced divisor in $|2\Theta|$.*

These two results, show that we have to find the difficulties and interesting examples like Chen-Hacon's surface in the non-abelian case. However, we expect that the key point to study the primitive varieties of general type and $q(X) = \dim X$ with non-birational bicanonical map, is the type of singularities that are allowed in the Stein factorization. So, we end with the following conjecture,

Conjecture V.6.17. *Let X be a primitive smooth complex variety of general type and $q(X) = \dim X$. Then, the following are equivalent,*

- *(a) the bicanonical map of X is non-birational,*
- *(b) the finite part of the Stein factorization of $\mathrm{alb}\colon X \to \mathrm{Alb}\, X$ has canonical singularities and $\chi(\omega_X) = 1$,*
- *(c) X is birationally equivalent to a double cover of an indecomposable principally polarized abelian variety (A, Θ), branched along a reduced divisor in $|2\Theta|$.*

CHAPTER

I

Fourier-Mukai Preliminaries

Introduction

The Fourier-Mukai transform was introduced by Mukai in his seminal paper [M2] to study the moduli of deformations of Picard sheaves. In general, an abelian variety and its dual are non-isomorphic. However, he introduced an equivalence of categories between their derived categories. This equivalence is based on the Poincaré line bundle and its properties (see Theorem I.2.1).

More generally, given two varieties X and Y, if we substitute the Poincaré line bundle by an object in the derived category of the product $X \times Y$, we can consider the generalized Fourier-Mukai transform between the derived categories of X and Y. In this sense, the Fourier-Mukai is the derived version of the notion of a correspondence, which has been studied for all kinds of cohomology theories (e.g. Chow groups, singular cohomology, etc.) for many decades.

In a sequence of articles (see [PP8, PP3, PP7]), Pareschi and Popa studied the generalized Fourier-Mukai transform from the derived category of an irregular variety and the derived category of its Picard variety (see §1 for a more precise definition). In particular, when X is an abelian variety they have the Mukai's original transform. Using results of commutative algebra they have given an interpretation of some (local) properties of the Fourier-Mukai transform in terms of the generic vanishing theorems introduced by Green and Lazarsfeld in [GL1, GL2]. On one hand, they have used the Auslander-Buchsbaum Theorem to show that we have an equivalence between the fact that a sheaf has non-negative generic vanishing index (see Definition I.1.9) and the fact that the Fourier-Mukai transform of its Grothendieck dual is a sheaf (see Theorem I.1.10). One implication was a conjecture of Green and Lazarsfeld [GL2, Prob. 6.2] that was already proven by Hacon [Ha, Thm. 1.5]. On the other hand they have found a dictionary between the non-negative values of the generic vanishing index and the local properties of the Fourier-Mukai transform of the dual sheaf in the sense of Grothendieck (see Theorem I.1.16), based on the Evans-Griffith Syzygy Theorem. These two theorems are essential for the main results of the subsequent chapters and will be referred frequently. However, although they provide a dictionary between properties, they have a mild geometrical content. As Pareschi and Popa have already done in their articles, we will use two geometric inputs to obtain geometrical results. On one hand, the original Mukai Equivalence Theorem I.2.1 in abelian varieties (that we recall in §2). And, on the other hand, the Generic Vanishing Theorems of Green and Lazarsfeld that will be recalled and used in Chapter V.

This chapter is of expository nature, so almost none of the results are proved and they are only stated for easy reference in the subsequent chapters. The exposition is influenced by the point of view of Giuseppe Pareschi and his work in collaboration with Mihnea Popa.

1 Generalized Fourier-Mukai transforms and generic vanishing

This preliminary chapter works in two different settings. We can suppose that X is a smooth[1] projective, reduced and irreducible variety over on an arbitrary algebraically closed field. Or we can assume that X is a compact, connected, Kähler manifold over the complex numbers $\mathbb{C}$.

Let X be a variety of dimension d, equipped with a morphism to an abelian variety (resp. complex torus) A,

$$a\colon X \to A.$$

Let $\mathrm{Pic}^0 X$ denote the identity component of the Picard group scheme of X and we will denote by $\mathrm{Alb}\, X$ the *Albanese variety* (resp. Albanese torus) of X and $\mathrm{alb} : X \to \mathrm{Alb}\, X$ the Albanese map. $\mathrm{Alb}\, X$ and $\mathrm{Pic}^0 X$ are dual to each other [BL, Prop. 11.11.6]. Its dimension is called the *irregularity* of X and it is denoted by $q(X) = \dim \mathrm{Alb}\, X = \dim \mathrm{Pic}^0 X$.

Let $\mathcal{P}$ be a Poincaré line bundle on $A \times \mathrm{Pic}^0 A$. We will denote

$$P_a = (a \times \mathrm{id}_{\mathrm{Pic}^0 A})^* \mathcal{P}, \tag{1.1}$$

the *induced Poincaré line bundle* in $X \times \mathrm{Pic}^0 A$ by a. When $a = \mathrm{alb}$, the Albanese map of X, then the map alb^* identifies $\mathrm{Pic}^0(\mathrm{Alb}\, X)$ to $\mathrm{Pic}^0 X$ and the line bundle P_{alb} is identified to the *Poincaré line bundle of* X. We will simply denote $P = P_{\mathrm{alb}}$.

In the sequel we will consider often the *derived category* of X, $\mathrm{D}^b(X)$ that by definition is the *bounded derived category* of the abelian category $\mathrm{Coh}(X)$, i.e. $\mathrm{D}^b(X) := \mathrm{D}^b(\mathrm{Coh}(X))$ (see [Hu, Ch. 1-3] for a short introduction to derived categories and specially to the derived category of coherent sheaves). We will use the abuse of notation $\mathcal{F} \in \mathrm{D}^b(X)$, meaning $\mathcal{F} \in \mathrm{Ob}(\mathrm{D}^b(X))$.

Letting p and q the two projections of $X \times \mathrm{Pic}^0 A$, we consider the left exact functor

$$\Phi_{P_a}(\mathcal{F}) = q_*(p^*\mathcal{F} \otimes P_a),$$

and its right derived functor between bounded from below derived categories,

$$\mathrm{R}\Phi_{P_a} : \mathrm{D}^+(X) \to \mathrm{D}^+(\mathrm{Pic}^0 A).$$

Since for every sheaf $\mathcal{F}$, $\mathrm{R}\Phi_{P_a}(\mathcal{F}) \in \mathrm{D}^b(\mathrm{Pic}^0 A)$, this right derived functor induces

[1] In fact, we could suppose only X being Cohen-Macaulay, since the Grothendieck-Verdier duality works for Cohen-Macaulay schemes (see [Co, Thm. 4.3.1]).

an exact functor (see [Hu, Cor. 2.68])

$$\mathrm{R}\Phi_{P_a} : \mathrm{D}^b(X) \to \mathrm{D}^b(\mathrm{Pic}^0 A). \tag{1.2}$$

We will simply call $\mathrm{R}\Phi_{P_a}$ the *Fourier-Mukai transform associated to* P_a. For some authors, this functor does not deserve the name of Fourier-Mukai transform since, in general, it is not an equivalence of categories. They would call $\mathrm{R}\Phi_{P_a}$ the integral transform with kernel P_a.

The functors $\mathrm{R}\Phi_{P_a}$ and $\mathrm{R}\Phi_{\mathcal{P}}$ are related by the following formula.

Proposition I.1.1. $\mathrm{R}\Phi_{P_a} \cong \mathrm{R}\Phi_{\mathcal{P}} \circ \mathrm{R}a_*$.

Proof. $$\mathrm{R}\Phi_{P_a}(\,\cdot\,) = \mathrm{R}q_*(p_X^*(\,\cdot\,) \otimes (a \times \mathrm{id})^*\mathcal{P}) \overset{L+PF}{\cong} \mathrm{R}q_*(R(a \times \mathrm{id})_*(p_X^*(\,\cdot\,)) \otimes \mathcal{P}) \cong$$
$$\overset{BC}{\cong} \mathrm{R}q_*(p_A^*(\mathrm{R}a_*(\,\cdot\,)) \otimes \mathcal{P}) = \mathrm{R}\Phi_{\mathcal{P}} \circ \mathrm{R}a_*(\,\cdot\,),$$

where: $L =$ Leray, $PF =$ projection formula and $BC =$ Base Change, in the derived category. $\square$

Definition I.1.2 (Base-change property)**.** Given $\mathcal{F} \in \mathrm{D}^b(X)$, we will say that the sheaf $R^i\Phi_{P_a}\mathcal{F}$ has the *base-change property* in a neighborhood W of $\alpha \in \mathrm{Pic}^0 A$, if it is locally free in W and

$$R^i\Phi_{P_a}\mathcal{F} \otimes k(\beta) \cong H^i(X, \mathcal{F} \otimes \beta) \qquad \text{for all } \beta \in W,$$

where $H^i(X, \,\cdot\,)$ is the i-th cohomology sheaf of the functor $\mathrm{R}\Gamma(X, \,\cdot\,)$.

We remark that for $\mathcal{F} \in \mathrm{D}^b(X)$, $H^i(X, \mathcal{F})$ will always mean the i-th cohomology sheaf of $\mathrm{R}\Gamma(X, \mathcal{F})$.

If $h^{i+1}(X, \mathcal{F} \otimes \alpha)$ is constant in a neighborhood of $\alpha \in \mathrm{Pic}^0 A$, then, both $R^{i+1}\Phi_{P_a}\mathcal{F}$ and $R^i\Phi_{P_a}\mathcal{F}$ have the base-change property in a neighborhood of α. When $\mathcal{F}$ is a sheaf this follows from [Mu2, Cor. 2, pg. 50]. The more general case $\mathcal{F} \in \mathrm{D}^b(X)$ follows from [EGA3, §7.7].

Sometimes we will have to consider the analogous derived functor $\mathrm{R}\Phi_{P_a^{-1}} : \mathrm{D}^b(X) \to \mathrm{D}^b(\mathrm{Pic}^0 A)$ as well. By the Seesaw Theorem [Mu2, Cor. 6, pg. 54], $\mathcal{P}^{-1} = (1_A \times (-1)_{\mathrm{Pic}^0 A})^*\mathcal{P}$ and we have the following immediate corollary.

Corollary I.1.3. $\mathrm{R}\Phi_{P_a^{-1}} = (-1_{\mathrm{Pic}^0 A})^* \circ \mathrm{R}\Phi_{P_a}$.

Changing the order of the projections we have another left exact functor

$$\Psi_{P_a}(\mathcal{F}) = p_*(q^*\mathcal{F} \otimes P_a)$$

and its induced exact functor between bounded derived categories is

$$\mathrm{R}\Psi_{P_a} : \mathrm{D}^b(\mathrm{Pic}^0 A) \to \mathrm{D}^b(X). \tag{1.3}$$

In what follows, we will adopt the following notation for the *dualizing functor*

$$\mathrm{R}\Delta\, \mathcal{F} = \mathrm{R}\mathcal{H}om(\mathcal{F}, \omega_X).$$

We will often use the *Grothendieck-Verdier duality* (see [Co, Thm. 4.3.1] for a complete proof in a more general case).

Theorem I.1.4 (Grothendieck-Verdier duality). *Let $f : X \to Y$ be a morphism of smooth schemes over a field k of relative dimension d. Then, we have the following functorial isomorphism of functors in the derived category,*

$$\mathrm{R}\Delta_Y \circ \mathrm{R}f_* \cong \mathrm{R}f_* \circ \mathrm{R}\Delta_X[d].$$

When $f : X \to \{\text{point}\}$ we get as a corollary the Grothendieck-Serre duality.

Corollary I.1.5 (Grothendieck-Serre duality). *Let X be a smooth scheme over a field k of dimension d. If $\mathcal{F} \in \mathrm{D}^b(X)$, then we have the following functorial isomorphism,*

$$H^i(X, \mathcal{F})^* \cong H^{d-i}(X, \mathrm{R}\Delta\, \mathcal{F}),$$

where $H^i(X, \cdot\,)$ is the i-th cohomology sheaf of the functor $\mathrm{R}\Gamma(X, \cdot\,)$ and * denotes the dual as a k-vector space.

However, when we refer to the Grothendieck-Verdier duality of the Fourier-Mukai transform we will basically mean the following functorial isomorphism, that we state for easy reference.

Proposition I.1.6 (Grothendieck-Verdier duality). *We have the following functorial isomorphism of functors in the derived category,*

$$\mathrm{R}\Delta \circ \mathrm{R}\Phi_{P_a} \cong \mathrm{R}\Phi_{P_a^{-1}} \circ \mathrm{R}\Delta[d],$$

where the left dualizing functor is $\mathrm{R}\mathcal{H}om(\,\cdot\,, \mathcal{O}_{\mathrm{Pic}^0 A})$ since $\omega_{\mathrm{Pic}^0 A} \cong \mathcal{O}_{\mathrm{Pic}^0 A}$, and the right one is $\mathrm{R}\mathcal{H}om(\,\cdot\,, \omega_X)$.

Remark I.1.7. *Given an object $\mathcal{F} \in \mathrm{D}^b(X)$, when we take cohomology on the isomorphism provided by the Grothendieck-Verdier duality we have the following isomorphism,*

$$\mathcal{E}xt^i(\mathrm{R}\Phi_{P_a}\, \mathcal{F}, \mathcal{O}_{\mathrm{Pic}^0 A}) \cong R^{d+i}\Phi_{P_a^{-1}}\, \mathrm{R}\Delta\, \mathcal{F}.$$

Definition I.1.8. Given a coherent sheaf $\mathcal{F}$ on X, its *i-th cohomological support locus with respect to a* is

$$V_a^i(\mathcal{F}) = \left\{\alpha \in \mathrm{Pic}^0 A \,\middle|\, h^i(\mathcal{F} \otimes a^*\alpha) > 0\right\}$$

Again, when a is the Albanese map of X, we will omit the subscript, simply writing $V^i(\mathcal{F})$.

By base change, these loci contain the set-theoretical support of $R^i\Phi_{P_a}\mathcal{F}$, i.e.

$$\operatorname{supp} R^i\Phi_{P_a}\mathcal{F} \subseteq V^i_a(\mathcal{F}).$$

A way to measure the size of all the $V^i_a(\mathcal{F})$'s is provided by the following invariant introduced by Pareschi and Popa.

Definition I.1.9 ([PP7, Def. 3.1])**.** Given a coherent sheaf $\mathcal{F}$ on X, the *generic vanishing index* of $\mathcal{F}$ (with respect to a) is

$$\operatorname{gv}_a(\mathcal{F}) := \min_{i>0}\left\{\operatorname{codim}_{\operatorname{Pic}^0 A} V^i_a(\mathcal{F}) - i\right\}.$$

The Auslander-Buchsbaum Theorem [Ma, Thm. 19.1], gives a first basic result that, in the most useful case, relates the positivity of the generic vanishing index to the fact that the Fourier-Mukai transform of its Grothendieck dual is a sheaf (in cohomological degree d).

Theorem I.1.10 ([PP8, Thm. A],[PP7, Thm. 2.2])**.** *Let $\mathcal{F}$ be a coherent sheaf on X. The following are equivalent,*

(a) $\operatorname{gv}_a(\mathcal{F}) \geqslant -k$ *for* $k \geqslant 0$*;*

(b) $R^i\Phi_{P_a}(\mathrm{R}\Delta\,\mathcal{F}) = 0$ *for all* $i \neq d-k, \dots, d$.

Definition I.1.11. If $\operatorname{gv}_a(\mathcal{F}) \geqslant 0$ the sheaf $\mathcal{F}$ is said to be a *GV-sheaf (generic vanishing sheaf).*

In [M2, Def. 2.3] Mukai introduced the following notation

Definition I.1.12. *(a)* We say that the *weak index theorem* holds for a coherent sheaf $\mathcal{F}$ on X or that $\mathcal{F}$ is a *WIT$_i$-sheaf* (with respect to a) if, and only if,

$$R^j\Phi_{P_a}\mathcal{F} = 0 \qquad \text{for all } j \neq i.$$

This i is called the index of $\mathcal{F}$. We denote the coherent sheaf $R^i\Phi_{P_a}\mathcal{F}$ by $\widehat{\mathcal{F}}$.

(b) We say that the *index theorem* holds for $\mathcal{F}$ or that $\mathcal{F}$ is a *IT$_i$-sheaf* (with respect to a) if, and only if,

$$V^j_a(\mathcal{F}) = \emptyset \qquad \text{for all } j \neq i.$$

By the base change theorem in cohomology, IT$_i$ implies WIT$_i$. Moreover the Fourier-Mukai transform of a sheaf satisfying IT is locally free.

With this nomenclature, Theorem I.1.10 says that, if $\mathcal{F}$ is a GV-sheaf, then the full transform $\mathrm{R}\Phi_{P_a}(\mathrm{R}\Delta\,\mathcal{F})$ is a sheaf concentrated in degree d, i.e. $\mathrm{R}\Delta\,\mathcal{F}$ is a WIT_d-sheaf, and we usually denote

$$\widehat{\mathrm{R}\Delta\,\mathcal{F}} = R^d\Phi_{P_a}(\mathrm{R}\Delta\,\mathcal{F}).$$

Observe that, when $\mathcal{F}$ is a coherent sheaf, since $R^j\Phi_{P_a}\,\mathcal{F}=0$ and $R^j\Phi_{P_a}(\mathrm{R}\Delta\,\mathcal{F})=0$ vanish for $j>d$, both $R^d\Phi_{P_a}\,\mathcal{F}=0$ and $R^d\Phi_{P_a}(\mathrm{R}\Delta\,\mathcal{F})=0$ have the base-change property at all $\alpha\in\operatorname{Pic}^0 A$.

Note that, if $\mathcal{F}$ is a GV-sheaf, then $H^i(\mathcal{F}\otimes a^*\alpha)=0$ for all $i>0$ and general $\alpha\in\operatorname{Pic}^0 A$. Therefore, by deformation-invariance of χ, the generic value of $h^0(\mathcal{F}\otimes a^*\alpha)$ equals $\chi(\mathcal{F})$, in particular $\chi(\mathcal{F})\geqslant 0$. Since, by base-change, the fiber of $\widehat{\mathrm{R}\Delta\,\mathcal{F}}$ at a general point $\alpha\in\operatorname{Pic}^0 A$ is isomorphic to $H^d(\mathrm{R}\Delta\,\mathcal{F}\otimes a^*\alpha)\cong H^0(\mathcal{F}\otimes a^*\alpha^{-1})^*$ (by Grothendieck-Serre duality I.1.5), the (generic) rank of $\widehat{\mathrm{R}\Delta\,\mathcal{F}}$ is

$$\operatorname{rk}(\widehat{\mathrm{R}\Delta\,\mathcal{F}}) = \chi(\mathcal{F}). \tag{1.4}$$

Via base-change, Theorem I.1.10 yields to,

Corollary I.1.13 ([Ha, Thm. 1.2], [PP8, Prop. 3.13]). *If* $\operatorname{gv}_a(\mathcal{F})\geqslant -k$, *for any* $k\geqslant 0$ *then,*

$$V_a^d(\mathcal{F})\subseteq\cdots\subseteq V_a^{k-1}(\mathcal{F})\subseteq V_a^k(\mathcal{F}).$$

From Grothendieck-Verdier duality I.1.6 and Theorem I.1.10 it follows that,

Corollary I.1.14 ([PP8, Rem. 3.12], [PP7, Pf. of Cor. 3.2]). *If* $\operatorname{gv}_a(\mathcal{F})\geqslant 0$ *then*

$$\mathcal{E}xt^i_{\mathcal{O}_{\operatorname{Pic}^0 A}}(\widehat{\mathrm{R}\Delta\,\mathcal{F}},\mathcal{O}_{\operatorname{Pic}^0 A})\cong R^i\Phi_{P_a^{-1}}(\mathcal{F})\cong(-1_{\operatorname{Pic}^0 A})^*\,R^i\Phi_{P_a}(\mathcal{F}).$$

The previous corollary and base-change leads to the following result of Pareschi-Popa that allows us to descend the irreducible components of $V_a^0(\mathcal{F})$ through the chain of inclusions of Corollary I.1.13. In [PP8, Prop. 3.15] this result appears with an unnecessary hypothesis as Pareschi pointed out to us in a private communication.

Corollary I.1.15. *Let* $\mathcal{F}$ *be a GV-sheaf. Let* W *be an irreducible component of* $V_a^0(\mathcal{F})$, *and let* $k=\operatorname{codim}_{\operatorname{Pic}^0 A}W$. *Then* W *is also a component of* $V_a^k(\mathcal{F})$. *In particular, it follows that* $k\leqslant d$.

Proof. Since $\widehat{\mathrm{R}\Delta\,\mathcal{F}}$ has the base-change property, it is supported at $V_a^d(\mathrm{R}\Delta\,\mathcal{F})=-V_a^0(\mathcal{F})$ (by Grothendieck-Serre duality I.1.5). Hence $-W$ is a component of the support of $\widehat{\mathrm{R}\Delta\,\mathcal{F}}$. Let α be a general point of W. By Corollary I.1.14, $R^i\Phi_{P_a}\,\mathcal{F}=(-1_{\operatorname{Pic}^0 A})^*\,\mathcal{E}xt^i(\widehat{\mathrm{R}\Delta\,\mathcal{F}},\mathcal{O}_{\operatorname{Pic}^0 A})$ and from well-known properties of $\mathcal{E}xt$'s it follows that, in a suitable neighborhood of $\alpha\in\operatorname{Pic}^0 A$, $R^i\Phi_{P_a}\,\mathcal{F}$ vanishes for $i<k$ and is supported at W for $i=k$. Therefore, by base-change, W is contained in $V_a^k(\mathcal{F})$ (and in fact it is a component since, again by Theorem I.1.10, $\operatorname{codim}V_a^k(\mathcal{F})\geqslant k$). □

A deeper result of Pareschi-Popa, based on the Evans-Griffith Syzygy Theorem [EG, Cor. 1.7], gives a dictionary between the value of $\mathrm{gv}_a(\mathcal{F})$ and the local properties of the transform $\widehat{\mathrm{R}\Delta\,\mathcal{F}}$.

Theorem I.1.16 ([PP7, Cor. 3.2]). *Assume that $\mathcal{F}$ is a GV-sheaf (with respect to a). Then the following are equivalent*

(a) $\mathrm{gv}_a(\mathcal{F}) \geqslant m$;

(b) $\widehat{\mathrm{R}\Delta\,\mathcal{F}}$ *is a m-th syzygy sheaf.*

We recall the definition of a m-th syzygy sheaf.

Definition I.1.17. A coherent sheaf $\mathcal{F}$ on X is called a *m-th syzygy sheaf* if locally there exists an exact sequence

$$0 \to \mathcal{F} \to \mathcal{E}_m \to \ldots \to \mathcal{E}_1 \to \mathcal{G} \to 0$$

with $\mathcal{E}_j$ locally free for all j. It is well-known for example that first syzygy sheaf is equivalent to torsion-free, and second syzygy sheaf is equivalent to reflexive. Every coherent sheaf is declared to be a 0-th syzygy sheaf, while a locally free sheaf is declared to be an ∞-syzygy sheaf.

Hence, the next corollary is a particular case of the previous theorem, but we emphasize it, as this will be the most used case (maybe the only one).

Corollary I.1.18. *Assume that $\mathcal{F}$ is a GV-sheaf (with respect to a). Then,*

$$\mathrm{gv}_a(\mathcal{F}) \geqslant 1 \quad \textit{if and only if,} \quad \widehat{\mathrm{R}\Delta\,\mathcal{F}} \textit{ is torsion-free}$$

2 Mukai's equivalence on abelian varieties

Assume that X coincides with the abelian variety or complex torus A (and the map a is the identity). In this special case, according to the notation introduced in §1, $P = P_a = \mathcal{P}$ denotes the Poincaré line bundle on $A \times \operatorname{Pic}^0 A$. Then Mukai's theorem asserts that $\mathrm{R}\Phi_{\mathcal{P}}$ is an equivalence of categories. More precisely, denoting $\mathrm{R}\Psi_{\mathcal{P}} : \mathrm{D}^b(\operatorname{Pic}^0 A) \to \mathrm{D}^b(A)$ the functor in (1.3) and $g = \dim A$, we have the following precise equivalence.

Theorem I.2.1 ([M2, Thm. 2.2])**.**

$$\mathrm{R}\Psi_{\mathcal{P}} \circ \mathrm{R}\Phi_{\mathcal{P}} = (-1)_A^*[g], \qquad \mathrm{R}\Phi_{\mathcal{P}} \circ \mathrm{R}\Psi_{\mathcal{P}} = (-1)_{\operatorname{Pic}^0 A}^*[g].$$

For the proof we refer to the original paper of Mukai or to [Hu, Thm. 9.19]. We remark that a key point in both proofs is the fact that $\mathrm{R}\Phi_{\mathcal{P}}\, \mathcal{O}_A = k(\hat{0})[g]$.

The Fourier-Mukai transform of ample line bundles in abelian varieties was explicitly computed by Mukai in [M2]

Proposition I.2.2. *Consider L a line bundle on A.*

(a) L is an ample line bundle if, and only if, L is an IT_0-sheaf.

(b) If L is an ample line bundle, then

$$\phi_L^* \widehat{L} \cong \bigoplus_{h^0(L)} L^{-1},$$

where $\phi_L \colon A \to \operatorname{Pic}^0 A$ is the isogeny $a \mapsto t_a^* L \otimes L^{-1}$.

(c) In particular, when L gives a principal polarization in A, i.e. L is an ample line bundle and $h^0(L) = 1$, then

$$\phi_L^* \widehat{L} \cong L^{-1},$$

and $\phi_L \colon A \xrightarrow{\sim} \operatorname{Pic}^0 A$ is an isomorphism. So we usually identify $A \cong \operatorname{Pic}^0 A$ via ϕ_L and abusing notation we simply write $\widehat{L} \cong L^{-1}$.

Remark I.2.3. *As we have noticed at the beginning, we remark that all the results of the present chapter are algebraic, and work on algebraically closed fields of any characteristic.*

Principally Polarized Abelian Varieties

CHAPTER

II

Principally Polarized Abelian Varieties Preliminaries

Introduction

Principally polarized abelian varieties are quite well-behaved and relatively easy to study. Since the polarization provides an isomorphism between the abelian variety and its dual, the Fourier-Mukai transform can be seen as an autoequivalence. More precisely, Mukai's has showed (see [Hu, §9.3]) that the elements of the group $\mathrm{Sl}_2(\mathbb{Z})$ act naturally as autoequivalences on the derived category of a principally polarized abelian variety (up to shifts). Moreover, Orlov has given a complete description of the group of all autoequivalences of the derived category of an abelian variety (see [Hu, Prop. 9.55]).

In this first section, §1 we will fix the notation for principally polarized abelian varieties that we will use in Chapters III and IV. In particular, we will consider the Fourier-Mukai transform as an autoequivalence, using the isomorphism provided by the principal polarization. We will also introduce the notation that we will use in Chapter IV for the relative Fourier-Mukai transform.

Finally, as we have already said, the principal polarization gives an isomorphism between the abelian variety and its dual. This allow us to study a duality theory similar to the duality theory of the projective space, where the role played by hyperplanes in the projective space is now played by divisors representing the principal polarization. That is, given a subvariety Z of a principally polarized abelian variety (A, Θ), we say that the theta-dual of Z is the set of all translates of Θ that contain Z. This classical point of view of the theta-duality mainly takes into account the set-theoretical structure of the theta-dual and the original variety Z. Our point of view is to consider the natural scheme-theoretic structure on the theta-dual provided by the Fourier-Mukai transform (see 2.1 for a precise definition). This definition was proposed by Pareschi and Popa in [PP6] and will be one of the leitmotif of Chapters III and IV.

1 Principally polarized abelian varieties

Throughout this chapter, (A,Θ) denotes a principally polarized abelian variety (ppav for short) over an algebraically closed field k. We choose a symmetric divisor Θ that represents the principal polarization and we assume that is irreducible. For each point $a\in A$ we denote by $t_a\colon A\to A$ the *translation map*.

$$\begin{array}{rccc} t_a: & A & \longrightarrow & A \\ & p & \longmapsto & p+a. \end{array}$$

The points in the dual abelian variety $\widehat{A}$ are denoted by Greek letters, and are identified with homogeneous line bundles on A. There is a canonical isomorphism

$$\begin{array}{rccc} \phi_\Theta: & A & \longrightarrow & \widehat{A} \\ & a & \longmapsto & \mathcal{O}_A(t_a^{-1}\Theta-\Theta). \end{array} \tag{1.1}$$

The dual abelian variety $\widehat{A}$ inherits a natural theta-divisor $\widehat{\Theta}$ that corresponds to Θ under ϕ_Θ.

For each homogeneous line bundle $\alpha\in\widehat{A}$, we let Θ_α denote the unique effective divisor whose associated line bundle is $\mathcal{O}_A(\Theta)\otimes\alpha$. If $\alpha=\phi_\Theta(a)$, then we have

$$\Theta_\alpha=t_a^{-1}\Theta. \tag{1.2}$$

Thus the collection of Θ_α for varying α is also the collection of Θ-translates. Since we will frequently identify $\widehat{A}$ and A via ϕ_Θ, we will also denote by Θ_a the *theta-translate* $t_a^{-1}\Theta$, where a is a point in A. We observe that in the literature Θ_a is sometimes denoted by $\Theta-a$ and, on the other hand, Θ_a sometimes denotes $\Theta+a$.

We will identify A with $\widehat{A}$ via the isomorphism $\phi_\Theta:A\to\widehat{A}$. Then, one may consider the *Fourier-Mukai transform* as an autoequivalence $\mathrm{D}^b(A)\to\mathrm{D}^b(A)$. When we use the identification of A with $\widehat{A}$ via ϕ_Θ, we will use the original Mukai notation $\mathrm{R}\mathcal{S}$. To be precise, $\mathrm{R}\mathcal{S}$ is the Fourier-Mukai transform with kernel the *Mumford line bundle* $\mathscr{M}=(\mathrm{id}\times\phi_\Theta)^*\mathcal{P}=m^*\mathcal{O}_A(\Theta)\otimes p^*\mathcal{O}_A(-\Theta)\otimes q^*\mathcal{O}_A(-\Theta)$, where $m\colon A\times A\to A$ is the *group law* on the abelian variety and p,q are the two projections. Equivalently,

$$\mathrm{R}\mathcal{S}=\phi_\Theta^*\circ\mathrm{R}\Phi_{\mathcal{P}}\colon\mathrm{D}^b(A)\to\mathrm{D}^b(A).$$

In this setting we have the following exchange property between the translation by an element $a\in A$ and the tensor product by elements a seen as an element in $\operatorname{Pic}^0 A$ via the identification ϕ_Θ.

Lemma 1.1 ([M2, (3.1)]).

$$\mathrm{R}\mathcal{S}\circ t_a^*\cong(\otimes\alpha)\circ\mathrm{R}\mathcal{S}\ \textit{ and }\ \mathrm{R}\mathcal{S}\circ(\otimes\alpha)\cong t_a^*\circ\mathrm{R}\mathcal{S}\qquad \textit{where } \alpha=\phi_\Theta(a).$$

With this notation Proposition I.2.2 has the following form.

Proposition 1.2 ([M2, Prop. 3.11]). $\mathrm{R}\mathcal{S}(\mathcal{O}_A(\Theta)) \cong \mathcal{O}_A(-\Theta)$.

1.1 Relative Fourier-Mukai transform

For an arbitrary base scheme S, we similarly define a *relative Fourier-Mukai transform* $\mathrm{R}\mathcal{S}_S : \mathrm{D}^b(A \times S) \to \mathrm{D}^b(A \times S)$, by

$$\mathrm{R}\mathcal{S}_S(\mathcal{F}) = \mathrm{R}p_{23*}(p_{13}^*\mathcal{F} \otimes p_{12}^*\mathscr{M}) \tag{1.3}$$

where p_{ij} denotes the projection from $A \times A \times S$ onto its i'th and j'th factors and $\mathscr{M} = (\mathrm{id} \times \phi_\Theta)^*\mathcal{P} = m^*\mathcal{O}_A(\Theta) \otimes p^*\mathcal{O}_A(-\Theta) \otimes q^*\mathcal{O}_A(-\Theta)$ is the Mumford line bundle. Then the weak index theorem and the Fourier-Mukai transform $\widehat{\mathcal{F}}$ of a WIT sheaf $\mathcal{F}$ on $A \times S$ can be defined as above with $\mathrm{R}\mathcal{S}$ replaced by $\mathrm{R}\mathcal{S}_S$. The base change theorem in cohomology yields to the following result.

Lemma 1.3. *Let $\mathcal{F}$ be a coherent sheaf on $A \times S$, and let i be the maximal number such that $R^i\mathcal{S}_S(\mathcal{F}) \neq 0$. Then, for any base extension $f \colon S' \to S$, we have*

$$f_{\widehat{A}}^*(R^i\mathcal{S}_S(\mathcal{F})) \cong R^i\mathcal{S}_{S'}(f_A^*\mathcal{F}),$$

where we write f_X for the product of the identity map on a scheme X with f.

Proof. This is again [Mu2, Cor. 2 and Cor. 3, pg. 50-52]. □

In particular, if we have a WIT sheaf $\mathcal{F}$ on $A \times S$, then for any closed point $s \in S$, the Fourier-Mukai transform of the fiber $\mathcal{F} \otimes k(s)$ is the fiber $\widehat{\mathcal{F}} \otimes k(s)$ of the Fourier-Mukai transform of $\mathcal{F}$.

2 Theta-duality

Let $Y \subset A$ be an arbitrary subscheme. The *set-theoretical theta-dual of* Y is,

$$\{a \in A \,|\, Y \subseteq t_a^{-1}\Theta\} \tag{2.1}$$

where the inclusion $Y \subseteq t_a^{-1}\Theta$ is required to hold scheme theoretically. When Y is reduced, the set $T(Y)$ is clearly an intersection of theta-divisors and hence, it is closed. We want to provide $T(Y)$ with a natural schematic structure.

Definition 2.1 ([PP6, §4])**.** Let $Y \subset A$ be an arbitrary subscheme. The *schematic theta-dual of* Y is,

$$T(Y) = \operatorname{supp}\left((-1_A)^* R^g\mathcal{S}(\mathrm{R}\Delta(\mathcal{I}_Y(\Theta)))\right),$$

where we denote by supp the Fitting support (e.g. [Ei]).

Let $\mathcal{F}$ is a coherent sheaf on a scheme X, and let

$$\mathcal{E}_1 \xrightarrow{\psi} \mathcal{E}_0 \to \mathcal{F} \to 0 \tag{2.2}$$

be an exact sequence with $\mathcal{E}_1 \cong \mathcal{O}_X^{\oplus m}$ and $\mathcal{E}_0 \cong \mathcal{O}_X^{\oplus n}$ free sheaves (when X is not locally noetherian, m could be infinite). Then recall that the *Fitting ideal* $\operatorname{Fitt}_i \mathcal{F}$ of the sheaf $\mathcal{F}$ is the ideal given locally as the ideal generated by the $(n-i)\times(n-i)$ minors of ψ for any presentation of $\mathcal{F}$ as above. By Fitting's Lemma (e.g. [Ei, Cor-Def. 20.4]) this is independent of the presentation (2.2) chosen. The zero locus of $\operatorname{Fitt}_0 \mathcal{F} \subseteq X$ is called the *Fitting support* of $\mathcal{F}$. It is easy to see that the annihilator support is contained in the Fitting support (e.g. [Ei, Prop. 20.7]) and, in general, they are not equal.

Basically, we prefer the Fitting support instead of the annihilator support to define the schematic theta-dual because the Fitting support commutes with arbitrary base-change.

Proposition 2.2 (e.g. [Ei, Cor. 20.5])**.** *Let $f\colon X \to Y$ morphism and let $\mathcal{F}$ be a coherent sheaf on Y. Then $\operatorname{Fitt}_i(f^*\mathcal{F}) = f^*\operatorname{Fitt}_i(\mathcal{F})$, so $\operatorname{supp} f^*\mathcal{F} = f^{-1}(\operatorname{supp}\mathcal{F})$ as schemes.*
In particular, if X is a subscheme of Y and f is the inclusion

$$\operatorname{Fitt}_i(\mathcal{F}\otimes\mathcal{O}_X) = \operatorname{Fitt}_i(\mathcal{F})\otimes\mathcal{O}_X \quad \textit{and} \quad \operatorname{supp}(\mathcal{F}\otimes\mathcal{O}_X) = \operatorname{supp}\mathcal{F}\cap X.$$

The previous definition 2.1 is motivated by the following lemma that shows that the underlying subset of $T(Y)$ coincides with the set defined in (2.1).

Lemma 2.3 ([PP6, Lemma 4.1])**.** *For any $a \in A$ there is a canonical identification*

$$(-1_A)^* R^g\mathcal{S}(\mathrm{R}\Delta(\mathcal{I}_Y(\Theta))) \otimes_{\mathcal{O}_{A,a}} k(a) \cong H^0(A, \mathcal{I}_Y(\Theta_a))^\vee.$$

Hence, in any case, $T(Y)$ is a closed subscheme of A.

Recall that $\mathcal{O}_A(\Theta_a) = \mathcal{O}(\Theta) \otimes \phi_\Theta(a)$ (see (1.2)).

More precisely, Pareschi and Popa show that $(-1_A)^* R^g\mathcal{S}(\mathrm{R}\Delta(\mathcal{I}_Y(\Theta)))$ is a line bundle on $T(Y)$.

Corollary 2.4 ([PP6, Cor. 4.3])**.** *Let Y be an arbitrary scheme, then*

$$(-1_A)^* R^g\mathcal{S}(\mathrm{R}\Delta(\mathcal{I}_Y(\Theta))) \cong \mathcal{O}_{T(Y)}(\Theta).$$

Remark 2.5. *From the previous corollary and using the Grothendieck-Verdier duality (see Remark I.1.7) and Corollary I.1.3, we have also*

$$\mathcal{H}om(\mathrm{R}\mathcal{S}(\mathcal{I}_Y(\Theta)), \mathcal{O}_A) \cong \mathcal{O}_{T(Y)}(\Theta).$$

Example 2.6. Set theoretically it is clear that the theta-dual $T(a)$ of a point $a \in A$ is the divisor Θ_a, that is, $t_a^{-1}(\Theta)$. Scheme theoretically, we deduce this equality by applying the Fourier-Mukai transform to the sequence

$$0 \to \mathcal{I}_a(\Theta) \to \mathcal{O}_A(\Theta) \to k(a) \to 0.$$

By Proposition 1.2, we get that $R^0\mathcal{S}\,\mathcal{I}_a(\Theta) \subset \mathcal{O}_A(-\Theta)$. Since $\mathcal{O}_A(-\Theta)$ is torsion-free and $R^0\mathcal{S}\,\mathcal{I}_a(\Theta)$ is a torsion sheaf, it must be zero. So, $\mathcal{I}_a(\Theta)$ is WIT_1 and

$$0 \to \mathcal{O}_A(-\Theta) \to \mathcal{P}_a \to \widehat{\mathcal{I}_a(\Theta)} \to 0,$$

where $\mathcal{P}_a$ is the homogeneous line bundle on A corresponding to a. Thus, $\widehat{\mathcal{I}_a(\Theta)} = \mathcal{O}_{\Theta_a} \otimes \mathcal{P}_a$ and by Corollary I.1.14,

$$(-1_A)^* R^g\mathcal{S}(\mathrm{R}\Delta(\mathcal{I}_a(\Theta))) \cong \mathcal{E}xt^1(\widehat{\mathcal{I}_a(\Theta)}, \mathcal{O}_A) \cong \mathcal{O}_{\Theta_a}(\Theta).$$

Observe also that $R^i\mathcal{S}(\mathrm{R}\Delta(\mathcal{I}_a(\Theta))) = 0$ for all $i \neq g$.

The schematic theta-dual respects the usual properties of the set-theoretical duality.

Lemma 2.7. *Let $Y \subseteq Y'$ be an inclusion of subschemes of A. Then $T(Y') \subseteq T(Y)$.*

Proof. We have the exact sequence $0 \to \mathcal{I}_{Y'}(\Theta) \to \mathcal{I}_Y(\Theta) \to \mathcal{I}_{Y/Y'}(\Theta) \to 0$. Then we have the following triangle in the derived category $\mathrm{R}\mathcal{S}\,\mathrm{R}\Delta\,\mathcal{I}_{Y'/Y}(\Theta) \to \mathrm{R}\mathcal{S}\,\mathrm{R}\Delta\,\mathcal{I}_Y(\Theta) \to \mathrm{R}\mathcal{S}\,\mathrm{R}\Delta\,\mathcal{I}_{Y'}(\Theta)$. Taking (hyper)cohomologies we have the following long exact sequence

$$\ldots \to R^g\mathcal{S}\,\mathrm{R}\Delta\,\mathcal{I}_Y(\Theta) \xrightarrow{\varrho} R^g\mathcal{S}\,\mathrm{R}\Delta\,\mathcal{I}_{Y'}(\Theta) \to R^{g+1}\mathcal{S}\,\mathrm{R}\Delta\,\mathcal{I}_{Y'/Y}(\Theta) \to \ldots.$$

The support of $R^{g+1}\mathcal{S}\,\mathrm{R}\Delta\,\mathcal{I}_{Y'/Y}(\Theta)$ is contained in $V^{g+1}(\mathrm{R}\Delta\,\mathcal{I}_{Y'/Y}(\Theta))$ that, by Grothendieck-Serre duality I.1.5, is $-V^{-1}(\mathcal{I}_{Y'/Y}(\Theta)) = \emptyset$. Hence, ϱ is surjective and

$$T(Y') = \operatorname{supp} R^g\mathcal{S}\,\mathrm{R}\Delta\,\mathcal{I}_{Y'}(\Theta) \subseteq \operatorname{supp} R^g\mathcal{S}\,\mathrm{R}\Delta\,\mathcal{I}_Y(\Theta) = T(Y).$$

$\square$

Given any subscheme $Y \subset A$, we want to prove that $Y \subseteq T(T(Y))$. This is a natural property for a duality theory. In order to prove it scheme-theoretically we will need some preliminary lemmas.

Lemma 2.8. *Let Y be a subscheme of A and consider $\Delta_Y \subset Y \times Y$ be the diagonal subscheme. Then $\mathcal{I}_{\Delta_Y/A\times Y}(p_1^{-1}\Theta)$ is a relative WIT$_1$ sheaf. More precisely,*

$$\mathrm{R}\mathcal{S}_Y\left(\mathcal{I}_{\Delta_Y/A\times Y}(p_1^{-1}\Theta)\right) \cong \mathcal{L}|_{m_Y^{-1}\Theta}[-1],$$

where $\mathcal{L}$ is the restriction to $A\times Y$ of the line bundle $m^*\mathcal{O}_A(\Theta)\otimes p_1^*\mathcal{O}_A(-\Theta)$ and $m_Y\colon A\times Y \to A$ is the restricted group law.

Proof. Consider the standard short exact sequence

$$0 \to \mathcal{I}_{\Delta_Y/A\times Y}(p_1^{-1}\Theta) \to \mathcal{O}_{A\times Y}(p_1^{-1}\Theta) \to \mathcal{O}_{\Delta_Y}(p_1^{-1}\Theta) \to 0,$$

and its Fourier-Mukai transform

$$0 \to p_1^*\mathcal{O}_A(-\Theta) \to R^0\mathcal{S}_Y\,\mathcal{O}_{\Delta_Y}(p_1^{-1}\Theta) \to R^1\mathcal{S}_Y\left(\mathcal{I}_{\Delta_Y/A\times Y}(p_1^{-1}\Theta)\right) \to 0,$$

where $R^0\mathcal{S}_Y\left(\mathcal{I}_{\Delta_Y/A\times Y}(p_1^{-1}\Theta)\right) = 0$ since it is a torsion sheaf included in a line bundle.

Observe that, using notation as in (1.3), we get

$$\begin{aligned} R^i\mathcal{S}_Y\,\mathcal{O}_{\Delta_Y}(p_1^{-1}\Theta) &= R^i p_{23*}(p_{13}^*\mathcal{O}_{\Delta_Y}(p_1^{-1}\Theta)\otimes p_{12}^*\mathscr{M}) \\ &\cong R^i p_{23*}(p_{13}^*\mathcal{O}_{\Delta_Y}\otimes p_{12}^* m^*\mathcal{O}_A(\Theta))\otimes p_1^*\mathcal{O}_A(-\Theta), \end{aligned} \tag{2.3}$$

where the subindices of the projections indicate the factor where we project. Now, consider the following commutative diagram

$$\begin{tikzcd} \Delta_{13} \arrow[r, hook, "j"'] \arrow[d] \arrow[rr, bend left, "\bar{p}_{23}", "\cong"'] & A\times A\times Y \arrow[r, "p_{23}"'] \arrow[d, "p_{13}"] & A\times Y \\ \Delta_Y \arrow[r, hook] & A\times Y. & \end{tikzcd}$$

Then, the previous chain of isomorphisms (2.3) continues

$$\begin{aligned} R^i\mathcal{S}_Y\,\mathcal{O}_{\Delta_Y}(p_1^{-1}\Theta) &\cong R^i p_{23*}(j_*\mathcal{O}_{\Delta_{13}}\otimes p_{12}^* m^*\mathcal{O}_A(\Theta))\otimes p_1^*\mathcal{O}_A(-\Theta) \\ &\cong R^i p_{23*}(j_*\mathcal{O}_{\Delta_{13}}\otimes p_{12}^* m^*\mathcal{O}_A(\Theta))\otimes p_1^*\mathcal{O}_A(-\Theta) \\ &\cong R^i \bar{p}_{23*}(j^* p_{12}^* m^*\mathcal{O}_A(\Theta))\otimes p_1^*\mathcal{O}_A(-\Theta). \end{aligned}$$

Observe that, $\bar{p}_{23} = p_{23} \circ j$ is an isomorphism and $p_{12} \circ j$ induces an isomorphism of Δ_{13} in its image $Y \times A$. The composition of both isomorphisms give $Y \times A \cong A \times Y$. Hence,

$$R^i \mathcal{S}_Y\, \mathcal{O}_{\Delta_Y}(p_1^{-1}\Theta) = \begin{cases} m_Y^* \mathcal{O}_A(\Theta) \otimes p_1^* \mathcal{O}_A(-\Theta) = \mathcal{L} & \text{if } i = 0 \\ 0 & \text{otherwise,} \end{cases}$$

since $m_Y^* \mathcal{O}_A(\Theta) = m^* \mathcal{O}_A(\Theta) \otimes \mathcal{O}_{A \times Y}$.

Then, $R^1 \mathcal{S}_Y \left(\mathcal{I}_{\Delta_Y / A \times Y}(p_1^{-1}\Theta)\right)$ is $\mathcal{L}$ restricted to the unique section of the following line bundle,

$$p_1^* \mathcal{O}_A(\Theta) \otimes \mathcal{L} \cong m^* \mathcal{O}_A(\Theta)|_{A \times Y} \cong \mathcal{O}_{A \times Y}(m_Y^{-1}\Theta). \qquad \square$$

Before proving $Y \subseteq T(T(Y))$, we prove the following preliminar inclusion.

Corollary 2.9. *$T(Y) \times Y \subseteq m_Y^{-1}\Theta$, where $m_Y : A \times Y \to A$ is the restricted group law.*

Proof. Let $\Delta_Y \subset Y \times Y$ be the diagonal subscheme.

$$0 \to p_1^*(\mathcal{I}_{Y/A}(\Theta)) \to \mathcal{I}_{\Delta_Y / A \times Y}(p_1^{-1}\Theta) \to \mathcal{I}_{\Delta_Y / Y \times Y}(p_1^{-1}\Theta) \to 0.$$

Then we have the following triangle in the derived category

$$\mathrm{R}\mathcal{S}_Y\, \mathrm{R}\Delta(\mathcal{I}_{\Delta_Y / Y \times Y}(p_1^{-1}\Theta)) \to \mathrm{R}\mathcal{S}_Y\, \mathrm{R}\Delta(\mathcal{I}_{\Delta_Y / A \times Y}(p_1^{-1}\Theta)) \to p_1^*\, \mathrm{R}\mathcal{S}\, \mathrm{R}\Delta(\mathcal{I}_{Y/A}(\Theta)).$$

Taking (hyper)cohomologies we have the following long exact sequence

$$\ldots \to R^g \mathcal{S}_Y\, \mathrm{R}\Delta(\mathcal{I}_{\Delta_Y / A \times Y}(p_1^{-1}\Theta)) \xrightarrow{\varrho} p_1^*\, R^g \mathcal{S}\, \mathrm{R}\Delta(\mathcal{I}_{Y/A}(\Theta)) \to$$
$$R^{g+1} \mathcal{S}_Y\, \mathrm{R}\Delta(\mathcal{I}_{\Delta_Y / Y \times Y}(p_1^{-1}\Theta)) \to \ldots.$$

The support of $R^{g+1} \mathcal{S}_Y\, \mathrm{R}\Delta(\mathcal{I}_{\Delta_Y / Y \times Y}(p_1^{-1}\Theta))$ is contained in

$$V^{g+1}(\mathrm{R}\Delta(\mathcal{I}_{\Delta_Y / Y \times Y}(p_1^{-1}\Theta))) = -V^{-1}((\mathcal{I}_{\Delta_Y / Y \times Y}(p_1^{-1}\Theta))) = \emptyset$$

by Grothendieck-Serre duality I.1.5. Hence, ϱ is surjective and

$$\begin{aligned} T(Y) \times Y &= \operatorname{supp} p_1^*\, R^g \mathcal{S}\, \mathrm{R}\Delta(\mathcal{I}_{Y/A}(\Theta)) \\ &\subseteq \operatorname{supp} R^g \mathcal{S}_Y\, \mathrm{R}\Delta(\mathcal{I}_{\Delta_Y / A \times Y}(p_1^{-1}\Theta)) \\ &\overset{GVd}{=} \operatorname{supp} \mathcal{H}om(\mathrm{R}\mathcal{S}_Y(\mathcal{I}_{\Delta_Y / A \times Y}(p_1^{-1}\Theta)), \mathcal{O}_{A \times Y}) \\ &= \operatorname{supp} \mathcal{E}xt^1(R^1 \mathcal{S}_Y(\mathcal{I}_{\Delta_Y / A \times Y}(p_1^{-1}\Theta)), \mathcal{O}_{A \times Y}) && \text{Lem. 2.8} \\ &= \operatorname{supp} \mathcal{E}xt^1(\mathcal{L}|_{m_Y^{-1}\Theta}, \mathcal{O}_{A \times Y}) && \text{Lem. 2.8} \\ &= m_Y^{-1}\Theta, \end{aligned}$$

where GVd stands for Grothendieck-Verdier duality I.1.6. $\square$

We will need the following spectral sequence.

Lemma 2.10. *Let Z be a subscheme of A. Then, for any sheaf $\mathcal{E}$ on A, there exists a fourth quadrant spectral sequence*

$$E_2^{i,j} = \mathcal{E}xt^i_A((-1_A)^* R^{-j}\mathcal{S}(\mathrm{R}\Delta\, \mathcal{I}_Z(\Theta)), \mathcal{E}) \Rightarrow R^{i+j+g}q_*(p^*\mathcal{I}_Z \otimes m^*\mathcal{O}_A(\Theta) \otimes q^*\mathcal{E}(-\Theta)).$$

Proof. Let $\mathcal{E}$ be a sheaf on A. We claim that, by Grothendieck-Verdier Duality I.1.4

$$\mathrm{R}\mathcal{H}om_A((-1_A)^*\, \mathrm{R}\mathcal{S}(\mathrm{R}\Delta\, \mathcal{I}_Z(\Theta)), \mathcal{E}) \cong \mathrm{R}q_*(p^*\mathcal{I}_Z \otimes m^*\mathcal{O}_A(\Theta) \otimes q^*\mathcal{E}(-\Theta))[g].$$

Indeed, if we denote $\mathscr{M} = m^*\mathcal{O}_A(\Theta) \otimes p^*\mathcal{O}_A(-\Theta) \otimes q^*\mathcal{O}_A(-\Theta)$ the kernel of $\mathrm{R}\mathcal{S}$, we have

$$\begin{aligned}
\mathrm{R}\mathcal{H}om_A((-1_A)^*\, \mathrm{R}\mathcal{S}(\mathrm{R}\Delta\, \mathcal{I}_Z(\Theta)), \mathcal{E}) &= \mathrm{R}\mathcal{H}om_A(\mathrm{R}q_*(p^*\, \mathrm{R}\Delta\, \mathcal{I}_Z(\Theta) \otimes \mathscr{M}^{-1}), \mathcal{E}) \\
&\overset{GVd}{\cong} \mathrm{R}q_*(\mathrm{R}\mathcal{H}om_{A\times A}(p^*\, \mathrm{R}\Delta\, \mathcal{I}_Z(\Theta) \otimes \mathscr{M}^{-1}, q^*\mathcal{E}[g])) \\
&\cong \mathrm{R}q_*(\mathcal{H}om_{A\times A}(\mathcal{O}_{A\times A}, p^*\mathcal{I}_Z(\Theta) \otimes \mathscr{M} \otimes q^*\mathcal{E}))[g] \\
&\cong \mathrm{R}q_*(p^*\mathcal{I}_Z \otimes m^*\mathcal{O}_A(\Theta) \otimes q^*\mathcal{E}(-\Theta))[g]
\end{aligned}$$

Therefore we have a fourth quadrant spectral sequence (see [Hu, Ex. 2.70 ii)])

$$E_2^{i,j} = \mathcal{E}xt^i_A((-1_A)^* R^{-j}\mathcal{S}(\mathrm{R}\Delta\, \mathcal{I}_Z(\Theta)), \mathcal{E}) \Rightarrow R^{i+j+g}q_*(p^*\mathcal{I}_Z \otimes m^*\mathcal{O}_A(\Theta) \otimes q^*\mathcal{E}(-\Theta)).$$

$\square$

Now we are ready to prove the desired inclusion.

Proposition 2.11. *For any subscheme $Y \subset A$, we have a schematic inclusion*

$$Y \subseteq T(T(Y)).$$

Proof. Firstly, suppose that Y is irreducible and let $\mathcal{E}$ be a sheaf on A. By Lemma 2.10 we have a fourth quadrant spectral sequence

$$E_2^{i,j} = \mathcal{E}xt^i_A((-1_A)^* R^{-j}\mathcal{S}(\mathrm{R}\Delta\, \mathcal{I}_Z(\Theta)), \mathcal{E}) \Rightarrow R^{i+j+g}q_*(p^*\mathcal{I}_Z \otimes m^*\mathcal{O}_A(\Theta) \otimes q^*\mathcal{E}(-\Theta)).$$

Clearly the term $E_2^{i,j}$ is non-zero only if $i \geqslant 0$. Assuming $i \geqslant 0$, in the case $i+j+g=0$, i.e. $-j = i+g$ we have that $R^{-j}\mathcal{S}(\mathrm{R}\Delta\, \mathcal{I}_Z(\Theta))$ is non-zero if and only if $-j = g$, i.e. $i = 0$. In conclusion for $i+j+g = 0$ the only non-zero E_2-term is $E_2^{0,-g} = \mathcal{H}om_A(R^g\mathcal{S}(\mathrm{R}\Delta\, \mathcal{I}_Z(\Theta)), \mathcal{E}) = \mathcal{H}om_A(\mathcal{O}_{T(Z)}(\Theta), \mathcal{E})$ (see Corollary 2.4). Since the differentials from and to $E_2^{0,-g}$ are zero, we get that

$$\mathcal{H}om_A(\mathcal{O}_{T(Z)}(\Theta), \mathcal{E}) = E_2^{0,-g} = E_\infty^{0,-g} \cong q_*(p^*\mathcal{I}_Z \otimes m^*\mathcal{O}_A(\Theta) \otimes q^*\mathcal{E}(-\Theta)).$$

Taking global sections, we get the isomorphism (functorial in $\mathcal{E}$)

$$\mathrm{Hom}_A(\mathcal{O}_{T(Z)}(\Theta), \mathcal{E}) \cong H^0(A \times A, p^*\mathcal{I}_Z \otimes m^*\mathcal{O}_A(\Theta) \otimes q^*\mathcal{E}(-\Theta)). \tag{2.4}$$

Let $Z = T(Y)$ and consider $\mathcal{E} = \mathcal{O}_Y(\Theta)$ in the previous functorial isomorphism. We claim that

$$\mathrm{Hom}_A(\mathcal{O}_{T(T(Y))}(\Theta), \mathcal{O}_Y(\Theta)) = \mathrm{Hom}_A(\mathcal{O}_{T(T(Y))}, \mathcal{O}_Y) \neq 0 \tag{2.5}$$

implies $Y \subseteq T(T(Y))$. Indeed, since we are assuming Y irreducible, $h^0(\mathcal{O}_Y) = 1$, and every non-zero map $\mathcal{O}_A \to \mathcal{O}_Y$ is surjective. Suppose that f is a non-zero map $\mathcal{O}_{T(T(Y))} \xrightarrow{f} \mathcal{O}_Y$. Then the composition $\mathcal{O}_A \twoheadrightarrow \mathcal{O}_{T(T(Y))} \xrightarrow{f} \mathcal{O}_Y$ is non-zero, hence surjective which implies f surjective and $Y \subseteq T(T(Y))$.

Due to the functorial isomorphism (2.4), the condition (2.5) is equivalent to $0 \neq H^0(A \times A, p^*\mathcal{I}_{T(Y)} \otimes m^*\mathcal{O}_A(\Theta) \otimes q^*\mathcal{O}_Y)) = H^0(A \times Y, \mathcal{I}_{T(Y)\times Y} \otimes m_Y^*\mathcal{O}_A(\Theta))$, i.e.

$$T(Y) \times Y \subseteq m_Y^{-1}(\Theta),$$

where $m_Y : A \times Y \to A$ is the *restricted group law*. This inclusion follows from Corollary 2.9.

Now, let $Y = \bigcup Z$ a decomposition of Y in irreducible components Z. By Lemma 2.7, $Z \subseteq T(T(Z)) \subseteq T(T(Y))$. Since $T(T(Y))$ contains all the irreducible components of Y,

$$Y = \bigcup Z \subseteq T(T(Y)). \qquad \square$$

CHAPTER

III

Jacobian and Prym Varieties

Introduction

Jacobian varieties JC of non-rational curves C provide positive-dimensional examples of principally polarized abelian varieties (see [BL, §11.1]). Jacobian varieties encode a lot of geometric information of the corresponding curve, so they are a good tool to study them. In particular, we have the Abel-Jacobi embedding $i\colon C \to JC$ that exhibits the curve canonically embedded (up to translation) in its Jacobian. Moreover, for any curve we can consider the d-th symmetric product of a curve $C^{(d)}$, whose image for the corresponding Abel-Jacobi map $\sigma_d\colon C^{(d)} \to JC$ are the simplest Brill-Noether loci $W_d(C)$, which encode the information about the positive dimensional complete linear series. For $d = 1$ we recover the curve embedded in JC. The loci $W_d(C)$ have a natural scheme structure provided by the Fitting ideals.

In the first section of this chapter we will study the schematic theta-dual of the Brill-Nother loci $W_d(C)$ (Proposition 1.9). This computation has already been done by Pareschi and Popa in [PP6], but our method is different since they use the cohomological classes of the Brill-Nother loci and we restrict ourselves to work with the ideal sheaves and their Fourier-Mukai transform instead. Our approach is based on the work of Polishchuk [Po2]. In fact, we use the work of Polishchuk to get explicitly the Fourier-Mukai transform of $\mathcal{I}_{W_d/JC}(\Theta)$ (see Proposition 1.10). As a byproduct we obtain the cohomological support loci of the ideal sheaves of the Brill-Nother loci in the Jacobian variety twisted by the principal polarization (see Proposition 1.11).

The Prym variety P associated to an étale double cover $\pi : \widetilde{C} \to C$ provides another classical example of principally polarized abelian variety (see [Mu3] or III.§2). When the curve C is not hyperelliptic (in particular, its genus is greater than 2), we have that the Abel-Prym map $j\colon \widetilde{C} \to P$ is an embedding that exhibits the curve canonically embedded (up to translation) in a principally polarized abelian variety.

In the second section of this chapter we will study the schematic theta-dual of the Abel-Prym curve $j(\widetilde{C})$ (Theorem 2.9). To complete the picture for the Abel-Prym curve we also include the computation of the cohomological support loci of the ideal sheaves of the Abel-Prym curve in the Prym variety twisted by the principal polarization Ξ (Proposition 2.7) and twice the principal polarization 2Ξ (Proposition 2.12). These two results has been obtained in collaboration with Sebastian Casalaina-Martin and Filippo Viviani in [CLV].

1 Jacobian Varieties

Throughout this section, we work over an algebraically closed field k of arbitrary characteristic.

Let C be a smooth curve of genus $g > 0$ and let (JC, Θ) be its Jacobian variety, where Θ is a divisor representing the natural principal polarization that arises from the intersection product on the integral homology of the curve (see [BL, §11.1]). In this case, the identification provided by ϕ_Θ (II.1.1) is known as the Abel Theorem that identifies $\operatorname{Pic}^0 C$ with JC. From now on, we will use this identification, so we will also identify the Poincaré line bundle on $JC \times \operatorname{Pic}^0 C$ with the Mumford line bundle in $JC \times JC$ or $\operatorname{Pic}^0 C \times \operatorname{Pic}^0 C$.

When working with Jacobian varieties is quite common to use the homogeneous varieties $\operatorname{Pic}^d C$ which are (non-canonically) isomorphic to $\operatorname{Pic}^0 C$. For example, one defines a Poincaré line bundle of degree d for C as a line bundle P_d on $C \times \operatorname{Pic}^d C$ which, for each L in $\operatorname{Pic}^d C$, restricts to L on $C \cong C \times \{L\}$.

Let L_d be any line bundle in $\operatorname{Pic}^d C$ and let

$$\begin{array}{rccc} t_{L_d^{-1}}: & \operatorname{Pic}^d C & \longrightarrow & \operatorname{Pic}^0 C \\ & L & \longmapsto & L \otimes L_d^{-1}. \end{array}$$

Then, we can take $P_d = (\mathrm{id}_C \times t_{L_d^{-1}})^* P_0 \otimes p^* L_d$, where P_0 is a Poincaré line bundle on $C \times \operatorname{Pic}^0 C$ and $p\colon C \times \operatorname{Pic}^0 C \to C$ is the first projection (see [ACGH, IV.§2]). Recall that we can take $P_0 = (i \times \mathrm{id})^* \mathcal{P}$ where

$$\begin{array}{rccc} i: & C & \longrightarrow & \operatorname{Pic}^0 C \\ & p & \longmapsto & \mathcal{O}_C(p - p_0) \end{array}$$

is the Abel-Jacobi embedding once we have identified $JC \cong \operatorname{Pic}^0 C$ and fixed an arbitrary point $p_0 \in C$. Thus, we have that P_0, and all the P_d, are normalized at the same point p_0, i.e. the point such that $i(p_0) = 0 \in \operatorname{Pic}^0 C$. Then,

Lemma 1.1. *$R\Phi_{P_d}(\,\cdot\,) \cong t^*_{L_d^{-1}} \circ R\Phi_{\mathcal{P}} \circ i_*(\,\cdot \otimes L_d)$ as functors from $\mathrm{D}^b(C)$ to $\mathrm{D}^b(\operatorname{Pic}^d C)$.*

Proof. This is basically the same proof as Propostion I.1.1,

$$\begin{aligned} R\Phi_{P_d}(\,\cdot\,) &= Rq_{d*}(p^*(\,\cdot\,) \otimes (i \times t_{L_d^{-1}})^* \mathcal{P} \otimes p^* L_d) \cong && \text{Leray and Proj. Form.} \\ &\cong Rq_{d*}(R(i \times t_{L_d^{-1}})_*(p^*(\,\cdot \otimes L_d)) \otimes \mathcal{P}) \cong && \text{Base Change} \\ &\cong t^*_{L_d^{-1}} Rq_*(p_A^*(Ri_*(\,\cdot \otimes L_d)) \otimes \mathcal{P}) = && i \text{ is closed immersion} \\ &\cong t^*_{L_d^{-1}} R\Phi_{\mathcal{P}} \circ i_*(\,\cdot \otimes L_d). && \square \end{aligned}$$

1.1 Symmetric products and Brill-Noether loci

Denote by P_d a fixed Poincaré line bundle of degree d for C, and by $q_d\colon C\times \operatorname{Pic}^d C \to \operatorname{Pic}^d C$ the second projection. In the language of Fitting ideals[1] we define

$$\mathcal{I}_{W_d^r(C)/\operatorname{Pic}^d C} = \operatorname{Fitt}_{g-d+r-1}(R^1 q_{d*} P_d). \tag{1.1}$$

This definition gives a natural schematic structure to the set-theoretic *Brill-Noether loci*

$$W_d^r(C) = \left\{ L \in \operatorname{Pic}^d C \,\middle|\, h^0(C,L) \geqslant r+1 \right\},$$

(see [ACGH, Rem. IV.3.2]).

It is common to denote simply $W_d(C) = W_d^0(C)$. If we consider the d-fold symmetric product $C^{(d)}$ which is a smooth projective variety (e.g. [Po1, Prop. 16.2]) and the usual map,

$$\begin{array}{rccc} \sigma_d : & C^{(d)} & \longrightarrow & \operatorname{Pic}^d C \\ & p_1+\ldots+p_d & \longmapsto & \mathcal{O}_C(p_1+\ldots+p_d), \end{array}$$

we have the following relation,

Proposition 1.2 ([ACGH, Prop. IV.3.4])**.** *As schemes,* $\sigma_d{}^{-1}(W_d) = C^{(d)}$.

Definition 1.3. *(a)* Given a line bundle L on C, we define the *symmetric power* $L^{(d)}$ *of* L on $C^{(d)}$ as the quotient of $L^{\boxtimes d}$ by the action of the symmetric group $\mathfrak{S}_d$.
That is, if $\pi_d\colon C^d \to C^{(d)}$ is the quotient by $\mathfrak{S}_d$, then $L^{(d)}$ is such that $\pi_d^* L^{(d)} \cong L^{\boxtimes d}$.

(b) We also define by $F_d(L)$ the derived push-forward of $L^{(d)}$ under the natural morphism $\sigma_d\colon C^{(d)} \to \operatorname{Pic}^d C$.

(c) Finally, we define R_p^d as the image of the following map

$$\begin{array}{rccc} s_p = s_p^d : & C^{(d-1)} & \longrightarrow & C^{(d)} \\ & D & \longmapsto & p+D. \end{array}$$

We have the following properties concerning the previous definitions,

Lemma 1.4. *(a)* $\mathcal{O}_{C^{(d)}}(mR_p^d) \cong (\mathcal{O}_C(mp))^{(d)}$ *for every* $m\in\mathbb{Z}$.

(b) [Po2, Lemma 3.2 ii)] If $\deg(L) \geqslant -1$ *then* $F_d(L)$ *is a sheaf on degree* 0 *in* $\operatorname{Pic}^d C$*, i.e.* $R^i\sigma_{d*}L^{(d)} = 0$ *for* $i>0$.

[1] We use the indexing system of [Ei], instead of those used in [ACGH]. Observe that there is a shift by 1 in the two different presentations. That is, [Ei]'s i-th Fitting ideal is [ACGH]'s $(i+1)$-th Fitting ideal.

Proof. (a) If $\pi_d \colon C^d \to C^{(d)}$ is the quotient by $\mathfrak{S}_d$, then it is easy to see that $\pi_d^* R_p^d \cong (\mathcal{O}_C(p))^{\boxtimes d}$. $\square$

We have the following description of the canonical line bundle on $C^{(d)}$.

Lemma 1.5 (e.g. [Po2, Lemma 3.3]). *For every $d \geqslant 1$ one has an isomorphism*

$$\omega_{C^{(d)}} \cong (\sigma_d)^*(\mathcal{O}(\Theta))((g-d-1)R_p^d),$$

where we have identified $\operatorname{Pic}^0 C$ *with* $\operatorname{Pic}^d C$ *via* $d \cdot p$.

The schemes W_d are reduced, irreducible, normal and Cohen-Macaulay (see [ACGH, Cor. IV.4.5]), hence it make sense to consider its canonical sheaf ω_{W_d}. Since $C^{(d)}$ is smooth, the map $\sigma_d : C^{(d)} \to W_d$ is a resolution of singularities. Moreover, the fibers of σ_d are projective spaces, so that $R^i \sigma_{d*} \mathcal{O}_{C^{(d)}} = 0$ for all $i > 0$ and W_d has rational singularities. Then

$$\begin{aligned} \omega_{W_d} &\cong (\sigma_d)_* \omega_{C^{(d)}} && \text{e.g. [K4, Cor. 11.9]} \\ &\cong F_d(\mathcal{O}_C((g-d-1)p))(\Theta) && \text{Lemma 1.4}\,(a). \end{aligned} \tag{1.2}$$

Hence we can interpret $F_d(\mathcal{O}_C((g-d-1)p))$ as $\omega_{W_d}(-\Theta)$. In fact, the sheaves $F_d(L)$ are intimately related to the geometry of W_d. The following result of Polishchuk shows that $F_d(\mathcal{O}_C((g-d)p))$ is a WIT_d sheaf with respect the Fourier-Mukai transform on JC and computes its transform.

Theorem 1.6 ([Po2, Thm. 0.2]). *Assume that $1 \leqslant d \leqslant g-1$. Then one has the following isomorphisms in the derived category $D^b(JC)$,*

$$(-1)^* \,\mathrm{R}\mathcal{S}(F_d(\mathcal{O}_C((g-d)p))) \cong F_{g-d}(\mathcal{O}_C(-p))(\Theta)[-d] \cong \mathrm{R}\Delta(F_{g-d}(\mathcal{O}_C(dp))),$$

where we have identified $\operatorname{Pic}^0 C$ *and* $\operatorname{Pic}^d C$ *via* $d \cdot p$.

Remark 1.7. *Observe that, in particular, the previous theorem includes two vanishing theorems since it says that $\mathcal{E}xt^i(F_{g-d}(\mathcal{O}_C(dp)), \mathcal{O}_{JC}) = 0$ for all $i \neq d$ and $F_d(\mathcal{O}_C((g-d)p))$ is a WIT_d sheaf.*

We have previously interpreted $F_d(\mathcal{O}_C((g-d-1)p))$ as $\omega_{W_d}(-\Theta)$. But the last theorem deals with $F_d(\mathcal{O}_C((g-d)p))$. The next proposition shows that the dual of $F_d(\mathcal{O}_C((g-d)p))$ is the ideal sheaf of W_{d-1} inside W_d translated by theta.

Proposition 1.8. *For every $1 \leqslant d \leqslant g-1$,*

$$\mathcal{I}_{W_{d-1}/W_d}(\Theta) \cong \mathrm{R}\Delta\left(F_d\left(\mathcal{O}_C\left((g-d)p\right)\right)\right)[g-d].$$

Moreover, $\mathcal{I}_{W_{d-1}/W_d}(\Theta)$ is a WIT_d sheaf and $R^d\mathcal{S}\,\mathcal{I}_{W_{d-1}/W_d}(\Theta) \cong F_{g-d}(\mathcal{O}_C(dp))$.

Proof. By the previous Theorem 1.6, $\mathrm{R}\Delta(F_d(\mathcal{O}_C((g-d)p)))$ is a sheaf in degree $g-d$. Hence it is enough to compute $\mathcal{E}xt^{g-d}(F_d(\mathcal{O}_C((g-d)p)), \mathcal{O}_{JC})$.

$$\begin{aligned}\mathcal{E}xt^{g-d}(F_d(\mathcal{O}_C((g-d)p)), \mathcal{O}_{JC}) &= \sigma_{d*}(\omega_{C^{(d)}}(-(g-d)R_p^d)) && \text{GVd and Lem. 1.4}\\ &= \sigma_{d*}(\mathcal{O}_{C^{(d)}}(-R_p^d))(\Theta) && \text{Lem. 1.5}\\ &= \sigma_{d*}(\mathcal{I}_{s_pC^{(d-1)}/C^{(d)}}) \otimes \mathcal{O}(\Theta) && \text{Def. of } R_p^d,\end{aligned}$$

where GVd stands for Grothendieck-Verdier duality I.1.7. Since W_d is normal (e.g. [ACGH, Cor. IV.4.5]) and $\sigma_d\colon C^{(d)} \to W_d$ is a resolution of singularities, we have that $\sigma_{d*}\mathcal{O}_{C^{(d)}} = \mathcal{O}_{W_d}$. So the standard short exact sequence shows that $\sigma_{d*}(\mathcal{I}_{s_pC^{(d-1)}/C^{(d)}}) = \mathcal{I}_{W_{d-1}/W_d}$. Hence,

$$\mathcal{E}xt^{g-d}(F_d(\mathcal{O}_C((g-d)p)), \mathcal{O}_{JC}) = \mathcal{I}_{W_{d-1}/W_d}(\Theta).$$

Moreover,

$$\begin{aligned}\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_{d-1}/W_d}(\Theta) &\cong \mathrm{R}\mathcal{S}\,\mathrm{R}\Delta(F_d(\mathcal{O}_C((g-d)p)))[g-d]\\ &= (-1_A)^*\,\mathrm{R}\mathcal{S}\,\mathrm{R}\mathcal{S}\,F_{g-d}(\mathcal{O}_C(dp))[g-d] && \text{by Thm. 1.6}\\ &= F_{g-d}(\mathcal{O}_C(dp))[-d] && \text{by Thm. I.2.1.}\end{aligned}$$

□

1.2 Theta-dual of the Brill-Nother loci

The following proposition describes the theta-dual of the Brill-Nother loci W_d. This computation was also done by Pareschi and Popa in [PP6, Ex. 4.5, §8.1], but they prove it in an indirect way by checking the equality of their cohomological classes. We directly study the sheaves involved in the definitions.

Proposition 1.9. *Let C be a smooth projective of genus $g > 0$. Then, for every $0 \leqslant d \leqslant g-1$,*

$$T\left(t_L^*(W_d)\right) = t_{\kappa-L}^* W_{g-d-1} \subset \operatorname{Pic}^0 C,$$

where κ is a theta-characteristic and L is any line bundle in $\operatorname{Pic}^d C$ *used to identify* $\operatorname{Pic}^d C$ *with* $\operatorname{Pic}^0 C$.

Proof. We have $\Theta = (W_{g-1})_\kappa = (W_d)_L + (W_{g-d-1})_{\kappa-L}$ in $\operatorname{Pic}^{g-1} C$ and it is easy to see that set-theoretically $T\left((W_d)_L\right) = (W_{g-d-1})_{\kappa-L}$ (e.g. [Mar, Lem. 2]). Now we deal with the scheme structure. We will omit the translates that allow us to work directly in $JC \cong \operatorname{Pic}^0 C$.

$$\begin{aligned}\mathcal{O}_{T(W_d)}(\Theta) &= (-1)^*\, R^g\mathcal{S}\,\mathrm{R}\Delta(\mathcal{I}_{W_d}(\Theta)) && \text{Cor. II.2.4}\\ &= \mathcal{H}om(\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_d}(\Theta), \mathcal{O}_{JC}) && \text{Rem. II.2.5.}\end{aligned}$$

Now, we proceed by induction on d. For $d = 0$, it is clear $T(W_0) \cong W_{g-1}$ (see Example 2.6). Consider the following exact sequence,

$$0 \to \mathcal{I}_{W_d}(\Theta) \to \mathcal{I}_{W_{d-1}}(\Theta) \to \mathcal{I}_{W_{d-1}/W_d}(\Theta) \to 0.$$

Then,

$$\begin{aligned} \dots \to \mathcal{H}om(\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_{d-1}/W_d}(\Theta), \mathcal{O}_{JC}) \to \mathcal{H}om(\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_{d-1}}(\Theta), \mathcal{O}_{JC}) \to \\ \to \mathcal{H}om(\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_d}(\Theta), \mathcal{O}_{JC}) \to \mathcal{E}xt^1(\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_{d-1}/W_d}(\Theta), \mathcal{O}_{JC}) \to \dots \end{aligned} \tag{1.3}$$

By induction we have

$$\mathcal{H}om(\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_{d-1}}(\Theta), \mathcal{O}_{JC}) \cong \mathcal{O}_{W_{g-d}}(\Theta),$$

and by Proposition 1.8, Grothendieck-Verdier duality I.1.6 and Theorem 1.6, we also have

$$\begin{aligned} \mathrm{R}\Delta\,\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_{d-1}/W_d}(\Theta) &= \mathrm{R}\Delta\,\mathrm{R}\mathcal{S}\,\mathrm{R}\Delta\left(F_d\left(\mathcal{O}_C\left((g-d)p\right)\right)\right)[d-g] && \text{Prop. 1.8} \\ &= (-1)^*\,\mathrm{R}\mathcal{S}\left(F_d\left(\mathcal{O}_C\left((g-d)p\right)\right)\right)[d] && \text{GV-duality I.1.6} \\ &= \mathrm{R}\Delta(F_{g-d}(\mathcal{O}_C(dp)))[d] && \text{Thm. 1.6} \\ &= \mathcal{I}_{W_{g-d-1}/W_{g-d}}(\Theta) && \text{Prop. 1.8.} \end{aligned} \tag{1.4}$$

Hence, (1.3) becomes

$$\begin{aligned} \dots \to \mathcal{E}xt^{-1}(\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_d}(\Theta), \mathcal{O}_{JC}) \to \\ \to \mathcal{I}_{W_{g-d-1}/W_{g-d}}(\Theta) \xrightarrow{\psi} \mathcal{O}_{W_{g-d}}(\Theta) \to \mathcal{H}om(\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_d}(\Theta), \mathcal{O}_{JC}) \to 0. \end{aligned}$$

Since ψ is generically surjective (set-theoretically we already know that the sheaf $\mathcal{H}om(\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_d}(\Theta), \mathcal{O}_{JC})$ is supported on W_{g-d-1}) and $\mathcal{I}_{W_{g-d-1}/W_{g-d}}(\Theta)$ is a torsion-free sheaf on W_{g-d} (irreducible and reduced), ψ is injective and

$$\mathcal{H}om(\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_d}(\Theta), \mathcal{O}_{JC}) \cong \mathcal{O}_{W_{g-d-1}}(\Theta). \qquad \square$$

1.3 The cohomological support loci of $\mathcal{I}_{W_d}(\Theta)$

The following proposition computes the Fourier-Mukai transform of the sheaves $\mathcal{I}_{W_d}(\Theta)$, where W_d are the Brill-Noether locus of a smooth projective curve C of genus $g > 0$.

Proposition 1.10. *For any $1 \leqslant d \leqslant g-1$, $\mathcal{I}_{W_d}(\Theta)$ is a GV-sheaf that satisfies WIT_{d+1} and*

$$R^{d+1}\mathcal{S}\,\mathcal{I}_{W_d}(\Theta) \cong \omega_{W_{g-d-1}}(-\Theta) \cong F_{g-d-1}(dp).$$

Proof. We can proceed by descending induction.
It is clear that for $d = g-1$, we have $\mathcal{I}_{W_{g-1}}(\Theta) \cong \mathcal{O}_{JC}$. Hence, $\mathcal{I}_{W_{g-1}}(\Theta)$ it is a GV-sheaf that satisfies WIT_g, and $R^d\mathcal{S}\,\mathcal{I}_{W_{g-1}}(\Theta) = k(0)$. Observe that,

$$\mathcal{E}xt^i(\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_{g-1}}(\Theta), \mathcal{O}_{JC}) = \mathcal{E}xt^{g+i}(R^g\mathcal{S}\,\mathcal{I}_{W_{g-1}}(\Theta), \mathcal{O}_{JC}) = 0 \quad \text{for all } i \neq 0. \tag{1.5}$$

By (1.4), we also have that $\mathcal{E}xt^i(\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_{d-1}/W_d}(\Theta), \mathcal{O}_{JC}) = 0$ for all $i \neq 0$.

Suppose the statement of the proposition true by d and we want to prove it for $d-1$.
By induction on the long exact sequence (1.3) we have that

$$\mathcal{E}xt^i(\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_{d-1}}(\Theta), \mathcal{O}_{JC}) = 0 \qquad \text{for all } i \neq 0.$$

By Grothendieck-Verdier duality I.1.6, this is equivalent to $R^{g+i}\mathcal{S}(\mathrm{R}\Delta\,\mathcal{I}_{W_{d-1}}(\Theta)) = 0$ for all $i \neq 0$, i.e. $\mathrm{R}\Delta\,\mathcal{I}_{W_{d-1}}(\Theta)$ is a WIT_g sheaf. On one hand, this implies that $\mathcal{I}_{W_{d-1}}(\Theta)$ is a GV-sheaf by Theorem I.1.10. On the other hand,

$$\begin{aligned}
\mathcal{O}_{W_{g-d}} &= \mathcal{O}_{T(W_{d-1})} && \text{Prop. 1.9}\\
&= (-1)^*\, R^g\mathcal{S}\,\mathrm{R}\Delta\,\mathcal{I}_{W_{d-1}}(\Theta) && \text{Cor. II.2.4}\\
&= (-1)^*\, \mathrm{R}\mathcal{S}\,\mathrm{R}\Delta\,\mathcal{I}_{W_{d-1}}(\Theta)[g] && \mathrm{R}\Delta\,\mathcal{I}_{W_{d-1}}(\Theta) \text{ is a } \mathrm{WIT}_g \text{ sheaf}\\
&= \mathrm{R}\Delta\,\mathrm{R}\mathcal{S}\,\mathcal{I}_{W_{d-1}}(\Theta) && \text{GV-duality I.1.6.}
\end{aligned}$$

Since $\mathrm{R}\Delta$ is an involution on $\mathrm{D}^b(JC)$, we deduce that $\mathrm{R}\Delta\,\mathcal{O}_{W_{g-d}}(\Theta) = \mathrm{R}\mathcal{S}\,\mathcal{I}_{W_{d-1}}(\Theta)$. By [ACGH, Cor. IV.4.5], W_{g-d} is Cohen-Macaulay and $g-d$ pure-dimensional, which means that

$$\begin{aligned}
R^i\mathcal{S}\,\mathcal{I}_{W_{d-1}}(\Theta) &= \mathcal{E}xt^i(\mathcal{O}_{W_{g-d}}(\Theta), \mathcal{O}_{JC}) = 0 \qquad \text{for } i \neq d && \text{and}\\
R^d\mathcal{S}\,\mathcal{I}_{W_{d-1}}(\Theta) &= \mathcal{E}xt^d(\mathcal{O}_{W_{g-d}}(\Theta), \mathcal{O}_{JC}) = \omega_{W_{g-d}}(-\Theta)\\
&= F_{g-d}((d-1)p) && \text{by (1.2).}
\end{aligned}$$

□

The following result was already obtained by Pareschi and Popa in [PP1, Thm. 4.1].

Proposition 1.11. *For any* $1 \leqslant d \leqslant g-1$,

(a) Set-theoretically, $V^i(\mathcal{I}_{W_d}(\Theta)) = W_{g-d-1}$ *for any* $0 \leqslant i \leqslant d+1$.

(b) $\mathcal{I}_{W_d}(n\Theta)$ *is* IT_0, *for every* $n \geqslant 2$.

Proof. From the definition of set-theoretical theta-dual it is clear that $V^0(\mathcal{I}_{W_d}(\Theta)) = W_{g-d-1}$. By Proposition 1.10 and base change, $V^{d+1}(\mathcal{I}_{W_d}(\Theta)) = W_{g-d-1}$. Hence, the first assertion follows from [PP8, Prop 3.13], because $\mathcal{I}_{W_d}(\Theta)$ is a GV-sheaf which implies that (see Corollary I.1.13)

$$V^{d+1}(\mathcal{I}_{W_d}(\Theta)) \subseteq V^d(\mathcal{I}_{W_d}(\Theta)) \subseteq \ldots \subseteq V^1(\mathcal{I}_{W_d}(\Theta)) \subseteq V^0(\mathcal{I}_{W_d}(\Theta)).$$

The second statement is simply [PP1, Thm. 4.1] or we can deduce it directly from the fact that $\mathcal{I}_{W_d}(\Theta)$ is a GV-sheaf using [PP6, Lemma 3.1]. $\square$

2 Prym Varieties

2.1 Notation and basic definitions

Throughout this section, we work over an algebraically closed field k of characteristic different from 2. The basic results of these preliminaries are due to Mumford [Mu3].

Let $\pi\colon \widetilde{C} \to C$ be an étale double cover of irreducible smooth projective curves of genus $\tilde{g}$ and g, respectively. By the Hurwitz formula, we get that $\tilde{g} = 2g - 1$. We denote by σ the involution on $\widetilde{C}$ associated to the above double cover. Consider the *norm map*

$$\begin{array}{rrcl} \mathrm{Nm}: & \mathrm{Pic}(\widetilde{C}) & \longrightarrow & \mathrm{Pic}(C) \\ & \mathcal{O}_{\widetilde{C}}(\sum_j r_j p_j) & \longmapsto & \mathcal{O}_C(\sum_j r_j \pi(p_j)), \end{array}$$

and observe that $\mathrm{Nm}(\mathrm{Pic}^d \widetilde{C}) = \mathrm{Pic}^d C$.

The kernel of the norm map has two connected components

$$\ker \mathrm{Nm} = P \cup P' \subset \mathrm{Pic}^0 \widetilde{C},$$

where P is the component containing the identity element and is, by definition, the *Prym variety associated to the étale double cover* π. The above components P and P' have the following explicit description

$$P = \left\{ \mathcal{O}_{\widetilde{C}}(D - \sigma D) \,\middle|\, D \in \mathrm{Div}^{2N}(\widetilde{C}), N \geqslant 0 \right\},$$

$$P' = \left\{ \mathcal{O}_{\widetilde{C}}(D - \sigma D) \,\middle|\, D \in \mathrm{Div}^{2N+1}(\widetilde{C}), N \geqslant 0 \right\}.$$

It is often useful to consider the inverse image of the canonical line bundle of C via the norm map. This also has two connected components

$$\mathrm{Nm}^{-1}(\omega_C) = P^+ \cup P^- \subset \mathrm{Pic}^{2g-2}(\widetilde{C}) = \mathrm{Pic}^{\tilde{g}-1}(\widetilde{C}),$$

which have the following explicit description

$$P^+ = \left\{ L \in \mathrm{Nm}^{-1}(\omega_C) \,\middle|\, h^0(L) \equiv 0 \mod 2 \right\},$$

$$P^- = \left\{ L \in \mathrm{Nm}^{-1}(\omega_C) \,\middle|\, h^0(L) \equiv 1 \mod 2 \right\}.$$

The above varieties P', P^+ and P^- are isomorphic to the Prym variety P. The translations given by fixed elements give the isomorphisms between P and its torsors. For example

$$\begin{array}{rccl} t_{L'_0}: & P & \longrightarrow & P' \\ & L & \longmapsto & L \otimes L'_0, \end{array}$$

for a fixed $L'_0 \in P'$ gives an isomorphism between P and P'.

Principal polarization of the Prym variety. There is a principal polarization $[\Xi] \in \mathrm{NS}(P)$ induced by the *principal polarization* $[\tilde{\Theta}] \in NS(J\widetilde{C})$. One of the primary motivations for considering P^+ is the existence of a canonically defined divisor Ξ^+ whose class in the Neron-Severi group of P is $[\Xi]$:

$$\Xi^+ = \left\{ L \in P^+ \subset \mathrm{Pic}^{\tilde{g}-1}(\widetilde{C}) \mid h^0(L) > 0 \right\} \subset P^+.$$

In fact, for some representative of divisor $\tilde{\Theta}$ over $\mathrm{Pic}^{\tilde{g}-1}(\widetilde{C})$ such that $[\tilde{\Theta}]$ is the principal polarization of $J\widetilde{C}$, we have $\tilde{\Theta}\Big|_{P^+} = 2\Xi^+$.

The Abel-Prym map. The *canonical Abel-Prym map* is defined as

$$\begin{array}{rccc} \jmath\colon \widetilde{C} & \longrightarrow & P' \\ p & \longmapsto & \mathcal{O}_{\widetilde{C}}(p - \sigma p). \end{array} \tag{2.1}$$

If $\widetilde{C}$ is hyperelliptic then the image of $\widetilde{C}$ via the Abel-Prym map is a smooth hyperelliptic curve D and the Prym variety P is isomorphic to the Jacobian JD of D ([BL, Cor. 12.5.7]). On the other hand, if C is hyperelliptic but $\widetilde{C}$ is not, then the Prym variety P is the product of two hyperelliptic Jacobians (see [Mu3]). Therefore, since we are mostly interested in the case of an irreducible non-Jacobian principally polarized abelian variety, we will assume throughout this chapter that C is not hyperelliptic (and in particular $g \geqslant 3$). Note that under this hypothesis, the Abel-Prym map (2.1) is an embedding ([BL, Cor. 12.5.6]).

Since the Abel-Prym curve $\widetilde{C} \subset P'$ and the canonical representative of the principal polarization $\Xi^+ \subset P^+$ lie canonically in different spaces, the cohomological support loci for the twisted ideal sheaf $\mathcal{I}_{\widetilde{C}}(n\Xi^+)$ is only defined up to a translation.

Fixing the isomorphism between torsors. Let us fix the isomorphisms that we will use to move from one torsor to another.

Terminology/Notation 2.1. *We fix $L_0^+ \in P^+$ such that $h^0(\widetilde{C}, L_0^+) = 0$ and we will denote $\Xi = \Xi^+_{L_0^+}$. We will also fix $L_0' = \mathcal{O}_{\widetilde{C}}(p_0 - \sigma p_0) \in P'$ such that we will embed $\widetilde{C}$ in P by composing the canonical Abel-Prym map with the translation by $t_{-L_0'}$, that is, by substracting L_0'. We will denote this* non-canonical Abel-Prym map *by*

$$\begin{array}{rccc} j: & \widetilde{C} & \longrightarrow & P \\ & p & \longmapsto & p - \sigma p - p_0 + \sigma p_0. \end{array} \tag{2.2}$$

Finally, let $L_0^- = L_0^+ \otimes {L_0'}^{-1} \in P^-$. Observe that $h^0(\widetilde{C}, L_0^-) = 1$.

Notation for the Fourier-Mukai transform on Prym varieties. We will simply denote by $\mathrm{R}\mathcal{S}$ the Fourier-Mukai transform

$$\mathrm{R}\mathcal{S}\colon \mathrm{D}^b(P) \to \mathrm{D}^b(P).$$

In this section when we need the Fourier-Mukai transform on $J\widetilde{C}$ we will denote it by $\mathrm{R}\mathcal{S}_{J\widetilde{C}}\colon \mathrm{D}^b(J\widetilde{C}) \to \mathrm{D}^b(J\widetilde{C})$.

2.2 Dimension of the theta-dual of an Abel-Prym curve $\widetilde{C}$

We want to study the theta-dual of $\widetilde{C}$ in the Prym variety P. Set-theoretically this can be identified canonically with the set:

$$T(\widetilde{C}) = \left\{ L \in P^- \,\middle|\, h^0(P', \mathcal{I}_{\widetilde{C}/P'}(\Xi_L^+)) > 0 \right\} \subset P^-,$$

that is, in P^-, $T(\widetilde{C})$ is independent of the choices (see Terminology/Notation 2.1).

Once we have fixed our isomorphisms between the torsors and the Prym variety, we recall that we have already endowed this set with a natural scheme structure (see Definition II.2.1). Up to translation from P to P^-,

$$T(\widetilde{C}) = \mathrm{supp}\left(R^{g-1}\mathcal{S}(\mathrm{R}\Delta(\mathcal{I}_{\widetilde{C}/P}(\Xi))) \right).$$

We will see that the theta-dual $T(\widetilde{C})$ can be described in terms of the following standard *Brill-Noether loci*:

Definition 2.2 ([W3, (1.2)])**.** We define $V^r \subset P^-$ (resp. $V^r \subset P^+$) if r is even (resp. odd) as

$$V^r := \left\{ L \in \mathrm{Nm}^{-1}(\omega_C) \,\middle|\, h^0(L) \geqslant r+1,\ h^0(L) \equiv r+1 \mod 2 \right\}.$$

These *Brill-Noether loci* can be endowed with a natural scheme structure. Following Welters (see [W3]) we define[2]

$$V^r = W^r_{\tilde{g}-1}(\widetilde{C}) \cap P^+ \qquad \text{if } r \text{ is odd},$$
$$V^r = W^r_{\tilde{g}-1}(\widetilde{C}) \cap P^- \qquad \text{if } r \text{ is even}.$$

For example, the first odd cases are $V^1 = \Xi^+ \subset P^+$ and $V^3 \subset P^+$, the stable singularities of Ξ (see [Mu3, pg. 343]). The first even cases are $V^0 = P^-$ and $V^2 = T(\widetilde{C}) \subset P^-$ as we will see next. First, we obtain the set-theoretical equality.

[2] We recall that De Concini and Pragacz [DP] define a more reduced natural schematic structure on V^r.

Lemma 2.3 ([CLV, Lem. 2.1]). *We have the set-theoretic equality*

$$T(\widetilde{C}) = V^2.$$

Proof. An element $L \in P^-$ belongs to $T(\widetilde{C})$ if and only if $\widetilde{C} \subset \Xi_L^+$, which, by the definition of $\widetilde{C} \subset P'$, is equivalent to $h^0(\widetilde{C}, L \otimes \mathcal{O}_{\widetilde{C}}(\sigma p - p)) > 0$ for every $p \in \widetilde{C}$. By Mumford's parity trick (see [Mu3]), this happens if and only if $h^0(\widetilde{C}, L) \geqslant 3$, that is $L \in V^2$. □

To compute the dimension of the theta-dual $T(\widetilde{C}) = V^2$ we will need the Jacobi Inversion Theorem for Prym varieties, which describes the restriction of the translates of the theta-divisor to the Abel-Prym curve. Although it is a classical result, we give a proof in our "canonical setting".

Proposition 2.4 (Jacobi Inversion Theorem for Prym varieties). *Given any $L \in P^-$, there is an isomorphism of line bundles*

$$\mathcal{O}_{P'}(\Xi_L^+)\big|_{\widetilde{C}} \cong \sigma^* L.$$

Moreover, if L is a closed point in $P^- \setminus V^2$, then we have an equality of divisors

$$\left(\Xi_L^+\right)_{|\widetilde{C}} = \widetilde{C} \cap \Xi_L^+ = \sigma D,$$

where D is the unique divisor in $|L|$,

Proof. Suppose first that L is a closed point in $P^- \setminus V^2$, i.e. $h^0(\widetilde{C}, L) = 1$. Write $|L| = D = p_1 + \ldots + p_{\tilde{g}-1}$, where $p_i \in \widetilde{C}$. Since p_i is a fixed point of the linear series $|L|$, we have that $h^0(\widetilde{C}, L \otimes \mathcal{O}_{\widetilde{C}}(-p_i + \sigma p_i)) = 2$, which implies that

$$\sigma D \subset \widetilde{C} \cap \Xi_L^+ = \left(\Xi_L^+\right)\big|_{\widetilde{C}}.$$

Using that $\widetilde{C} \cdot \Xi_L^+ = \tilde{g} - 1$, we get the desired second equality. Now consider the maps

$$\widetilde{C} \times P^- \xrightarrow{(\jmath, \mathrm{id})} P' \times P^- \xrightarrow{\mu} P^+,$$

where $\jmath$ is the Abel-Prym map and μ is the multiplication map. Let $\mathcal{P}$ be the Poincaré line bundle on $\widetilde{C} \times P^-$, trivialized over the section $\{p\} \times P^-$ for some $p \in \widetilde{C}$. Consider the line bundle on $\widetilde{C} \times P^-$ given by $\mathcal{L} := (\jmath \times \mathrm{id})^*(\mu^* \mathcal{O}_{P^+}(\Xi^+))$. We can trivialize $\mathcal{L}$ along the given section $\{p\} \times P^-$ by tensoring with the pull back from P^- of the divisor $\Xi^+_{\sigma p - p}$. It is easy to check that the fibers of $(\sigma \times \mathrm{id})^* \mathcal{P}$ and $\mathcal{L}$ over $\widetilde{C} \times \{L\}$ are given by

$$\begin{cases} (\sigma \times \mathrm{id})^* \mathcal{P}|_{\widetilde{C} \times \{L\}} = \sigma^* L, \\ \qquad\qquad \mathcal{L}|_{\widetilde{C} \times \{L\}} = \mathcal{O}_{P'}(\Xi_L^+)\big|_{\widetilde{C}}. \end{cases}$$

By what was proved above, if L is a closed point in $P^- \setminus V^2$ then the two fibers agree. By the Seesaw Theorem [Mu2, Cor. 6, pg. 54], $(\sigma \times \mathrm{id})^* \mathcal{P} \cong \mathcal{L}$ and we get the desired first equality. □

Now we want to study the dimension of $T(\widetilde{C}) = V^2$.

Theorem 2.5 ([CLV, Thm. 2.2]). *For any étale double cover $\widetilde{C} \to C$ as above with C non-hyperelliptic of genus g, it holds that*

$$\dim(V^2) = \dim(P) - 3 = g - 4.$$

For $g = 3$, the Theorem says that $V^2 = \emptyset$. We start with the following Lemma, which is similar to [Mu3, Lemma p. 345].

Lemma 2.6 ([CLV, Lem. 2.3]). *If $Z \subseteq V^2$ is an irreducible component, and $\dim Z \geq g - 3$, then for a general line bundle $L \in Z$, there is a line bundle M on C with $h^0(M) \geq 2$, and an effective divisor F on $\widetilde{C}$ such that $L \cong \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(F)$.*

Proof. Let Z and L be as in the statement. Suppose that $h^0(L) = r + 1$ for $r \geqslant 2$ even, so that $L \in W^r_{\tilde{g}-1} - W^{r+1}_{\tilde{g}-1}$. From the hypothesis, we get that

$$\dim T_L W^r_{\tilde{g}-1} \cap T_L P^- \geq g - 3 = \dim(P^-) - 2. \tag{2.3}$$

The Zariski tangent space to $W^r_{\tilde{g}-1}$ at L is given by the orthogonal complement to the image of the Petri map (e.g. [ACGH, Prop. 4.2]):

$$H^0(\widetilde{C}, L) \otimes H^0(\widetilde{C}, \sigma^* L) \to H^0(\widetilde{C}, \omega_{\widetilde{C}}),$$

where we have used that $\omega_{\widetilde{C}} = \pi^*(\omega_C) = L \otimes \sigma^* L$. On the other hand, the tangent space to the Prym is by definition $T_L P^- = H^0(\widetilde{C}, \omega_{\widetilde{C}})^-$, the (-1)-eigenspace of $H^0(\widetilde{C}, \omega_{\widetilde{C}})$ relative to the involution σ. Therefore, it is easy to see that the intersection of the Zariski tangent spaces $T_L W^r_{\tilde{g}-1} \cap T_L P^-$ is given as the orthogonal complement to the image of the map

$$v_0 \colon \wedge^2 H^0(\widetilde{C}, L) \to H^0(\widetilde{C}, \omega_{\widetilde{C}})^-$$

defined by $v_0(s_i \wedge s_j) = s_i \sigma^* s_j - s_j \sigma^* s_i$.
The inequality (2.3) is equivalent to $\operatorname{codim}(\ker v_0) \leq 2$. On the other hand, the decomposable forms in $\wedge^2 H^0(\widetilde{C}, L)$ form a subvariety of dimension $2r - 1 \geqslant 3$, and so there is a decomposable vector $s_i \wedge s_j$ in $\ker v_0$. This means that $s_i \sigma^* s_j - s_j \sigma^* s_i = 0$, or in other words that $\frac{s_j}{s_i}$ defines a rational function h in C. We conclude by taking $M = \mathcal{O}_C((h)_0)$ and F the be the maximal common divisor between $(s_i)_0$ and $(s_j)_0$. □

Proof of Theorem 2.5. The dimension of $T(\widetilde{C}) = V^2$ is at least $g - 4$ by the theorem of Bertram ([Be], see also [DP]). Suppose, by contradiction, that there is an irreducible component $Z \subseteq V^2$ such that $\dim Z = m \geq g-3$. Then, by applying the preceding Lemma 2.6 for the general element $L \in Z$,

$$L \cong \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(B)$$

where M is an invertible sheaf on C such that $h^0(M) \geqslant 2$, and B is an effective divisor on $\widetilde{C}$ such that $\mathrm{Nm}(B) \in \left|K_C \otimes M^{\otimes -2}\right|$. The family of such pairs (M, B) is a finite cover of the set of pairs $\{M, F\}$ where:

- M is an invertible sheaf on C of degree $d \geqslant 2$ such that $h^0(M) \geqslant 2$,
- F is an effective divisor on C of degree $2g-2-2d \geqslant 0$, such that $F \in \left|K_C \otimes M^{\otimes -2}\right|$.

By Marten's theorem applied to the non-hyperelliptic curve C (see [ACGH, Pag. 192]), the dimension of the above family of line bundles M is bounded above by

$$\dim(W_d^1) < d-2. \tag{2.4}$$

Fixing a line bundle M as above, the dimension of possible F satisfying the second condition is bounded by Clifford's theorem,

$$h^0(K_C \otimes M^{\otimes -2}) - 1 \leqslant g-1-d, \tag{2.5}$$

By putting together the inequalities (2.4) and (2.5), we get that the dimension m of our family of pairs $\{M, F\}$ is bounded above by $m < d-2+g-1-d = g-3$, contradicting our hypothesis. $\square$

2.3 The cohomological support loci of $\mathcal{I}_{\widetilde{C}}(\Xi)$

In this section we compute the cohomological support loci for the ideal sheaf $\mathcal{I}_{\widetilde{C}}(\Xi)$, which can be identified with the auxiliary canonical loci

$$\tilde{V}^i(\mathcal{I}_{\widetilde{C}}(\Xi^+)) = \left\{L \in P^- \,\middle|\, h^i(P', \mathcal{I}_{\widetilde{C}}(\Xi^+_L)) > 0\right\} \subset P^-, \tag{2.6}$$

where we denote by $\Xi^+_L \subset P'$ the translate $t^*_L \Xi^+$ of the canonical theta-divisor Ξ^+ as in (II.1.2).

The relation between $\tilde{V}^i(\mathcal{I}_{\widetilde{C}}(\Xi^+))$ and the non-canonical loci $V^i(\mathcal{I}_{\widetilde{C}/P}(\Xi))$ is easy to work out. Recall that in Terminology/Notation 2.1, we have fixed $L_0^+ \in P^+$ such that $h^0(\widetilde{C}, L_0^+) = 0$ and $L_0' = \mathcal{O}_{\widetilde{C}}(\sigma p_0 - p_0) \in P'$. Moreover, we denote by $L_0^- = L_0^+ \otimes {L_0'}^{-1} \in P^-$ and $\Xi = \Xi^+_{L_0^+}$. We use the *non-canonical Abel-Prym map*

$$\begin{array}{rccc} j: & \widetilde{C} & \longrightarrow & P \\ & p & \longmapsto & p - \sigma p - p_0 + \sigma p_0. \end{array} \tag{2.7}$$

Then, it is easy to see that the isomorphism $t_{L_0^-} : P \to P^-$ gives

$$V^i(\mathcal{I}_{\widetilde{C}}(\Xi)) = t_{L_0^-}^{-1}(\tilde{V}^i(\mathcal{I}_{\widetilde{C}}(\Xi^+))).$$

Proposition 2.7 ([CLV, Thm. 3.1]). *The cohomological support loci for $\mathcal{I}_{\widetilde{C}}(\Xi)$ are*

(a) $\tilde{V}^0(\mathcal{I}_{\widetilde{C}}(\Xi^+)) = \tilde{V}^1(\mathcal{I}_{\widetilde{C}}(\Xi^+)) = T(\widetilde{C})$ *and*

(b) $\tilde{V}^2(\mathcal{I}_{\widetilde{C}}(\Xi^+)) = P^-$.

Proof. The equality $\tilde{V}^0(\mathcal{I}_{\widetilde{C}}(\Xi^+)) = T(\widetilde{C})$ is just the definition of the theta-dual of $\widetilde{C}$. Consider the exact sequence defining the ideal sheaf $\mathcal{I}_{\widetilde{C}}$ twisted by the divisor Ξ_L^+, for $L \in P^-$:

$$0 \to \mathcal{I}_{\widetilde{C}}(\Xi_L^+) \to \mathcal{O}_{P'}(\Xi_L^+) \to j_*\mathcal{O}_{\widetilde{C}}(\Xi_L^+) \to 0.$$

By taking cohomology and using the vanishing $H^j(P', \mathcal{O}_{P'}(\Xi_L^+)) = 0$ for $j > 0$, we get the emptiness of $\tilde{V}^i(\mathcal{I}_{\widetilde{C}}(\Xi^+))$ for $i \geqslant 3$ and the two exact sequences,

$$0 \to H^0(\mathcal{I}_{\widetilde{C}}(\Xi_L^+)) \to H^0(\mathcal{O}_{P'}(\Xi_L^+)) \stackrel{\psi_L}{\to} H^0(\mathcal{O}_{\widetilde{C}}(\Xi_L^+)) \to H^1(\mathcal{I}_{\widetilde{C}}(\Xi_L^+)) \to 0,$$
$$0 \to H^1(\mathcal{O}_{\widetilde{C}}(\Xi_L^+)) \to H^2(\mathcal{I}_{\widetilde{C}}(\Xi_L^+)) \to 0.$$

Using the second exact sequence and the Proposition 2.4, we get that

$$\tilde{V}^2(\mathcal{I}_{\widetilde{C}}(\Xi^+)) = \left\{ L \in P^- \,\middle|\, h^1(\widetilde{C}, \sigma^* L) > 0 \right\}.$$

Since $L \otimes \sigma^* L = \omega_{\widetilde{C}}$, by Serre duality we have that $h^1(\widetilde{C}, \sigma^* L) = h^0(\widetilde{C}, L)$, which is greater than 0 for all $L \in P^-$ by the definition of P^-. This proves assertion *(b)*.

Consider now the first above exact sequence. Since $H^0(P', \mathcal{O}_{P'}(\Xi_L^+)) = 1$ and $\tilde{V}^1(\mathcal{I}_{\widetilde{C}}(\Xi^+))$ consists of the elements L such that the map ψ_L is not surjective, we get using again Propostion 2.4 and Lemma 2.3

$$\begin{aligned}\tilde{V}^1(\mathcal{I}_{\widetilde{C}}(\Xi^+)) &= \left\{ L \in P^- \,\middle|\, h^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(\Xi_L^+)) > 1 \right\} \cup \left\{ L \in P^- \mid h^0(P', \mathcal{I}_{\widetilde{C}}(\Xi_L^+)) = 1 \right\} \\ &= V^2 \cup T(\widetilde{C}) = T(\widetilde{C}).\end{aligned}$$

□

2.4 Schematic theta-dual of an Abel-Prym curve $\widetilde{C}$

To prove the schematic equality $T(\widetilde{C}) = V^2$, we will also need the following Lemma that is of independent interest (see for example [Na, Prop. 3.1]).

Lemma 2.8. *For every $L^+ \in P^+$, its direct image by the non-canonical Abel-Prym map j_*L^+ is a WIT$_1$ sheaf with respect to the Fourier-Mukai transform*

$\mathrm{R}\mathcal{S}\colon \mathrm{D}^b(P) \to \mathrm{D}^b(P)$. *Moreover, we can relate its Fourier-Mukai transform with the Fourier-Mukai transform in the Jacobian of* $\widetilde{C}$. *More precisely,*

$$R^1\mathcal{S}(j_*L^+) \cong \left(R^1\mathcal{S}_{J\widetilde{C}}(i_*L^+)\right)\big|_P ,$$

where $j\colon \widetilde{C} \to P$ *is an Abel-Prym map,* $j(p) = p - \sigma p - p_0 + \sigma p_0$ *and* $i\colon \widetilde{C} \to J\widetilde{C}$ *is an Abel-Jacobi map* $i(p) = p - p_0$.

Proof. First we will see that j_*L^+ is WIT_1. We choose an effective divisor D on $\widetilde{C}$ of sufficiently high degree $m \gg 0$ and we consider the short exact sequence

$$0 \to j_*(L^+) \to j_*(L^+)(D) \to j_*\ (L^+)(D)\big|_D \to 0.$$

Its Fourier-Mukai transform is

$$0 \to R^0\mathcal{S}\, j_*(L^+) \to R^0\mathcal{S}\left(j_*(L^+)(D)\right) \to R^0\mathcal{S}\left(j_*\ (L^+)(D)\big|_D\right) \to R^1\mathcal{S}\, j_*(L^+) \to 0.$$

For D of sufficiently high degree $m \gg 0$, the two middle sheaves are locally free. On the other hand, for a general $\alpha \in P$, more precisely for $\alpha \notin \Xi_{L^+}$, $h^0(\widetilde{C}, L^+ \otimes \alpha) = 0$. Hence $R^0\mathcal{S}\, j_*(L^+)$ is a torsion-sheaf. So $R^0\mathcal{S}\, j_*(L^+) = 0$, and $j_*(L^+)$ is a WIT_1 sheaf.

The same argument shows that $i_*(L^+)$ satisfies WIT_1 with respect to $\mathrm{R}\mathcal{S}_{J\widetilde{C}}$. Indeed, the Fourier-Mukai transform of the short exact sequence $0 \to i_*(L^+) \to i_*(L^+)(D) \to i_*\ (L^+)(D)|_D \to 0$ is

$$\begin{aligned} 0 \to R^0\mathcal{S}_{J\widetilde{C}}\, i_*(L^+) &\to R^0\mathcal{S}_{J\widetilde{C}}\left(i_*(L^+)(D)\right) \to \\ &\to R^0\mathcal{S}_{J\widetilde{C}}\left(i_*\ (L^+)(D)\big|_D\right) \to R^1\mathcal{S}_{J\widetilde{C}}\, i_*(L^+) \to 0. \end{aligned}$$

For D of sufficiently high degree $m \gg 0$, the two middle sheaves are locally free. On the other hand, for a general $\alpha \in J\widetilde{C}$, more precisely for $\alpha \notin \widetilde{\Theta}_{L^+}$, $h^0(\widetilde{C}, L^+ \otimes \alpha) = 0$. Hence $R^0\mathcal{S}_{J\widetilde{C}}\, i_*(L^+)$ is a torsion-sheaf. So $R^0\mathcal{S}_{J\widetilde{C}}\, i_*(L^+) = 0$, and $i_*(L^+)$ is a WIT_1 sheaf with respect to $\mathrm{R}\mathcal{S}_{J\widetilde{C}}$.

Claim. In the derived category $\mathrm{D}^b(J\widetilde{C})$, we have

$$\mathrm{R}\mathcal{S} \circ \mathrm{R}j_* \cong (\mathcal{O}_P \underline{\otimes}) \circ \mathrm{R}\mathcal{S}_{J\widetilde{C}} \circ \mathrm{R}i_*.$$

Admitting the claim for the moment we finish the proof of the Proposition. The claim applied to L^+ says that $\mathrm{R}\mathcal{S}(j_*L^+) \cong \mathcal{O}_P \underline{\otimes} \mathrm{R}\mathcal{S}_{J\widetilde{C}}(i_*L^+)$. Then,

$$\begin{aligned} R^1\mathcal{S}(j_*L^+) &\cong \mathrm{R}\mathcal{S}(j_*L^+)[1] \cong \left(\mathcal{O}_P \underline{\otimes} R^1\mathcal{S}_{J\widetilde{C}}(i_*L^+)[-1]\right)[1] \\ &\cong \mathcal{O}_P \otimes R^1\mathcal{S}_{J\widetilde{C}}(i_*L^+) \cong R^1\mathcal{S}_{J\widetilde{C}}(i_*L^+)\big|_P . \end{aligned}$$

Proof of the Claim. By the projection formula in the derived category the right-hand side is $\mathrm{R}\mathcal{S}_{J\widetilde{C}} \circ (q^*\mathcal{O}_P \underline{\otimes}) \circ \mathrm{R}i_*$. So by Proposition I.1.1 we have to prove that

$$\mathrm{R}\Phi_{P_j} \cong \mathrm{R}\Phi_{P_i},$$

where $P_j = (j \times \mathrm{id})^* \tilde{\mathcal{P}}$ ($\tilde{\mathcal{P}}$ is the Poincaré line bundle in $P \times P$) and $P_i = (i \times \imath)^* \mathcal{P}$ ($\mathcal{P}$ is the Poincaré line bundle in $\mathrm{Pic}^0 C \times \mathrm{Pic}^0 C$ and $\imath : P \hookrightarrow \mathrm{Pic}^0 C$ is the inclusion). We want to compare P_j and P_i.

- For $L \in P$, we have $P_j|_{\widetilde{C} \times \{L\}} = j^* L = L$ because $j^* \colon P \to \mathrm{Pic}^0 \widetilde{C}$ is the natural inclusion.
- For $L \in P$, we have $P_i|_{\widetilde{C} \times \{L\}} = i^* L = L$ since i is the Albanese map of $\widetilde{C}$, so i^* is the identity.
- $P_j|_{\{p_0\} \times P} = \tilde{\mathcal{P}}\Big|_{\{j(p_0)\} \times P} = \mathcal{O}_P$.
- $P_i|_{\{p_0\} \times P} = \imath^*(\mathcal{P}|_{\{i(p_0)\} \times P}) = \imath^* \mathcal{O}_{\mathrm{Pic}^0 C} = \mathcal{O}_P$.

Therefore, the claim follows from the Seesaw Theorem [Mu2, Cor. 6, pg. 54]. $\square$

Now, we are ready to prove the main result of this section.

Theorem 2.9. *We have the scheme-theoretic equality*

$$T(\widetilde{C}) = V^2.$$

Proof. By Lemma 2.3 we already know the set-theoretic equality that holds in P^- without any choices. Now, in order to avoid all the translations that could complicate the Fourier-Mukai argument, we will work up to translation on P using the fixed isomorphisms like in Terminology/Notation 2.1.

By Remark II.2.5,

$$\mathcal{O}_{T(\widetilde{C})}(\Xi) = \mathcal{H}om(\mathrm{R}\mathcal{S}(\mathcal{I}_{\widetilde{C}}(\Xi)), \mathcal{O}_P),$$

so we want to compute this last sheaf, or construct a comprehensible short exact sequence where it appears.

Consider p_0 the point in $\widetilde{C}$ that we have used to define the non-canonical Abel-Prym map (2.2) and the following short exact sequence in P

$$0 \to \mathcal{I}_{\widetilde{C}/P}(\Xi) \to \mathcal{I}_{0/P}(\Xi) \to j_* \mathcal{I}_{p_0/\widetilde{C}}(\Xi) \to 0, \tag{2.8}$$

where we have used that $j(p_0) = 0 \in P$. We will work out the Fourier-Mukai transform of this exact sequence and its dual. The transforms of the leftmost sheaf will provide the sheaves that we want to understand. The middle sheaf is easy to work with. And the rightmost sheaf can be work out in the Abel-Prym curve.

First, let us study the Fourier-Mukai transform of $j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi)$. Recall that our choices were $L'_0 = \mathcal{O}_{\widetilde{C}}(p_0 - \sigma p_0) \in P'$, $L_0^+ \in P^+$ and then, $L_0^+ = L_0^- \otimes L'_0$ (see Terminology/Notation 2.1). First, we observe that by the Jacobi Inversion Theorem 2.4 $j_*\mathcal{O}_{\widetilde{C}}(\Xi) = j_*(\sigma^* L_0^- \otimes (L'_0)^{-1})$. Since $(L'_0)^{-1} = \sigma^* L'_0$, we have $j_*\mathcal{O}_{\widetilde{C}}(\Xi) = j_*(\sigma^* L_0^+)$ Hence,

$$j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi) = j_*(\sigma^* L_0^+ \otimes \mathcal{O}_{\widetilde{C}}(-p_0)). \tag{2.9}$$

Claim 1. $j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi)$ is a WIT_1 sheaf.
Using (2.9) we construct the following exact sequence in P,

$$0 \to j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi) \to j_*(\sigma^* L_0^+) \to k(0) \to 0, \tag{2.10}$$

where we have used that j_* is an exact functor and $j(p_0) = 0$. By the previous Lemma 2.8, $j_*(\sigma^* L_0^+)$ is a WIT_1 sheaf (observe that $\sigma^* L_0^+ \in P^+$, since σ is an isomorphism, hence it does not change the parity of the number of sections). This proves Claim 1, i.e. $j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi)$ is a WIT_1 sheaf.

Now, apply the Fourier-Mukai transform to the short exact sequence (2.8) using Example II.2.6 to describe $R^1\mathcal{S}(\mathcal{I}_{0/P}(\Xi))$ as $\mathcal{O}_\Xi$,

$$0 \to R^1\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)) \to \mathcal{O}_\Xi \to R^1\mathcal{S}(j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi)) \to R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)) \to 0.$$

The set-theoretic support of $R^1\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi))$ is included in $V^1(\mathcal{I}_{\widetilde{C}/P}(\Xi))$ that has codimension 3 in P (see Proposition 2.7 and Theorem 2.5). Hence, from the exact sequence we get $R^1\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)) = 0$. Thus, we have already computed that the Fourier-Mukai transform of the exact sequence (2.8) is simply

$$0 \to \mathcal{O}_\Xi \to R^1\mathcal{S}(j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi)) \to R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)) \to 0. \tag{2.11}$$

In particular, we have seen that $\mathcal{I}_{\widetilde{C}/P}(\Xi)$ satisfies WIT_2, so

$$\mathrm{R}\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)) = R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi))[-2]. \tag{2.12}$$

Now, we apply the functor $\mathrm{R}\mathcal{H}om(\,\cdot\,, \mathcal{O}_P)$ to (2.11). We get

$$\mathcal{H}om(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)), \mathcal{O}_P) \cong \mathcal{H}om(R^1\mathcal{S}(j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi)), \mathcal{O}_P) \tag{2.13}$$

and,

$$\begin{aligned} 0 \to \mathcal{E}xt^1(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)), \mathcal{O}_P) \to \mathcal{E}xt^1(R^1\mathcal{S}(j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi)), \mathcal{O}_P) \to \mathcal{O}_\Xi(\Xi) \to \\ \to \mathcal{E}xt^2(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)), \mathcal{O}_P) \to \mathcal{E}xt^2(R^1\mathcal{S}(j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi)), \mathcal{O}_P) \to 0. \end{aligned} \tag{2.14}$$

Let us compute some terms of this last long exact sequence.

Claim 2. $\mathcal{E}xt^2(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)),\mathcal{O}_P)=\mathcal{O}_{T(\widetilde{C})}(\Xi)$.
By Remark II.2.5,

$$\begin{aligned}\mathcal{O}_{T(\widetilde{C})}(\Xi) &\cong \mathcal{H}om(\mathrm{R}\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)),\mathcal{O}_P)\\ &\cong \mathcal{H}om(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi))[-2],\mathcal{O}_P) \qquad \text{by (2.12), } \mathcal{I}_{\widetilde{C}/P}(\Xi) \text{ is WIT}_2\\ &\cong \mathcal{E}xt^2(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)),\mathcal{O}_P). \end{aligned} \tag{2.15}$$

In particular, by Theorem 2.5, we have

$$\operatorname{codim}_P \mathcal{E}xt^2(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)),\mathcal{O}_P)\geqslant 3. \tag{2.16}$$

Claim 3. $\mathcal{E}xt^1(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)),\mathcal{O}_P)=0$.
Observe that, $\mathcal{E}xt^1(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)),\mathcal{O}_P)=\mathcal{E}xt^{-1}(\mathrm{R}\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)),\mathcal{O}_P)$ that by Grothendieck-Verdier duality I.1.6 is $R^{g-1}\mathcal{S}\,\mathrm{R}\Delta(\mathcal{I}_{\widetilde{C}/P}(\Xi))$. By base change, we have that $\operatorname{supp} R^{g-1}\mathcal{S}\,\mathrm{R}\Delta(\mathcal{I}_{\widetilde{C}/P}(\Xi))\subseteq V^{g-1}(\mathrm{R}\Delta(\mathcal{I}_{\widetilde{C}/P}(\Xi)))=-V^1(\mathcal{I}_{\widetilde{C}/P}(\Xi))$ (the last equality, follows from Grothendieck-Serre duality I.1.5), so

$$\operatorname{codim}_P \mathcal{E}xt^1(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)),\mathcal{O}_P)\geqslant 3, \tag{2.17}$$

by Proposition 2.7 and Theorem 2.5.
Since P is smooth, the functor $\mathrm{R}\mathcal{H}om(\cdot,\mathcal{O}_P)$ is an involution on $\mathrm{D}^b(P)$. Thus there is a fourth quadrant spectral sequence

$$E_2^{i,j}:=\mathcal{E}xt^i\Big((\mathcal{E}xt^{-j}(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)),\mathcal{O}_P),\mathcal{O}_P\Big)\Rightarrow H^{i+j}=\begin{cases}R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)) & \text{if } i+j=0\\ 0 & \text{otherwise.}\end{cases}$$

We have the following

$$\begin{aligned}\operatorname{codim}_P \operatorname{supp}\mathcal{E}xt^1(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)),\mathcal{O}_P)&\geqslant 3, &&\text{by (2.17)}\\ \operatorname{codim}_P \operatorname{supp}\mathcal{E}xt^2(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)),\mathcal{O}_P)&\geqslant 3, &&\text{by (2.16)}\\ \mathcal{E}xt^i(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)),\mathcal{O}_P)&=0 \quad \text{for all } i>2.\end{aligned}$$

Recall that $\mathcal{E}xt^l(\mathcal{F},\mathcal{O}_P)=0$ for all $l<\operatorname{codim}_P\operatorname{supp}\mathcal{F}$. Then the previous spectral sequence yields to the following short exact sequence

$$\begin{aligned}0\to R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi))\to \mathcal{H}om(\mathcal{H}om(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)),\mathcal{O}_P),\mathcal{O}_P)\to\\ \to \mathcal{E}xt^3(\mathcal{O}_{T(\widetilde{C})}(\Xi),\mathcal{O}_P)\to 0.\end{aligned} \tag{2.18}$$

By (2.10) and Claim 1,

$$0\to\mathcal{O}_P\to R^1\mathcal{S}(j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi))\to R^1\mathcal{S}(j_*(L_0^+))\to 0. \tag{2.19}$$

And supp $R^1\mathcal{S}(j_*(L_0^+)) \subseteq V^1(j_*(L_0^+)) = \Xi$. Therefore, $R^1\mathcal{S}(j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi))$ has (generic) rank 1. By (2.13), $\mathcal{H}om(\mathcal{H}om(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)), \mathcal{O}_P), \mathcal{O}_P)$ is a reflexive sheaf of rank 1, hence a line bundle. Thus, dualizing (2.18) we get Claim 3, i.e.

$$\mathcal{E}xt^1(R^2\mathcal{S}(\mathcal{I}_{\widetilde{C}/P}(\Xi)), \mathcal{O}_P) = 0.$$

Claim 4. $\mathcal{E}xt^2(R^1\mathcal{S}\, j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi), \mathcal{O}_P) = 0$.
We will prove the claim by constructing quite big diagrams that will be useful in the sequel. Consider D an effective divisor on $\widetilde{C}$ of sufficiently high degree $m \gg 0$ and the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi) \arrow[r] \arrow[d] & j_*(\sigma^* L_0^+ \otimes \mathcal{O}_{\widetilde{C}}(D - p_0)) \arrow[r] \arrow[d] & j_*\mathcal{O}_D \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & j_*(L_0^+) \arrow[r] \arrow[d] & j_*(\sigma^* L_0^+ \otimes \mathcal{O}_{\widetilde{C}}(D)) \arrow[r] \arrow[d] & j_*\mathcal{O}_D \arrow[r] & 0 \\
 & k(0) \arrow[r, equal] \arrow[d] & k(0) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Observe that $j_*\mathcal{O}_D$ and $k(0)$ are IT_0 sheaves since they are supported in dimension 0. For D of sufficiently high degree $m \gg 0$, the sheaves $j_*(\sigma^* L_0^+ \otimes \mathcal{O}_{\widetilde{C}}(D - p_0))$ and $j_*(\sigma^* L_0^+ \otimes \mathcal{O}_{\widetilde{C}}(D))$ are IT_0. If we apply the Fourier-Mukai transform to the previous diagram we obtain the following commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \mathcal{O}_P \arrow[d] & \\
0 \arrow[r] & \mathcal{E}_2 \arrow[r, "M"] \arrow[d] & \mathcal{E}_1 \arrow[r] \arrow[d, equal] & R^1\mathcal{S}\,\mathcal{I}_{p_0/\widetilde{C}}(\Xi) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}_0 \arrow[r, "N"] \arrow[d] & \mathcal{E}_1 \arrow[r] & R^1\mathcal{S}\, j_*(\sigma^* L_0^+) \arrow[r] \arrow[d] & 0 \\
 & \mathcal{O}_P \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd} \tag{2.20}$$

where the sheaves $\mathcal{E}_i$ are locally free of rank $\operatorname{rk}\mathcal{E}_0 = \operatorname{rk}\mathcal{E}_1 = m$ and $\operatorname{rk}\mathcal{E}_2 = m-1$. The top short exact sequence implies Claim 4, i.e.

$$\mathcal{E}xt^2(R^1\mathcal{S}\, j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi), \mathcal{O}_P) = 0.$$

Combining Claims 2, 3 and 4 with (2.14) we get

$$0 \to \mathcal{E}xt^1(R^1\mathcal{S}(j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi)), \mathcal{O}_P) \to \mathcal{O}_\Xi(\Xi) \to \mathcal{O}_{T(\widetilde{C})}(\Xi) \to 0.$$

Hence $\mathcal{E}xt^1(R^1\mathcal{S}(j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi)), \mathcal{O}_P) \cong \mathcal{I}_{T(\widetilde{C})/\Xi}(\Xi)$. Now, we will focus on the equations of $T(\widetilde{C})$ inside Ξ, that is, we will look for $\mathcal{I}_{T(\widetilde{C})/\Xi}(\Xi)$ instead of $\mathcal{O}_{T(\widetilde{C})}(\Xi)$. If we dualize the diagram (2.20) we obtain

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & & & 0 \arrow[d] & \mathcal{H}om(R^1\mathcal{S}\,\mathcal{I}_{p_0/\widetilde{C}}(\Xi), \mathcal{O}_P) \arrow[d] & \\
 & & & \mathcal{O}_P \arrow[r, equal] \arrow[d] & \mathcal{O}_P \arrow[d] & \\
 & 0 \arrow[r] & \mathcal{E}_1^\vee \arrow[r, "N^t"] \arrow[d, equal] & \mathcal{E}_0^\vee \arrow[r] \arrow[d] & \mathcal{E}xt^1(R^1\mathcal{S}\, j_*(\sigma^* L_0^+), \mathcal{O}_P) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{H}om(R^1\mathcal{S}\,\mathcal{I}_{p_0/\widetilde{C}}(\Xi), \mathcal{O}_P) \arrow[r] & \mathcal{E}_1^\vee \arrow[r, "M^t"] & \mathcal{E}_2^\vee \arrow[r] \arrow[d] & \mathcal{I}_{T(\widetilde{C})/\Xi}(\Xi) \arrow[r] \arrow[d] & 0 \\
 & & & 0 & 0 &
\end{tikzcd} \tag{2.21}$$

From the last row of the previous diagram (2.21) we have the following exact sequence on Ξ,

$$\mathcal{E}_1^\vee \otimes \mathcal{O}_\Xi \xrightarrow{\;M^t|_\Xi\;} \mathcal{E}_2^\vee \otimes \mathcal{O}_\Xi \longrightarrow \mathcal{I}_{T(\widetilde{C})/\Xi}(\Xi) \longrightarrow 0.$$

Recall that $\operatorname{rk}\mathcal{E}_1 = \operatorname{rk}\mathcal{E}_2 + 1 = m$. Hence $T(\widetilde{C})$ corresponds to the determinantal scheme where $M^t|_\Xi$ drops rank. That is,

$$\begin{aligned}
\mathcal{I}_{T(\widetilde{C})/\Xi} &\cong \operatorname{Fitt}_1\left(\mathcal{I}_{T(\widetilde{C})/\Xi}(\Xi)\right) = \operatorname{Fitt}_1\left(\mathcal{E}xt^1(R^1\mathcal{S}\, j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi), \mathcal{O}_P) \otimes \mathcal{O}_\Xi\right) \\
&\cong \operatorname{Fitt}_1\left(\mathcal{E}xt^1(R^1\mathcal{S}\, j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi), \mathcal{O}_P)\right) \otimes \mathcal{O}_\Xi \qquad \text{by Prop. II.2.2.}
\end{aligned}$$

The determinants of M^t are the same of those of M. So, going back to the first horizontal short exact sequence in (2.20) we get $\mathrm{Fitt}_1\left(\mathcal{E}xt^1(R^1\mathcal{S}\,j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi),\mathcal{O}_P)\right) = \mathrm{Fitt}_2\left(R^1\mathcal{S}\,j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi)\right)$ i.e.

$$\mathcal{I}_{T(\widetilde{C})/\Xi} = \mathrm{Fitt}_2\left(R^1\mathcal{S}\,j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi)\right)\otimes\mathcal{O}_\Xi.$$

The Fitting ideals are computed locally (in fact, they can be computed locally because they commute with arbitrary base change, see Proposition II.2.2). Choose a covering of P by open affine subsets such that, in any of these open subsets, the first vertical short exact sequence of diagram (2.20) splits. Then, we can choose basis such that in these open sets the matrices representing M and N are

$$N = \left(\begin{array}{c|c} M & * \\ \hline 0\cdots 0 & 1\end{array}\right).$$

Hence, diagram (2.20) shows that $\mathrm{Fitt}_2\left(R^1\mathcal{S}\,j_*\mathcal{I}_{p_0/\widetilde{C}}(\Xi)\right) = \mathrm{Fitt}_2\left(R^1\mathcal{S}\,j_*(\sigma^*L_0^+)\right)$ i.e.

$$\mathcal{I}_{T(\widetilde{C})/\Xi} \cong \mathrm{Fitt}_2\left(R^1\mathcal{S}\,j_*(\sigma^*L_0^+)\right)\otimes\mathcal{O}_\Xi.$$

Therefore, we have $\mathcal{I}_{T(\widetilde{C})/\Xi}$ in terms of L^+ that by Lemma 2.8 allows us to jump to the Jacobian. More precisely, by Lemma 2.8 $R^1\mathcal{S}\,j_*(\sigma^*L_0^+)\cong R^1\mathcal{S}_{J\widetilde{C}}\,i_*(\sigma^*L_0^+)\otimes\mathcal{O}_P$, so

$$\begin{aligned}
\mathcal{I}_{T(\widetilde{C})/\Xi} &\cong \left(\mathrm{Fitt}_2(R^1\mathcal{S}_{J\widetilde{C}}\,i_*(\sigma^*L_0^+))\otimes_{\mathcal{O}_{J\widetilde{C}}}\mathcal{O}_P\right)\otimes_{\mathcal{O}_P}\mathcal{O}_\Xi && \text{by Prop. II.2.2}\\
&\cong \mathcal{I}_{(W^2_{\tilde{g}-1}(\widetilde{C})\cap P)/P}\otimes_{\mathcal{O}_P}\mathcal{O}_\Xi && \text{by Def. (1.1)}\\
&\cong \mathcal{I}_{V^2/\Xi} && \text{by Def. 2.2.}
\end{aligned}$$

So we have proved the schematic equality $V^2 = T(\widetilde{C})$ (up to translation). □

2.5 The cohomological support loci of $\mathcal{I}_{\widetilde{C}}(2\Xi)$

In this section we determine the cohomological support loci of the ideal sheaf $\mathcal{I}_{\widetilde{C}}(2\Xi^+)$. In the proof of the next Theorem, we will need to know that the set $S(\widetilde{C})$ of *theta-characteristics of* $\widetilde{C}$ has a point in P^-. It is not much more work to show the stronger lemma below.

Lemma 2.10 ([CLV, Lem. 4.1]). *A theta-characteristic $L\in S(\widetilde{C})$ belongs to $\mathrm{Nm}^{-1}(\omega_C) = P^+\cup P^-$ if and only if it is the pull-back of a theta-characteristic M on C. Moreover, it holds that*

$$\left|S(\widetilde{C})\cap P^-\right| = \left|S(\widetilde{C})\cap P^+\right| = 2^{2g-1}.$$

Proof. If M is a theta-characteristic on C, then $\pi^*(M) \in S(\widetilde{C}) \cap \mathrm{Nm}^{-1}(\omega_C)$ as

$$\begin{cases} \mathrm{Nm}(\pi^*(M)) = M^{\otimes 2} = \omega_C, \\ \pi^*(M)^{\otimes 2} = \pi^*(M^{\otimes 2}) = \pi^*(\omega_C) = \omega_{\widetilde{C}}. \end{cases}$$

Conversely, if $L^{\otimes 2} = \omega_{\widetilde{C}}$ and $\mathrm{Nm}(L) = \omega_C$, then

$$L \otimes L = \omega_{\widetilde{C}} = \pi^*(\omega_C) = \pi^*(\mathrm{Nm}(L)) = L \otimes \sigma^* L,$$

which implies that $L = \pi^*(M)$ for some $M \in \mathrm{Pic}(C)$. By applying the Norm map, we obtain $\omega_C = \mathrm{Nm}(L) = \mathrm{Nm}(\pi^*(M)) = M^{\otimes 2}$.
Moreover, if we denote by η_0 the line bundle of order two on C satisfying $\pi_*(\mathcal{O}_{\widetilde{C}}) = \mathcal{O}_C \oplus \eta_0$, then the pull-back $\pi^*(M)$ of a theta-characteristic M on C belongs to P^- if and only if $h^0(C, M) \not\equiv h^0(C, M \otimes \eta_0) \mod 2$ (and to P^+ otherwise), as follows from the formula $H^0(\widetilde{C}, \pi^*(M)) = H^0(C, M) \oplus H^0(C, M \otimes \eta_0)$.
Now, fix a theta-characteristic M_0 of C. Then all the theta-characteristics of C are of the form $M_0 \otimes \eta$ for a unique $\eta \in J_2(C)$, where $J_2(C)$ is the group of the 2^{2g} line bundles of C whose square is trivial. Consider the following map

$$\begin{array}{rccl} q_0: & J_2(C) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ & \eta & \longmapsto & h^0(M_0 \otimes \eta \otimes \eta_0) - h^0(M_0 \otimes \eta) \mod 2 \end{array}$$

The Riemann-Mumford relation (see [Mu1, p. 182]) yields

$$q_0(\eta) = q_0(\eta_0) + \ln e_2(\eta, \eta_0),$$

where $e_2 \colon J_2(C) \times JC \to \{\pm 1\}$ is the Riemann skew-symmetric bilinear form (recall that $\mathrm{char}(k) \neq 2$) and ln is defined by $\ln(+1) = 0$ and $\ln(-1) = 1$. From the non-degeneracy of the form e_2 and the fact that $\eta_0 \neq \mathcal{O}_C$, it follows that the function

$$\begin{array}{ccc} J_2(C) & \longrightarrow & \{\pm 1\} \\ \eta & \longmapsto & e_2(\eta_0, \eta) \end{array}$$

takes half the times the value $+1$ and half the value -1. This concludes our proof. □

The next proposition is a generalization of [IvS, Lemma 2.4], where Izadi and van Straten prove it for the case of genus four curves. They attribute the idea of the proof to Beauville.

Proposition 2.11 ([CLV, Prop. 4.3]). *If $L \in P^-$ is not a theta-characteristic, then the restriction map*

$$\phi_L \colon H^0(P', \mathcal{O}_{P'}(2\Xi_L^+)) \to H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(2\Xi_L^+))$$

is surjective.

Proof. We start by proving that a general $\alpha \in \Xi_L^+ \subset P'$ satisfies

(a) $(-1)^*(\alpha) = \alpha^{-1} \notin \Xi_L^+$,

(b) $|L \otimes \alpha|$ is a pencil without base points.

The first assertion follows from the fact that Ξ_L^+ is not symmetric. Indeed, Ξ_L^+ is symmetric if and only if L is a theta-characteristic (see [BL, Thm. 11.2.4]), which we have excluded by hypothesis.
The fact that the complete linear series $|L \otimes \alpha|$ is a pencil for a general $\alpha \in \Xi_L^+$ follows from the fact that a generic element $L \in \Xi^+$ has $h^0(L) = 2$.
Now we want to see that the linear system $|L \otimes \alpha|$ has no base points for a general $\alpha \in \Xi_L^+$. Consider the incidence variety

$$\begin{tikzcd}
 & \Xi_L^+ \times \widetilde{C} & \\
 & I = \{(\alpha, p) \mid p \text{ is a base point of } |L \otimes \alpha|\} \arrow[u, hook] \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
\Xi_L^+ & & \widetilde{C}
\end{tikzcd}$$

For every point $q \in \widetilde{C}$, the following injection

$$\begin{array}{ccc} p_2^{-1}(q) & \longrightarrow & V^2 \\ \alpha & \longmapsto & \alpha \otimes L \otimes \mathcal{O}(-q + \sigma q) \end{array}$$

is well-defined since q is a fixed point of $|L \otimes \alpha|$. Therefore, by Theorem 2.5, the fibers of p_2 have dimension at most $g-4$ and hence I has dimension at most $g-3$. Since the dimension of Ξ_L^+ is $g-2$, this implies that the first projection is not dominant and hence the conclusion.
Now we want to find elements in $H^0(P', \mathcal{O}_{P'}(2\Xi_L^+))$ that form a basis when we restrict them to $H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(2\Xi_L^+))$. From Proposition 2.4 and the fact that L is not a theta-characteristic, we get that $h^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(2\Xi_L^+) = h^0(\widetilde{C}, \sigma^* L^{\otimes 2}) = \tilde{g} - 1$. We begin by choosing an $\alpha \in \Xi_L^+ \subset P^-$ satisfying the two conditions above. In particular, from the condition (2), we can choose an effective divisor $L_\alpha = \sum_{1 \leqslant i \leqslant \tilde{g}-1} x_i \in |L \otimes \alpha|$ such that all the points x_i are distinct.[3]
We define the following effective divisors in P^-

$$L_{\alpha,j} := L_\alpha - x_j + \sigma x_j = \sigma x_j + \sum_{i \neq j} x_i \qquad \text{for } j = 1, \dots, \tilde{g} - 1.$$

[3] The multiple points correspond to ramification points of the morphism to $\mathbb{P}^1$ induced by a pencil inside $|L \otimes \alpha|$.

By condition (2), we get that $h^0(P', \mathcal{O}_{P'}(L_{\alpha,j})) = 1$ and therefore, by Proposition 2.4, $\left(\Xi^+_{L_{\alpha,j}}\right)_{|\widetilde{C}} = \sigma L_{\alpha,j}$.

Next, consider the line bundle $L \otimes \alpha^{-1} \otimes \mathcal{O}_{\widetilde{C}}(x_j - \sigma x_j) \in P^-$. Since $h^0(\widetilde{C}, L \otimes \alpha^{-1}) = 0$ by condition (1), using Mumford's parity trick we deduce that

$$h^0(\widetilde{C}, L \otimes \alpha^{-1} \otimes \mathcal{O}_{\widetilde{C}}(x_j - \sigma x_j)) = 1.$$

Define $L'_{\alpha,j}$ to be the unique effective divisor of $\left|L \otimes \alpha^{-1} \otimes \mathcal{O}_{\widetilde{C}}(x_j - \sigma x_j)\right|$. By Proposition 2.4, we get that $\left(\Xi^+_{L'_{\alpha,j}}\right)\Big|_{\widetilde{C}} = \sigma L'_{\alpha,j}$.

Summing up, we have constructed $\tilde{g} - 1$ couples of divisors $(L_{\alpha,j}, L'_{\alpha,j})$ satisfying

$$\begin{cases} \mathcal{O}_{P'}(\Xi^+_{L_{\alpha,j}} + \Xi^+_{L'_{\alpha,j}}) \cong \mathcal{O}_{P'}(2\Xi^+_L), \\ \left(\Xi^+_{L_{\alpha,j}} + \Xi^+_{L'_{\alpha,j}}\right)\Big|_{\widetilde{C}} = \sigma L_{\alpha,j} + \sigma L'_{\alpha,j}. \end{cases}$$

It remains to show that the $\tilde{g} - 1$ divisors $\sigma L_{\alpha,j} + \sigma L'_{\alpha,j}$ corresponds to independent sections on $H^0(\widetilde{C}, 2 \cdot \sigma L)$. This will follow from the next,

Claim: $\sigma x_j \notin \sigma L_{\alpha,j} + \sigma L'_{\alpha,j}$ and $\sigma x_j \in \sigma L_{\alpha,k} + \sigma L'_{\alpha,k}$ for every $k \neq j$.

By the definition of the $L_{\alpha,j}$ and using that σ has no fixed points, we get that $\sigma x_j \notin \sigma L_{\alpha,j}$ while $\sigma x_j \in \sigma L_{\alpha,k}$ for every $k \neq j$. Finally, observe that $\sigma^* \mathcal{O}_{\widetilde{C}}(L'_{\alpha,j}) \otimes \mathcal{O}_{\widetilde{C}}(x_j - \sigma x_j) = \sigma^*(L \otimes \alpha^{-1})$ which by condition (1) has no sections. This can happen only if $\sigma L'_{\alpha,j} - \sigma x_j$ is not effective, or in other words $\sigma x_j \notin \sigma L'_{\alpha,j}$. $\square$

Now we are ready to compute the cohomological support loci for the ideal sheaf $\mathcal{I}_{\widetilde{C}}$ twisted by 2Ξ. In this case is not so clear what should be the canonical cohomological support loci of $\mathcal{I}_{\widetilde{C}}(2\Xi)$.

Proposition 2.12 ([CLV, Thm. 4.2]). *The cohomological support loci for the ideal sheaf $\mathcal{I}_{\widetilde{C}}(2\Xi)$ are*

(a) $V^0(\mathcal{I}_{\widetilde{C}}(2\Xi)) = P$ if $g \geqslant 4$ and $V^0(\mathcal{I}_{\widetilde{C}}(2\Xi))$ is a point $q_0 = \omega_{\widetilde{C}} \otimes \sigma^*(L_0^-)^{-2}$ if $g = 3$.

(b) $V^1(\mathcal{I}_{\widetilde{C}}(2\Xi)) = V^2(\mathcal{I}_{\widetilde{C}}(2\Xi)) = \{q_0\}$.

Proof. Consider the exact sequence defining the ideal sheaf $\mathcal{I}_{\widetilde{C}}$ twisted by the divisor $2\Xi^+_L$ with $L \in P^-$:

$$0 \to \mathcal{I}_{\widetilde{C}}(2\Xi^+_L) \to \mathcal{O}_{P'}(2\Xi^+_L) \to \jmath_* \mathcal{O}_{\widetilde{C}}(2\Xi^+_L) \to 0.$$

By taking cohomology and using the vanishing $H^j(P', \mathcal{O}_{P'}(2\Xi^+_L)) = 0$ for $j > 0$, we get the emptiness of $V^i(\mathcal{I}_{\widetilde{C}}(2\Xi))$ for $i \geqslant 3$ and two exact sequences

$$0 \to H^0(\mathcal{I}_{\widetilde{C}}(2\Xi_L^+)) \to H^0(\mathcal{O}_{P'}(2\Xi_L^+)) \xrightarrow{\phi_L} H^0(\mathcal{O}_{\widetilde{C}}(2\Xi_L^+)) \to H^1(\mathcal{I}_{\widetilde{C}}(2\Xi_L^+)) \to 0,$$
$$0 \to H^1(\mathcal{O}_{\widetilde{C}}(2\Xi_L^+)) \to H^2(\mathcal{I}_{\widetilde{C}}(2\Xi_L^+)) \to 0.$$

By Proposition 2.4, we have that $\mathcal{O}_{\widetilde{C}}(2\Xi_L^+) = \sigma^* L^2$ and therefore the second exact sequence implies that

$$V^2(\mathcal{I}_{\widetilde{C}}(2\Xi)) = \left\{ L \in P \,\middle|\, h^1(\widetilde{C}, \sigma^*(L_0^-)^2 \otimes L) > 0 \right\} = \left\{ L \in P \,\middle|\, \omega_{\widetilde{C}} = \sigma^*(L_0^-)^2 \otimes L) \right\},$$

that is non-empty because $S(\widetilde{C}) \cap P^- \neq \emptyset$ according to Lemma 2.10. Hence we denote $q_0 = \omega_{\widetilde{C}} \otimes \sigma^*(L_0^-)^{-2}$. Moreover, from the Proposition 2.11 and the first exact sequence, we have that $V^1(\mathcal{I}_{\widetilde{C}}(2\Xi)) \subseteq \{q_0\}$.
In order to determine $V^0(\mathcal{I}_{\widetilde{C}}(2\Xi))$, we consider the first above exact sequence. If $g \geqslant 4$, then we have the inequality

$$h^0(P, \mathcal{O}_P(2\Xi)) = 2^{g-1} > 2g - 1 = \tilde{g} \geqslant h^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(2\Xi)),$$

from which we conclude that

$$V^0(\mathcal{I}_{\widetilde{C}}(2\Xi)) = P \text{ if } g \geqslant 4.$$

On the other hand, if $g = 3$ and L is not a theta-characteristic, then the map ϕ_L is a surjection (see Proposition 2.11) between two spaces of the same dimension and therefore an isomorphism. This implies that

$$V^0(\mathcal{I}_{\widetilde{C}}(2\Xi)) \subseteq \left\{ L \in P \,\middle|\, \omega_{\widetilde{C}} = \sigma^*(L_0^-)^2 \otimes L) \right\} \text{ if } g = 3.$$

We deduce that in any case the ideal $\mathcal{I}_{\widetilde{C}}(2\Xi)$ is a GV-sheaf and hence, by Corollary I.1.13 we get that

$$\{q_0\} = V^2(\mathcal{I}_{\widetilde{C}}(2\Xi)) \subseteq V^1(\mathcal{I}_{\widetilde{C}}(2\Xi)) \subseteq V^0(\mathcal{I}_{\widetilde{C}}(2\Xi)),$$

which gives the desired conclusion. $\square$

CHAPTER

IV

A Geometric Schottky Problem

Introduction

There is a classical result in projective geometry, due to Castelnuovo, saying that a finite collection of points in $\mathbb{P}^r$ which is in linearly general position, but in special position with respect to quadrics, is contained in a unique rational normal curve.

Pareschi and Popa [PP4] have discovered an analogy for a g-dimensional principally polarized abelian varieties (A, Θ), where, as for the theta-duality, divisors algebraically equivalent to Θ play the role of hyperplanes, and divisors algebraically equivalent to 2Θ play the role of quadrics. The Castelnuovo result of Pareschi and Popa says that if we have $g+2$ distinct points on A, in a suitable sense general with respect to Θ, but special with respect to 2Θ, then A is the Jacobian of a curve C, and the $g+2$ points are contained in an Abel-Jacobi curve, i.e. a translate of C embedded into its Jacobian. Thus, Abel-Jacobi curves play the role of rational normal curves, and the analogue of Castelnuovo's result contains a Schottky statement. Moreover, as a corollary they also get a Torelli statement.

We extend this Pareschi and Popa result [PP4] to possibly non-reduced subschemes (see Theorem 6.1). We remark that, already the fact that a subscheme Γ as in the theorem is contained in a non-singular curve (i.e. Γ is curvilinear) is not obvious. On the other hand, the converse to the theorem is easy, as we explain in Section 3.1:

A finite degree $g+2$ subscheme Γ of a curve C imposes less than $g+2$ conditions on general 2Θ-translates in the Jacobian.

In the first section of this chapter we introduce some preliminaries on finite schemes fixing our notation and terminology. In Section §2, we study the theta-dual of a finite scheme, giving special attention to the case of the theta-dual of a non-reduced point of degree 2 in an arbitrary principally polarized abelian variety and in the particular case of Jacobian varieties. We also define the theta-general position as an analogue of linearly general position on projective spaces.

In Section §3, we start the study of the dependence locus, i.e. the locus parameterizing the α's such that a fixed finite subscheme Γ does not impose independent conditions on $|2\Theta \otimes \alpha|$. We say that Γ is superabundant when this locus is the whole variety A. We prove that if Γ is superabundant and in theta-general position, it has at least degree $g+2$.

The fourth section is the technical core of this chapter. We show that theta-general superabundant finite schemes Γ of degree $g+2$ are very special. In particular, we prove that they are curvilinear. The way to study them is to establish schematic inclusions among theta-duals and dependence loci of subschemes of Γ. In Section §5 we give another schematic inclusion that yields to the construction of a 1-dimensional family of trisecants.

Finally, in Section §6 we state and prove the above mentioned main theorem of this chapter. Its Schottky part depends on the characterization of Jacobians by degenerate trisecants to the Kummer variety due to Welters [W2]. The Castelnuovo part follows from one of the schematic inclusions proved in Section §4. The Torelli statement is consequence of the previous ones by the original argument of Pareschi and Popa.

Our approach differs from that of Pareschi and Popa by our systematic use of the Fourier-Mukai transform and the control of the schematic structures involved in the proofs.

The results of this chapter of the Thesis has been obtained in collaboration with Martin Gulbrandsen and are still unpublished [GL].

1 Preliminaries of finite schemes

In this chapter the ambient space will be a principally polarized abelian variety (A, Θ) over an algebraically closed field k of characteristic different from 2. The dimension of A will always be denoted by g.

Γ will usually denote a *finite subscheme* in A. Hence, $\Gamma = \operatorname{Spec} R$, where R is an Artin ring. By the structure theorem for Artin rings, $R = \bigoplus R_{\mathfrak{m}}$, where the sum is taken over all the maximal ideals $\mathfrak{m}$ of R. We will denote the *degree of* Γ by $\deg \Gamma$, i.e the dimension of R as a k-vector field. Since R is an Artin ring and k is an algebraically closed field, Γ has a *composition series*, i.e. a series of subschemes

$$\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_d = \Gamma,$$

where Γ_i has degree i (e.g. [Ei, Thm. 2.13]).

Suppose that $(R, \mathfrak{m})$ is a local Artin ring. We will denote by $\operatorname{Soc}(R) = (0 : \mathfrak{m})$ the *socle of* R, i.e. the elements in R annihilated by its maximal ideal $\mathfrak{m}$. We will say that $\operatorname{Spec} R$ is Gorenstein if, and only if, R is Gorenstein, i.e. the socle $\operatorname{Soc}(R)$ is one dimensional as a vector space (e.g. [Ei, Thm. 21.5]).

We will say that an arbitrary finite scheme Γ, not necessarily supported in a unique point, is *Gorenstein* if, and only if, any component of Γ is Gorenstein. In other words, Gorenstein means that the choice of a closed point $b \in \Gamma$ uniquely determines a subscheme $\Gamma' \subset \Gamma$, such that $\deg \Gamma' + 1 = \deg \Gamma$ and $\operatorname{supp} \mathcal{I}_{\Gamma'/\Gamma} = b$.

A finite scheme $\Gamma = \operatorname{Spec} R$ supported at a unique closed point is called *curvilinear* if the maximal ideal of the (necessarily local) ring R is generated by one element; or, equivalently, if its Zariski tangent space has dimension zero or one. (The name comes from the fact that these are exactly the schemes that can be contained in a non-singular curve).

We will say that an arbitrary finite scheme Γ, not necessarily supported in a unique point, is curvilinear if, and only if, any component of Γ is curvilinear.

Eisenbud and Harris prove in [EH, Lemma 1.4], that a local Gorenstein Artin ring $(R, \mathfrak{m})$ gives a curvilinear finite scheme $\Gamma = \operatorname{Spec} R$ if, and only, if $\dim_{k(\mathfrak{m})}(0 : \mathfrak{m}^2) \leqslant 2$. Geometrically, this corresponds to the fact that a finite scheme is curvilinear if, and only if,

- it is Gorenstein, i.e. the choice of a closed point $b \in \Gamma$ uniquely determines a subscheme $\Gamma' \subset \Gamma$, such that $\deg \Gamma' + 1 = \deg \Gamma$ and $\operatorname{supp} \mathcal{I}_{\Gamma'/\Gamma} = b$,
- and, any subscheme Γ' of Γ of codegree 1 is Gorenstein, i.e. the choice of a

closed point (possibly equal to b) $a \in \Gamma'$ uniquely determines another subscheme $\Gamma'' \subset \Gamma'$, such that $\deg \Gamma'' + 1 = \deg \Gamma'$ and $\operatorname{supp} \mathcal{I}_{\Gamma''/\Gamma'} = a$.

We will usually deal with coherent sheaves supported on a finite subscheme. In particular, since the finite subschemes are affine, these sheaves will not feel the twist by a line bundle. Hence, when a sheaf is supported in a finite subscheme, we will systematically omit the eventual twisting line bundle.

1.1 Residual subschemes

First, we want to study some simple properties of finite schemes.

Definition 1.1. Let Γ be a finite scheme and $\Gamma' \subset \Gamma$ a subscheme. The *residual subscheme* of Γ' in Γ is the support

$$\Gamma'' = \operatorname{supp} \mathcal{I}_{\Gamma'/\Gamma}$$

of the ideal of Γ' in Γ. If $\mathcal{I}_{\Gamma'/\Gamma}$ is a principal ideal, then we say that Γ'' is *well-formed*.

Remark 1.2. *When the residual subscheme is well-formed, we may identify $\mathcal{I}_{\Gamma'/\Gamma}$ with the structure sheaf $\mathcal{O}_{\Gamma''}$, so have a short exact sequence*

$$0 \to \mathcal{O}_{\Gamma''} \to \mathcal{O}_{\Gamma} \to \mathcal{O}_{\Gamma'} \to 0.$$

Indeed, suppose that A is the artinian module $\mathcal{O}_\Gamma$ and $I = (f)$ is the principally generated ideal $\mathcal{I}_{\Gamma'/\Gamma}$. Then there is a natural isomorphism $I \cong A/\operatorname{Ann}(I)$, coming form the usual exact sequence

$$0 \to \operatorname{Ann}(I) \to A \xrightarrow{f\cdot} I \to 0.$$

Hence $\mathcal{O}_{\Gamma''} \cong \mathcal{I}_{\Gamma'/\Gamma}$. In particular, there is an equality

$$[\Gamma] = [\Gamma'] + [\Gamma''] \tag{1.1}$$

between the underlying zero-cycles.

Remark 1.3. *(a) If the union $\Gamma' \cup \Gamma''$ denotes the subscheme defined by the product of the corresponding ideals, then it is immediate from the definition of the residual subscheme (not necessarily well-formed) that*

$$\Gamma \subset \Gamma' \cup \Gamma''.$$

(b) In particular, if D is an effective divisor containing Γ', then there is an inclusion of ideals

$$\mathcal{I}_{\Gamma''}(-D) \subset \mathcal{I}_{\Gamma}.$$

That is, if D is an effective divisor containing Γ', then there is a non-trivial section of $H^0(A, \mathcal{I}_{\Gamma'}(D))$ which defines an inclusion $\mathcal{O}_A \subset \mathcal{I}_{\Gamma'}(D)$. Tensoring by $\mathcal{O}_A(-D)$ and multiplying by $\mathcal{I}_{\Gamma''}$ on both sides of the inclusion, we obtain

$$\mathcal{I}_{\Gamma''}(-D) \subset \mathcal{I}_{\Gamma''} \cdot \mathcal{I}_{\Gamma'} \subset \mathcal{I}_{\Gamma}.$$

Example 1.4. If Γ' has degree $\deg \Gamma - 1$, then the ideal $\mathcal{I}_{\Gamma'/\Gamma}$ is isomorphic to the residue field $k(x)$ of the unique closed point x where Γ' and Γ differ. Thus x is the *residual point* of Γ' in Γ, and it is well-formed.

Example 1.5. If Γ is curvilinear, and $\Gamma' \subset \Gamma$ is arbitrary, then the residual subscheme Γ'' is well-formed. In fact, it is uniquely determined by (1.1).

Example 1.6. If $\Gamma = \operatorname{Spec} k[x,y]/(x^2, xy, y^2)$ and Γ' is the origin, then it is clear that the data given does not distinguish any degree 2 subscheme of Γ. Indeed, the residual subscheme is just the origin again, so it is not well-formed.

Lemma 1.7. *If Γ is Gorenstein, and $\Gamma' \subset \Gamma$ has degree $\deg \Gamma - 2$, then the residual subscheme Γ'' of Γ' in Γ is well-formed.*

Proof. If Γ' and Γ differ at two distinct points x and y, then the ideal of Γ' in Γ is just $k(x) \oplus k(y)$ and thus $\Gamma'' = \{x, y\}$.
On the other hand, if Γ'' and Γ differ at a single point x, then locally at x, we have

$$\Gamma = \operatorname{Spec}(R)$$

for a local Artin Gorenstein ring R. The ideal of Γ'' in R is two dimensional as a vector space. Hence it is either a principal ideal, or it is generated by two linearly independent elements of the socle of R. The Gorenstein assumption rules out the latter possibility, so the ideal of Γ'' in R is principal. □

Remark 1.8. *Already in the slightly simple case in Example 1.4, the viewpoint, that the residual point is defined as the support of the relative ideal, has the advantage that the definition works well in families: Thus, if S is an arbitrary base scheme, and $Z' \subset Z$ are two flat and finite schemes over S of degree $d-1$ and d, then the support of the ideal $\mathcal{I}_{Z'/Z}$ defines a section $S \to Z$ of the structure map. We call this the* residual section *of Z' in Z.*

We view the residual subscheme Γ'' as a complement to Γ' in Γ. The following proposition, where Γ'' does not need to be well-formed, fits well with this intuition.

Proposition 1.9. *Let $\Gamma' \subset \Gamma$ be finite subschemes with residual subscheme Γ'', and let D be an effective divisor satisfying $D \cap \Gamma = \Gamma'$. Then, for any other effective divisor E, we have*

$$\Gamma \subset D + E \quad \text{if and only if} \quad \Gamma'' \subset E.$$

Proof. The statement is local, so let I and I' be the ideals of Γ and Γ' in some affine chart. The quotient I'/I is the relative ideal of Γ' in Γ. Let d and e be local equations for D and E. We want to show that $de \in I$ if and only if e annihilates I'/I. The hypothesis $D \cap \Gamma = \Gamma'$ says that

$$(d) + I = I'$$

and so (the image of) d generates I'/I. Thus e annihilates I'/I if and only if the image of ed in I'/I vanishes, which means that $ed \in I$ as claimed. □

1.2 Intermediate subschemes

Let $\Gamma'' \subset \Gamma$ be finite subschemes of degrees d and $d+2$, with well-formed residual subscheme S, of degree 2. If Γ were a disjoint union $\Gamma = \Gamma'' \cup S$, then there is an obvious family of subschemes parameterizing all the degree $d+1$ subschemes of Γ that contain Γ'', namely

$$\Gamma''_S \subset (\Gamma'' \times S) \cup S_1 \subset \Gamma_S$$

where we write Γ_S for the product $\Gamma \times S$ and $S_1 \subset S \times S$ is the diagonal. Such a family exists in general: Denoting by $S_2 \subset S \times S$ the residual section to the diagonal $S_1 \subset S \times S$, we have the following:

Proposition 1.10. *Let $\Gamma'' \subset \Gamma$ be finite schemes of degrees d and $d+2$, and assume the residual subscheme S is well-formed. Then there exists Z an intermediate S-scheme,*

$$\Gamma''_S \subset Z \subset \Gamma_S$$

which is flat and finite of degree $d+1$ over S, and such that S_1 is the residual section of Γ''_S in Z, and S_2 is the residual section of Z in Γ_S.

Proof. Let A be the affine coordinate ring of Γ, and let $I = \mathcal{I}_{\Gamma'/\Gamma} \subset A$ be the ideal corresponding to Γ''. Thus the subscheme $S \subset \Gamma$ corresponds to $\mathcal{I}_{S/\Gamma} = \operatorname{Ann}(I)$, and Γ_S has coordinate ring $\mathcal{O}_{\Gamma_S} = A \otimes_k (A/\operatorname{Ann}(I))$. Let J denote the kernel of the multiplication map

$$I \otimes_k (A/\operatorname{Ann}(I)) \to I.$$

Then the family Z defined by the ideal J has the required properties: Let $B = (A \otimes_k (A/\operatorname{Ann}(I)))/J$ be the coordinate ring of Z. Then there is the following commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & J \arrow[r, equal] \arrow[d] & J \arrow[d] & & \\
0 \arrow[r] & I \otimes A/\operatorname{Ann}(I) \arrow[r] \arrow[d] & A \otimes A/\operatorname{Ann}(I) \arrow[r] \arrow[d] & A/I \otimes A/\operatorname{Ann}(I) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & A/I \otimes A/\operatorname{Ann}(I) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

with exact rows and columns. By well-formedness (see Remark 1.2), $I \cong A/\operatorname{Ann}(I) = \mathcal{O}_S$, hence J can be seen also as the kernel of the multiplication map $\mathcal{O}_S \otimes_k \mathcal{O}_S \to \mathcal{O}_S$, so J is the ideal of the diagonal inside $S \times S$ (or $\Gamma \times S$).
B is a flat $A/\operatorname{Ann}(I)$-module since I is principal, hence flat and $A/I \otimes A/\operatorname{Ann}(I)$ is trivially flat. It follows that B is flat of degree $d+1$ as an $A/\operatorname{Ann}(I)$-module. The rest is straight forward. $\square$

2 Theta-dual of finite schemes and theta-general position

Let Γ be a finite subscheme of a principally polarized abelian variety (A, Θ). We want to study in detail the theta-dual $T(\Gamma)$ (see section II.2 to recall its definition).

Consider the short exact sequence

$$0 \to \mathcal{I}_\Gamma(\Theta) \to \mathcal{O}_A(\Theta) \to \mathcal{O}_\Gamma \to 0.$$

By dimension reasons, $\mathcal{O}_\Gamma$ is IT_0. By Proposition 1.2, the sheaf $\mathcal{O}_A(\Theta)$ is also IT_0 and its Fourier-Mukai transform is the line bundle $\mathcal{O}_A(-\Theta)$. Since $R^0\mathcal{S}(\mathcal{I}_\Gamma(\Theta))$ is a torsion sheaf included in a line bundle it is 0. Hence, $\mathcal{I}_\Gamma(\Theta)$ satisfies WIT_1. Thus the Fourier-Mukai transform gives a short exact sequence

$$0 \to \mathcal{O}_A(-\Theta) \xrightarrow{F_\Gamma} \widehat{\mathcal{O}_\Gamma} \to \widehat{\mathcal{I}_\Gamma(\Theta)} \to 0. \tag{2.1}$$

We recall that $T(\Gamma)$ is the Fitting support $\operatorname{supp} \mathcal{E}xt^1(\widehat{\mathcal{I}_\Gamma(\Theta)}, \mathcal{O}_A)$ (see Definition II.2.1 and Remark II.2.5). By base change, since $\mathcal{O}_\Gamma$ is IT_0, $\widehat{\mathcal{O}_\Gamma}$ is locally free. Hence $T(\Gamma)$ is precisely the locus where $\widehat{\mathcal{I}_\Gamma(\Theta)}$ fails to be locally free, or equivalently, the zero locus of F_Γ. Therefore, for a finite subscheme we have the following equivalent definition of theta-dual.

Definition 2.1. The *theta-dual* $T(\Gamma)$ *of a finite subscheme* $\Gamma \subset A$ is the closed subscheme of A defined by the vanishing of F_Γ in (2.1).

Example 2.2. When $\Gamma = \{p_1, \dots, p_d\}$ is finite collection of distinct points, $T(\Gamma) = \bigcap_{i=1}^d T(p_i) = \bigcap_{i=1}^d \Theta_{p_i}$.

Remark 2.3. *Let Γ a finite scheme of degree d. Since F_Γ in (2.1) is locally a matrix $d \times 1$, and $T(\Gamma)$ is defined as the vanishing locus of F, it is clear that $T(\Gamma)$ is locally defined by d equations. This implies, for example, that if S is a finite scheme of degree 2, $T(S)$ has codimension 2.*
Moreover if, $\Gamma \subset \Gamma'$ is a subscheme of degree $d+1$, then $T(\Gamma') \subseteq T(\Gamma)$ (compare with Lemma II.2.7) and for each reduced and irreducible component W of $T(\Gamma)$, the intersection $T(\Gamma') \cap W$ either equals W or has codimension 1 in W, since it is locally defined by one equation.
In particular if Γ is a finite scheme of degree $d \leqslant g-1$, every component of $T(\Gamma)$ has at least dimension 1, and for $\Gamma \subset \Gamma'$, $T(\Gamma) \setminus T(\Gamma')$ is empty or it has at least dimension 1 too.

Theta-dual of a non-reduced scheme of degree 2. Already the first example of theta-dual of a non-reduced subscheme is interesting.

Example 2.4. Let S be a non-reduced scheme of degree 2 supported at b. Then $T(S)$ is the scheme of zeroes of the corresponding section $s \in H^0(\mathcal{O}_{\Theta_b}(\Theta_b))$. Indeed, $T(S)$ is precisely the locus where $\widehat{\mathcal{I}_S(\Theta)}$ fails to be locally free. From the typical exact sequence $0 \to \mathcal{I}_S(\Theta) \to \mathcal{I}_b(\Theta) \to k(b) \to 0$, we have the following exact sequence:

$$0 \to \widehat{k(b)} \to \widehat{\mathcal{I}_S(\Theta)} \to \widehat{\mathcal{I}_b(\Theta)} \to 0.$$

That is, by Example II.2.6,

$$0 \to \mathcal{P}_b \xrightarrow{\psi} \widehat{\mathcal{I}_S(\Theta)} \to \mathcal{O}_{\Theta_b} \otimes \mathcal{P}_b \to 0. \tag{2.2}$$

Hence, the extension $\widehat{\mathcal{I}_S(\Theta)}$ correspond to a non-zero (by the non-reduceness of S) section of $\mathrm{Ext}^1(\mathcal{O}_{\Theta_b}, \mathcal{O}_A) \cong H^0(\mathcal{E}xt^1(\mathcal{O}_{\Theta_b}, \mathcal{O}_A)) \cong H^0(\mathcal{O}_{\Theta_b}(\Theta_b))$, and the locus where $\widehat{\mathcal{I}_S(\Theta)}$ fails to be locally free correspond to the locus where the section vanishes. We can consider the tangent vector $v \in H^0(T_A) \cong H^1(\mathcal{O}_A) \cong H^0(\mathcal{O}_\Theta(\Theta)) \cong H^0(\mathcal{O}_{\Theta_b}(\Theta_b))$ and the section s is the image of v through this sequence of isomorphisms.

Let us compute precisely $\widehat{\mathcal{I}_S(\Theta)}$. From exact sequence (2.1) and Remark 2.3, observe that

$$\mathrm{codim}_A\, \mathcal{E}xt^i(\widehat{\mathcal{I}_S(\Theta)}, \mathcal{O}_A) \geqslant i+1 \text{ for all } i > 0.$$

By a well-known result (e.g. [PP7, Lem. A.5]), $\widehat{\mathcal{I}_S(\Theta)}$ is a torsion-free sheaf of generic rank 1, hence an ideal sheaf. By Definition 2.1, the locus where $\widehat{\mathcal{I}_S(\Theta)}$ fails to be locally free is exactly $T(S)$, so we have that $\widehat{\mathcal{I}_S(\Theta)} = \mathcal{I}_{T(S)} \otimes \mathcal{L}$, where $\mathcal{L}$ is a line bundle.

By the previous exact sequence (2.2), it is clear that ψ is the multiplication by a section in $H^0(\widehat{\mathcal{I}_S(\Theta)} \otimes \mathcal{P}_b^{-1})$ that vanishes of Θ_b. Hence $\mathcal{L} \otimes \mathcal{P}_b^{-1} \cong \mathcal{O}(\Theta_b)$ and

$$\widehat{\mathcal{I}_S(\Theta)} = \mathcal{I}_{T(S)}(\Theta_{2b}). \tag{2.3}$$

In fact, the exact sequence (2.2) twisted by $\mathcal{P}_b^{-2}$ corresponds to the exact sequence,

$$0 \to \mathcal{I}_{\Theta_b/A}(\Theta) \to \mathcal{I}_{T(S)/A}(\Theta) \to \mathcal{I}_{T(S)/\Theta_b}(\Theta) \to 0. \tag{2.4}$$

Moreover, this result is coherent with the theta-dual of two distinct points $S = \{a, b\}$. In this case the Fourier-Mukai transform of $0 \to \mathcal{I}_S(\Theta) \to \mathcal{O}(\Theta) \to k(a) \oplus k(b) \to 0$, is the Koszul exact sequence,

$$0 \to \mathcal{O}(-\Theta) \to \mathcal{P}_a \oplus \mathcal{P}_b \to \widehat{\mathcal{I}_S(\Theta)} \to 0.$$

So we also have $\widehat{\mathcal{I}_S(\Theta)} = \mathcal{I}_{\Theta_a \cap \Theta_b}(\Theta_{a+b}) = \mathcal{I}_{T(S)}(\Theta_{a+b})$.

When (A, Θ) is the Jacobian of a curve C with the canonical polarization we have an explicit description of the theta-dual of an arbitrary subscheme of degree 2.

Example 2.5. Let (A, Θ) be the Jacobian of a curve C of genus $g = g(C)$. For any two distinct points p, q in $C \hookrightarrow \operatorname{Pic}^1 C$, we have the following scheme-theoretic equality in $\operatorname{Pic}^{g-1}(C)$

$$W_{g-1} \cap (W_{g-1})_{q-p} = (W_{g-2})_{-p} \cup (-W_{g-2})_{q-K},$$

where K is a canonical divisor of C. Recall that we are using $D_a = t_a^* D = D - a$ and we use the same convention for any subvariety in A (compare with [Mu4, pg. 77]). Hence, when $S = \{a, b\}$ is a reduced scheme of degree 2 supported in two distinct points

$$T(\{a, b\}) = (W_{g-2})_\alpha \cup (-W_{g-2})_{-\beta},$$

with α and β depending linearly on a, b and the theta-characteristic chosen to translate the previous inclusion to $\operatorname{Pic}^0 C$.
When S is a non-reduced scheme of degree 2 supported at b and included in C, consider the flat family over C of the schemes $T(\{a, b\})$ varying $a \in C$. For $a \neq b$ we have seen that the fiber over b is $(W_{g-2})_\alpha \cup (-W_{g-2})_\beta$. Then, taking the flat limit we get,

$$T(S) = (W_{g-2})_\beta \cup (-W_{g-2})_{-\beta},$$

where β depends linearly on b and the theta-characteristic chosen to translate the previous inclusion to $\operatorname{Pic}^0 C$.

Theta-general position Given a pair $\Gamma' \subset \Gamma$ of finite subschemes of A, we define $T(\Gamma', \Gamma)$ as the scheme-theoretic closure of $T(\Gamma) \setminus T(\Gamma')$ in $T(\Gamma)$, i.e.

$$T(\Gamma', \Gamma) = \overline{T(\Gamma) \setminus T(\Gamma')}.$$

Thus, the underlying set of $T(\Gamma', \Gamma)$ is the closure of the underlying set of $T(\Gamma') \setminus T(\Gamma)$ which consists of the points $a \in A$ satisfying

$$\Gamma' \subset \Theta_a \not\supset \Gamma.$$

Next we define theta-genericity: Recall that a finite subscheme Γ in $\mathbb{P}^r$ is in linearly general position if every subscheme $\Gamma' \subseteq \Gamma$ of degree $d \leqslant r + 1$ spans a linear subspace of dimension $d - 1$. Equivalently, for any pair of $\Gamma'' \subset \Gamma'$ of subschemes of Γ, such that

$$\deg \Gamma' - 1 = \deg \Gamma'' \leqslant r,$$

there exists a hyperplane containing Γ'' but not Γ'. Phrased in this way, the condition of linear independence can be carried over to (A, Θ), with theta-translates replacing hyperplanes.

Definition 2.6. A finite subscheme Γ is *theta-general* if, for every pair $\Gamma'' \subset \Gamma'$ of subschemes of Γ satisfying

$$\deg \Gamma' - 1 = \deg \Gamma'' \leqslant g,$$

there exists a theta-translate containing Γ'' but not Γ'.

Note that the condition on $\Gamma'' \subset \Gamma'$ is just that $T(\Gamma'', \Gamma')$ is non-empty.

3 Superabundance and dependence loci

Let $D \subset A$ be an ample divisor. In later sections, D will be taken to be 2Θ. We are concerned with the *number of independent conditions* imposed by a finite subscheme Γ on the linear system D. By this we mean the codimension of $H^0(A, \mathcal{I}_\Gamma(D))$ in $H^0(A, \mathcal{O}_A(D))$. The expected number of conditions imposed is the degree of Γ. Since D is ample, its higher cohomology spaces vanish, so there is an exact sequence

$$0 \to H^0(A, \mathcal{I}_\Gamma(D)) \to H^0(A, \mathcal{O}_A(D)) \to H^0(A, \mathcal{O}_\Gamma) \to H^1(A, \mathcal{I}_\Gamma(D)) \to 0,$$

which shows that the $H^1(A, \mathcal{I}_\Gamma(D))$ measures the failure of Γ to impose $\deg \Gamma$ independent conditions.

We will in fact study the number of independent conditions imposed by Γ on all the linear systems associated to $H^0(A, \mathcal{O}_A(D) \otimes \alpha)$ for $\alpha \in \widehat{A}$. Note that, if $\alpha = \phi_D(a)$, then this is the translated linear system $|t_a^* D|$.

Definition 3.1. The *superabundance of a finite subscheme* $\Gamma \subset A$ with respect to D is the value

$$\omega(\Gamma, D) = \dim H^1(A, \mathcal{I}_\Gamma(D) \otimes \alpha)$$

for general $\alpha \in \widehat{A}$. Equivalently, it is the minimal value of the right-hand side, over $\alpha \in \widehat{A}$. The subscheme Γ is *superabundant* if its superabundance is non-zero.

It is also useful to study the locus of points $\alpha \in \widehat{A}$ such that Γ does not impose independent conditions on $H^0(A, \mathcal{O}_A(D) \otimes \alpha)$.

Definition 3.2. The *dependence locus* $\Delta(\Gamma, D)$ is the support of

$$R^1\mathcal{S}(\mathcal{I}_\Gamma(D)).$$

Remark 3.3. *Note that $R^i\mathcal{S}(\mathcal{I}_\Gamma(D))$ vanishes for all $i > 1$. Hence, by base change, the fiber of $R^1\mathcal{S}(\mathcal{I}_\Gamma(D))$ at a is*

$$H^1(A, \mathcal{I}_\Gamma(D) \otimes \phi_\Theta(a))$$

which is non-zero precisely when Γ fails to impose independent conditions on $t_a^{-1}D$.

Remark 3.4. *Using the relative Fourier-Mukai transform over a base S, we can replace Γ by a family $Z \subset A \times S$, flat and finite over S, and define the relative dependence locus $\Delta_S(Z, D)$ as the support of $R^1\mathcal{S}_S(\mathcal{I}_Z(p_1^* D))$. By the Base Change Lemma II.1.3, the fiber of $\Delta_S(Z, D)$ over a point $s \in S$ is just the dependence locus $\Delta(Z_s, D)$ of the fiber Z_s.*

Consider a pair of finite subschemes $\Gamma' \subset \Gamma$ of degrees d' and d. Hence, we have an exact sequence

$$0 \to \mathcal{I}_\Gamma(D) \to \mathcal{I}_{\Gamma'}(D) \to \mathcal{I}_{\Gamma'/\Gamma} \to 0, \tag{3.1}$$

where the rightmost sheaf has finite support, and hence doesn't feel the twist by D. Then, twisting by α and taking cohomologies we have that,

$$h^0(\mathcal{I}_{\Gamma'}(D) \otimes \alpha) - h^0(\mathcal{I}_{\Gamma}(D) \otimes \alpha) = (d - d') + h^1(\mathcal{I}_{\Gamma'}(D) \otimes \alpha) - h^1(\mathcal{I}_{\Gamma}(D) \otimes \alpha)$$

Therefore, if Γ fails to impose d independent conditions on $H^0(A, \mathcal{O}_A(D) \otimes \alpha)$ (i.e. $h^1(\mathcal{I}_\Gamma(D) \otimes \alpha) > 0$), then either already Γ' fails to impose d' independent conditions, or we have

$$h^0(\mathcal{I}_{\Gamma'}(D) \otimes \alpha) - h^0(\mathcal{I}_{\Gamma}(D) \otimes \alpha) < d - d'. \tag{3.2}$$

We formalize this observation in the schematic setting as follows. Apply the Fourier-Mukai transform to the exact sequence (3.1). We obtain a right exact sequence

$$\widehat{\mathcal{I}_{\Gamma'/\Gamma}} \xrightarrow{\phi} R^1\mathcal{S}(\mathcal{I}_\Gamma(D)) \to R^1\mathcal{S}(\mathcal{I}_{\Gamma'}(D)) \to 0. \tag{3.3}$$

Then, we make the following schematic definition of the locus of the α that fulfills condition (3.2).

Definition 3.5. Given a pair $\Gamma' \subset \Gamma$ of finite subschemes, their *mutual dependence locus* is the closed subscheme

$$\Delta(\Gamma', \Gamma, D) = \operatorname{supp}\left(\ker\left(R^1\mathcal{S}(\mathcal{I}_\Gamma(D)) \twoheadrightarrow R^1\mathcal{S}(\mathcal{I}_{\Gamma'}(D))\right)\right) \subseteq A,$$

or $\operatorname{supp}(\operatorname{Im}(\phi))$ where ϕ is the map in (3.3).

Remark 3.6. *Using the base change theorem in cohomology, one can check that as a topological space, the mutual dependence locus $\Delta(\Gamma', \Gamma, D)$ is the closure of $\phi_\Theta^{-1}\left(\left\{\alpha \in \widehat{A} \,\middle|\, \alpha \text{ satisfies } (3.2)\right\}\right)$.*

Remark 3.7. *Observe that $\Delta(\emptyset, \Gamma, D)$ is just the dependence locus $\Delta(\Gamma, D)$. If $\Gamma' \subset \Gamma$, then we have $R^1\mathcal{S}(\mathcal{I}_\Gamma(D)) \twoheadrightarrow R^1\mathcal{S}(\mathcal{I}_{\Gamma'}(D))$ (see (3.3)), so*

$$\overline{\Delta(\Gamma, D) \setminus \Delta(\Gamma', D)} \subseteq \Delta(\Gamma', \Gamma, D).$$

Lemma 3.8. *Let $\Gamma'' \subset \Gamma' \subset \Gamma$ be a triple of finite subschemes. Then we have*

$$\Delta(\Gamma'', \Gamma, D) \subseteq \Delta(\Gamma'', \Gamma', D) \cup \Delta(\Gamma', \Gamma, D).$$

In particular, in the case $\Gamma'' = \emptyset$, we have

$$\Delta(\Gamma, D) \subseteq \Delta(\Gamma', D) \cup \Delta(\Gamma', \Gamma, D).$$

Proof. Apply the Fourier-Mukai transform to the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \mathcal{I}_{\Gamma'/\Gamma} \arrow[d] & \\
0 \arrow[r] & \mathcal{I}_{\Gamma}(D) \arrow[r] \arrow[d] & \mathcal{I}_{\Gamma''}(D) \arrow[r] \arrow[d, equal] & \mathcal{I}_{\Gamma''/\Gamma} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_{\Gamma'}(D) \arrow[r] \arrow[d] & \mathcal{I}_{\Gamma''}(D) \arrow[r] & \mathcal{I}_{\Gamma''/\Gamma'} \arrow[r] \arrow[d] & 0 \\
 & \mathcal{I}_{\Gamma'/\Gamma} \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

to obtain,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \ker \nu_1 \arrow[r, equal] \arrow[d] & \ker \nu_1 \arrow[d] & & \\
0 \arrow[r] & \ker \nu_2 \arrow[r] \arrow[d] & R^1\mathcal{S}(\mathcal{I}_{\Gamma}(D)) \arrow[r, "\nu_2"] \arrow[d, "\nu_1"] & R^1\mathcal{S}(\mathcal{I}_{\Gamma''}(D)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker \nu_3 \arrow[r] \arrow[d] & R^1\mathcal{S}(\mathcal{I}_{\Gamma'}(D)) \arrow[r, "\nu_3"] \arrow[d] & R^1\mathcal{S}(\mathcal{I}_{\Gamma''}(D)) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where the left vertical short exact sequence relates the three sheaves whose support define $\Delta(-,-,D)$. □

Lemma 3.9. *Let $\Gamma \subset A$ be a finite subscheme, and let $\Gamma(a)$ denote the component of Γ supported in a. Then we have an inclusion of schemes*

$$\Delta(\Gamma, D) \subseteq \bigcup_{a \in \Gamma} \Delta(\Gamma \setminus \Gamma(a), \Gamma, D),$$

where the union is taken in the sense of product of ideals. Moreover, this inclusion is an equality of the underlying sets.

Proof. The structure sheaf of Γ has a direct sum decomposition

$$\mathcal{O}_\Gamma = \bigoplus_a \mathcal{O}_{\Gamma(a)}$$

and the relative ideal sheaf $\mathcal{I}_{\Gamma\setminus\Gamma(a)/\Gamma}$ can be identified with the summand $\mathcal{O}_{\Gamma(a)}$. Apply the Fourier-Mukai transform to the short exact sequence

$$0 \to \mathcal{I}_\Gamma(D) \to \mathcal{O}(D) \to \mathcal{O}_\Gamma \to 0$$

to obtain the surjective boundary map

$$\begin{tikzcd}
\widehat{\mathcal{O}_\Gamma} \arrow[r] \arrow[d, equal] & R^1\mathcal{S}(\mathcal{I}_\Gamma(D)) \arrow[r] & 0 \\
\bigoplus_a \widehat{\mathcal{O}_{\Gamma(a)}} \arrow[ur, "\sum_a \psi_a"'] & &
\end{tikzcd}$$

with a decomposition as shown. Now $\Delta(\Gamma, D)$ is the Fitting support of $R^1\mathcal{S}(\mathcal{I}_\Gamma(D))$, whereas $\Delta(\Gamma \setminus \Gamma(a), \Gamma, D)$ is the support of $\mathrm{Im}\,(\psi_a)$. The claim follows. □

3.1 Theta-general superabundant subschemes of minimal degree

From here on, we fix $D = 2\Theta$ and drop any reference to D from the notations. Thus we let $\Delta(\Gamma) = \Delta(\Gamma, 2\Theta)$, $\omega(\Gamma) = \omega(\Gamma, 2\Theta)$ etc.

It is to be expected that the superabundance $\omega(\Gamma)$ vanishes as long as Γ has small degree. We begin by establishing that the minimal degree of a theta-general superabundant subscheme is $g + 2$.

Proposition 3.10. *Let $\Gamma \subset A$ be a theta-general finite subscheme of degree at most $g + 1$. Then $\omega(\Gamma) = 0$.*

Proof. Choose a composition series of Γ, i.e. a series of subschemes

$$\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_d = \Gamma,$$

where Γ_i has degree i. By theta-genericity there exists, for each i, a theta-translate Θ_a containing Γ_{i-1} but not Γ_i. Thus, for general $b \in \widehat{A}$, the divisor $\Theta_a + \Theta_b$ also contains Γ_{i-1}, but not Γ_i. This shows that for general x $(= a + b)$, the inclusion

$$H^0(A, \mathcal{I}_{\Gamma_i}(2\Theta) \otimes \mathcal{P}_x) \subset H^0(A, \mathcal{I}_{\Gamma_{i-1}}(2\Theta) \otimes \mathcal{P}_x)$$

is strict, and so $H^0(A, \mathcal{I}_\Gamma(2\Theta) \otimes \mathcal{P}_x)$ indeed has codimension d in $H^0(A, \mathcal{O}_A(2\Theta) \otimes \mathcal{P}_x)$. $\square$

Remark 3.11. *The above bound is sharp: On a Jacobian A there exist superabundant subschemes of degree $g+2$. In fact, by Riemann-Roch, an Abel-Jacobi curve $C \subset A$ imposes $g+1$ independent conditions on $H^0(A, \mathcal{O}_A(2\Theta) \otimes \mathcal{P}_x)$ for any x. Hence a finite subscheme Γ of C, no matter how big, cannot impose more than $g+1$ conditions. See Pareschi-Popa [PP4, Example 3.7] for a more precise statement. Our main Theorem 6.1 below says that subschemes of Abel-Jacobi curves are the only (theta-general) examples of superabundant subschemes of degree $g+2$.*

Corollary 3.12. *Let $\Gamma \subset A$ be a theta-general, superabundant finite subscheme of degree $g+2$, and let $\Gamma' \subset \Gamma$ have degree $g+1$. Then any theta-translate containing Γ' also contains Γ, i.e. $T(\Gamma', \Gamma) = \emptyset$.*

Proof. By the same argument as in the proof of the proposition, the existence of a theta-translate Θ_a containing Γ' but not Γ, would imply that Γ imposes one more condition on general 2Θ-translates than Γ', hence Γ could not be superabundant. $\square$

Corollary 3.13. *Let $\Gamma \subset A$ be a theta-general, superabundant finite subscheme of degree $g+2$. Then Γ is Gorenstein, i.e. each component of Γ is the spectrum of a Gorenstein ring.*

Proof. Let $\Gamma_0 \subset \Gamma$ be a component, so $\Gamma_0 = \operatorname{Spec} R$ for local Artin ring R. We need to show that the socle $\operatorname{Soc}(R)$, i.e. the elements in R annihilated by its maximal ideal, is one dimensional as a vector space. For contradiction, assume $f, g \in \operatorname{Soc}(R)$ are linearly independent elements. Noting that the ideal generated by any collection of socle elements coincides with the vector space they span, the ideals (f, g) and (f) in R determine subschemes

$$\Gamma'' \subset \Gamma'$$

in Γ, of degree g and $g+1$, respectively (precisely, Γ' is the union of $\Gamma \setminus \Gamma_0$ with the subscheme of Γ_0 defined by (f), and similarly for Γ''). By theta-genericity, there exists a theta-translate Θ_a containing Γ'' but not Γ'. Let $\vartheta \in R$ be a local equation for Θ_a. Then ϑ is a socle element, since $\vartheta \in (f, g)$, and hence defines a subscheme $Z \subset \Gamma$ of degree $g+1$. But then Z is contained in Θ_a, and Γ is not, contradicting the previous corollary. $\square$

4 Dependence loci and theta-duality

In the previous section we have seen that a theta-general finite subscheme of degree at most $g+1$ is not superabundant (Proposition 3.10) and that this bound is sharp, because any theta-general subscheme Γ of degree $\geqslant g+2$ inside an Abel-Jacobi curve is superabundant (see Remark 3.11). To prove that any theta-general, superabundant finite subscheme of degree $g+2$ has to be inside an Abel-Jacobi curve we will have to prove strong relations among theta-duals $T(\Gamma')$ and dependence loci $\Delta(\Gamma'')$ of finite subschemes $\Gamma', \Gamma'' \subseteq \Gamma$. This section contains the technical core of the chapter.

The easiest relation comes from the following lemma.

Lemma 4.1. *Let $\Gamma' \subset \Gamma$ be finite subschemes of A of such that $\deg \Gamma' = \deg \Gamma - 1$, and let a denote the residual point. Then we have an inclusion of schemes*

$$\Delta(\Gamma', \Gamma) \subseteq \Theta_{a-y}$$

for all closed point $y \in T(\Gamma') \setminus T(\Gamma)$.

Proof. Since Θ_y contains Γ', but not Γ, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_\Gamma \arrow[r] & \mathcal{I}_{\Gamma'} \arrow[r] & k(a) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{I}_a(-\Theta_y) \arrow[r] \arrow[u, hook] & \mathcal{O}(-\Theta_y) \arrow[r] \arrow[u, hook] & k(a) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

with exact rows (where the middle vertical inclusion follows from Remark 1.3*(b)*). Twist with 2Θ, use that $2\Theta - \Theta_y$ is linearly equivalent to Θ_{-y}, and apply the Fourier-Mukai transform to arrive at the commutative diagram

$$\begin{tikzcd}
\mathcal{P}_a \arrow[r, "\phi"] & R^1\mathcal{S}(\mathcal{I}_\Gamma(2\Theta)) \arrow[r] & R^1\mathcal{S}(\mathcal{I}_{\Gamma'}(2\Theta)) \arrow[r] & 0 \\
\mathcal{P}_a \arrow[r] \arrow[u, equal] & \mathcal{F} \arrow[rr] \arrow[u] & & 0
\end{tikzcd}$$

with exact rows, and where $\mathcal{F}$ is the Fourier-Mukai transform of the WIT_1 sheaf $\mathcal{I}_a(\Theta_{-y})$. By Example II.2.6, we have that $\mathcal{F} \cong \mathcal{P}_a|_{\Theta_{a-y}}$. By definition, $\Delta(\Gamma', \Gamma)$ is the (Fitting) support of $\operatorname{Im}\phi$. Since the image of ϕ is a quotient of $\mathcal{F}$, it follows that the support of ϕ is contained in the support of $\mathcal{F}$, which gives the claim. $\square$

A first consequence is the following lemma that bounds the codimension of $\Delta(\Gamma)$ for a theta-general Γ of degree at most g.

Lemma 4.2 (Compare with [PP4, Lem. 3.12]). *Let Γ be a theta-general finite subscheme of A of degree at most g. Then $\Delta(\Gamma)$ has codimension at least 2.*

Proof. We will prove it by induction on the degree $d > 0$ of Γ. Let $\Gamma' \subset \Gamma$ be subscheme of degree $d-1$. Its theta-dual $T(\Gamma')$ is locally defined by $d-1$ equations, hence has positive dimension everywhere. The inclusion $T(\Gamma) \subset T(\Gamma')$ is strict by theta-genericity, so $T(\Gamma') \setminus T(\Gamma)$ has positive dimension. By Lemma 4.1, it follows that $\Delta(\Gamma', \Gamma)$ has codimension at least 2. Then, we apply induction using inclusion $\Delta(\Gamma) \subseteq \Delta(\Gamma') \cup \Delta(\Gamma', \Gamma)$ from Lemma 3.8. The first step of the inclusion follows by definition, since $\Delta(\{a\}) = \Delta(\emptyset, \{a\})$. $\square$

4.1 Curvilinearity

Any theta-general, superabundant finite subscheme of degree $g + 2$ is curvilinear which is a consequence of the following lemma. Eventhough, the Lemma gives more information than that.

Lemma 4.3. *Assume Γ is a theta-general and superabundant finite subscheme of A of degree $g + 2$. Let $\Gamma_g \subset \Gamma_{g+1} \subset \Gamma$ be any finite subschemes of Γ of degrees indicated by the subscripts. Then the following hold.*

(a) There exists a unique theta-translate Θ_x containing Γ_g but not Γ.

(b) The dependence locus $\Delta(\Gamma_{g+1})$ is set-theoretically Θ_{b-x}, where x is as above and b is the residual point of $\Gamma_g \subset \Gamma_{g+1}$. Moreover, the schematic divisorial part of $\Delta(\Gamma_{g+1})$ is Θ_{b-x}.

Remark 4.4. *By Corollary 3.12, the theta-translate Θ_x in part (a) cannot contain Γ_{g+1}. Thus Θ_x is also the unique theta-translate containing Γ_g but not Γ_{g+1}.*

First, we prove the announced consequences.

Corollary 4.5. *Let $\Gamma \subset A$ be a theta-general, superabundant finite subscheme of degree $g + 2$. Then Γ is curvilinear.*

Proof. By Corollary 3.13 Γ is Gorenstein, so it suffices to show that every subscheme $\Gamma_{g+1} \subset \Gamma$ of degree $g+1$ is also Gorenstein. In other words, Gorenstein means that the choice of a closed point $b \in \Gamma$ uniquely determines a subscheme $\Gamma_{g+1} \subset \Gamma$ with residual point b. If also the choice of a closed residual point $a \in \Gamma_{g+1}$ uniquely determines $\Gamma_g \subset \Gamma_{g+1}$, then Γ is curvilinear [EH, Lemma 1.4].
Thus we suppose that Γ_g^1 and Γ_g^2 are two subschemes of Γ_{g+1} of degree g with residual point b. By the first part of the Lemma, there are unique points x_1 and x_2 such that Γ_g^i is contained in Θ_{x_i}, but Γ_{g+1} is not. By the second part of the lemma, we have set-theoretically

$$\Delta(\Gamma_{g+1}) = \Theta_{b-x_1} = \Theta_{b-x_2}$$

and so $x_1 = x_2$. Call this point x. Then Θ_x contains both Γ_g^1 and Γ_g^2, but not Γ_{g+1}, which is impossible unless $\Gamma_g^1 = \Gamma_g^2$. $\square$

The proof of Lemma 4.3 takes up the rest of this subsection. We will begin by establishing the following relative statement. By Corollary 3.13 Γ is Gorenstein, so by Lemma 1.7, the residual scheme S' of $\Gamma_g \subset \Gamma$ is well-formed. This allows us to consider Z the intermediate scheme between $\Gamma_g \times S'$ and $\Gamma \times S'$ constructed in Proposition 1.10.

Lemma 4.6. *Let Γ be as in Lemma 4.3. Consider $\Gamma_g \subset \Gamma_{g+1} \subset \Gamma$ any subschemes of degrees indicated by the subscripts and Θ_x a theta-translate containing Γ_g, but not Γ. Then, we have the following schematic inclusions,*

$$m_{S'}^{-1}(\Theta_{-x}) \subseteq \Delta_{S'}(Z) \subseteq m_{S'}^{-1}(\Theta_{-x}) \cup p_1^{-1}\Delta(\Gamma_g),$$

where S' is the residual scheme of Γ_g in Γ, $m_{S'}\colon A \times S' \to A$ is the restricted group law, $p_1\colon A \times S' \to A$ the first projection and Z is the intermediate scheme between $\Gamma_g \times S'$ and $\Gamma \times S'$ constructed in Proposition 1.10. The union on the right-hand side is scheme-theoretically defined by taking the product of the corresponding ideals.

Proof. As before, we let $S_1 \subset S' \times S'$ be the diagonal, with residual section S_2, all considered as subschemes of $A_{S'} = A \times S'$. Since Γ is Gorenstein (see Corollary 3.13), S' is well-formed by Lemma 1.7 and S_1 are also well-formed (see Example 1.4).

So we have the following short exact sequence (see Proposition 1.10)

$$0 \to \mathcal{I}_{\Gamma\times S'} \to \mathcal{I}_Z \to \mathcal{O}_{S_2} \to 0.$$

By Corollary 3.12, we have $\Gamma \cap \Theta_x = \Gamma_g$. So by Remark 1.3*(b)*, we have the following inclusion,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}_{\Gamma\times S'} \arrow[r] & \mathcal{I}_Z \arrow[r] & \mathcal{O}_{S_2} \arrow[r] & 0 \\ & & \mathcal{I}_{S_1}(-p_1^*\Theta_x) \arrow[u, hook] & & \end{tikzcd} \tag{4.1}$$

and the composition $\mathcal{I}_{S_1}(-p_1^*\Theta_x) \hookrightarrow \mathcal{I}_Z \to \mathcal{O}_{S_2}$ is surjective, otherwise it would be 0 and $\mathcal{I}_{S_1}(-p_1^*\Theta_x)$ would be included in $\mathcal{I}_{\Gamma\times S'}$. This contradicts Corollary 3.12, since it implies that $Z \subset p_1^{-1}\Theta_x \cap \Gamma \times S$. One deduces that there is an exact commutative diagram

$$
\begin{tikzcd}
 & 0 & 0 & & \\
 & p_1^*\mathcal{I}_{\Gamma_g/\Theta_x} \arrow[u] \arrow[r, equal] & p_1^*\mathcal{I}_{\Gamma_g/\Theta_x} \arrow[u] & & \\
0 \arrow[r] & \mathcal{I}_{\Gamma_{S'}} \arrow[u] \arrow[r] & \mathcal{I}_Z \arrow[u] \arrow[r] & \mathcal{O}_{S_2} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{I}_{S'\times S'}(-p_1^*\Theta_x) \arrow[u] \arrow[r] & \mathcal{I}_{S_1}(-p_1^*\Theta_x) \arrow[u] \arrow[r] & \mathcal{O}_{S_2} \arrow[u, equal] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}
$$

Now twist this last diagram by 2Θ, note that $2\Theta - \Theta_x$ is linearly equivalent to Θ_{-x}, and apply the Fourier-Mukai transform. This gives the following exact commutative diagram

$$
\begin{tikzcd}
 & & 0 & 0 & \\
 & & p_1^*(R^1\mathcal{S}(\mathcal{I}_{\Gamma_g/\Theta_x}(2\Theta))) \arrow[u] \arrow[r, equal] & p_1^*(R^1\mathcal{S}(\mathcal{I}_{\Gamma_g/\Theta_x}(2\Theta))) \arrow[u] & \\
 & \widehat{\mathcal{O}_{S_2}} \arrow[r, "\tau"] & p_1^*(R^1\mathcal{S}(\mathcal{I}_{\Gamma}(2\Theta))) \arrow[u] \arrow[r, "\rho"] & R^1\mathcal{S}_{S'}(\mathcal{I}_Z(2p_1^*\Theta)) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \widehat{\mathcal{O}_{S_2}} \arrow[u, equal] \arrow[r, "\sigma"] & p_1^*(\widehat{\mathcal{I}_{S'}(\Theta_{-x})}) \arrow[u, "\mu"] \arrow[r] & \mathcal{L}|_{m_{S'}^{-1}(\Theta_{-x})} \arrow[u, "\nu"] \arrow[r] & 0
\end{tikzcd}
\tag{4.2}
$$

where $\mathcal{L}|_{m_{S'}^{-1}(\Theta_{-x})}$ is the Fourier-Mukai transform of $\mathcal{I}_{S_1}(p_1^*\Theta_{-x})$ by Lemma II.2.8. The Fitting supports of the two sheaves occurring as the domain and codomain of ρ are the dependence loci $\Delta_{S'}(\Gamma_{S'})$ and $\Delta_{S'}(Z)$. The first of these equals all of $A_{S'}$, as Γ is superabundant, and the latter is the locus we want to understand.

Since $\widehat{\mathcal{O}_{S_2}}$ is a torsion-free sheaf of rank one (in fact, it is an invertible sheaf) and the map τ is generically surjective, we have that τ is injective. By Example 2.4 and Lemma II.1.1, $\widehat{\mathcal{I}_{S'}(\Theta_{-x})}$ is $\mathcal{I}_{T(S')_{-x}}(\Theta_r)$ for some r. In particular, it is torsion-free of (generic) rank 1, so the same argument as for τ, shows that μ is injective and chasing through the diagram we have that also ν is injective.

Now, the right columns yields to

$$m_{S'}^{-1}(\Theta_{-x}) \subseteq \Delta_{S'}(Z) \subseteq m_{S'}^{-1}(\Theta_{-x}) \cup \operatorname{supp} p_1^* R^1\mathcal{S}(\mathcal{I}_{\Gamma_g/\Theta_x}(2\Theta)).$$

By Corollary 3.12, we have $\Gamma \cap \Theta_x = \Gamma_g$. That is

$$0 \to \mathcal{O}_A(-\Theta_x) \to \mathcal{I}_{\Gamma_g} \to \mathcal{I}_{\Gamma_g/\Theta_x} \to 0.$$

Twisting this exact sequence by 2Θ and applying the Fourier-Mukai transform over S', we obtain

$$R^1\mathcal{S}(\mathcal{I}_{\Gamma_g}(2\Theta)) \cong R^1\mathcal{S}(\mathcal{I}_{\Gamma_g/\Theta_{-x}}(2\Theta)),$$

since $R^1\mathcal{S}(\mathcal{O}_A(\Theta_x)) = 0$. Thus the second inclusion is also proved. □

Proof of part (a) of Lemma 4.3. Let Θ_x be a theta-translate containing Γ_g but not Γ, and choose any intermediate subscheme Γ_{g+1}. The Lemma says in particular that we have inclusions

$$\Theta_{b-x} \subseteq \Delta(\Gamma_{g+1}) \subseteq \Theta_{b-x} \cup \Delta(\Gamma_g), \tag{4.3}$$

where b is the residual point of Γ_g in Γ_{g+1}. Since $\Delta(\Gamma_g)$ has codimension at least 2 in A (see Lemma 4.2), this says that $\Delta(\Gamma_{g+1})$ consists of the divisor Θ_{b-x} possibly together with components of higher codimension. In particular, we can recover x from the given data Γ and it is unique. □

Having established part *(a)* of Lemma 4.3, and in view of Remark 4.4, we know that $T(\Gamma_g, \Gamma_{g+1})$ contains a unique closed point x.

Lemma 4.7. *Let Γ be as in Lemma 4.3, let*

$$\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma$$

be a composition series of Γ, and let x be the unique closed point in $T(\Gamma_g, \Gamma_{g+1})$. Then x is in $T(\Gamma_i, \Gamma_{i+1})$ for all $i \leqslant g$.

Proof. For each reduced and irreducible component W of $T(\Gamma_i)$, the intersection $T(\Gamma_{i+1}) \cap W$ either equals W or has codimension 1 in W (see Remark 2.3). The closure of $T(\Gamma_i) \setminus T(\Gamma_{i+1})$ is the union of those components W of $T(\Gamma_i)$ that are not completely contained in $T(\Gamma_{i+1})$. But since set-theoretically $T(\Gamma_g, \Gamma_{g+1})$ is a point x, i.e. it has codimension g in A, it follows that there exists a component $W_i \subset T(\Gamma_i)$ for each i, such that

$$W_1 \supset W_2 \supset \cdots \supset W_g = \{x\}$$

and such that W_{i+1} has codimension 1 in W_i. Thus W_i is also a component of the closure of $T(\Gamma_i) \setminus T(\Gamma_{i+1})$, and it does contain x. □

Proof of part (b) of Lemma 4.3. For each point p in the support of Γ_{g+1}, let $\Gamma_{g+1}(p)$ denote the component supported at p. By Lemma 3.9, we have

$$\Delta(\Gamma_{g+1}) \subseteq \bigcup_{p\in\Gamma_{g+1}} \Delta(\Gamma_{g+1} \setminus \Gamma_{g+1}(p), \Gamma_{g+1}). \tag{4.4}$$

We claim that each $\Delta(\Gamma_{g+1} \setminus \Gamma_{g+1}(p), \Gamma_{g+1})$ is a theta-translate. Let d be the degree of $\Gamma_{g+1}(p)$. Choose a composition series that starts (from above) by removing the component $\Gamma_{g+1}(p)$, i.e.

$$\Gamma_1 \subset \cdots \subset \Gamma_g \subset \Gamma_{g+1},$$

where $\Gamma_1 = \Gamma_{g+1} \setminus \Gamma_{g+1}(p)$ and such that the residual point of Γ_j in Γ_{j+1} is p for $j > g - d$. By Lemma 3.8, we have

$$\Delta(\Gamma_{g+1} \setminus \Gamma_{g+1}(p), \Gamma_{g+1}) \subseteq \bigcup_{j>g-d} \Delta(\Gamma_j, \Gamma_{j+1})$$

and by Lemma 4.1,

$$\Delta(\Gamma_j, \Gamma_{j+1}) \subseteq \Theta_{p-y} \qquad \text{for every closed point } y \in T(\Gamma_j) \setminus T(\Gamma_{j+1}).$$

Since x is in the closure of all the $T(\Gamma_j) \setminus T(\Gamma_{j+1})$, we conclude that we have the following set-theoretical inclusion

$$\Delta(\Gamma_{g+1} \setminus \Gamma_{g+1}(p), \Gamma_{g+1}) \subseteq \Theta_{p-x}.$$

Now we have inclusions

$$\begin{aligned} \Theta_{b-x} \subseteq \Delta(\Gamma_{g+1}) &\subseteq \Delta(\Gamma_{g+1} \setminus \Gamma_{g+1}(p)) \cup \Delta(\Gamma_{g+1} \setminus \Gamma_{g+1}(p), \Gamma_{g+1}) \\ &\subseteq \Delta(\Gamma_{g+1} \setminus \Gamma_{g+1}(p)) \cup \Theta_{p-x} \end{aligned}$$

and all components of the dependence locus $\Delta(\Gamma_{g+1} \setminus \Gamma_{g+1}(p))$ have codimension greater than 1. It follows that the reduced structure of $\Delta(\Gamma_{g+1} \setminus \Gamma_{g+1}(p), \Gamma_{g+1})$ is the theta-translate Θ_{b-x}. As this holds for all p, we see by (4.4) that the reduced structure of $\Delta(\Gamma_{g+1})$ is contained in Θ_{b-x}. And by (4.3) it is clear that the schematic divisorial part of $\Delta(\Gamma_{g+1})$ is Θ_{b-x} since all the components of $\Delta(\Gamma_g)$ have codimension greater than 1. $\square$

4.2 Sums of theta-duals and dependence loci

The following Lemma will be the key step to proof the Castelnuovo statement in Theorem 6.1.

Lemma 4.8. *Assume that $\Gamma \subset A$ is theta-general and superabundant finite subscheme of degree $g+2$. Let $\Sigma_k \subset \Gamma_{g+1} \subset \Gamma$ be two finite subschemes of degree indicated by the subscripts, and let x be the residual point to Γ_{g+1} in Γ. Now,*

(a) Let $l = g+1-k$. Then, there exists unique subschemes Σ_{k+1}, Λ_l and $\Lambda_{l+1} \subset \Gamma$ (not necessarily contained in Γ_{g+1}) such that the underlying zero cycles satisfy

$$[\Gamma_{g+1}] = [\Sigma_k] + [\Lambda_l] \qquad [\Sigma_{k+1}] = [\Sigma_k] + x \qquad [\Lambda_{l+1}] = [\Lambda_l] + x.$$

(b) Moreover, there is an inclusion of schemes

$$T(\Sigma_k, \Sigma_{k+1}) + T(\Lambda_l, \Lambda_{l+1}) \subseteq \Delta(\Gamma_{g+1}),$$

where the left-hand side denotes the scheme-theoretic image of $T(\Sigma_k, \Sigma_{k+1}) \times T(\Lambda_l, \Lambda_{l+1})$ under the group law $m\colon A \times A \to A$.

Proof. (a) The equalities of zero cycles define the various finite subschemes uniquely, as Γ is curvilinear by Corollary 4.5.

(b) Since formation of Fitting ideals commute with Base Change II.2.2, it suffices to show that

$$T(\Sigma_k, \Sigma_{k+1}) \times T(\Lambda_l, \Lambda_{l+1}) \subseteq \operatorname{supp}(\nu^* R^1\mathcal{S}(\mathcal{I}_{\Gamma_{g+1}}(2\Theta))), \tag{4.5}$$

where

$$\nu\colon T(\Sigma_k) \times T(\Lambda_l) \to A$$

is the restriction of the group law.

To understand the right-hand side of (4.5), we begin with the short exact sequence

$$0 \to \mathcal{I}_\Gamma(2\Theta) \to \mathcal{I}_{\Gamma_{g+1}}(2\Theta) \to k(x) \to 0. \tag{4.6}$$

Instead of first applying Fourier-Mukai, and then pulling back by ν, we encode both operations in the functor $\mathcal{T}$ sending a sheaf $\mathcal{F}$ on A to the sheaf

$$\mathcal{T}(\mathcal{F}) = p_{23*}(p_1^*(\mathcal{F}) \otimes (1 \times \nu)^*\mathscr{M})$$

on $T(\Sigma_k) \times T(\Lambda_l)$, where $\mathscr{M} = (\mathrm{id} \times \phi_\Theta)^*\mathcal{P} = m^*\mathcal{O}_A(\Theta) \otimes p^*\mathcal{O}_A(-\Theta) \otimes q^*\mathcal{O}_A(-\Theta)$ is the Mumford line bundle. In standard terminology, $\mathcal{T}$ (or its total derived functor) is the Fourier-Mukai transform with kernel

$$(1 \times \nu)^*\mathscr{M} \cong p_{12}^*(\mathscr{M}|_{A \times T(\Sigma_k)}) \otimes p_{13}^*(\mathscr{M}|_{A \times T(\Lambda_l)}).$$

Applying $\mathcal{T}$ to (4.6), we get a long exact sequence

$$\begin{aligned} 0 \to \mathcal{T}(\mathcal{I}_\Gamma(2\Theta)) \to \mathcal{T}(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) \to \mathcal{T}(k(x)) \to \\ \to R^1\mathcal{T}(\mathcal{I}_\Gamma(2\Theta)) \to R^1\mathcal{T}(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) \to \dots \end{aligned}$$

Observe that, if $\mathcal{F}$ is a sheaf on A and p is maximal such that $R^p\mathcal{S}(\mathcal{F}) \neq 0$, then Base Change II.1.3 shows that $\nu^*\, R^p\mathcal{S}(\mathcal{F}) \cong R^p\mathcal{T}(\mathcal{F})$ and $R^j\mathcal{T}(\mathcal{F}) = 0$ for all $j > p$. Using this, we can rewrite the last few terms in the long exact sequence, and obtain

$$\begin{aligned} 0 \to \mathcal{T}(\mathcal{I}_\Gamma(2\Theta)) \to \mathcal{T}(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) \xrightarrow{\phi} \nu^*\mathcal{P}_x \to \\ \to \nu^*\, R^1\mathcal{S}(\mathcal{I}_\Gamma(2\Theta)) \to \nu^*\, R^1\mathcal{S}(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) \to 0. \end{aligned} \tag{4.7}$$

As Γ is superabundant, the support of $R^1\mathcal{S}(\mathcal{I}_\Gamma(2\Theta))$ is all of A, so the support of its pullback by ν is all of $T(\Sigma_k) \times T(\Lambda_l)$.

We want to show that the homomorphism labelled ϕ in the long exact sequence is surjective over $(T(\Sigma_k) \setminus T(\Sigma_{k+1})) \times (T(\Lambda_l) \setminus T(\Lambda_{l+1}))$, that is $\operatorname{supp} \operatorname{coker} \phi \subseteq T(\Sigma_{k+1}) \times T(\Lambda_{l+1})$. Then, since

$$0 \to \operatorname{coker} \phi \to \nu^*\, R^1\mathcal{S}(\mathcal{I}_\Gamma(2\Theta)) \to \nu^*\, R^1\mathcal{S}(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) \to 0$$

and $\operatorname{supp} \nu^*\, R^1\mathcal{S}(\mathcal{I}_\Gamma(2\Theta)) = T(\Gamma_i) \times T(\Lambda_j)$, it will follow that

$$(T(\Sigma_k) \setminus T(\Sigma_{k+1})) \times (T(\Lambda_l) \setminus T(\Lambda_{l+1})) \subseteq \operatorname{supp} \nu^*\, R^1\mathcal{S}(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)),$$

which will imply the desired inclusion (4.5) by definition.

Claim 1. The morphism labelled ϕ in the long exact sequence (4.7) is surjective over $(T(\Sigma_k) \setminus T(\Sigma_{k+1})) \times (T(\Lambda_l) \setminus T(\Lambda_{l+1}))$.

Proof of Claim 1. By the definition of Σ_k and Λ_l and Remark 1.3 we have the commutative diagram

$$\begin{tikzcd} \mathcal{I}_{\Sigma_k}(\Theta) \otimes \mathcal{I}_{\Lambda_l}(\Theta) \arrow[r, hook] \arrow[d, two heads] & \mathcal{I}_{\Gamma_{g+1}}(2\Theta) \arrow[d, two heads] \\ k(x) \otimes k(x) \arrow[r, equal] & k(x), \end{tikzcd}$$

where the vertical arrows are the evaluation maps at x as in (4.6). If we apply the functor $\mathcal{T}$ to the previous diagram, we get

$$\begin{tikzcd} \mathcal{T}\left(\mathcal{I}_{\Sigma_k}(\Theta) \otimes \mathcal{I}_{\Lambda_l}(\Theta)\right) \arrow[r] \arrow[d] & \mathcal{T}\left(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)\right) \arrow[d] \\ \mathcal{T}\left(k(x) \otimes k(x)\right) \arrow[r, equal] & \nu^*\mathcal{P}_x. \end{tikzcd} \tag{4.8}$$

Recall that, considering projections from $A \times T(\Sigma_k) \times T(\Gamma_j)$,

$$
\begin{aligned}
&\mathcal{T}(\mathcal{I}_{\Sigma_k}(\Theta) \otimes \mathcal{I}_{\Lambda_l}(\Theta)) = \\
&\qquad = p_{23*}\left(p_1^*\mathcal{I}_{\Sigma_k}(\Theta) \otimes p_{12}^* \mathscr{M}|_{A\times T(\Sigma_k)} \otimes p_1^*\mathcal{I}_{\Lambda_l}(\Theta) \otimes p_{13}^* \mathscr{M}|_{A\times T(\Lambda_l)}\right) \\
&\qquad = p_{23*}\left(\mathcal{G}_1 \otimes \mathcal{G}_2\right), \qquad (4.9)
\end{aligned}
$$

where $\mathcal{G}_1 = p_{12}^*(p_1^*\mathcal{I}_{\Sigma_k}(\Theta) \otimes \mathscr{M}|_{A\times T(\Sigma_k)})$ and $\mathcal{G}_2 = p_{13}^*(p_1^*\mathcal{I}_{\Lambda_l}(\Theta) \otimes \mathscr{M}|_{A\times T(\Lambda_l)})$.

Consider now the following natural map

$$p_{23*}(\mathcal{G}_1) \otimes p_{23*}(\mathcal{G}_2) \xrightarrow{\varrho} p_{23*}(\mathcal{G}_1 \otimes \mathcal{G}_2). \qquad (4.10)$$

To understand the domain of ϱ, we define the functor $\mathcal{T}'$, sending a sheaf $\mathcal{F}$ on A to the sheaf

$$\mathcal{T}'(\mathcal{F}) = p_{2*}(p_1^*(\mathcal{F}) \otimes \mathscr{M}|_{A\times T(\Sigma_k)}),$$

in other words $\mathcal{T}'$ is the Fourier-Mukai transformation with kernel $\mathscr{M}|_{A\times T(\Sigma_k)}$ and analogously the functor $\mathcal{T}''$ as the Fourier-Mukai transformation with kernel $\mathscr{M}|_{A\times T(\Lambda_l)}$. With this notation we can rewrite the left-hand side sheaf in (4.10) as $p_1^*\mathcal{T}'(\mathcal{I}_{\Sigma_k}(\Theta)) \otimes p_2^*\mathcal{T}''(\mathcal{I}_{\Lambda_l}(\Theta))$ and using (4.9), the morphism ϱ becomes

$$p_1^*\mathcal{T}'(\mathcal{I}_{\Sigma_k}(\Theta)) \otimes p_2^*\mathcal{T}''(\mathcal{I}_{\Lambda_l}(\Theta)) \xrightarrow{\varrho} \mathcal{T}(\mathcal{I}_{\Sigma_k}(\Theta) \otimes \mathcal{I}_{\Lambda_l}(\Theta)).$$

Analogously we have a natural morphism,

$$p_1^*(\mathcal{P}_x|_{T(\Sigma_k)}) \otimes p_2^*(\mathcal{P}_x|_{T(\Lambda_l)}) = p_1^*\mathcal{T}'(k(x)) \otimes p_2^*\mathcal{T}''(k(x)) \xrightarrow{\varrho'} \mathcal{T}(k(x) \otimes k(x)).$$

Since $\mathcal{T}(k(x)\otimes k(x)) = \nu^*\mathcal{P}_x$, ϱ' is the restriction of the isomorphism $p_1^*\mathcal{P}_x \otimes p_2^*\mathcal{P}_x \to m^*\mathcal{P}_x$.

Therefore, if we consider diagram (4.8) composed with the natural maps ϱ and ϱ' we get the following commutative diagram of sheaves on $T(\Sigma_k) \times T(\Lambda_l)$

$$
\begin{tikzcd}
p_1^*\mathcal{T}'(\mathcal{I}_{\Sigma_k}(2\Theta)) \otimes p_2^*\mathcal{T}''(\mathcal{I}_{\Lambda_l}(2\Theta)) \arrow[r] \arrow[d] & \mathcal{T}(\mathcal{I}_{\Gamma_{g+1}}(2\Theta)) \arrow[d, "\phi"] \\
p_1^*(\mathcal{P}_x|_{T(\Sigma_k)}) \otimes p_2^*(\mathcal{P}_x|_{T(\Lambda_l)}) \arrow[r, equal] & \nu^*\mathcal{P}_x.
\end{tikzcd}
\qquad (4.11)
$$

We want to see that the left vertical arrow is surjective on $(T(\Sigma_k) \setminus T(\Sigma_{k+1})) \times (T(\Lambda_l) \setminus T(\Lambda_{l+1}))$ proving thus the claim. More precisely, we will see that

$$\mathcal{T}'(\mathcal{I}_{\Sigma_k}(2\Theta)) \to \mathcal{P}_x|_{T(\Sigma_k)}$$

is surjective on $T(\Sigma_k) \setminus T(\Sigma_{k+1})$ and analogously $\mathcal{T}''(\mathcal{I}_{\Lambda_l}(2\Theta)) \to \mathcal{P}_x|_{T(\Lambda_l)}$ is surjective on $T(\Lambda_l) \setminus T(\Lambda_{l+1})$.

Claim 2. $\mathcal{T}'(\mathcal{I}_{\Sigma_k}(2\Theta)) \to \mathcal{P}_x|_{T(\Sigma_k)}$ is surjective on $T(\Sigma_k) \setminus T(\Sigma_{k+1})$.

Proof of Claim 2. Associated to $\Sigma_k \subset \Sigma_{k+1}$ there is a commutative diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \mathcal{I}_{\Sigma_{k+1}}(\Theta) \arrow[r] \arrow[d] & \mathcal{I}_{\Sigma_k}(\Theta) \arrow[r] \arrow[d] & k(x) \arrow[r] & 0 \\
 & & \mathcal{O}(\Theta) \arrow[r, equal] \arrow[d] & \mathcal{O}(\Theta) \arrow[d] & & \\
0 \arrow[r] & k(x) \arrow[r] & \mathcal{O}_{\Sigma_{k+1}} \arrow[r] \arrow[d] & \mathcal{O}_{\Sigma_k} \arrow[r] \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}
$$

If we apply functor $\mathcal{T}'$ to the previous diagram we get the following commutative diagram,

$$
\begin{tikzcd}
 & 0 \arrow[r] & \mathcal{T}'(\mathcal{I}_{\Sigma_{k+1}}(\Theta)) \arrow[r] \arrow[d] & \mathcal{T}'(\mathcal{I}_{\Sigma_k}(\Theta)) \arrow[d, "\cong"] \arrow[ddddll, bend left] & \\
 & & \widehat{\mathcal{O}(\Theta)}\Big|_{T(\Sigma_k)} \arrow[r, equal] \arrow[d, "\psi_{k+1}"'] \arrow[dl, dashed] & \widehat{\mathcal{O}(\Theta)}\Big|_{T(\Sigma_k)} \arrow[d, "\psi_k"', "0"] & \\
0 \arrow[r] & \mathcal{P}_x|_{T(\Sigma_k)} \arrow[r] \arrow[d, equal] & \widehat{\mathcal{O}_{\Sigma_{k+1}}}\Big|_{T(\Sigma_k)} \arrow[r] \arrow[d] & \widehat{\mathcal{O}_{\Sigma_k}}\Big|_{T(\Sigma_k)} \arrow[r] \arrow[d] & 0 \\
 & \mathcal{P}_x|_{T(\Sigma_k)} \arrow[r] & \widehat{\mathcal{I}_{\Sigma_{k+1}}(\Theta)}\Big|_{T(\Sigma_k)} \arrow[r] \arrow[d] & \widehat{\mathcal{I}_{\Sigma_k}(\Theta)}\Big|_{T(\Sigma_k)} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
$$

Recall that $T(\Sigma_j)$ is the vanishing locus of $\widehat{\mathcal{O}(\Theta)} \to \widehat{\mathcal{O}_{\Sigma_j}}$ by Definition 2.1. Hence the two vertical maps ψ_{k+1} and ψ_k vanish respectively along $T(\Sigma_{k+1})$ and $T(\Sigma_k)$. Since we are restricted to $T(\Sigma_k)$, $\psi_k = 0$. Therefore, there is an induced (dashed in the diagram) map $\widehat{\mathcal{O}(\Theta)}\Big|_{T(\Sigma_k)} \to \mathcal{P}_x|_{T(\Sigma_k)}$ whose vanishing locus is still $T(\Sigma_{k+1})$. Moreover, this latter map is identified with the "snake map" in the above diagram. We conclude that there is a map

$$\mathcal{T}'(\mathcal{I}_{\Sigma_k}(\Theta)) \to \mathcal{P}_x|_{T(\Sigma_k)}$$

between line bundles on $T(\Sigma_k)$. Since $\mathcal{P}_x|_{T(\Sigma_k)}$ is a line bundle over $T(\Sigma_k)$, the fact that the map vanishes precisely along $T(\Sigma_{k+1})$ is equivalent to say that it is surjective over $T(\Sigma_k) \setminus T(\Sigma_{k+1})$.

Similarly, defining $\mathcal{T}''$ as the Fourier-Mukai transform with kernel $\mathscr{M}|_{A\times T(\Lambda_l)}$, we obtain a map

$$\mathcal{T}''(\mathcal{I}_{\Lambda_l}(\Theta)) \to \mathcal{P}_x|_{T(\Lambda_l)}$$

between line bundles on $T(\Lambda_l)$, which is surjective over $T(\Lambda_l)\setminus T(\Lambda_{l+1})$. The tensor product of these two maps fits into diagram (4.11) proving Claim 1, i.e that ϕ is surjective over $(T(\Sigma_k)\setminus T(\Sigma_{k+1}))\times(T(\Lambda_l)\setminus T(\Lambda_{l+1}))$, so the lemma is established. $\square$

5 Trisecants

We want to construct trisecants to the Kummer variety associated to A. I.e. let

$$\psi : A \to \mathbb{P}^N, \qquad \text{with } N = 2^g - 1, \tag{5.1}$$

be the map corresponding to the linear system $|2\Theta|$. If l is a line in $\mathbb{P}^N$, we will say that l is a *trisecant to the Kummer variety associated to A* if $\psi^{-1}(l)$ contains a finite subscheme of degree 3.

In fact, we will construct a 1-dimensional family of trisecants, that is, a 1-dimensional subset of

$$V = \left\{2\xi \mid \xi + Y \subset \psi^{-1}(l) \text{ for some line } l \subset \mathbb{P}^N\right\}$$

where Y is a specific subscheme of degree 3 that we will specify below. Then, the well-known Gunning-Welters criterion will lead us to a Schottky-type result and the multiplication by 2 in V is just to interpret V as an Abel-Jacobi curve (see [W2, Thm. 0.5]).

To obtain the trisecants, we need first to relate the dependence loci $\Delta(\Gamma_{g-1}, \Gamma_{g+1})$ and $\Delta(\Gamma_{g-1}, \Gamma_g)$. Recall that by Remark 2.3, $T(\Gamma_{g-1})\backslash T(\Gamma_g)$ has at least dimension 1.

Lemma 5.1. *Let $\Gamma_{g-1} \subset \Gamma_g \subset \Gamma_{g+1}$ be finite subschemes of degrees indicated by the subscripts, such that the residual subscheme S of Γ_{g-1} in Γ_{g+1} is well-formed. For each closed point $y \in T(\Gamma_{g-1}) \setminus T(\Gamma_g)$ there is an inclusion of schemes*

$$\Theta_{a-y} \cap \Delta(\Gamma_{g-1}, \Gamma_{g+1}) \subset T(S)_{-y} \cup \Delta(\Gamma_{g-1}, \Gamma_g)$$

where the union is defined by the products of the corresponding ideals and a is the residual point of Γ_{g-1} in Γ_g.

Proof. It is rather easy to check that the inclusion holds set theoretically. To prove it scheme-theoretically, we need to relate the maps of locally free sheaves whose degeneracy loci define the various schemes involved. This forces us to draw rather large diagrams, but it is relatively straight forward.

Let a be the residual point of Γ_{g-1} in Γ_g and c the residual point of Γ_g in Γ_{g+1}. Since $\Gamma_{g-1} \subset \Theta_y$ we have, by Remark 1.3, inclusions $\mathcal{I}_S(-\Theta_y) \subset \mathcal{I}_{\Gamma_{g+1}}$ and $\mathcal{I}_a(-\Theta_y) \subset$

$\mathcal{I}_{\Gamma_g}$. These give rise to a commutative diagram

$$\begin{tikzcd}[column sep=small]
& 0 \arrow[rr] & & \mathcal{I}_{\Gamma_{g+1}} \arrow[rr] \arrow[dd] & & \mathcal{I}_{\Gamma_{g-1}} \arrow[rr] \arrow[dd, equal] & & \mathcal{O}_S \arrow[rr] \arrow[dd, two heads] & & 0 \\
0 \arrow[rr] & & \mathcal{I}_S(-\Theta_y) \arrow[rr] \arrow[ur, hook] \arrow[dd, hook] & & \mathcal{O}(-\Theta_y) \arrow[rr] \arrow[ur, hook] \arrow[dd, equal] & & \mathcal{O}_S \arrow[rr] \arrow[ur, equal] \arrow[dd, two heads] & & 0 & \\
& 0 \arrow[rr] & & \mathcal{I}_{\Gamma_g} \arrow[rr] & & \mathcal{I}_{\Gamma_{g-1}} \arrow[rr] & & k(a) \arrow[rr] & & 0 \\
0 \arrow[rr] & & \mathcal{I}_a(-\Theta_y) \arrow[rr] \arrow[ur, hook] & & \mathcal{O}(-\Theta_y) \arrow[rr] \arrow[ur, hook] & & k(a) \arrow[rr] \arrow[ur, equal] & & 0 &
\end{tikzcd}$$

with exact rows.

Now twist the diagram with 2Θ, use that $2\Theta - \Theta_y$ is linearly equivalent to Θ_{-y}, and apply the Fourier-Mukai transform. This produces a similar diagram

$$\begin{tikzcd}[column sep=small]
& \widehat{\mathcal{O}_S} \arrow[rr, "M"] \arrow[dd, two heads] & & R^1\mathcal{S}(\mathcal{I}_{g+1}(2\Theta)) \arrow[rr] \arrow[dd, two heads] & & R^1\mathcal{S}(\mathcal{I}_{\Gamma_{g-1}}(2\Theta)) \arrow[r] \arrow[dd, equal] & 0 \\
\widehat{\mathcal{O}_S} \arrow[rr] \arrow[ur, equal] \arrow[dd] & & \widehat{\mathcal{I}_S(\Theta_{-y})} \arrow[rr] \arrow[ur] \arrow[dd, two heads] & & 0 & & \\
& \mathcal{P}_a \arrow[rr, "N"] & & R^1\mathcal{S}(\mathcal{I}_g(2\Theta)) \arrow[rr] & & R^1\mathcal{S}(\mathcal{I}_{\Gamma_{g-1}}(2\Theta)) \arrow[r] & 0 \\
\mathcal{P}_a \arrow[rr] \arrow[ur, equal] & & \widehat{\mathcal{I}_a(\Theta_{-y})} \arrow[rr] \arrow[ur] & & 0. & &
\end{tikzcd}$$

where the kernels of the vertical maps from $\widehat{\mathcal{O}_S}$ and $\widehat{\mathcal{I}_S(\Theta_{-y})}$ are $\mathcal{P}_c$.
If we consider locally free $\mathcal{F}$ such that the following diagram preserves the exactness of the rows[1],

$$\begin{tikzcd}[column sep=small]
& \mathcal{F} \arrow[rr, "f_1"] \arrow[dd, equal] & & \widehat{\mathcal{O}_S} \arrow[rr, "M"] \arrow[dd, two heads] & & \operatorname{Im} M \arrow[rr] \arrow[dd, two heads] & & 0 \\
\mathcal{O}(\Theta) \otimes \mathcal{P}_{-y} \arrow[rr, "f_3"] \arrow[ur] \arrow[dd, equal] & & \widehat{\mathcal{O}_S} \arrow[rr] \arrow[ur, equal] \arrow[dd, two heads] & & \widehat{\mathcal{I}_S(\Theta_{-y})} \arrow[rr] \arrow[ur, two heads] \arrow[dd, two heads] & & 0 & \\
& \mathcal{F} \arrow[rr, "f_2"'] & & \mathcal{P}_a \arrow[rr, "N"] & & \operatorname{Im} N \arrow[rr] & & 0 \\
\mathcal{O}(\Theta) \otimes \mathcal{P}_{-y} \arrow[rr, "f_4"'] \arrow[ur] & & \mathcal{P}_a \arrow[rr] \arrow[ur, equal] & & \widehat{\mathcal{I}_a(\Theta_{-y})} \arrow[rr] \arrow[ur, two heads] & & 0 &
\end{tikzcd}$$

we can verify the inclusion by locally representing the maps f_i by matrices, since $\Delta(\Gamma_{g-1}, \Gamma_{g+1})$ and $\Delta(\Gamma_{g-1}, \Gamma_g)$ are defined by the maximal minors of f_1 and f_2,

[1]We can take $\mathcal{F}$ to be locally the generators of $R^0\mathcal{S}(\mathcal{I}_{g-1}(2\Theta))$.

respectively, whereas $T(S)_{-y}$ and Θ_{a-y} are, respectively, the loci where f_3 and f_4 vanish.

$$\begin{tikzcd}
 & \mathcal{F} \arrow[rr, "\begin{pmatrix} r_1 & \dots & r_n \\ s_1 & \dots & s_n \end{pmatrix}"] \arrow[dd, equal] & & \widehat{\mathcal{O}_S} \arrow[dd, "(a_1\ a_2)"] \\
\mathcal{O}(\Theta) \otimes \mathcal{P}_{-y} \arrow[ur, "\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}"] \arrow[rr, "\binom{b_1}{b_2}"] \arrow[dd, equal] & & \widehat{\mathcal{O}_S} \arrow[ur, equal] \arrow[dd] & \\
 & \mathcal{F} \arrow[rr, "(t_1\ \dots\ t_n)"'] & & \mathcal{P}_a \\
\mathcal{O}(\Theta) \otimes \mathcal{P}_{-y} \arrow[ur] \arrow[rr, "(h)"'] & & \mathcal{P}_a \arrow[ur, equal] &
\end{tikzcd}$$

We want to see that the generators of the product of ideals $b_k \cdot t_l$ are contained in the ideal generated by h and the 2×2 minors of f_1. For example,

$$\begin{aligned}
b_1 \cdot t_1 &= b_1(a_1 r_1 + a_2 s_1) \\
&= (u_1 r_1 + \ldots + u_n r_n)(a_1 r_1 + a_2 s_1) \\
&= (u_1 r_1 + \ldots + u_n r_n) a_1 r_1 + (u_1 r_1 + \ldots + u_n r_n) a_2 s_1 \\
&= (u_1 r_1 + \ldots + u_n r_n) a_1 r_1 + (u_1 s_1 + \ldots + u_n s_n) a_2 r_1 + a_2 \sum_{i=1}^{n} u_i (r_i s_1 - r_1 s_i) \\
&= r_1(a_1 b_1 + a_2 b_2) + a_2 \sum_{i=1}^{n} u_i (r_i s_1 - r_1 s_i) \\
&= r_1 h + a_2 \sum_{i=1}^{n} u_i (r_i s_1 - r_1 s_i). \qquad \square
\end{aligned}$$

The following Corollary is a direct consequence of Lemmas 4.1, 4.2, 4.3 and 5.1. Recall that by Remark 2.3, $T(\Gamma_{g-1}) \setminus T(\Gamma_g)$ has at least dimension 1.

Corollary 5.2. *Assume Γ is a theta-general and superabundant finite subscheme of A. Then for any subschemes*

$$\Gamma_{g-1} \subset \Gamma_g \subset \Gamma_{g+1} \subset \Gamma$$

of degrees indicated by the subscripts, consider S the residual schemes of Γ_{g-1} in Γ_{g+1} and a (resp. b) the residual point of Γ_{g-1} in Γ_g (resp. Γ_g in Γ_{g+1}). Then, for every pair of closed points $y, y' \in T(\Gamma_{g-1}) \setminus T(\Gamma_g)$ we have

$$\Theta_{a-y} \cap \Theta_{b-x} \subseteq T(S)_{-y} \cup \Theta_{a-y'}, \tag{5.2}$$

where x is the only closed point in $T(\Gamma_g, \Gamma_{g+1})$.

Proof. On one hand, by Remark 3.7, $\overline{\Delta(\Gamma_{g+1}) \setminus \Delta(\Gamma_{g-1})} \subseteq \Delta(\Gamma_{g-1}, \Gamma_{g+1})$. Since by Lemma 4.3 $\Delta(\Gamma_{g+1})$ contains Θ_{b-x} and by Lemma 4.2 $\Delta(\Gamma_{g-1})$ has at least codimension 2, we have that $\Theta_{b-x} \subseteq \Delta(\Gamma_{g-1}, \Gamma_{g+1})$.
On the other hand, by Lemma 4.1 $\Delta(\Gamma_{g-1}, \Gamma_g) \subset \Theta_{a-y'}$ for every $y' \in T(\Gamma_{g-1}) \setminus T(\Gamma_g)$.
Then, inclusion (5.2) follows from Lemma 5.1. □

Now we are ready to obtain a unidimensional family of trisecants in our principally polarized abelian variety.

Proposition 5.3. *Assume Γ is a theta-general and superabundant finite subscheme of A. With the same notation as the previous Corollary 5.2 and for any fixed closed point $y' \in T(\Gamma_{g-1}) \setminus T(\Gamma_g)$ such that $y' \neq x + b - a$, consider $Y = \{a + (x - y'), S\}$. Then, we have the following set-theoretical inclusion,*

$$-\gamma - (T(\Gamma_{g-1}) \setminus T(\Gamma_g)) \subset V = \left\{2\xi \mid \xi + Y \subset \psi^{-1}(l) \text{ for some line } l \subset \mathbb{P}^N\right\}$$

where $\gamma = a + b - y'$ and $\psi : A \to \mathbb{P}^N$, with $N = 2^g - 1$, is the map corresponding to the linear system $|2\Theta|$ as in (5.1).

Proof. The residual scheme S is well-formed by Corollary 3.13. Let $y' \in T(\Gamma_{g-1}) \setminus (T(\Gamma_g) \cup \{x + b - a\})$ Since $y' \neq x \in T(\Gamma_g)$ and $b \neq a + (x - y')$, we have that $S \cap \{a + (x - y')\} = \emptyset$. Hence, Y is well-defined and has at least to different closed points. Thus we deal separately with two possible cases (observe that we can avoid the most technical case $Y \cong \operatorname{Spec} k[\varepsilon]/\varepsilon^3$),

i) $Y \cong \sum_{i=1}^{3} \operatorname{Spec} k$, that is $a \neq b$.
In this case, $Y = \{a, b, a + (x - y')\}$, hence $T(S) \subseteq \Theta_a \cap \Theta_b$. So the inclusion (5.2) becomes
$$\Theta_{a-y} \cap \Theta_{b-x} \subseteq \Theta_{b-y} \cup \Theta_{a-y'},$$
that implies the following 3 points in the Kummer variety: $\psi\left(a + \frac{1}{2}(-y - \gamma)\right)$, $\psi\left(b + \frac{1}{2}(-y - \gamma)\right)$ and $\psi\left(a + (x - y') + \frac{1}{2}(-y - \gamma)\right)$ are collinear (see [BL, Prop. 11.9.3]), where $\gamma = a + b - y'$.

ii) $Y \cong \operatorname{Spec}(k[\varepsilon]/\varepsilon^2) + \operatorname{Spec} k$, i.e. S is a non-reduced scheme supported in $a = b$.
In this situation, the inclusion (5.2) becomes $\Theta_{a-y} \cap \Theta_{a-x} \subseteq T(S)_{-y} \cup \Theta_{a-y'}$, or equivalently
$$\Theta \cap \Theta_{y-x} \subseteq T(S)_{-a} \cap \Theta_{y-y'}.$$
Let $s \in H^0(\mathcal{O}_\Theta(\Theta))$ be the section corresponding to $T(S)_{-a}$ (see Example 2.4). We will also denote θ_t a section of $H^0(\mathcal{O}(\Theta_t))$. Then $s \cdot \theta_{y-y'}$ vanish on $\Theta \cap \Theta_{y-x}$. By the usual exact sequence,
$$0 \to H^0(\mathcal{O}_\Theta(\Theta_{x-y'})) \overset{\theta_{y-x}}{\to} H^0(\mathcal{O}_\Theta(\Theta + \Theta_{y-y'})) \to H^0(\mathcal{O}_{\Theta_{y-x} \cap \Theta}(\Theta + \Theta_{y-y'})),$$

we get that in $H^0(\mathcal{O}_\Theta(\Theta + \Theta_{y-y'}))$,

$$\theta_{x-y'} \cdot \theta_{y-x} = (\text{const.})\theta_{y-y'} \cdot s.$$

Thus, using [W2, proof of Thm. 0.5, case ii)] we get that, the line l passing through $\psi\left(a + \frac{1}{2}(-y-\gamma)\right)$ and $\psi\left(a + (x-y') + \frac{1}{2}(-y-\gamma)\right)$ is tangent at $\psi\left(a + \frac{1}{2}(-y-\gamma)\right)$, where $\gamma = 2a - y'$. More precisely, $\frac{1}{2}(-y-\gamma) + S \subset \psi^{-1}(l)$. Since $Y = \{a + (x-y'), S\}$, this leads us to $\frac{1}{2}(-y-\gamma) + Y \subset \psi^{-1}(l)$. $\square$

Remark 5.4. *Observe that, since we are avoiding to use case iii) in the proof of [W2, Thm. 0.5], everything works over a algebraically closed field of characteristic different from 2 by [W2, Rem. 0.7].*

6 A Schottky-Castelnuovo-Torelli theorem

The main result of this chapter is the following theorem.

Theorem 6.1. *Let $\Gamma \subset A$ be a theta-general finite subscheme of degree $g+2$, imposing less than $g+2$ conditions on general 2Θ-translates. Then the following holds:*

- *(a)* ***Schottky:*** *The principally polarized abelian variety (A, Θ) is isomorphic to a Jacobian $J(C)$ of a curve C, with its canonical polarization.*
- *(b)* ***Castelnuovo:*** *The subscheme Γ is contained in an Abel-Jacobi curve, i.e. the image of an Abel-Jacobi map $C \to J(C)$.*
- *(c)* ***Torelli:*** *The curve C equals the scheme-theoretic intersection of all 2Θ-translates containing Γ.*

First we prove the Schottky statement,

Proof of part (a) of Theorem 6.1. Since $T(\Gamma_{g-1}, \Gamma_g)$ is of dimension at least 1 (see Remark 2.3), the Gunning-Welters Criterion (see [W2, Thm. 0.5]) implies that the algebraic subset V of Proposition 5.3 is a smooth irreducible curve, and A is its Jacobian (see also Remark 5.4). □

Then we prove the Castelnuovo statement,

Proof of part (b) of Theorem 6.1. By part *(a)* of Theorem 6.1, we know that $A = J(C)$ is a Jacobian of some curve C. Let i be maximal such that there exists a degree i subscheme $\Gamma_i \subset \Gamma_{g+2} = \Gamma$, which is contained in a translate of $\pm C$. Replace C with this translate of $\pm C$, and fix such $\Gamma_i \subset C$. By [W1, Thm 3.1], $i \geqslant 2$ and the claim is that $i = g+2$.

By contradiction, assume $i \leqslant g+1$. Then Γ_i is contained in a degree $g+1$ subscheme $\Gamma_{g+1} \subset \Gamma$, which we fix. Let x be the residual point to Γ_{g+1} in Γ, and

- let $\Gamma'_{i+1} \subset \Gamma$ be a subscheme (we call it Γ'_{i+1} to remark that it is not necessarily contained in Γ_{g+1}),
- and let $\Lambda_j \subset \Lambda_{j+1} \subset \Gamma$ be subschemes where $j = g+1-i$,

such that the underlying zero cycles satisfy,

$$[\Gamma_{g+1}] = [\Gamma_i] + [\Lambda_j] \qquad [\Gamma'_{i+1}] = [\Gamma_i] + x \qquad [\Lambda_{j+1}] = [\Lambda_j] + x,$$

see Lemma 4.8*(a)*. Then by Lemma 4.8*(b)* there is an inclusion of schemes

$$T(\Gamma_i, \Gamma_{i+1}) + T(\Lambda_j, \Lambda_{j+1}) \subseteq \Delta(\Gamma_{g+1}). \tag{6.1}$$

Observe that $j = g+1-i \leqslant g-1$. Now we use that C and W_{g-2} are theta-duals of each other (see Proposition III.1.9). Both C and W_{g-2} are defined up to translation; as we have fixed C, we may just as well fix W_{g-2} as $T(C)$. From Lemma II.2.7 and Proposition II.2.11, we infer that the inclusion $\Gamma_i \subset C$ is equivalent to $W_{g-2} \subseteq T(\Gamma_i)$. So $T(\Gamma_i, \Gamma_{i+1})$ contains W_{g-2}.

If $i > 2$, or equivalently $j < g-1$, then $T(\Lambda_j, \Lambda_{j+1})$ has dimension at least 2. Then $W_{g-2} + T(\Lambda_j, \Lambda_{j+1})$ necessarily equals all of A. Indeed, it is well-known that W_{g-2} is geometrically non-degenerate (see [Ra, §.II]), i.e. W_{g-2} intersects every 2-dimensional subvariety of A. Thus the left-hand side in (6.1) is A, which is a contradiction, since $\Delta(\Gamma_{g+1})$ has codimension one.

The case $i = 2$, $j = g-1$ is more subtle. By Example 2.5, the theta-dual $T(\Gamma_2)$ is the union of two translates of $\pm W_{g-2}$ (the two copies may coincide, when C is hyperelliptic, in which case $T(\Gamma_2)$ is a multiplicity two scheme structure on this W_{g-2}). More precisely,

$$T(\Gamma_2) \cong (W_{g-2})_\alpha \cup (-W_{g-2})_{-\beta},$$

for some α and β depending linearly on Γ_2. By minimality of i, neither of these are contained in $T(\Gamma_3)$, so $T(\Gamma_2, \Gamma_3)$ equals $T(\Gamma_2)$.

By Proposition 5.3 and the proof of part *(a)* of Theorem 6.1, the locus $T(\Lambda_{g-1}, \Lambda_g)$ is a translate of $-C$. Let $W = (W_{g-2})_\alpha$ and $W' = (-W_{g-2})_{-\beta}$. Thus, the left-hand side of (6.1) contains a translate of the divisor $(W \cup W') - C$. By Lemma 4.3, the divisorial part of the right-hand side of (6.1) is a theta-translate. So we have an inclusion

$$(W \cup W') - C \subseteq \Theta_{-d}$$

for some point d, which says that $(W \cup W')_d$ is contained in $T(-C)$. But, by Proposition III.1.9, the latter is just $-W_{g-2}$, which is integral, so it cannot contain a translate of $W \cup W'$, and we have a contradiction. $\square$

Finally, we observe that the proof of part *(c)* of Theorem 6.1 is exactly the same as [PP4, Cor. 4.3]. $\square$

Irregular varieties

CHAPTER

V

Bicanonical map

Introduction

Pluricanonical maps are a basic tool to study varieties of general type. The purpose of this chapter is to study the birationality of the bicanonical map of irregular varieties of higher dimension.

In the first section, we recall the generic vanishing theorems of Green and Lazarsfeld ([GL1, GL2]), some results of Kollár concerning the higher direct images of the canonical sheaf ([K1, K2]) and the continuously globally generation introduced by Pareschi and Popa in a sequence of articles (see for example, [PP1, PP3]). The conclusions of the generic vanishing theorems are expressed with a single invariant, the generic vanishing index of the canonical sheaf (see Definition 1.3). Using the dictionary established by Pareschi and Popa (see Chapter I) we can interpret the generic vanishing properties of the canonical sheaf as local properties of the generalized Fourier-Mukai transform of the structural sheaf.

In section §2 we prove a criterion for the global generation of the product of a line bundle and a coherent sheaf. This is based on the Pareschi and Popa notion of continuous global generation (Definition 2.2). In section §3 we compute the top Fourier-Mukai transform of the canonical sheaf of an irregular variety X (Proposition 3.2). When X itself is an abelian variety (or a complex torus) we recover a well-known result, proven by Mumford in [Mu2, pg. 128], that is crucial in the proof of the Mukai Equivalence Theorem. In fact, this result substitute as a geometric input for a general irregular variety, the Mukai Equivalence Theorem. As a first consequence we obtain a characterization of abelian varieties (see Propostion 3.9).

Combining the results of the previous section, in section §4 we prove a birationality criterion for the bicanonical map asserting, roughly speaking, that the non-birationality of the bicanonical map implies that, for general $\alpha \in \operatorname{Pic}^0 X$, the linear series $|\omega_X \otimes \alpha|$ has a base divisor (see Theorem 4.9 and Corollary 4.11). Using this base divisors, we construct an idempotent endomorphism of $\operatorname{Pic}^0 X$. This gives strong geometric constraints to $\operatorname{Pic}^0 X$ in the case that this endomorphism is not the identity.

The consequences of the birationality criterion allow us to give a numerical criterion based on the generic vanishing index for the birationality of the bicanonical map of an irregular variety (see Theorem 5.1). This numerical criterion implies also that varieties of maximal Albanese dimension and non-birational bicanonical map admit a fibration to a maximal Albanese dimension variety (possibly a point) whose fibers map onto a subvariety of codimesion at most 1 in a fixed abelian variety (Corollary 5.2).

In section §6, we focus our attention on primitive varieties (Definition 6.1). The fibrations to a maximal Albanese dimension variety that primitive varieties admit are very special. Namely, they have to be surjective onto an abelian variety and the kernel of the restriction map from $\operatorname{Pic}^0 X$ to the Pic^0 of the general fiber has to be connected (see Proposition 6.3). Then, we can characterize varieties birationally equivalent to a theta-divisor in a principally polarized abelian variety, as those primitive varieties that have $\chi(\omega_X) = 1$ and irregularity bigger than the dimension (Proposition 6.4). This result extends a cohomological characterization of theta-divisors due to Hacon and Pardini [HP1, Prop. 4.2] and has been proved independently, with a different proof, by Lazarsfeld and Popa [LP, Prop. 3.13]. We use this characterization to show that a primitive variety with irregularity bigger than the dimension, has non-birational bicanonical map if, and only if, it is birationally equivalent to a theta-divisor in a principally polarized abelian variety (Theorem 6.7).

Finally we study the case of primitive varieties with irregularity equal to the dimension and non-birational bicanonical map. Under these hypotheses, the Albanese map is surjective and generically finite. Moreover, an example of Chen and Hacon (see Example 6.10) shows that in this case the problem is more subtle since we cannot expect that the classification of varieties of non-birational bicanonical map coincides with those of varieties with $\chi(\omega_X) = 1$.

When the Stein factorization of the Albanese map factors through an Galois abelian cover, the Stein factorization has only rational singularities and the Albanese variety is simple, we show that the non-birationality of the bicanonical map of X implies that X must be birationally equivalent to a double cover over a principally polarized abelian variety branched along a reduced divisor in the linear series $|2\Theta|$ (Theorem 6.13). We also show that in this particular case, this example is also the only one with $\chi(\omega_X) = 1$ (see Proposition 6.16). We end up with a conjecture about what we expect on the general case with $q(X) = \dim X$ (see Conjecture 6.17).

Sections §1-4 and also §6.1 and §6.2 are joint work with Miguel Ángel Barja, Joan Carles Naranjo and Giuseppe Pareschi and are included in the preprint [BLNP].

1 Irregular varieties

The following will be the main setting of this chapter. Except otherwise stated, we will work over the complex numbers $\mathbb{C}$.

Definition 1.1. Let X be a compact, connected, Kähler manifold of dimension d and let $\mathrm{Pic}^0 X$ denote the identity component of the Picard group of X. We will denote by $\mathrm{Alb}\, X$ the Albanese torus of X and $\mathrm{alb} : X \to \mathrm{Alb}\, X$ the Albanese map. $\mathrm{Alb}\, X$ and $\mathrm{Pic}^0 X$ are dual to each other [BL, Prop. 11.11.6]. Its dimension is called the *irregularity* of X and it is denoted by $q(X) = \dim \mathrm{Alb}\, X = \dim \mathrm{Pic}^0 X$.

(a) X is *irregular* if $q(X) > 0$.

(b) X is of *maximal Albanese dimension* if $\mathrm{alb}\colon X \to \mathrm{Alb}\, X$ has generically finite fibers.

In addition, in some results we will require X to be projective, specially when we use Kollár's Theorems 1.8. Also the cohomological criterion for continuously global generation (see Theorem 2.6) in abelian varieties needs the projective assumption.

When X is *irregular*, i.e. $\mathrm{Pic}^0 X$ is not the trivial group, Green and Lazarsfeld [GL1, GL2] showed that the cohomological support loci of the canonical line bundle (see Def. I.1.8) are relevant invariants in the study of the geometry of X. We recall here the definition for the convenience of the reader.

Definition 1.2. Let X be a Kähler compact manifold equipped with a morphism to a complex torus, $a\colon X \to A$. We define the *cohomological support loci* of X with respect to a as

$$V^i_a(\omega_X) = \left\{ \alpha \in \mathrm{Pic}^0 A \,\middle|\, h^i(X, \omega_X \otimes a^*\alpha) > 0 \right\}.$$

Following the criterion of section I.§1, when A is the Albanese torus $\mathrm{Alb}\, X$ and a the Albanese map alb, we will omit it and we will simply write $V^i(\omega_X)$.

We also particularize the generic vanishing index to the canonical line bundle (see Definition I.1.9). Pareschi and Popa showed that this invariant encodes a lot of the information coming from the morphism $a : X \to A$. In particular, when a is the Albanese map, encodes some geometric structure of X.

Definition 1.3. Let X be a Kähler compact manifold equipped with a morphism to a complex torus, $a\colon X \to A$. We define the *generic vanishing index* of X with respect to a as

$$\mathrm{gv}_a(\omega_X) := \min_{i>0} \left\{ \mathrm{codim}_{\mathrm{Pic}^0 A} V^i(\omega_X) - i \right\}.$$

As above, when A is the Albanese torus $\operatorname{Alb} X$ and a is the Albanese map alb we will omit it, and we will simply write $\operatorname{gv}(\omega_X)$.

The next theorem is due to Green and Lazarsfeld [GL1, Thm. 1]. We state it using the generic vanishing index notation and the remark of Ein and Lazarsfeld [EL, Rem. 1.6].

Theorem 1.4. *Let X be a compact, connected, Kähler manifold and assume that the generic fiber of $a\colon X \to A$ has dimension k. Then $\operatorname{gv}_a(\omega_X) \geqslant -k$.*

When $a\colon X \to A$ is a generically finite morphism onto its image, the previous theorem says that $\operatorname{gv}_a(\omega_X) \geqslant 0$. This result has been generalized by Hacon-Pardini to the direct images of the canonical sheaf through a fibration, although here we need the projectivity assumption.

Theorem 1.5 ([HP2, Thm. 2.2])**.** *Let X and Y be smooth projective varieties. Let moreover $f\colon X \to Y$ be a surjective morphism and $a\colon Y \to A$ a generically finite morphism. Then $\operatorname{gv}_a(R^i f_*(\omega_X \otimes \alpha)) \geqslant 0$, for all $i \geqslant 0$ and all torsion point $\alpha \in \operatorname{Pic}^0 X$.*

The following theorem is due to Green and Lazarsfeld [GL2, Thm 0.1] with an important addition due to Simpson [S, §4,6,7]

Theorem 1.6. *Let X be a compact Kähler manifold, and W an irreducible component of $V^i(\omega_X)$ for some i. Then,*

(a) There exists a torsion point $\beta \in \operatorname{Pic}^0 X$ and a subtorus B of $\operatorname{Pic}^0 X$ such that $W = t_\beta^* B$.

(b) There exists a normal variety Y of dimension $\leqslant d - i$, such that any smooth model of Y has maximal Albanese dimension and a morphism with connected fibres $f\colon X \to Y$ such that B is contained in $f^* \operatorname{Pic}^0 Y$.

Remark 1.7. *It is useful to recall that the morphism $f\colon X \to Y$ in the second part of the previous theorem, arises as the Stein factorization of the morphism $\pi \circ \operatorname{alb}\colon X \to \operatorname{Pic}^0 W$, where $\pi\colon \operatorname{Alb} X \to \operatorname{Pic}^0 W$ is the dual map of the inclusion $W \subseteq \operatorname{Pic}^0 X$. Hence, the key point of the second part of the theorem is the dimensional bound for Y.*

The following theorem is a summary of three theorems due to Kollár ([K1, Thm. 2.1 and Prop. 7.6], [K2, Thm 3.1]). Here we need X and Y to be projective.

Theorem 1.8. *Let $f\colon X \to Y$ a surjective map between to complex projective varieties of dimension d and $d-k$ and assume that X is smooth and Y is reduced. Then*

(a) $R^i f_* \omega_X$ is torsion-free for all $i \geqslant 0$.

(b) $R^i f_* \omega_X = 0$ *for all* $i > k$.

(c) In the derived category of Y,

$$\mathrm{R}f_*\omega_X \cong \bigoplus_{i=0}^{k} R^i f_* \omega_X[-i].$$

(d) When Y *is also smooth and* f *has connected fibers,* $R^k f_* \omega_X = \omega_Y$.

Remark 1.9. *Hacon-Pardini pointed out in [HP2, Thm 2.1] that the previous Theorem also holds when we replace* ω_X *by* $\omega_X \otimes \alpha$*, where* $\alpha \in \operatorname{Pic}^0 X$ *is a torsion point.*

The following result of Pareschi shows that the converse of Theorem 1.4 is also true

Proposition 1.10 ([BLNP, Prop. 1.9])**.** *Let* X *be a smooth complex projective* d*-dimensional variety equipped with a morphism to a complex torus,* $a\colon X \to A$*. Then*

$$\mathrm{gv}_a(\omega_X) \geqslant -k \quad \textit{if, and only if,} \quad \dim a(X) \geqslant d - k.$$

Such statement was proved independently by Lazarsfeld-Popa (compare with [LP, Prop. 1.5]).

Proof. The left implication is Green-Lazarsfeld Generic Vanishing Theorem 1.4. Now, let $e = d - \dim a(X)$. Let

$$X \xrightarrow{b} Y \xrightarrow{c} A$$

be the Stein factorization of a. Since the $V_a^i(\omega_X)$ are birational invariants, we can assume that Y is smooth. By Kollár's theorem 1.8*(c)*, the Leray spectral sequence of b splits:

$$H^i(\omega_X \otimes b^*(c^*\alpha)) \cong \bigoplus_{l=0}^{i} H^l(R^{i-l}b_*(\omega_X) \otimes c^*\alpha). \tag{1.1}$$

Moreover, again by a result of Kollár (Theorem 1.8*(d)*),

$$R^e b_*(\omega_X) = \omega_Y. \tag{1.2}$$

Assume that $\dim Y < \dim X - k$, i.e. $e > k$, and we want to arrive to contradiction. From (1.1) for $i = e$ and (1.2) it follows that $V_a^e(\omega_X)$ contains $V_c^0(\omega_Y)$. The fact that $\mathrm{gv}_a(\omega_X) \geqslant -k$ implies that $\operatorname{codim} V_c^0(\omega_Y) \geqslant e - k > 0$. Since $c\colon Y \to A$ is generically finite, the above mentioned Green-Lazarsfeld Generic Vanishing Theorem yields to $\mathrm{gv}_c(\omega_Y) \geqslant 0$. Hence, by Theorem I.1.10, $\mathrm{R}\Phi_{P_c}(\mathcal{O}_Y)$ is a sheaf (in cohomological degree equal to $\dim Y$), denoted $\widehat{\mathcal{O}_Y}$. Since $V_c^0(\omega_Y)$ is a *proper* subvariety of $\operatorname{Pic}^0 A$, $\widehat{\mathcal{O}_Y}$ must be a torsion sheaf. Therefore, by Corollary I.1.18, there is a $i > 0$ such that $\operatorname{codim}_{\operatorname{Pic}^0 A} V_c^i(\omega_Y) = i$. Since, again by (1.1) and (1.2), $V_c^i(\omega_Y)$ is contained in $V_a^{e+i}(\omega_X)$, it follows that $\operatorname{codim}_{\operatorname{Pic}^0 A} V_a^{e+i}(\omega_X) = i < e - k + i$, a contradiction. $\square$

2 Continuous global generation

The notion of *continuous global generation* is a useful notion related to global generation. It was introduced by Pareschi-Popa in [PP1] and applied to various geometric problems in [PP1, PP2, PP3]. See also the survey [PP5].

We will use the following basic notation,

Terminology/Notation 2.1. *(a) Given a line bundle L,* $\mathrm{Bs}(L)$ *will denote its base locus.*

(b) Given a coherent sheaf $\mathcal{F}$, $\tau(\mathcal{F})$ will denote the torsion part of $\mathcal{F}$, that is the maximal torsion subsheaf of $\mathcal{F}$.

Let $\mathcal{F}$ be a coherent sheaf on an irregular variety X, and let T be a subset of $\mathrm{Pic}^0 A$. Then we have the *continuous evaluation map associated to the pair* $(\mathcal{F}, T)$:

$$ev_{T,\mathcal{F}}\colon \bigoplus_{\alpha\in T} H^0(X, \mathcal{F}\otimes a^*\alpha^{-1})\otimes a^*\alpha \to \mathcal{F}.$$

Definition 2.2. Let p be a point of X. The sheaf $\mathcal{F}$ is said to be *continuously globally generated at p (CGG at p for short), with respect to the morphism a*, if the map $ev_{U,\mathcal{F}}$ is surjective *at p*, for all open sets $U \subseteq \mathrm{Pic}^0 A$. When possible, we will omit the reference to the morphism a, and say simply *CGG* at p.

Remark 2.3. *If L is a line bundle, then L is CGG at p if, and only if, the locus of $\alpha \in \mathrm{Pic}^0 A$ such that p is not a base point of $L\otimes a^*\alpha$ is a Zariski open subset of $\mathrm{Pic}^0 A$.*

We will also need the following weaker version. This is in fact a variant of the *weak continuous global generation* of [PP2]. However, for technical reasons we prefer to give the following definition, which is natural in view of Proposition 2.5*(b)* and Theorem 2.6*(b)* below.

Definition 2.4. Let p be a point of X and let $\mathcal{F}$ be a GV-sheaf. Let $\widehat{\mathrm{R}\Delta\mathcal{F}}$ be the transform of the dual of $\mathcal{F}$ and let $\tau(\widehat{\mathrm{R}\Delta\mathcal{F}})$ be its *torsion sheaf*. Then $\mathcal{F}$ is said to be *essentially continuously globally generated at p (ECGG at p for short), with respect to the morphism a*, if the map $ev_{T,\mathcal{F}}$ is surjective *at p*, for all subsets of the form $T = U\cup S$, where U is a non-empty Zariski open subset of $\mathrm{Pic}^0 A$ and S is the underlying subset of $\operatorname{supp}\tau(\widehat{\mathrm{R}\Delta\mathcal{F}})$. As above, when possible, we will omit the reference to the morphism a, and say simply *ECGG* at p.

Obviously *CGG* at p implies *ECGG* at p. Moreover, we will say simply that a sheaf is *CGG* (resp. *ECGG*), when it is *CGG* (resp. *ECGG*) for all p.

The structure sheaf of an abelian variety X is an easy example of a line bundle which is not *CGG* at any point, since for any open subset U of $\mathrm{Pic}^0 X$ not containing the identity point $\hat{0}$, the map $ev_{U,\mathcal{O}_X}$ is zero, but it is *ECGG*. In fact it is well-known that in this case $\widehat{\mathcal{O}_X} = \mathbb{C}(\hat{0})$ ([Mu2, III.§13]). Hence the underlying subset of the support of $\tau(\widehat{\mathcal{O}_X}) = \mathbb{C}(\hat{0})$ is the identity point $\{\hat{0}\}$, and all evaluation maps $ev_{U\cup\{\hat{0}\},\mathcal{O}_X}$ are trivially surjective at all points. A generalization of this example is provided by Corollary 3.8*(b)* below.

A useful relation between continuous global generation and the usual *global generation* is provided by the following.

Proposition 2.5 ([PP2, Prop. 2.4])**.** *Let $\mathcal{F}$ and L be respectively a coherent sheaf and a line bundle on X, and let p be a point of X.*

(a) If both $\mathcal{F}$ and L are continuously globally generated at p then $\mathcal{F} \otimes L \otimes a^*\beta$ is globally generated at p for any $\beta \in \mathrm{Pic}^0 A$.

(b) Assume that L is CGG at p and that $\mathcal{F}$ is ECGG at p. Let $\tau(\widehat{\mathrm{R}\Delta\,\mathcal{F}})$ be the torsion sheaf of $\widehat{\mathrm{R}\Delta\,\mathcal{F}}$, and assume that the underlying set S of the support of $\tau(\widehat{\mathrm{R}\Delta\,\mathcal{F}})$ is finite. For any $\beta \in \mathrm{Pic}^0 A$, if $p \notin \bigcup_{\alpha\in S_{+\beta}} \mathrm{Bs}(L\otimes a^*\alpha)$, then $\mathcal{F} \otimes L \otimes a^*\beta$ is globally generated at p.

Proof. *(a)* By Remark 2.3 there is an open subset V_p of $\mathrm{Pic}^0 A$ such that p is not a base point of $L \otimes a^*\alpha$ for all $\alpha \in V_p$, i.e. the evaluation map at p: $H^0(L \otimes a^*\alpha) \to (L\otimes a^*\alpha)_p$ is surjective for all $\alpha \in V_p$. Since $\mathcal{F}$ is *CGG* at p, it follows that the map

$$ev_{V_p,\mathcal{F}}\colon \oplus_{\alpha\in V_p} H^0(\mathcal{F} \otimes a^*\beta \otimes a^*\alpha^{-1}) \otimes H^0(L \otimes a^*\alpha) \to (\mathcal{F} \otimes L \otimes a^*\beta)_p$$

is surjective. This proves the assertion, since the above map factors trough $H^0(\mathcal{F} \otimes L \otimes a^*\beta)$.

(b) The proof is the same with the difference that now if we use the continuous evaluation map ev_{T_p} with $T_p = V_p \cup (S_{+\beta})$. If $p \notin \bigcup_{\alpha\in S_{+\beta}} \mathrm{Bs}(L \otimes a^*\alpha)$, then the evaluation map $H^0(L \otimes a^*\alpha) \to (L \otimes a^*\alpha)_p$ is also surjective for all $\alpha \in S_{+\beta}$. □

As for the usual global generation, in many applications it is useful to have a criterion ensuring that, if the higher cohomology of a given sheaf $\mathcal{F}$ satisfies certain vanishing conditions, then $\mathcal{F}$ is *CGG* or *ECGG*. The following criterion, which applies to *sheaves on abelian varieties*, is due to Pareschi-Popa. In the reference a sheaf $\mathcal{F}$ on an abelian variety A such that $\mathrm{gv}(\mathcal{F}) \geqslant 1$ is called *M-regular*. Bearing in mind this terminology, the first part of the following theorem is [PP1, Prop. 2.13] or [PP3, Cor. 5.3]. The proof of the second part essentially follows the proof of [PP2, Thm. 4.1].

Theorem 2.6 ([BLNP, Thm. 4.5]). *Let $\mathcal{F}$ be a sheaf on an abelian variety A.*

(a) If $\mathrm{gv}(\mathcal{F}) \geqslant 1$ then $\mathcal{F}$ is CGG.

(b) If $\mathrm{gv}(\mathcal{F}) \geqslant 0$ and $\operatorname{supp} \tau(\widehat{\mathrm{R}\Delta\,\mathcal{F}})$ is a reduced scheme[1]*, then $\mathcal{F}$ is ECGG.*

The main point is the following,

Lemma 2.7 ([BLNP, Lem. 4.6]). *Let $\mathcal{F}$ be a GV-sheaf on an abelian variety A. Let $\widehat{\mathrm{R}\Delta\,\mathcal{F}}$ be the transform of the dual of $\mathcal{F}$ and let $\tau(\widehat{\mathrm{R}\Delta\,\mathcal{F}})$ be its torsion sheaf. Let L be an ample line bundle on A. Then, for all sufficiently high $n \in \mathbb{N}$, and for any subset $T \subseteq \operatorname{Pic}^0 A$, the Fourier-Mukai transform $\Phi_{\mathcal{P}}$ induces a canonical isomorphism*

$$H^0(A, \operatorname{coker} ev_{T,\mathcal{F}} \otimes L^n) \cong (\ker \psi_{T,\mathcal{F}})^*,$$

where ψ is the natural evaluation map,

$$\psi_{T,\mathcal{F}} \colon \operatorname{Hom}(\widehat{L^n}, \widehat{\mathrm{R}\Delta\,\mathcal{F}}) \to \prod_{\alpha \in T} \mathcal{H}om(\widehat{L^n}, \widehat{\mathrm{R}\Delta\,\mathcal{F}}) \otimes \mathbb{C}(\alpha). \tag{2.1}$$

Proof. Let $T \subseteq \operatorname{Pic}^0 A$ be any subset. The map $H^0(ev_T \otimes L^n)$ is the "continuous multiplication map of global sections":

$$m^T_{\mathcal{F},L^n} \colon \bigoplus_{\alpha \in T} H^0(\mathcal{F} \otimes a^*\alpha^{-1}) \otimes H^0(L^n \otimes a^*\alpha) \to H^0(\mathcal{F} \otimes L^n).$$

A standard argument with Serre vanishing shows that, if n is big enough,

$$H^0(\operatorname{coker}(ev_{T,\mathcal{F}}) \otimes L^n) \cong \operatorname{coker}(m^T_{\mathcal{F},L^n}).$$

By Grothendieck-Serre duality I.1.5, the dual of $m^T_{\mathcal{F},L^n}$ is

$$\operatorname{Ext}^q(L^n, \mathrm{R}\Delta\,\mathcal{F}) \to \prod_{\alpha \in T} \operatorname{Hom}_{\mathbb{C}}\big(H^0(L^n \otimes a^*\alpha), H^q((\mathrm{R}\Delta\,\mathcal{F}) \otimes a^*\alpha)\big). \tag{2.2}$$

Let us interpret such map via the Fourier-Mukai transform. Concerning the source, Mukai's Theorem I.2.1 provides the isomorphism

$$\begin{aligned} \operatorname{Hom}_{\mathrm{D}^b(A)}(L^n, \mathrm{R}\Delta\,\mathcal{F}[q]) &\cong \operatorname{Hom}_{\mathrm{D}^b(\operatorname{Pic}^0 A)}(\mathrm{R}\Phi_{\mathcal{P}}(L^n), \mathrm{R}\Phi_{\mathcal{P}}(\mathrm{R}\Delta\,\mathcal{F})[q]) \\ &\cong \operatorname{Hom}_{\mathrm{D}^b(\operatorname{Pic}^0 A)}(\widehat{L^n}, \widehat{\mathrm{R}\Delta\,\mathcal{F}}) \end{aligned}$$

(here $q = \dim A$) i.e., since $\widehat{L^n}$ is locally free,

$$\operatorname{Ext}^q_A(L^n, \mathrm{R}\Delta\,\mathcal{F}) \cong \operatorname{Hom}_{\operatorname{Pic}^0 A}(\widehat{L^n}, \widehat{\mathrm{R}\Delta\,\mathcal{F}}). \tag{2.3}$$

[1] Here we can consider the annihilator support, instead of the Fitting support (e.g. [Ei]). Thus, the hypothesis is weaker.

Note that, besides $\widehat{L^n}$, also $\widehat{\mathrm{R}\Delta\,\mathcal{F}}$ has the base change property, since $H^{d+1}(\mathcal{F}\otimes a^*\alpha)=0$ for all $\alpha\in\operatorname{Pic}^0 A$, see [Mu2, Cor. 3, pg. 53]. This means that, in the target of the map (2.2), we have that $H^0(L^n\otimes a^*\alpha)$ (respectively $H^q((\mathrm{R}\Delta\,\mathcal{F})\otimes a^*\alpha))$ are isomorphic to the fiber at the point α of $\widehat{L^n}$ (resp. of $\widehat{\mathrm{R}\Delta\,\mathcal{F}}$). Hence also the sheaf $\mathcal{H}om(\widehat{L^n},\widehat{\mathrm{R}\Delta\,\mathcal{F}})$ has the base change property and the Fourier-Mukai isomorphism (2.3) identifies the map (2.2) to the evaluation map of the sheaf $\mathcal{H}om(\widehat{L^n},\widehat{\mathrm{R}\Delta\,\mathcal{F}})$ at points in T:

$$\operatorname{Hom}(\widehat{L^n},\widehat{\mathrm{R}\Delta\,\mathcal{F}})\to\prod_{\alpha\in T}\mathcal{H}om(\widehat{L^n},\widehat{\mathrm{R}\Delta\,\mathcal{F}})\otimes\mathbb{C}(\alpha).\qquad\square$$

Now we are ready to proof the previous theorem.

Proof of Theorem 2.6. (a) By Theorem I.1.10 we can assume that $\widehat{\mathrm{R}\Delta\,\mathcal{F}}$ is torsion-free. Then the evaluation map $\psi_{U,\mathcal{F}}$ is injective for *all* open subsets $U\subseteq\operatorname{Pic}^0 A$, so $H^0(\operatorname{coker} ev_{U,\mathcal{F}}\otimes L^n)=0$ for $n\gg 0$. From Serre's theorem it follows that $\operatorname{coker} ev_{U,\mathcal{F}}=0$.

(b) By Nakayama's Lemma, given a non-zero global section $s\in\operatorname{Hom}(\widehat{L^n},\widehat{\mathrm{R}\Delta\,\mathcal{F}})$, we have that $s(\alpha)\in\mathcal{H}om(\widehat{L^n},\widehat{\mathrm{R}\Delta\,\mathcal{F}})\otimes\mathbb{C}(\alpha)$ vanishes for all α in an dense subset $T\subset\operatorname{Pic}^0 A$, only if T does not meet a component of the support of the torsion part $\tau(\widehat{\mathrm{R}\Delta\,\mathcal{F}})$ or this support is non-reduced. The second possibility is excluded by hypothesis. So if we consider subsets of the form $T=U\cup\operatorname{supp}\tau(\widehat{\mathrm{R}\Delta\,\mathcal{F}})$, the map $\psi_{T,\mathcal{F}}$ is injective. Therefore $H^0(\operatorname{coker} ev_{T,\mathcal{F}}\otimes L^n)=0$. Hence, by Serre's theorem, $\operatorname{coker} ev_{T,\mathcal{F}}=0$. $\square$

To apply Theorem 2.6 to geometric situations we will need to control the torsion sheaf of the Fourier-Mukai transform of a sheaf $\mathcal{F}$ such that $\operatorname{gv}_a(\mathcal{F})=0$. This control is provided by the following proposition.

Proposition 2.8. *Let X be a d-dimensional variety, equipped with a surjective morphism to an abelian variety $a\colon X\to A$. Suppose that $\mathcal{F}$ is a coherent sheaf such that:*

(a) $\operatorname{codim} V^i_a(\mathcal{F})\geqslant i+1$ *for all i such that $0<i<d$;*

(b) $V^d_a(\mathcal{F})$ *is a finite set.*

Then $\mathrm{R}\Delta\,\mathcal{F}$ is a WIT_d object in $\mathrm{D}^b(X)$. Moreover, if $\tau(\widehat{\mathrm{R}\Delta\,\mathcal{F}})$ is the torsion subsheaf of $\widehat{\mathrm{R}\Delta\,\mathcal{F}}$, we have a surjective morphism

$$\mathcal{E}xt^d(R^d\Phi_{P_a}\,\mathcal{F},\mathcal{O}_{\operatorname{Pic}^0 A})\twoheadrightarrow\tau(\widehat{\mathrm{R}\Delta\,\mathcal{F}}).$$

Proof. Hypothesis *(a)* and *(b)* ensure that $\mathrm{gv}_a(\mathcal{F}) \geqslant 0$. Therefore, by Theorem I.1.10, $\mathrm{R}\Delta\,\mathcal{F}$ is a WIT_d object, that is, its transform $\widehat{\mathrm{R}\Delta\,\mathcal{F}}$ is concentrated in degree d.

Now we want to control $\tau(\widehat{\mathrm{R}\Delta\,\mathcal{F}})$. The argument is similar to the proof of [PP3, Prop. 2.8]. Since $\mathrm{Pic}^0 A$ is smooth, the functor $\mathrm{R}\mathcal{H}om(\,\cdot\,,\mathcal{O}_{\mathrm{Pic}^0 A})$ is an involution on $\mathrm{D}^b(\mathrm{Pic}^0 A)$. Thus there is a fourth quadrant spectral sequence

$$E_2^{i,j} := \mathcal{E}xt^i\Big((\mathcal{E}xt^{-j}(\widehat{\mathrm{R}\Delta\,\mathcal{F}},\mathcal{O}_{\mathrm{Pic}^0 A}),\mathcal{O}_{\mathrm{Pic}^0 A}\Big) \Rightarrow$$
$$H^{i+j} = \mathcal{H}^{i+j}\widehat{\mathrm{R}\Delta\,\mathcal{F}} = \begin{cases} \widehat{\mathrm{R}\Delta\,\mathcal{F}} & \text{if } i+j=0 \\ 0 & \text{otherwise.} \end{cases}$$

By duality (Corollary I.1.14), $\dim\operatorname{supp}\big(\mathcal{E}xt^i(\widehat{\mathrm{R}\Delta\,\mathcal{F}},\mathcal{O}_{\mathrm{Pic}^0 A})\big) = \dim\operatorname{supp} R^i\Phi_{P_a}\mathcal{F}$ and, by base-change, $\operatorname{supp} R^i\Phi_{P_a}\mathcal{F} \subseteq V_a^i(\mathcal{F})$. Thus, assumption *(a)* implies that for all l such that $0<l<d$ we have $\operatorname{codim}\operatorname{supp}\big(\mathcal{E}xt^l(\widehat{\mathrm{R}\Delta\,\mathcal{F}},\mathcal{O}_{\mathrm{Pic}^0 A})\big) > l$. Therefore $\mathcal{E}xt^i\Big(\mathcal{E}xt^{-j}(\widehat{\mathrm{R}\Delta\,\mathcal{F}},\mathcal{O}_{\mathrm{Pic}^0 A}),\mathcal{O}_{\mathrm{Pic}^0 A}\Big) = 0$ for all (i,j) such that $i+j \leq 0$, except for $(i,j)=(0,0)$ and $(i,j)=(d,-d)$. Hence the only non-zero $E_\infty^{i,-i}$ terms are $E_\infty^{d,-d}$ and $E_\infty^{0,0}$, and we have the exact sequence

$$0 \to E_\infty^{d,-d} \to H^0 = \widehat{\mathrm{R}\Delta\,\mathcal{F}} \to E_\infty^{0,0} \to 0. \tag{2.4}$$

The differentials coming into $E_p^{0,0}$ are always zero, so we get an inclusion,

$$E_\infty^{0,0} \subseteq E_2^{0,0} = \widehat{\mathrm{R}\Delta\,\mathcal{F}}^{**}$$

and (2.4) is identified to the natural exact sequence

$$0 \to \tau(\widehat{\mathrm{R}\Delta\,\mathcal{F}}) \to \widehat{\mathrm{R}\Delta\,\mathcal{F}} \to \widehat{\mathrm{R}\Delta\,\mathcal{F}}^{**}.$$

Hence the torsion $\tau(\widehat{\mathrm{R}\Delta\,\mathcal{F}})$ is canonically isomorphic to $E_\infty^{d,-d}$. Since the differentials going out of $E_p^{d,-d}$ are always zero, we get a surjection

$$E_2^{d,-d} \cong \mathcal{E}xt^d(R^d\Phi_{P_a}\mathcal{F},\mathcal{O}_{\mathrm{Pic}^0 A}) \twoheadrightarrow E_\infty^{d,-d} \cong \tau(\widehat{\mathrm{R}\Delta\,\mathcal{F}}). \qquad \square$$

3 Applications to the canonical sheaf

3.1 The top Fourier-Mukai transform of the canonical sheaf

As we have seen in the previous section, to control the torsion part of a GV-sheaf $\mathcal{F}$ under hypothesis of the Proposition 2.8, it would be very useful to compute exactly $R^d\Phi_{P_a}(\mathcal{F})$. We will do this computation when $\mathcal{F} = \omega_X$ and X is any irregular variety over an arbitrary algebraically closed field k.

Lemma 3.1. *Let X be a smooth variety of dimension d, equipped with a non-trivial morphism to an abelian variety $a\colon X \to A$ (over any algebraically closed field k). Then, there exists a fourth quadrant spectral sequence*

$$E_2^{i,j} = \mathcal{E}xt^i_{\mathrm{Pic}^0 A}(R^{-j}\Phi_{P_a}(\omega_X), \mathcal{E}) \Rightarrow R^{i+j+d}q_*(\mathcal{H}om_{X\times \mathrm{Pic}^0 A}(P_a, q^*\mathcal{E})).$$

Proof. Let $\mathcal{E}$ be a sheaf on $\mathrm{Pic}^0 A$. By Grothendieck-Verdier duality I.I.1.4,

$$\mathrm{R}\mathcal{H}om_{\mathrm{Pic}^0 A}(\mathrm{R}\Phi_{P_a}(\omega_X), \mathcal{E}) \cong \mathrm{R}q_*(\mathcal{H}om_{X\times \mathrm{Pic}^0 A}(P_a, q^*\mathcal{E}))[d]. \tag{3.1}$$

Indeed,

$$\begin{aligned} \mathrm{R}\mathcal{H}om_{\mathrm{Pic}^0 A}(\mathrm{R}\Phi_{P_a}(\omega_X), \mathcal{E}) &= \mathrm{R}\mathcal{H}om_{\mathrm{Pic}^0 A}(\mathrm{R}q_*(p^*\omega_X \otimes P_a), \mathcal{E}) \\ &\overset{GVd}{\cong} \mathrm{R}q_*(\mathrm{R}\mathcal{H}om_{X\times \mathrm{Pic}^0 A}(p^*\omega_X \otimes P_a, p^*\omega_X \otimes q^*\mathcal{E}[d])) \\ &\cong \mathrm{R}q_*(\mathcal{H}om_{X\times \mathrm{Pic}^0 A}(P_a, q^*\mathcal{E}))[d]. \end{aligned}$$

Therefore we have a fourth quadrant spectral sequence (see [Hu, Ex. 2.70 ii)])

$$E_2^{i,j} = \mathcal{E}xt^i_{\mathrm{Pic}^0 A}(R^{-j}\Phi_{P_a}(\omega_X), \mathcal{E}) \Rightarrow R^{i+j+d}q_*(\mathcal{H}om_{X\times \mathrm{Pic}^0 A}(P_a, q^*\mathcal{E})). \qquad \square$$

Proposition 3.2 ([BLNP, Prop. 6.1])**.** *Let X be a smooth variety of dimension d, equipped with a non-trivial morphism to an abelian variety $a\colon X \to A$ (over any algebraically closed field k) such that the map $a^*\colon \mathrm{Pic}^0 A \to \mathrm{Pic}^0 X$ is an embedding. Then*

$$R^d\Phi_{P_a}(\omega_X) \cong k(\hat{0}),$$

where $P_a = (a \times \mathrm{id})^*\mathcal{P}$ and $\mathcal{P}$ is the Poincaré line bundle on $A \times \mathrm{Pic}^0 A$.

Remark 3.3. *When X itself is an abelian variety (or a complex torus) we recover a well-known result (see [BL, Cor. 14.1.6] for an elementary proof in the complex case and [Hu, pg. 202], [Mu2, pg. 128] for arbitrary characteristic). This fact is crucial in the proof of Mukai Equivalence Theorem I.2.1.*

Proof. The top cohomological support locus is $V^d(\omega_X) = \{\hat{0}\}$. By base change

[Mu2, Cor. 3, pg. 53], it follows that, for $\alpha \in \operatorname{Pic}^0 A$,

$$R^d\Phi_{P_a}(\omega_X) \otimes k(\hat{\alpha}) \cong \begin{cases} k(\hat{0}) & \text{if } \alpha = \hat{0} \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

It follows that $R^d\Phi_{P_a}(\omega_X)$ is a sheaf set-theoretically supported at $\hat{0}$. It remains to prove that the schematic support is also $\hat{0}$.

Let $B = \mathcal{O}_{\operatorname{Pic}^0 A, \hat{0}}$ and $\mathfrak{m}$ its maximal ideal. By Nakayama's lemma, (3.2) implies that $R^d\Phi_{P_a}(\omega_X)$ is supported only at $\hat{0}$ and that $R^d\Phi_{P_a}(\omega_X) \cong B/J$, where J is a $\mathfrak{m}$-primary ideal.

Claim. $P_a|_{X \times \operatorname{Spec} B/J}$ *is trivial.*

Proof of the Claim. Let $\mathcal{E}$ be a sheaf on $\operatorname{Pic}^0 A$. By Lemma 3.1 we have a fourth quadrant spectral sequence

$$E_2^{i,j} = \mathcal{E}xt^i_{\operatorname{Pic}^0 A}(R^{-j}\Phi_{P_a}(\omega_X), \mathcal{E}) \Rightarrow R^{i+j+d}q_*(\mathcal{H}om_{X \times \operatorname{Pic}^0 A}(P_a, q^*\mathcal{E})).$$

Clearly the term $E_2^{i,j}$ is non-zero only if $i \geqslant 0$. Assuming $i \geqslant 0$, in the case $i + j + d = 0$, i.e. $-j = i + d$ we have that $R^{-j}\Phi_{P_a}(\omega_X)$ is non-zero if and only if $-j = d$, i.e. $i = 0$. In conclusion for $i + j + d = 0$ the only non-zero E_2-term is $E_2^{0,-d} = \mathcal{H}om_{\operatorname{Pic}^0 A}(R^d\Phi_{P_a}(\omega_X), \mathcal{E})$. Since the differentials from and to $E_2^{0,-d}$ are zero, we get that

$$\mathcal{H}om_{\operatorname{Pic}^0 A}(R^d\Phi_{P_a}(\omega_X), \mathcal{E}) = E_2^{0,-d} = E_\infty^{0,-d} \cong q_*(\mathcal{H}om_{X \times \operatorname{Pic}^0 A}(P_a, q^*\mathcal{E})).$$

Taking global sections, we get the isomorphism (functorial in $\mathcal{E}$)

$$\operatorname{Hom}_{\operatorname{Pic}^0 A}(R^d\Phi_{P_a}(\omega_X), \mathcal{E}) \cong \operatorname{Hom}_{X \times \operatorname{Pic}^0 A}(P_a, q^*\mathcal{E}). \tag{3.3}$$

Using the previous isomorphism twice, once for $\mathcal{E} = B/J$ and the other for $\mathcal{E} = k(\hat{0})$, by functoriality we get the commutative diagram

$$\begin{tikzcd} B/J \arrow[r, equal] \arrow[d] & \operatorname{Hom}(B/J, B/J) \arrow[r, equal] \arrow[d] & \operatorname{Hom}(P_a|_{X \times \operatorname{Spec} B/J}, \mathcal{O}_{X \times \operatorname{Spec} B/J}) \arrow[d] \\ k(\hat{0}) \arrow[r, equal] & \operatorname{Hom}(B/J, k(\hat{0})) \arrow[r, equal] & \operatorname{Hom}(P_a|_{X \times \{\hat{0}\}}, \mathcal{O}_{X \times \{\hat{0}\}}). \end{tikzcd} \tag{3.4}$$

Since $P_a|_{X \times \{\hat{0}\}}$ is trivial, we can take an isomorphism $h \in \operatorname{Hom}(P_a|_{X \times \{\hat{0}\}}, \mathcal{O}_{X \times \{\hat{0}\}})$. By the diagram above, h lifts to a morphism $\bar{h} \colon P_a|_{X \times \operatorname{Spec} B/J} \to \mathcal{O}_{X \times \operatorname{Spec} B/J}$. Since $\bar{h}$ is a map between invertible sheaves on $X \times \operatorname{Spec} B/J$ which is an isomorphism when restricted to $X \times \{\hat{0}\}$, $\bar{h}$ is an isomorphism. Therefore $P_a|_{X \times \operatorname{Spec} B/J}$

is trivial. The Claim is proved.

At this point the Proposition follows since *the smooth point* $\hat{0}$ *of* $\operatorname{Pic}^0 A$ *is the maximal subscheme* Z *of* $\operatorname{Pic}^0 A$ *such that* $P_a|_{X\times Z}$ *is trivial* ([Mu2] §10). This in turn follows from the well-known fact ([Mu2] §13) that the smooth point $\hat{0}$ of $\operatorname{Pic}^0 A$ is the maximal subscheme Z of $\operatorname{Pic}^0 A$ such that $\mathcal{P}|_{A\times Z}$ is trivial, combined with the fact that $a^* \operatorname{Pic}^0 A \to \operatorname{Pic}^0 X$ is an embedding. However, we provide an equivalent but self-contained argument. Let us consider the functor $\mathrm{R}\Psi_{P_a}\colon \mathrm{D}^b(\operatorname{Pic}^0 A) \to \mathrm{D}^b(X)$, defined by $\mathrm{R}\Psi_{P_a}(\cdot) = \mathrm{R}p_*(P_a \otimes q^*(\cdot))$ in (I.1.3). Since $P_a = (a\times \mathrm{id}_{\operatorname{Pic}^0 A})^*\mathcal{P}$, it follows that

$$\mathrm{R}\Psi_{P_a} \cong \mathrm{L}a^* \circ \mathrm{R}\Psi_{\mathcal{P}}\,. \tag{3.5}$$

The Claim implies, by the Künneth formula, that

$$\mathrm{R}\Psi_{P_a}(B/J) = R^0\Psi_{P_a}(B/J) = \mathcal{O}_X^{\oplus r}, \tag{3.6}$$

where $r = \operatorname{length} B/J$. On the other hand, by [M1, Lemma 4.8], $\mathrm{R}\Psi_{\mathcal{P}}(B/J) = R^0\Psi_{\mathcal{P}}(B/J) := U$, where U is a *unipotent* vector bundle on A of rank r, i.e. a vector bundle having a filtration $0 = U_0 \subset U_1 \subset \cdots \subset U_{r-1} \subset U_r = U$, such that $U_i/U_{i-1} \cong \mathcal{O}_A$. By (3.5) and (3.6) it follows that a^*U is trivial. The filtration of U, pulled back via a, induces the filtration of the trivial bundle:

$$0 \subset a^*U_1 \subset \cdots \subset a^*U_{r-1} \subset a^*U_r = \mathcal{O}_X^{\oplus r},$$

where $a^*U_i/a^*U_{i-1} \cong \mathcal{O}_X$. Since $h^0(X, a^*U_i) \le i$ for all i, the fact that a^*U_r is trivial implies easily, by descending induction on i, that

$$h^0(X, a^*U_i) = i \quad \text{for all } i. \tag{3.7}$$

This in turn implies that the sequence

$$0 \to a^*U_{i-1} \to a^*U_i \to \mathcal{O}_X \to 0$$

splits for all i (the coboundary map $H^0(\mathcal{O}_A) \to H^1(U_{i-1})$ is zero). In particular the extension

$$0 \to \mathcal{O}_X \to a^*U_2 \to \mathcal{O}_X \to 0$$

is split. But the natural pullback map

$$H^1(\mathcal{O}_A) \cong \operatorname{Ext}^1(\mathcal{O}_A, \mathcal{O}_A) \to \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) \cong H^1(\mathcal{O}_X) \tag{3.8}$$

is identified with the differential at $\hat{0}$ of the map $a^*\colon \operatorname{Pic}^0 A \to \operatorname{Pic}^0 X$. Since a^* is assumed to be an embedding, (3.8) is injective. Hence also the extension

$$0 \to \mathcal{O}_A \to U_2 \to \mathcal{O}_A \to 0$$

is split. This yields to $h^n(U) \geqslant 2$. But this is impossible since, by Mukai's inversion,

$\mathrm{R}\Phi_{\mathcal{P}}(U) = \hat{U}[-n] \cong (-1_{\mathrm{Pic}^0 A})^* B/J[-n]$, and therefore, by base change, $H^n(U) \cong (B/J) \otimes k(\hat{0}) \cong k(\hat{0})$. In conclusion $r = \mathrm{length}(B/J) = 1$, i.e. $B/J \cong k(\hat{0})$. □

3.2 Torsion part of the Fourier-Mukai of the structural sheaf

By Corollary I.1.18 when $\mathrm{gv}_a(\omega_X) \geqslant 1$, $\widehat{\mathcal{O}_X}$ is torsion-free. However, even when $\mathrm{gv}_a(\omega_X) = 0$ we can control the torsion part of $\widehat{\mathcal{O}_X}$ in some particular cases. More precisely, since we have already computed $R^d\Phi_{P_a}\,\omega_X$ for an irregular variety (Theorem 3.2), to compute $\tau(\widehat{\mathcal{O}_X})$ we will need only to impose the hypothesis of Proposition 2.8 adapted to the canonical sheaf.

Hypothesis 3.4. *Let X be a variety of dimension d, equipped with a morphism to an abelian variety $a\colon X \to A$ such that:*

(a) $\operatorname{codim} V_a^i(\omega_X) \geqslant i+1$ *for all i such that $0 < i < d$;*

(b) the map $a^* \operatorname{Pic}^0 A \to \operatorname{Pic}^0 X$ is an embedding.

Remark 3.5. *Note that the hypothesis (a) is slightly weaker than* $\mathrm{gv}_a(\omega_X) \geqslant 1$. *In fact, Hypothesis (a) is equivalent to* $\mathrm{gv}_a(\omega_X) \geqslant 1$ *unless* $\dim X = \dim A$, *i.e. the morphism a is surjective. Observe also that in any case hypothesis (a) implies that* $\mathrm{gv}_a(\omega_X) \geqslant 0$. *Hence, by Proposition 1.10 $a\colon X \to A$ is generically finite onto its image and X is of maximal Albanese dimension.*

Then, the following result is a corollary of Proposition 2.8 and Theorem 3.2.

Corollary 3.6 ([BLNP, Prop. 4.8])**.** *Under Hypothesis 3.4, assume moreover that* $\dim X = \dim A$, *i.e. that the morphism a is surjective. Then $\mathcal{O}_X$ is a WIT_d sheaf and $\widehat{\mathcal{O}_X}$ has a non-trivial torsion subsheaf. More precisely, the torsion part $\tau(\widehat{\mathcal{O}_X})$ is isomorphic to the one-dimensional skyscraper sheaf $\mathbb{C}(\hat{0})$.*

Proof. We can apply Proposition 2.8 to $\mathcal{F} = \omega_X$ noting that $\mathrm{R}\Delta\,\omega_X = \mathcal{O}_X$. Since $\dim X = \dim A$, Hypothesis 3.4*(b)* implies that $\mathrm{gv}_a(\mathcal{F}) < 1$ so, by Corollary I.1.18, the sheaf $\widehat{\mathcal{O}_X}$ has a non-trivial torsion subsheaf. Now, by Proposition 3.2, $R^d\Phi_{P_a}(\omega_X) \cong \mathbb{C}(\hat{0})$. Hence $E_2^{d,-d} = \mathcal{E}xt^d(R^d\Phi_{P_a}\,\omega_X, \mathcal{O}_{\mathrm{Pic}^0 A}) \cong \mathcal{E}xt^d(\mathbb{C}(\hat{0}), \mathcal{O}_{\mathrm{Pic}^0 A}) = \mathbb{C}(\hat{0})$ and

$$\mathcal{E}xt^d(R^d\Phi_{P_a}\,\omega_X, \mathcal{O}_{\mathrm{Pic}^0 A}) = \mathbb{C}(\hat{0}) \twoheadrightarrow \tau(\widehat{\mathcal{O}_X}).$$

As we already know that $\tau(\widehat{\mathcal{O}_X}) \neq 0$, it follows that $\tau(\widehat{\mathcal{O}_X}) \cong \mathbb{C}(\hat{0})$. □

Remark 3.7. *If we take the Fourier-Mukai transform $R^q\Psi_{\mathcal{P}}$ of the injection* $\mathbb{C}(\hat{0}) = \tau(\widehat{a_*\mathcal{O}_X}) \hookrightarrow \widehat{a_*\mathcal{O}_X}$, *we recover the natural injection* $\mathcal{O}_A \to a_*\mathcal{O}_X$.

Corollary 3.8 ([BLNP, Cor. 4.11])**.** *Assume Hypothesis 3.4 and that X is of general type.*

(a) *If* $\dim X < \dim A$ *then* $a_*\omega_X$ *is CGG.*

(b) *If* $\dim X = \dim A$*, i.e.* a *is surjective, then* $a_*\omega_X$ *is ECGG, with* $\{\hat{0}\}$ *as underlying subset of* $\operatorname{supp}\tau(\mathrm{R}\Delta(a_*\omega_X))$.

Proof. By Grauert-Riemenschneider vanishing (and projection formula), $R^i a_*(\omega_X \otimes a^*\alpha) = 0$ for $i > 0$. Therefore the Leray spectral sequence degenerates giving

$$V_a^i(\omega_X) = V^i(a_*\omega_X). \tag{3.9}$$

(a) If $\dim X < \dim A$ then $\operatorname{gv}(a_*\omega_X) \geqslant 1$. Therefore, $a_*\omega_X$ is CGG by Theorem 2.6*(a)*. As remarked before, this part of the result was already well-known [PP3, Prop. 5.5].

(b) If $\dim X = \dim A$ then $\tau(\widehat{\mathcal{O}_X}) \cong \mathbb{C}(\hat{0})$ (Corollary 3.6). Hence, by Theorem 2.6*(b)*, $a_*\omega_X$ is ECGG. □

3.3 Characterization of abelian varieties

As a first consequence we get a characterization of abelian varieties (this is mainly [BLNP, Prop. 4.10]).

Proposition 3.9. *The following are equivalent:*

(a) X *is a smooth projective variety of maximal Albanese dimension such that* $\dim V^0(\omega_X) = 0$*;*

(b) *Hypothesis 3.4 holds, and* X *is not of general type;*

(c) *Hypothesis 3.4 holds, and* $\chi(\omega_X) = 0$*;*

(d) *the morphism* $a\colon X \to A$ *is birational.*

Proof. It is clear that item *(d)* implies the other items.

$(a) \Rightarrow (c)$**:** Since X is of maximal Albanese dimension by Theorem 1.4, $\operatorname{gv}(\omega_X) \geqslant 0$. By Corollary I.1.13, Hypothesis 3.4 are clearly satisfied.

$(b) \Rightarrow (c)$**:** The morphism a is generically finite (Prop. 1.10). In this case it is well-known that $\chi(\omega_X) \geqslant 1$ implies that X is of general type (by [CH1, Thm. 4.4], one knows even that the tricanonical map of X is birational).

$(c) \Rightarrow (d)$**:** Since $\chi(\omega_X) = 0$ then, by (I.1.4), $\widehat{\mathcal{O}_X}$ is a torsion sheaf. By Corollary I.1.18 it follows that $\operatorname{gv}(\omega_X) = 0$. We already know that this, together with Hypothesis 3.4, is equivalent to the fact that the morphism a is surjective. Therefore, by Corollary 3.6, $\widehat{\mathcal{O}_X} = \mathbb{C}(\hat{0})$. By Proposition I.1.1

$$\mathbb{C}(\hat{0}) = \widehat{\mathcal{O}_X} = \widehat{\mathrm{R}a_*\mathcal{O}_X},$$

where the hat on the left is the transform $\mathrm{R}\Phi_{P_a}$ (from X to $\operatorname{Pic}^0 A$) and the hat on the right is the transform $\mathrm{R}\Phi_{\mathcal{P}}$ (from A to $\operatorname{Pic}^0 A$). By Mukai's

Theorem I.2.1

$$\mathcal{O}_A = \mathrm{R}\Psi_{\mathcal{P}}(\mathbb{C}(\hat{0})) = \mathrm{R}\Psi_{\mathcal{P}}(\mathrm{R}\Phi_{\mathcal{P}}(\mathrm{R}a_*\mathcal{O}_X)[d]) \overset{I.2.1}{=} (-1)_A^*\mathrm{R}a_*\mathcal{O}_X$$

whence $\mathrm{R}a_*\mathcal{O}_X = \mathcal{O}_A$. In particular, a has degree 1. $\square$

4 Birationality of the bicanonical map

We recall the following theorems:

Theorem 4.1 ([CH1, Thm. 4.4]). *Let X be of general type and maximal Albanese dimension. If $\chi(\omega_X) > 0$, then the rational map associated to ω_X^3 is birational.*

Theorem 4.2 ([PP3, Thm. 6.1, Rem. 6.5]). *Let X be a smooth projective complex variety with $\mathrm{gv}(\omega_X) \geqslant 1$. Then $\omega_X^3 \otimes \alpha$ is very ample away from the exceptional locus of the Albanese map for every $\alpha \in \mathrm{Pic}^0 X$.*

Remark 4.3. *In the articles of Ein, Lazarsfeld, Pareschi and Popa it is common to express the condition that $\mathrm{gv}(\omega_X) \geqslant 1$ by saying that X is of maximal Albanese dimension and the Albanese image of X is not ruled by tori. If X is a maximal Albanese dimension variety with $\mathrm{gv}(\omega_X) < 1$, then by Theorem 1.4 $\mathrm{gv}(\omega_X) = 0$. Let $T \subseteq V^k(\omega_X)$ an irreducible component of codimension k. Then by Theorem 1.6, the general fiber of $a(X) \to \mathrm{Pic}^0 T$ is a subtorus of $\mathrm{Alb}\, X$. Hence, if X is of maximal Albanese dimension and the Albanese image of X is not ruled by tori, in particular, $\mathrm{gv}(\omega_X) \geqslant 1$.*
To compare Theorem 4.2 with its original formulation in [PP3, Thm. 6.1, Rem. 6.5] we also point out that the hypothesis $\mathrm{gv}(\omega_X) \geqslant 1$ already implies that X is of general type (Proposition 3.9) and maximal Albanese dimension (Proposition 1.10).

These two results are sharp, since there are varieties of general type and maximal Albanese dimension, such that $\mathrm{gv}(\omega_X) \geqslant 1$ (hence $\chi(\omega_X) > 0$), such that the bicanonical map is not birational. Let us show the three paradigmatic constructions.

The first two examples is what one expects to be the building blocks of all other cases.

Example 4.4. Let (A, Θ) be an indecomposable principally polarized abelian variety, and let $X \to \Theta$ be a desingularization of Θ. Without loss of generality, we can assume that Θ is *symmetric*, i.e. $\Theta = (-1)^*\Theta$. The restriction map $H^0(A, \mathcal{O}_A(2\Theta)) \to H^0(\Theta, \mathcal{O}_\Theta(2\Theta)) = H^0(\Theta, \omega_\Theta^2)$ is surjective. Hence the projective map associated to ω_Θ^2 has degree two, since x and $-x$ have the same image. By a result of Ein-Lazarsfeld, Θ is normal and has rational singularities [EL, Thm. 1]. Hence Θ has canonical singularities (e.g. [K4, Thm. 11.1(1)]), and therefore $H^0(X, \omega_X^2) \cong H^0(\Theta, \omega_\Theta^2)$. It follows that the bicanonical map of X has degree 2 and it is not birational.
Observe that, $\chi(\omega_X) = \mathrm{gv}(\omega_X) = 1$ and X is clearly of maximal Albanese dimension and general type (since ω_X^3 is birational).

The second example verifies $\mathrm{gv}(\omega_X) = 0$ but it fulfills Hypothesis 3.4 and $\chi(\omega_X) = 1$.

Example 4.5. Let (A, Θ) be an indecomposable principally polarized abelian variety. Without loss of generality, we can assume as above that Θ is symmetric, i.e. $\Theta = (-1)^*\Theta$. Consider a covering $h\colon \tilde{X} \to A$ (finite and surjective morphism) of A branched along a reduced divisor $B \in |2\Theta|$. Observe that, since $B \in |2\Theta|$ is a reduced divisor, $\tilde{X}$ is a normal variety and $\omega_{\tilde{X}} = h^*\mathcal{O}_A(\Theta)$ is locally free. By [K3, Thm 17.13], (A, Θ) is a log canonical pair. Then by [KM, Prop. 5.20] $(\tilde{X}, 0)$ is also a log canonical pair, that is, $\tilde{X}$ has canonical singularities. Therefore, for any smooth variety X birationally equivalent to $\tilde{X}$, $H^0(X, \omega_X^2) \cong H^0(\tilde{X}, \omega_{\tilde{X}}^2)$ and we can check the birationality of the bicanonical map directly on $\tilde{X}$. On the other hand, the universal property of $\operatorname{Alb} X$ implies clearly that $A = \operatorname{Alb} X$.

Now consider the involution $\imath$ given by multiplication by -1 in A. Observe that, as the image of $\varphi_{|2\Theta|}$ is isomorphic to the Kummer variety (i.e the quotient $A/\imath$), any $B \in |2\Theta|$ is symmetric with respect to $\imath$. This implies that the involution $\imath$ lifts to an involution $\jmath$ on $\tilde{X}$ that commutes with the involution σ defined by the double cover. That is, we can think locally $\tilde{X}$ inside $A \times \mathbb{A}^1$ as defined by the equation $t^2 - s = 0$ where $(s)_0 = B$ and observe that $\imath^*s = s$. Or we can recall the global construction of $\tilde{X}$ as the divisor in $\mathbf{L} = \operatorname{Spec}_{\mathcal{O}_A} \operatorname{Sym} L^{-1}$ (where L is $\mathcal{O}_A(\Theta)$ possibly translated by a 2-torsion point) of the section $T^2 - p^*s \in H^0(\mathbf{L}, p^*\mathcal{O}(B))$.

Moreover, $\omega_{\tilde{X}} = \mathcal{O}(R)$, where R is the ramification locus, and $\omega_{\tilde{X}}^{\otimes 2} = h^*\mathcal{O}(B)$. So $H^0(\tilde{X}, \omega_{\tilde{X}}^{\otimes 2}) \cong H^0(A, \mathcal{O}(B) \oplus L)$. Hence the invariant and antiinvariant eigenspaces of $H^0(\tilde{X}, \omega_{\tilde{X}}^2)$ with respect to the involution σ are:

$$H^0(\tilde{X}, \omega_{\tilde{X}}^2)^+ \cong H^0(A, \mathcal{O}_A(\Theta)) \qquad \text{and} \qquad H^0(\tilde{X}, \omega_{\tilde{X}}^2)^- \cong H^0(A, \mathcal{O}_A(2\Theta)).$$

And both are invariant by $\jmath$. Hence the bicanonical map of $\tilde{X}$ factors through the quotient by $\jmath$ and is not birational. Indeed we have the following commutative diagram

$$\begin{tikzcd}
\tilde{X} \arrow[loop left, "\jmath"] \arrow[r, two heads, "\varphi_{|\omega_{\tilde{X}}^2|}", "(2:1)"'] \arrow[d, "h", "(2:1)"'] & \tilde{X}/\jmath \subseteq \mathbb{P}^{N_1} = \left|\omega_{\tilde{X}}^2\right|^\vee \arrow[d, "(2:1)"] \\
A \arrow[loop left, "\imath"] \arrow[r, two heads, "(2:1)", "\varphi_{|B|}"'] & K = A/\imath \subseteq \mathbb{P}^{N_2} = |2\Theta|^\vee .
\end{tikzcd}$$

It follows that the bicanonical map of X has degree 2 and it is not birational. Observe that, $\chi(\omega_X) = 1$ and $\operatorname{gv}(\omega_X) = 0$, since $\dim X = q(X) = \operatorname{codim}_{\operatorname{Pic}^0 X} \{\hat{0}\} = \operatorname{codim}_{\operatorname{Pic}^0 X} V^{\dim X}(\omega_X)$. However, $V^i(\omega_X) = \{\hat{0}\}$ for all $0 < i \leqslant \dim X$, so $\operatorname{codim} V^i(\omega_X) = q(X) = \dim X \geqslant i + 1$ for all i such that $0 < i < \dim X$ and Hypothesis 3.4 are fulfilled. It is clear that X has maximal Albanese dimension and it is of general type, since the canonical sheaf it is the pull-back of an ample line bundle Θ by a finite morphism.

The third example shows how can we construct other examples from smaller building blocks.

Example 4.6. Let $f : X \to Y$ be a fibration, i.e. a surjective morphism with connected fibers, and suppose that the general fiber F has non-birational bicanonical map. Since the bicanonical map of X restricts to a subsystem of the bicanonical map of F, the bicanonical map of X cannot be birational.
Thus, we can say that if a variety X has a fibration, whose general fiber has non-birational bicanonical map, then X has non-birational bicanonical map.

In the case of surfaces we have the following theorem, which shows that the previous three examples are the only possible cases that can occur when $q(S) \geqslant 2$.

Theorem 4.7. *(a)* ([CM1, Thm. 1.1]). *Let S be a complex irregular surface of general type with $\chi(\omega_S) \geqslant 2$ and non-birational bicanonical map. Then S has a pencil of curves of genus 2.*

(b) (Corollary of [CFM, Thm. A]). *Let S be a complex irregular surface of general type with $q(S) \geqslant 3$ and non-birational bicanonical map. Suppose that S has not a pencil of curves of genus 2. Then S is birational to a theta-divisor in a principally polarized abelian threefold.*

(c) ([CM2, Thm. 1.1]). *Let S be a complex irregular surface of general type with $q(S) = 2$ and non-birational bicanonical map. Suppose that S has not a pencil of curves of genus 2. Then S is birational to a double cover of a principally polarized abelian surface (A, Θ) branched along a smooth divisor $B \in |2\Theta|$.*

Observe that curves of genus 2 are also theta-divisors in a principally polarized abelian surface (the Jacobian of the curve of genus 2). Thus, in the case of surfaces, the previous results show that when $q(S) \geqslant 2$ and S does not present the case of Example 4.6, we are in the case of Example 4.4 or Example 4.5.

Summarizing:

- The behavior of the tricanonical map for varieties of maximal Albanese dimension varieties and arbitrary dimension (see Theorems 4.2 and 4.1),
- the behavior of the bicanonical map in the case of surfaces (see Theorem 4.7) and
- Examples 4.4, 4.5 and 4.6 in arbitrary dimension,

justify our aim to study and classify varieties with $\mathrm{gv}(\omega_X) \geqslant 1$ (or, slightly more general, varieties that fulfill Hypothesis 3.4) and such that the rational map associated to $\omega_X^2 \otimes \alpha$ is not birational for some $\alpha \in \mathrm{Pic}^0 X$.

We introduce a piece of notation that we will use in the sequel.

Terminology/Notation 4.8. *(a) We denote U_0 the complement in $\operatorname{Pic}^0 A$ of the closed subset $V_a^1(\omega_X)$. Since, by Corollary I.1.13, $V_a^1(\omega_X) \supseteq \cdots \supseteq V_a^d(\omega_X)$, it follows that, for all $\alpha \in U_0$, $h^0(\omega_X \otimes a^*\alpha)$ takes the minimal value, i.e. $\chi(\omega_X)$.*

(b) Given a point $p \in X$, we denote $\mathcal{B}_a(p)$ the subset (closed in U_0)

$$\mathcal{B}_a(p) = \{\alpha \in U_0 \mid p \in \operatorname{Bs}(\omega_X \otimes a^*\alpha)\}.$$

(c) We will say that a line bundle "is birational" to mean that the associated rational map to projective space is birational onto its image.

4.1 Birationality criterion

The statement we are aiming at is,

Theorem 4.9 ([BLNP, Thm. 4.13]). *Let X be a variety of general type satisfying Hypothesis 3.4. Suppose that for general p in X, $\operatorname{codim}_{\operatorname{Pic}^0 A} \mathcal{B}_a(p) \geqslant 2$, then $\omega_X^2 \otimes a^*\alpha$ is birational for all $\alpha \in \operatorname{Pic}^0 A$. In particular, ω_X^2 is birational.*

Remark 4.10. *When $\dim X < \dim A$ the general type assumption is a consequence of Hypothesis 3.4 by Remark 3.5 and Theorem 4.2 that, not only implies the ampleness of ω_X, but gives an effective bound.*
When $\dim X = \dim A$, recall that by Proposition 3.9, being of general type is equivalent to $\chi(\omega_X) > 0$.

Proof of Theorem 4.9. We first recall some basic facts about the Fourier-Mukai transform of the sheaves $\mathcal{I}_p \otimes \omega_X$. First of all, if p does not belong to $exc(a)$ then

$$R^i a_*(\mathcal{I}_p \otimes \omega_X \otimes a^*\alpha) = 0 \qquad \text{for } i > 0. \tag{4.1}$$

This follows immediately from the exact sequence

$$0 \to \mathcal{I}_p \otimes \omega_X \to \omega_X \to \mathcal{O}_p \otimes \omega_X \to 0 \tag{4.2}$$

and the Grauert-Riemenschneider vanishing theorem. Hence, as for the canonical sheaf (see the proof of Corollary 3.6), the Leray spectral sequence yields to

$$V_a^i(\mathcal{I}_p \otimes \omega_X) = V^i(a_*(\mathcal{I}_p \otimes \omega_X)). \tag{4.3}$$

By sequence (4.2), tensored by $a^*\alpha$, it follows that

$$V_a^i(\mathcal{I}_p \otimes \omega_X) = V_a^i(\omega_X) \qquad \text{for all } i \geqslant 2. \tag{4.4}$$

Concerning the case $i = 1$ we have the surjection

$$H^1(\mathcal{I}_p \otimes \omega_X \otimes a^*\alpha) \twoheadrightarrow H^1(\omega_X \otimes a^*\alpha),$$

which is an isomorphism if, and only if, p is not a base point of $\omega_X \otimes a^*\alpha$. In other words

$$V_a^1(\mathcal{I}_p \otimes \omega_X) = \mathcal{B}_a(p) \cup V_a^1(\omega_X).$$

Therefore the hypothesis about $\mathcal{B}_a(p)$ ensures that

$$\operatorname{codim} V_a^1(\mathcal{I}_p \otimes \omega_X) \geqslant 2. \tag{4.5}$$

Now we distinguish two cases:

(a) $\dim X < \dim A$. In this case, by Hypothesis 3.4, together with (4.3), (4.4) and (4.5), $\operatorname{gv}(a_*(\mathcal{I}_p \otimes \omega_X)) \geqslant 1$. Hence, by Theorem 2.6*(a)*, $a_*(\mathcal{I}_p \otimes \omega_X)$ is CGG. Therefore $\mathcal{I}_p \otimes \omega_X$ itself is CGG outside $exc(a)$ (with respect to a). Since the same is true for ω_X (Corollary 3.8*(a)*), it follows from Proposition 2.5*(a)* that, for general $p \in X$ and for all $\alpha \in \operatorname{Pic}^0 A$, $\mathcal{I}_p \otimes \omega_X^2 \otimes a^*\alpha$ is globally generated outside $exc(a)$. This means that the projective map associated to $\omega_X^2 \otimes a^*\alpha$ is birational.

(b) $\dim X = \dim A$, i.e. the map a is surjective. Again by Hypotheses 3.4, (4.4) and (4.5), the sheaf $\mathcal{I}_p \otimes \omega_X$ satisfies,

$$\operatorname{codim} V^i(\mathcal{I}_p \otimes \omega_X) \geqslant i+1 \quad \text{for all } i \text{ such that } 0 < i < d$$

while

$$V^d(\mathcal{I}_p \otimes \omega_X) = \{\hat{0}\} \quad \text{and} \quad R^d\Phi_{P_a}(\mathcal{I}_p \otimes \omega_X) = R^d\Phi_{P_a}(\omega_X) = \mathbb{C}(\hat{0}).$$

Therefore we can apply Proposition 2.8, using the same arguments as in Corollary 3.6 and proving that the torsion of $\mathrm{R}\mathcal{H}om(a_*(\mathcal{I}_p), \mathcal{O}_A)$ is $\mathbb{C}(\hat{0})$. Hence, by Theorem 2.6*(b)*, $a_*(\mathcal{I}_p \otimes \omega_X)$ is ECGG. It follows that, for p not belonging to $exc(a)$, the sheaf $\mathcal{I}_p \otimes \omega_X$ is *ECGG* away of $exc(a)$. Let W be the non-empty open set of points $p \in X$ such that ω_X is *CGG* at p. In view of Remark 2.3, W is the complement of the intersection of all base loci $\operatorname{Bs}(\omega_X \otimes a^*\alpha^{-1})$, for $\alpha \in \operatorname{Pic}^0 A$ such that $h^0(\omega_X \otimes a^*\alpha^{-1})$ is minimal, i.e. equal to $\chi(\omega_X)$. Now let $\alpha \in \operatorname{Pic}^0 A$. It follows from Proposition 2.5*(b)* that, if q is not a base point of $\omega_X \otimes a^*\alpha$ (and does not lie in $exc(a)$), then $\mathcal{I}_p \otimes \omega_X^2 \otimes a^*\alpha$ is globally generated at q. Denoting U_α the complement of $exc(a) \cup \operatorname{Bs}(\omega_X \otimes a^*\alpha)$, we conclude that for all $p \in U_\alpha \cap W$ the sheaf $\mathcal{I}_p \otimes \omega_X^2 \otimes a^*\alpha$ is globally generated at all points of $U_\alpha \cap W$. As above, this means that the projective map associated to $\omega_X^2 \otimes a^*\alpha$ is an isomorphism on $U_\alpha \cap W$. □

Consider the relative base locus

$$\mathcal{B}_a = \{(p, \alpha) \in X \otimes U_0 \,|\, p \text{ is a base point of } \omega_X \otimes a^*\alpha\}.$$

equipped with the projections on the two factors, p and q. $\mathcal{B}_a$ has a natural subscheme structure given by the image of the relative evaluation map $q^*(q_*\mathcal{L}) \otimes \mathcal{L}^{-1} \to$

$\mathcal{O}_{X\times U_0}$, where $\mathcal{L} = (p^*\omega_X \otimes P_a)|_{X\times U_0}$.

Corollary 4.11 ([BLNP, Cor. 4.14]). *Let X be a variety of general type satisfying Hypothesis 3.4. Suppose that there exists $\alpha \in \operatorname{Pic}^0 A$ such that $\omega_X^2 \otimes a^*\alpha$ is not birational.*
Then, for every $\beta \in U_0$, the fiber of $q\colon \mathcal{B}_a \to U_0$, $\operatorname{Bs}(\omega_X \otimes a^*\beta)$, has codimension one. Moreover X is covered by the divisorial components of $\operatorname{Bs}(\omega_X \otimes a^*\beta)$, for β varying in U_0.

Note that it makes sense to speak of the base locus of $\omega_X \otimes a^*\alpha$ since, by Proposition 3.9, the hypotheses of both the Theorem and the Corollary imply that $\chi(\omega_X) > 0$, whence $h^0(\omega_X \otimes a^*\alpha) > 0$ for all $\alpha \in \operatorname{Pic}^0 A$.

Proof of Corollary 4.11. Let $\mathcal{B}_a$ the closed subvariety of $X \times U_0$ defined as

$$\mathcal{B}_a = \{(p,\alpha) \in X \otimes U_0 \,|\, p \text{ is a base point of } \omega_X \otimes a^*\alpha\},$$

with the scheme structure given by the relative evaluation map. By base change, in any fiber over $\alpha \in U_0$, we have the evaluation map $H^0(\omega_X \otimes a^*\alpha) \otimes (\omega_X \otimes a^*\alpha)^{-1} \to \mathcal{O}_X$. Hence any fiber of $\mathcal{B}_a \to U_0$ is of codimension at least 1 because $h^0(\omega_X \otimes a^*\alpha) \geqslant \chi(\omega_X) > 0$.
On the other hand, since we are assuming that $\omega_X^2 \otimes a^*\alpha$ is not birational for some $\alpha \in \operatorname{Pic}^0 X$, then for a general $p \in X$ we have that $\operatorname{codim} \mathcal{B}_a(p) = 1$ by Theorem 4.9. Since $\mathcal{B}_a(p)$ is the fiber of the projection $p\colon \mathcal{B}_a \to X$ it follows that $\operatorname{codim}_{X\times U_0} \mathcal{B}_a = 1$. By semicontinuity of the fiber dimension, we have that all the fibers of the other projection $q\colon \mathcal{B}_a \to \operatorname{Pic}^0 A$ have at least codimension 1 in X. $\square$

Recall that by Theorem 4.9, under the hypothesis of the previous Corollary 4.11, $\mathcal{B}_a$ has codimension 1 and that its divisorial part is dominant on X and surjects on U_0 via the projections p and q.

Terminology/Notation 4.12. *Under the hypothesis of the previous Corollary 4.11, let $\mathcal{Y}$ be the subscheme of the relative base locus*

$$\mathcal{Y} \subseteq \mathcal{B}_a = \{(p,\alpha) \in X \otimes U_0 \,|\, p \textit{ is a base point of } \omega_X \otimes a^*\alpha\},$$

defined as the union of the divisorial components of $\mathcal{B}_a$ that dominate U_0. Let $\overline{\mathcal{Y}}$ be its closure in $X \times \operatorname{Pic}^0 X$.

(a) For $\alpha \in U_0$, we will denote by F_α the (scheme-theoretic) fiber of the projection $\overline{\mathcal{Y}} \to U_0$ (In fact, it coincides with the fiber of $q\colon \mathcal{Y} \to U_0$).
(b) For $p \in X$ let $\mathcal{D}_p$ be the fiber of the projection $\overline{\mathcal{Y}} \to X$.

Since we have restricted $\mathcal{Y}$ to be the union of the divisorial components of $\mathcal{B}_a$ that dominate U_0 we have the following interpretation of the fibers of the projections p and q restricted to $\mathcal{Y}$.

Remark 4.13. *Under the hypothesis of the previous Corollary 4.11 and using Terminology/Notation 4.12 observe that:*

(a) $|\omega_X \otimes a^*\alpha|$ *has a base divisor for all* $\alpha \in U_0$*. More precisely, at a* general *point* $\alpha \in U_0$*,* F_α *is the fixed divisor of* $\omega_X \otimes a^*\alpha$*:*

$$|\omega_X \otimes a^*\alpha| = |M_\alpha| + F_\alpha$$

where $|M_\alpha|$ *is the (possibly empty) mobile part.*

(b) $\mathcal{D}_p$ *is the closure of the union of the divisorial components of the locus of* $\alpha \in U_0$ *such that* $p \in \mathrm{Bs}(\omega_X \otimes a^*\alpha)$*.*

4.2 Decomposition

Keeping the Terminology/Notation 4.8 and 4.12, we have the following strong constraint on the Albanese and Picard variety of X on varieties of non-birational bicanonical map. Since, by Hypothesis 3.4, $a^* : \mathrm{Pic}^0 A \to \mathrm{Pic}^0 X$ is an embedding, in the next two lemmas, we will drop a^* from the notation.

Lemma 4.14. *Under the hypotheses of Corollary 4.11 and using Terminology/Notation 4.8 and 4.12, consider a fixed point* $\alpha_0 \in U_0$*, and the map*

$$f_{\alpha_0} \colon U_0 \to \mathrm{Pic}^0 X \qquad \alpha \mapsto \mathcal{O}_X(F_\alpha - F_{\alpha_0}).$$

It induces an idempotent homomorphism $f \colon \mathrm{Pic}^0 A \to \mathrm{Pic}^0 X$*, i.e.*

$$f^2 = f \quad \text{ and } \quad \mathrm{Pic}^0 X \cong \ker f \times \ker(\mathrm{id} - f).$$

Moreover $\dim \ker(\mathrm{id} - f) > 0$.

Proof. We consider the Abel-Jacobi map

$$f_{\alpha_0} \colon U_0 \to \mathrm{Pic}^0 X \qquad \alpha \mapsto \mathcal{O}_X(F_\alpha - F_{\alpha_0})$$

where α_0 is fixed in U_0. Since it is a map between abelian varieties, it extends to a morphism from $\mathrm{Pic}^0 A \to \mathrm{Pic}^0 X$. By rigidity,

$$f := f_{\alpha_0} - f_{\alpha_0}(\hat{0}) \colon \mathrm{Pic}^0 A \to \mathrm{Pic}^0 X$$

is a homomorphism. Note that f does not depend on α_0 since, given another suitably general $\alpha_1 \in U_0$, $f_{\alpha_1} - f_{\alpha_0}$ is a translation.

It remains to be proven that f is an idempotent homomorphism. Let $\alpha, \beta \in U_0$. We have that

$$\mathcal{O}_X(M_\alpha) \otimes \mathcal{O}_X(F_\beta) = \omega_X \otimes \alpha \otimes f(\beta \otimes \alpha^{-1}). \tag{4.6}$$

This follows by definition of f since the left-hand side is isomorphic to

$$\mathcal{O}_X(M_\alpha + F_\alpha) \otimes \mathcal{O}_X(F_\beta - F_\alpha) = \omega_X \otimes \alpha \otimes f(\beta \otimes \alpha^{-1}).$$

Let $W \subseteq U_0$ the open set such that for every $\alpha \in W$, F_α is the fixed divisor of $|\omega_X \otimes \alpha|$ (see Remark 4.13). Consider the map $(\alpha, \beta) \stackrel{\bar{f}}{\mapsto} \alpha \otimes f(\beta \otimes \alpha^{-1})$ and $(\alpha, \beta) \in \bar{f}^{-1}(W) \cap p_1^{-1}(W)$. Then, the fixed divisors of $\omega_X \otimes \alpha$ and $\omega_X \otimes \alpha \otimes f(\beta \otimes \alpha^{-1})$ are, respectively, F_α and $F_{\alpha \otimes f(\beta \otimes \alpha^{-1})}$ (i.e. $|M_\alpha|$ and $\left|M_{\alpha \otimes f(\beta \otimes \alpha^{-1})}\right|$ have no base divisors). It follows from (4.6) that $|M_\alpha| \stackrel{\cdot F_\beta}{\hookrightarrow} \left|\omega_X \otimes \alpha \otimes f(\beta \otimes \alpha^{-1})\right|$, so

$$F_\beta = F_{\alpha \otimes f(\beta \otimes \alpha^{-1})}.$$

Hence $f(\beta) = f(\alpha \otimes f(\beta \otimes \alpha^{-1}))$. This means that $f(f(\beta \otimes \alpha^{-1})) = f(\beta \otimes \alpha^{-1})$ for *general* α and β in U_0, hence $f^2 = f$. This gives the splitting of the exact sequence $\hat{0} \to \ker f \to \operatorname{Pic}^0 A \to \operatorname{Im} f \to \hat{0}$ and the identification $\operatorname{Im} f \cong \ker(\mathrm{id} - f)$. Moreover, the abelian subvariety $\ker(\mathrm{id} - f)$ is positive-dimensional since otherwise the fixed divisor of $\omega_X \otimes \alpha$ would be constant for general $\alpha \in U_0$, contradicting the last sentence in Corollary 4.11. $\square$

We will use the Greek letter α to denote an element in $\operatorname{Pic}^0 X$ and $\beta \otimes \gamma$ to denote an element in $\operatorname{Pic}^0 X$ seen as an element in $\operatorname{Pic}^0 X \cong \ker f \times \ker(\mathrm{id} - f)$, that is, $\beta \in \ker f$ and $\gamma \in \ker(\mathrm{id} - f)$. We will recall the definition each time it appears but we hope that this notation make the reading easier.

Lemma 4.15. *Under the hypotheses of Corollary 4.11, consider the decomposition* $\operatorname{Pic}^0 X \cong \ker f \times \ker(\mathrm{id} - f)$ *defined in Lemma 4.14, then there are two (effective) divisors M and F on X such that*

$$\omega_X = \mathcal{O}_X(M + F),$$

and the following properties:

(a) For all $(\beta, \gamma) \in \ker f \times \ker(\mathrm{id} - f)$ such that $\beta \otimes \gamma \in U_0$, $|\mathcal{O}_X(F) \otimes \gamma|$ is contained in the fixed divisor of $\omega_X \otimes \beta \otimes \gamma$;

(b) for (β, γ) such that $\beta \otimes \gamma$ is sufficiently general in U_0, $|\mathcal{O}_X(F) \otimes \gamma|$ is the fixed divisor of $|\omega_X \otimes \beta \otimes \gamma|$. Hence, for such (β, γ), $|\mathcal{O}_X(M) \otimes \beta|$ is the mobile part of $|\omega_X \otimes \beta \otimes \gamma|$.

(c) Suppose, moreover that $f(V_a^1(\omega_X)) \neq \ker(\mathrm{id} - f)$. Then $h^0(\mathcal{O}_X(M) \otimes \beta) = \chi(\omega_X)$ for all $\beta \in \ker f$.

Proof. Fix $\bar{\beta}$ such that $U_0 \cap \left(\{\bar{\beta}\} \times \ker(\mathrm{id} - f)\right)$ is non-empty. Then, for $\bar{\beta} \otimes \gamma \in U_0 \cap \left(\{\bar{\beta}\} \times \ker(\mathrm{id} - f)\right)$ the line bundle $\mathcal{O}_X(F_{\bar{\beta} \otimes \gamma}) \otimes \gamma^{-1} =: \mathcal{O}_X(F)$ does not depend on γ. We also define $\mathcal{O}_X(M) := \omega_X(-F)$.

(a) For $(\beta,\gamma) \in \ker f \times \ker(\mathrm{id} - f)$ such that $\beta \otimes \gamma \in U_0$, let $E \in |\mathcal{O}_X(F) \otimes \gamma|$. Then $\mathcal{O}_X(F_{\beta\otimes\gamma} - E) \cong \mathcal{O}_X(F_{\beta\otimes\gamma} - F_{\bar\beta\otimes\gamma}) = f(\beta \otimes \bar\beta^{-1}) = \mathcal{O}_X$. Since $F_{\beta\otimes\gamma}$ is a fixed divisor of $|\omega_X \otimes \beta \otimes \gamma|$, also $E = F_{\bar\beta\otimes\gamma}$ is a fixed divisor in $|\omega_X \otimes \beta \otimes \gamma|$.

(b) If $\beta \otimes \gamma$ is sufficiently general in U_0, we know that there are no other base divisors, i.e. $F_{\bar\beta\otimes\gamma} = F_{\beta\otimes\gamma}$ is the fixed divisor of $|\omega_X \otimes \beta \otimes \gamma|$ (see Remark 4.13).
Hence, $|\mathcal{O}_X(M) \otimes \beta| = \left|\omega_X \otimes \beta \otimes \gamma \otimes \mathcal{O}_X(-F_{\bar\beta\otimes\gamma})\right|$ that is the mobile part of $|\omega_X \otimes \beta \otimes \gamma|$, since $F_{\bar\beta\otimes\gamma}$ is its fixed divisor.

(c) By assumption we can choose $\bar\gamma \in \ker(\mathrm{id} - f)$ such that $\ker f \times \{\bar\gamma\} \subseteq U_0$. Therefore $h^0(\omega_X \otimes \beta \otimes \bar\gamma) = \chi(\omega_X)$ for all $\beta \in \ker f$. And by item *(a)*, $h^0(\omega_X \otimes \beta \otimes \bar\gamma) = h^0(\mathcal{O}_X(M) \otimes \beta)$. $\square$

We could have seen the decomposition given in Lemma 4.14 in the Albanese variety.

Lemma 4.16. *Since the Albanese map is defined up to a translation in* $\mathrm{Alb}\, X$*, we can assume that there is a point, say* $\bar p$*, such that* $\mathrm{alb}(\bar p) = 0$ *in* $\mathrm{Alb}\, X$*. Under the hypotheses of Corollary 4.11, consider the homomorphism*

$$g\colon \mathrm{Alb}\, X \to A$$

arising from the universal property of the Albanese variety from the map $p \mapsto \mathcal{O}_{\mathrm{Pic}^0 A}(\mathcal{D}_p - \mathcal{D}_{\bar p})$*, where* $\mathcal{D}_p$ *is the divisor in* $\mathrm{Pic}^0 A$ *defined in Terminology/Notation 4.12.*
The homomorphism $g\colon \mathrm{Alb}\, X \to A$ *is the dual homomorphism of* $f\colon \mathrm{Pic}^0 A \to \mathrm{Pic}^0 X$ *of Lemma 4.14. Hence,*

$$g^2 = g \qquad \textit{and} \qquad \mathrm{Alb}\, X \cong \ker g \times \ker(\mathrm{id} - g).$$

Proof. Given $Y \subset X_1 \times X_2$ be of pure dimension $\dim X_1 + \dim X_2 - 1$ dominant on each factor. The natural morphisms given by the fibers of Y $\mathrm{Alb}\, X_1 \to \mathrm{Pic}^0 X_2$ and $\mathrm{Alb}\, X_2 \to \mathrm{Pic}^0 X_1$ are determined by the linear maps at the level of tangent spaces: $h_1\colon H^0(X_1, \Omega^1_{X_1})^* \to H^1(X_2, \mathcal{O}_{X_2})$ and $h_2\colon H^0(X_2, \Omega^1_{X_2})^* \to H^1(X_1, \mathcal{O}_{X_1})$. Both morphisms are defined using Hodge theory and Künneth formula as follows: the class $[Y] \in H^2(X_1 \times X_2, \mathbb{Z})$ represents in $H^2(X_1 \times X_2, \mathbb{C})$ an element η belonging to the $(1,1)$ part of the Künneth decomposition

$$\eta = \eta_1 + \eta_2 \in (H^{1,0}(X_1) \otimes H^{0,1}(X_2)) \oplus H^{0,1}(X_1) \otimes H^{1,0}(X_2)).$$

Then the map h_1 is induced by

$$\eta_1 \in (H^{1,0}(X_1)^{**} \otimes H^{0,1}(X_2)) = \mathrm{Hom}(H^{1,0}(X_1)^*, H^{0,1}(X_2)),$$

and analogously for h_2. Since η is an integral class, is invariant by conjugation, so $\eta_2 = \overline{\eta_1}$. This yields $h_2 = \overline{h_1}$ which implies the statement. $\square$

Remark 4.17. *We have seen that* $\ker(\mathrm{id}-g) \cong \mathrm{Pic}^0(\ker(\mathrm{id}-f))$ *and* $\ker g \cong \mathrm{Pic}^0(\ker f)$.

It follows from Lemma 4.16 that

$$\mathrm{Alb}\, X \cong B \times C,$$

where we will call $B = \mathrm{Pic}^0(\ker f) = \ker g$ and $C = \mathrm{Pic}^0(\ker(\mathrm{id}-f)) = \mathrm{Im}\, g$. Hence the Poincaré line bundle $\mathcal{P}$ on $\mathrm{Alb}\, X \times \mathrm{Pic}^0 X$ is

$$\mathcal{P} \cong \mathcal{P}_B \boxtimes \mathcal{P}_C. \tag{4.7}$$

Keeping in mind that P, the Poincaré line bundle on $X \times \mathrm{Pic}^0 X$ is $(\mathrm{alb} \times \mathrm{id}_{\mathrm{Pic}^0 X})^* \mathcal{P}$, the next lemma provides a description of

$$((g \circ \mathrm{alb}) \times f)^* \mathcal{P}_C = (\mathrm{alb} \times \mathrm{id}_{\mathrm{Pic}^0 X})^* (\mathcal{O}_{B \times \mathrm{Pic}^0 B} \boxtimes \mathcal{P}_C),$$

which is “half” Poincaré line bundle.

Since the Albanese map is defined up to a translation in $\mathrm{Alb}\, X$, we can assume that there is one point, say $\bar{p}$, such that $\mathrm{alb}(\bar{p}) = 0$ in $\mathrm{Alb}\, X$.

Lemma 4.18 ([BLNP, Lem. 5.2]). *Under the hypotheses of Corollary 4.11, consider the projections f and g as defined in Lemmas 4.14 and 4.16. Then we have the following explicit description of the pull-back of the Poincaré line bundle $\mathcal{P}_C$ on* $C = \mathrm{Im}\, g = \mathrm{Pic}^0(\ker(\mathrm{id}-f))$,

$$((g \circ \mathrm{alb}) \times f)^* \mathcal{P}_C \cong \mathcal{O}_{X \times \mathrm{Pic}^0 X}(\overline{\mathcal{Y}}) \otimes p^* \mathcal{O}_X(-F) \otimes q^* \mathcal{O}_{\mathrm{Pic}^0 X}(-\mathcal{D}_{\bar{p}}),$$

where $\mathcal{D}_p$ are defined in Notation/Terminology 4.12, F is defined in Lemma 4.15 and $\bar{p}$ is such that $\mathrm{alb}(\bar{p}) = 0$ *in* $\mathrm{Alb}\, X$.

Proof. By the definition of $\overline{\mathcal{Y}}$ and Lemma 4.15 we have that the line bundle

$$\mathcal{O}_{X \times \mathrm{Pic}^0 X}(\overline{\mathcal{Y}}) \otimes p^* \mathcal{O}_X(-F) \otimes q^* \mathcal{O}_{\mathrm{Pic}^0 X}(-\mathcal{D}_{\bar{p}}),$$

- restricted to $X \times \{\beta \otimes \gamma\}$ is isomorphic to

  $$\mathcal{O}_X(F_{\beta \otimes \gamma} - F) = \mathcal{O}_X(F) \otimes \gamma \otimes \mathcal{O}_X(-F) = \gamma$$

  for all $(\beta, \gamma) \in \ker f \times \ker(\mathrm{id}-f)$ sufficiently general in U_0;
- restricted to $\{\bar{p}\} \times \mathrm{Pic}^0 X$ is isomorphic to $\mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_p) \otimes \mathcal{O}_{\mathrm{Pic}^0 X}(-\mathcal{D}_p)$, i.e. trivial.

On the other hand,

$$((g \circ \mathrm{alb}) \times f)^* \mathcal{P}_C$$

- restricted to $X \times \{\beta \otimes \gamma\}$ is isomorphic to

  $$(g \circ \mathrm{alb})^* \left. \mathcal{P}_C \right|_{C \times \{f(\beta \otimes \gamma)\}} = (g \circ \mathrm{alb})^* \left. \mathcal{P}_C \right|_{C \times \{\gamma\}} = (g \circ \mathrm{alb})^* \gamma = \gamma$$

for all $(\beta, \gamma) \in \ker f \times \ker(\mathrm{id} - f)$;

- restricted to $\{\bar{p}\} \times \mathrm{Pic}^0 X$ is isomorphic to $f^* \mathcal{P}_C|_{\{0\}\times \mathrm{Pic}^0 C} = f^* \mathcal{O}_{\mathrm{Pic}^0 C} = \mathcal{O}_{\mathrm{Pic}^0 X}$, i.e. trivial.

Then, the Lemma follows from the see-saw principle. $\square$

5 The bicanonical map of irregular varieties

The next theorem gives a sufficient numerical condition for the birationality of the bicanonical map, analogous to Pareschi-Popa Theorem 4.2 for the tricanonical map.

Theorem 5.1. *Let X be a smooth projective complex variety. If $\mathrm{gv}(\omega_X) \geqslant 2$, then the rational map associated to $\omega_X^2 \otimes \alpha$ is birational onto its image for every $\alpha \in \mathrm{Pic}^0 X$.*

As a first corollary we have the following result.

Corollary 5.2. *Let X be a smooth projective complex variety of maximal Albanese dimension such that the bicanonical map is not birational. Then $0 \leqslant \mathrm{gv}(\omega_X) \leqslant 1$. Moreover, it admits a fibration onto a normal projective variety Y with $0 \leqslant \dim Y < \dim X$, any smooth model $\tilde{Y}$ of Y is of maximal Albanese dimension, and*

- *either, the general fibers map onto divisors in a fixed abelian variety*
- *or the general fibers map onto a fixed abelian variety.*

In any case,

$$q(X) - \dim X \leqslant q(\tilde{Y}) - \dim Y + \mathrm{gv}(\omega_X).$$

Before giving the proof of the previous results we would like to see how they fit into the case of surfaces and their relation with other related results.

In the case of surfaces, we classically know the case when $\dim Y > 0$ as the standard case, i.e. we have a fibration by curves of genus 2 (see Theorem 4.7*(a)*). These curves are either mapped into their Jacobian as divisors or they are coverings branched at two points of a fixed elliptic curve. When $\dim Y = 0$ we have that

- X is mapped to a divisor in its Albanese variety like in Theorem 4.7*(b)* or Example 4.4 or
- X is mapped onto its Albanese variety like in Theorem 4.7*(c)* or Example 4.5.

Anyway, the case of surfaces is easier mainly because the only possible fibrations to positive dimensional varieties are fibrations onto curves. In this case, we have the following remark.

Remark 5.3. *Let X be a smooth variety that admits a relatively minimal non-isotrivial fibration $f : X \to B$ onto a curve B of genus $g \geqslant 3$. Suppose now that the bicanonical map of X is non-birational and let F be the general fiber of the fibration. Then the bicanonical map of F is non-birational.*

Proof of the Remark. Suppose that the bicanonical map of X is non-birational and $X \to B$ is a relatively minimal fibration which is not isotrivial. Since $X \to B$ is a relatively minimal fibration, $\omega_{X/B}$ is a nef line bundle (see [Oh, Thm. 1.4]) and, since it is not isotrivial, $\omega_{X/B}$ is big (see [Oh, Cor. 1.5]). Then, consider the following short exact sequence,

$$0 \to \omega_X^2 \otimes \mathcal{O}(-F) \to \omega_X^2 \to \omega_F^2 \to 0.$$

Observe that $\omega_X^2 \otimes \mathcal{O}(-F) \cong \omega_X \otimes \omega_{X/B} \otimes \mathcal{O}((2g-3)F)$. F is a fiber, so $\mathcal{O}(F)$ is nef. By the previous discussion $\omega_{X/B} \otimes \mathcal{O}((2g-3)F)$ is a nef and big line bundle. Then by Kawamata-Viehweg Vanishing Theorem $H^1(\omega_X^2 \otimes \mathcal{O}(-F)) = 0$, so

$$H^0(\omega_X^2) \twoheadrightarrow H^0(\omega_F^2). \tag{5.1}$$

By Fujita's Theorem [Fu], if $q(X) > g$, there exists an inclusion $\mathcal{O}_B \hookrightarrow f_*\omega_{X/B}$. Observe that in our situation $q(X) > g$ holds trivially, because if not, $\operatorname{Alb} X \to JB$ would be an isogeny and the fibers of X would not be of maximal Albanese dimension contradicting Proposition 3.9. So, $h^0(f_*\omega_{X/B}) = h^0(\omega_{X/B}) > 0$. Hence also $h^0(\omega_{X/B}^2) > 0$ and there exists an inclusion $f^*\omega_B^2 \hookrightarrow \omega_X^2$. Therefore,

$$H^0(f^*\omega_B^2) \hookrightarrow H^0(\omega_X^2),$$

and $\varphi_{|\omega_X^2|}$ factors through $\varphi_{|f^*\omega_B^2|}$. Even more, we have the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, dashed, two heads, "\varphi_{|\omega_X^2|}"] \arrow[d, "f"'] \arrow[dr, dashed, two heads, "\varphi_{|f^*\omega_B^2|}"'] & \tilde{X} \subseteq \mathbb{P}^{N_1} = \left|\omega_X^2\right|^\vee \arrow[d, "\tilde{f}"] \\
B \arrow[r, dashed, two heads, "\varphi_{|\omega_B^2|}"'] & \tilde{B} \subseteq \mathbb{P}^{N_2} = \left|\omega_B^2\right|^\vee .
\end{tikzcd}$$

Then, since ω_B^2 is birational (because $g(B) \geqslant 3$) and ω_X^2 is non-birational, the rational map induced by the subsystem $\operatorname{Im}(H^0(\omega_X^2) \to H^0(\omega_F^2))$ cannot be birational. By (5.1), $H^0(\omega_F^2) = \operatorname{Im}(H^0(\omega_X^2) \to H^0(\omega_F^2))$, so the bicanonical map of the general fiber F cannot be birational. $\square$

This kind of results shows that, roughly speaking, the only problematic fibrations in the case of surfaces are the fibrations by curves of genus 2. However, in higher dimensions the results on the positiveness of the higher direct images of the canonical line bundle are not so strong as the Fujita's Theorem is and this direct approach to the problem is not so easy.

When we consider the known results in surfaces, another question that arises in higher dimensions is whether the Euler characteristic $\chi(\omega_X)$ gives conditions on the birationality of the bicanonical map (see Theorem 4.7*(a)* for the case of surfaces). The next result of Pareschi and Popa shows that the generic vanishing index gives a lower bound to Euler characteristic.

Theorem 5.4 ([PP7, Cor. 4.1]). *Let X be a compact Kähler manifold of maximal Albanese dimension. Then*

$$\chi(\omega_X) \geqslant \mathrm{gv}(\omega_X).$$

Thus, the class of varieties with $0 \leqslant \mathrm{gv}(\omega_X) \leqslant 1$ contains all maximal Albanese dimension varieties that, either have non-birational bicanonical map, or $\chi(\omega_X) \leqslant 1$. Maximal Albanese dimension varieties of general type and $\chi(\omega_X) = 0$ are quite special and were discovered by Ein and Lazarsfeld [EL]. They do not exist in dimension $\leqslant 2$, showing also that the higher dimensional case should be more difficult in general.

What do we know about maximal Albanese dimension varieties of general type and $\mathrm{gv}(\omega_X) = \chi(\omega_X) = 1$? Theta-divisors, i.e. a divisor representing a principal polarization in an Abelian variety, provide examples of varieties with $\mathrm{gv}(\omega_X) = \chi(\omega_X) = 1$ in any dimension. In fact, the products of theta-divisors are the only known examples of such varieties. The following result of Hacon and Pardini gives constraints to varieties of maximal Albanese dimension and $\chi(\omega_X) = 1$.

Theorem 5.5 ([HP2, Thm. 3.1]). *Let X be a smooth projective variety of maximal Albanese dimension. If $\chi(\omega_X) = 1$, then $q(X) \leqslant 2\dim X$. If in addition $q(X) = 2\dim X$, then X is birational to a product of curves of genus 2.*

We have the analogous result for fibrations onto curves (compare with Remark 5.3). That is, suppose that X is a smooth variety with $\chi(\omega_X) = 1$ that admits a fibration onto a curve of maximal Albanese dimension. Then, if the base curve has genus $\geqslant 2$, the general fiber F reproduces the behavior of X, i.e. $\chi(\omega_F) = 1$.

Remark 5.6. *Let X be a maximal Albanese dimension smooth variety with $\chi(\omega_X) = 1$, that admits a fibration $f : X \to B$ onto a curve B of genus $g \geqslant 2$ and let F be its general fiber. Then $\chi(\omega_F) = 1$.*

Proof. By [HP2, Thm 2.4],

$$\chi(\omega_X) \geqslant \chi(\omega_B) \cdot \chi(\omega_F).$$

Therefore, if $\chi(\omega_X) = 1$, B must be a curve of genus 2 and $\chi(\omega_F) \leqslant 1$. Since, F is of maximal Albanese dimension by Green-Lazarsfeld Vanishing Theorem 1.4, $\chi(\omega_F) \geqslant 0$. By [HP2, Prop. 2.5], if $\chi(\omega_F) = 0$, then $\chi(\omega_X) = 0$, contradicting the hypothesis. Therefore, $\chi(\omega_F) = 1$. □

So there is a parallelism in the behavior of smooth varieties with $\mathrm{gv}(\omega_X) = 1$ and non-birational bicanonical map and varieties with $\mathrm{gv}(\omega_X) = \chi(\omega_X) = 1$. However, there clearly exist varieties with maximal Albanese dimension, non-birational bicanonical map and $\chi(\omega_X) > 1$ or $q(X) > 2\dim X$.

Example 5.7. We can consider for example the product $X = \Theta \times Y$ of a theta-divisor Θ with any variety Y chosen appropriately in order to increase arbitrarily either $\chi(\omega_X)$ or $q(X) - 2\dim X$, but the bicanonical map of X would never be birational (see Examples 4.4 and 4.6).

Example 5.8. We could have also considered a product of $X = Z \times Y$, where Z is a double covering of a principally polarized abelian variety (A, Θ) branched along a reduced divisor in $|2\Theta|$ as in Example 4.5, and Y is any variety chosen appropriately in order to increase arbitrarily either $\chi(\omega_X)$ or $q(X) - 2\dim X$. As above, by 4.6 the bicanonical map of X would never be birational.

However, these varieties admit a fibration onto a lower-dimensional variety Y of maximal Albanese dimension. We will see in the next section §6 that, when we do not allow this kind of fibrations, varieties with $\mathrm{gv}(\omega_X) = 1$ and non-birational bicanonical map have $\chi(\omega_X) = 1$ (proof of Theorem 6.7 below) and that, when $q(X) > \dim X$, we have only one class of such varieties, namely theta-divisors in principally polarized abelian varieties (see Proposition 6.4 below). So, in absence of fibrations and $q(X) > \dim X$ the problem of classifying the boundary examples of Theorem 5.1 and $\chi(\omega_X) = 1$ in Theorem 5.4, are equivalent. The main subject of the following section §6 is to study what happens in absence of fibrations.

Proofs

We go back to the proofs of Theorem 5.1 and Corollary 5.2.

Proof of Theorem 5.1. Assume that $\mathrm{gv}(\omega_X) \geqslant 1$ and the bicanonical map is not birational and we will see that $\mathrm{gv}(\omega_X) = 1$.
Since $\mathrm{gv}(\omega_X) \geqslant 1$, then Hypothesis 3.4 are fullfilled. Furthermore, the assumption that there exists $\alpha \in \mathrm{Pic}^0 X$ such that $\omega_X^{\otimes 2} \otimes \alpha$ is non-birational, places ourselves in the hypotheses of Corollary 4.11. Thus we can freely use Notation/Terminology 4.12 and Lemma 4.15. Moreover, Lemma 4.18 yields to the following short exact sequence on $X \times \mathrm{Pic}^0 X$

$$0 \to ((g\circ\mathrm{alb})\times f)^*\mathcal{P}_C^{-1} \xrightarrow{\cdot\overline{\mathcal{Y}}} p^*\mathcal{O}_X(F)\otimes q^*\mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar{p}}) \to (p^*\mathcal{O}_X(F)\otimes q^*\mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar{p}}))|_{\overline{\mathcal{Y}}} \to 0,$$

where p, q are the projections of $X \times \mathrm{Pic}^0 X$.
Recall that $((g \circ \mathrm{alb}) \times f)^*\mathcal{P}_C = (\mathrm{alb} \times \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{O}_{B\times \mathrm{Pic}^0 B} \boxtimes \mathcal{P}_C)$. We apply the functor $R^d q_*(\cdot \otimes (\mathrm{alb} \times \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C\times \mathrm{Pic}^0 C}))$, that is, we tensor by the other "half" Poincaré line bundle (see (4.7)) and we take the top direct image. We get

$$\begin{aligned} \cdots \to\ & R^d\Phi_{P^{-1}}(\mathcal{O}_X) \to \\ \to\ & R^d q_* \left(p^*\mathcal{O}_X(F) \otimes (\mathrm{alb} \times \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C\times \mathrm{Pic}^0 C})\right) \otimes \mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar{p}}) \to \\ \to\ & R^d q_*\left((p^*\mathcal{O}_X(F)\otimes q^*\mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar{p}}))|_{\overline{\mathcal{Y}}} \otimes (\mathrm{alb}\times \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C\times \mathrm{Pic}^0 C})\right) \to 0. \end{aligned}$$

Recall that $\mathrm{R}\Phi_{P^{-1}} \cong (-1)^*_{\mathrm{Pic}^0 X}\mathrm{R}\Phi_P$ (see Corollary I.1.3). Then we have the following short exact sequence,

$$0 \to (-1)^*_{\mathrm{Pic}^0 X}\widehat{\mathcal{O}_X} \xrightarrow{\mu} \mathcal{E}(\mathcal{D}_{\bar{p}}) \to \mathcal{T} \to 0 \tag{5.2}$$

where:

(a) By base change, $\mathcal{E} = R^d q_*(p^*\mathcal{O}_X(F) \otimes (\mathrm{alb} \times \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C\times \mathrm{Pic}^0 C}))$ is a coherent sheaf of rank $h^0(\mathcal{O}_X(M) \otimes \beta)$ by a general $\beta \in \ker f$, that is, by Lemma 4.15*(b)*, $\chi(\omega_X)$.
Recall that $B = \ker g$ and $b : X \to B$ is $b = (\mathrm{id} - g) \circ \mathrm{alb}$. We have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}^0 X \arrow[d, "p_{\hat{b}}"'] & X \times \mathrm{Pic}^0 X \arrow[l, "q"'] \arrow[r, "\mathrm{alb}\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times p_{\hat{b}}"] & \mathrm{Alb}\, X \times \mathrm{Pic}^0 X \arrow[d, "p_b \times p_{\hat{b}}"] \\
\mathrm{Pic}^0 B & X \times \mathrm{Pic}^0 B \arrow[l, "q"] \arrow[r, "b\times\mathrm{id}"'] & B \times \mathrm{Pic}^0 B
\end{tikzcd}$$

where we have denoted $p_b : \mathrm{Alb}\, X \to B$ and $p_{\hat{b}} : \mathrm{Pic}^0 X \to \mathrm{Pic}^0 B$ the corresponding projections (see lemmas 4.14 and 4.16). Abusing notation, we call q either the projection $X \times \mathrm{Pic}^0 X \to \mathrm{Pic}^0 X$ or $X \times \mathrm{Pic}^0 B \to \mathrm{Pic}^0 B$ and p the projections $X \times \mathrm{Pic}^0 X \to X$ or $X \times \mathrm{Pic}^0 B \to X$. Then

$$\begin{aligned}
\mathcal{E} &= R^d q_*(p^*\mathcal{O}_X(F) \otimes (\mathrm{alb}\times\mathrm{id})^*(\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C\times\mathrm{Pic}^0 C})) \\
&= R^d q_*(p^*\mathcal{O}_X(F) \otimes (\mathrm{alb}\times\mathrm{id})^*(p_b \times p_{\hat{b}})^*\mathcal{P}_B^{-1} \\
&= R^d q_*(p^*\mathcal{O}_X(F) \otimes (\mathrm{id}\times p_{\hat{b}})^*(b\times\mathrm{id})^*\mathcal{P}_B^{-1} && \text{comm. on the right square} \\
&= R^d q_*(\mathrm{id}\times p_{\hat{b}})^*(p^*\mathcal{O}_X(F) \otimes (b\times\mathrm{id})^*\mathcal{P}_B^{-1}) && \text{abuse of notation on } p \\
&= p_{\hat{b}}^* R^d q_*(p^*\mathcal{O}_X(F) \otimes (b\times\mathrm{id})^*\mathcal{P}_B^{-1}) && \text{flat base change} \\
&= p_{\hat{b}}^*\, R^d\Phi_{P_b^{-1}}(\mathcal{O}_X(F)),
\end{aligned}$$

following the notation of (I.1.1) and (I.1.2).

(b) The map μ is injective since it is a generically surjective map of sheaves of the same rank (recall that $\mathrm{rk}\,\widehat{\mathcal{O}_X} = \chi(\omega_X)$) and, as $\mathrm{gv}(\omega_X) \geqslant 1$, the source $\widehat{\mathcal{O}_X}$ is torsion-free (Corollary I.1.18).

(c) $\mathcal{T} = R^d q_*\big((p^*\mathcal{O}_X(F) \otimes q^*\mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar{p}}))|_{\overline{\mathcal{Y}}} \otimes (\mathrm{alb}\times\mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C\times\mathrm{Pic}^0 C})\big)$ is supported at the locus of the $\alpha \in \mathrm{Pic}^0 X$ such that the fiber of the projection $q\colon \overline{\mathcal{Y}} \to \mathrm{Pic}^0 X$ has dimension d, i.e. it coincides with X. Such locus is contained in $V^1(\omega_X)$, therefore, since $\mathrm{gv}(\omega_X) \geqslant 1$, $\mathrm{codim}\,\mathrm{supp}\,\mathcal{T} \geqslant 2$.

(d) μ is $R^d q_*(m_s)$, where m_s is the multiplication by the section defining $\overline{\mathcal{Y}}$. Hence by base change $R^d q_*(m_s) \otimes \mathbb{C}(\alpha) = H^d(m_s|_{q^{-1}\{\alpha\}})$ where q is the projection

$q\colon \overline{\mathcal{Y}} \to \operatorname{Pic}^0 X$. When $q^{-1}\{\alpha\} = X$, $m_s|_{q^{-1}\{\alpha\}} = 0$, so in these points $R^d q_*(m_s) \otimes \mathbb{C}(\alpha) = 0$.

Claim $\mathcal{T} \neq 0$.
Suppose that $\mathcal{T} = 0$, so μ is an isomorphism. Taking $\mathcal{E}xt^d(\,\cdot\,, \mathcal{O}_{\operatorname{Pic}^0 X})$ we get

$$\begin{aligned} k(\hat{0}) &= R^d\Phi_{P_a}\,\omega_X && \text{Prop. 3.2} \\ &= \mathcal{E}xt^d(\mathcal{E}, \mathcal{O}_{\operatorname{Pic}^0 X}) \otimes \mathcal{O}(-\mathcal{D}_{\bar{p}}) && \mathcal{E}xt^d(\mu, \mathcal{O}_{\operatorname{Pic}^0 X}) \text{ and Cor. I.1.14} \\ &= p_{\hat{b}}^*\,\mathcal{E}xt^d(R^d\Phi_{P_b}(\mathcal{O}_X(F)), \mathcal{O}_{\operatorname{Pic}^0 B}) \otimes \mathcal{O}(-\mathcal{D}_{\bar{p}}) && \text{item } \textit{(a)} \text{ and [Hu, (3.17)]}, \end{aligned}$$

which implies that $\operatorname{codim}_{\operatorname{Alb} X} B = \dim\ker(\mathrm{id} - f) = 0$ contradicting Lemma 4.14.

Hence we can assume that $\mathcal{T} \neq 0$. Let $\tau(\mathcal{E}(\mathcal{D}_{\bar{p}}))$ be the torsion part of $\mathcal{E}(\mathcal{D}_{\bar{p}})$ and $\widetilde{\mathcal{E}(\mathcal{D}_{\bar{p}})}$ the quotient of $\mathcal{E}(\mathcal{D}_{\bar{p}})$ by its torsion part. Hence $\widetilde{\mathcal{E}(\mathcal{D}_{\bar{p}})}$ is torsion-free. Now consider the following compostion

$$\begin{tikzcd} (-1)^*_{\operatorname{Pic}^0 X}\widehat{\mathcal{O}_X} \arrow[r, "\mu"] \arrow[dr, "\tilde{\mu}"'] & \mathcal{E}(\mathcal{D}_{\bar{p}}) \arrow[d] \\ & \widetilde{\mathcal{E}(\mathcal{D}_{\bar{p}})}. \end{tikzcd}$$

Since $\tilde{\mu}$ is generically surjective and $(-1)^*_{\operatorname{Pic}^0 X}\widehat{\mathcal{O}_X}$ is torsion-free (recall that, by assumption, $\operatorname{gv}(\omega_X) \geqslant 1$), we have that $\tilde{\mu}$ is injective. Completing the diagram we get,

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & \tau(\mathcal{E}(\mathcal{D}_{\bar{p}})) \arrow[r, equal] \arrow[d] & \tau(\mathcal{E}(\mathcal{D}_{\bar{p}})) \arrow[d] & \\ 0 \arrow[r] & (-1)^*_{\operatorname{Pic}^0 X}\widehat{\mathcal{O}_X} \arrow[r, "\mu"] \arrow[d, equal] & \mathcal{E}(\mathcal{D}_{\bar{p}}) \arrow[r] \arrow[d] & \mathcal{T} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (-1)^*_{\operatorname{Pic}^0 X}\widehat{\mathcal{O}_X} \arrow[r, "\tilde{\mu}"] & \widetilde{\mathcal{E}(\mathcal{D}_{\bar{p}})} \arrow[r] \arrow[d] & \widetilde{\mathcal{T}} \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & \end{tikzcd} \tag{5.3}$$

Claim $\widetilde{\mathcal{T}} \neq 0$.
If $\widetilde{\mathcal{T}} = 0$, then the middle horizontal short exact sequence splits. But, for α a closed point in the support of $\mathcal{T}$ (by the previous claim we know that $\mathcal{T} \neq 0$), $\mu \otimes \mathbb{C}(\alpha) = 0$ by item *(d)*, so μ cannot split. Therefore $\widetilde{\mathcal{T}} \neq 0$.

Let $k = \operatorname{codim}_{\operatorname{Pic}^0 X} \operatorname{supp} \widetilde{\mathcal{T}} \geqslant 2$ (see item *(c)*). Then,

$$\operatorname{codim}_{\operatorname{Pic}^0 X} \operatorname{supp} \mathcal{E}xt^k(\widetilde{\mathcal{T}}, \mathcal{O}_{\operatorname{Pic}^0 X}) = k.$$

Now, we take $\mathcal{E}xt^i(\,\cdot\,, \mathcal{O}_{\operatorname{Pic}^0 X})$ to the bottom row of (5.3) using Corollary I.1.14

$$\ldots \to R^{k-1}\Phi_{P_a}\,\omega_X \to \mathcal{E}xt^k(\widetilde{\mathcal{T}}, \mathcal{O}_{\operatorname{Pic}^0 X}) \to \mathcal{E}xt^k(\widetilde{\mathcal{E}(\mathcal{D}_{\bar{p}})}, \mathcal{O}_{\operatorname{Pic}^0 X}) \to \ldots$$

Since $\widetilde{\mathcal{E}(\mathcal{D}_{\bar{p}})}$ is torsion-free, $\operatorname{codim}_{\operatorname{Pic}^0 X} \operatorname{supp} \mathcal{E}xt^k(\widetilde{\mathcal{E}(\mathcal{D}_{\bar{p}})}, \mathcal{O}_{\operatorname{Pic}^0 X}) > k$. Therefore, we must have $\operatorname{codim}_{\operatorname{Pic}^0 X} \operatorname{supp} R^{k-1}\Phi_{P_a}\,\omega_X = k$ and $\operatorname{gv}(\omega_X) \leqslant 1$. □

Proof of Corollary 5.2. By the previous Theorem 5.1 and Proposition 1.10 it is clear that $0 \leqslant \operatorname{gv}(\omega_X) \leqslant 1$. Now, the proof is the same as the proof of [PP7, Thm. B]. Let $i > 0$ such that $\operatorname{codim}_{\operatorname{Pic}^0 X} V^i(\omega_X) - i = \operatorname{gv}(\omega_X)$, and let V be an irreducible component of $V^i(\omega_X)$ of maximal dimension, by Green-Lazarsfeld's Theorem 1.6*(a)*, V is translate of an abelian subvariety T. Then, let Y be the normal variety constructed in Remark 1.7 and let $\tilde{Y}$ be any desingularization. The general fiber F of $X \to Y$, maps to T via the Albanese map of X. Thus, by Green-Lazarsfeld's Theorem 1.6*(b)*, we have $q(X) - q(\tilde{Y}) \leqslant \operatorname{codim}_{\operatorname{Pic}^0 X} V^i(\omega_X) = \dim T$ and $\dim F = \dim X - \dim Y \geqslant i$. In conclusion, since $\operatorname{codim}_{\operatorname{Pic}^0 X} V^i(\omega_X) - i = \operatorname{gv}(\omega_X)$,

$$q(X) - \dim X \leqslant q(\tilde{Y}) - \dim Y + 1$$

and

$$i \leqslant \dim F \leqslant \dim T = \operatorname{codim} V^i(\omega_X),$$

so the general fiber, either maps to a divisor on T or surjectively to T. □

6 Primitive varieties

Theorem 1.6 and Remark 1.7 show that the existence of a positive dimensional component in some $V^i(\omega_X)$ for $i > 0$, implies the existence of a fibration onto a smooth lower-dimensional variety Y of maximal Albanese dimension. Choosing the fibration suitably, we can suppose that $\dim V^i(\omega_Y) = 0$ for all $i > 0$. This justifies the following definition due to Catanese [Ca, Def. 1.24].

Definition 6.1. An irregular Kähler manifold such that $\dim V^i(\omega_X) = 0$ for all $i > 0$ is called *primitive*.

Remark 6.2. *If X is a primitive variety* $\mathrm{gv}(X) = q(X) - \dim X$*. So, by Proposition 1.10, a primitive variety is of maximal Albanese dimension if, and only if, $q(X) \geqslant \dim X$. Moreover,*

$$\mathrm{gv}(\omega_X) \geqslant 1 \Leftrightarrow q(X) > \dim X$$
$$\mathrm{gv}(\omega_X) = 0 \Leftrightarrow q(X) = \dim X.$$

The following Proposition shows that a primitive variety only admits very special fibrations onto varieties of maximal Albanese dimension.

Proposition 6.3. *Let $f\colon X \to Y$ be a fibration from a primitive variety X to a maximal Albanese dimension variety Y. Then*

(a) Y is birational to an abelian variety.

(b) Let F be a general smooth fiber and $\rho\colon \mathrm{Pic}^0 X \to \mathrm{Pic}^0 F$ the restriction map. Then

$$\ker \rho = f^* \mathrm{Pic}^0 Y.$$

Proof. Suppose that the general fiber of $f\colon X \to Y$ is of dimension k.

(a) By Kollár's theorem 1.8*(c)*, the Leray spectral sequence of f splits, which yields to $f^*V^0(R^k f_*\omega_X) \subseteq V^k(\omega_X)$. Since X is primitive, $\dim V^0(R^k f_*\omega_X) = 0$. After passing to a desingularization of Y we assume that Y is smooth, and then by another result of Kollár $R^k f_*\omega_X \cong \omega_Y$ (Theorem 1.8*(d)*). Hence Y is a maximal Albanese dimension with $\dim V^0(\omega_Y) = 0$ and, by Proposition 3.9, is birational to an abelian variety.

(b) We know that $\ker\rho$ is the disjoint union of torsion translates of $f^* \mathrm{Pic}^0 Y$. Suppose that $\alpha \in \ker\rho - f^* \mathrm{Pic}^0 Y$. We can assume that α is torsion. Since $\alpha \in \ker\rho$, $h^k(F, (\omega_X \otimes \alpha)|_F) = h^k(F, \omega_F) \cong h^0(F, \mathcal{O}_F) \neq 0$. Hence $R^k f_*(\omega_X \otimes \alpha) \neq 0$ and, by Theorem 1.5, $\mathrm{gv}(R^k f_*(\omega_X \otimes \alpha)) \geqslant 0$. Hence, $V^0(R^k f_*(\omega_X \otimes \alpha))$ is non-empty since, if it were empty, by Corollary I.1.13, all the cohomological support loci would be empty and the Mukai Equivalence Theorem I.2.1 will lead us to $R^k f_*(\omega_X \otimes \alpha) = 0$. If $V^0(R^k f_*(\omega_X \otimes \alpha))$ were positive dimensional,

by Hacon's Remark 1.9 on Kollár's splitting, $V^k(\omega_X \otimes \alpha)$ would be also positive dimensional contradicting the assumption that X is primitive. Then, there exists $\beta \in V^0(R^k f_*(\omega_X \otimes \alpha))$ an isolated point. By Corollary I.1.15, since $\dim Y = q(Y)$ we have that $\beta \in V^{q(Y)}(R^k f_*(\omega_X \otimes \alpha))$. Let $d = \dim X = q(Y) + k$. Again by Remark 1.9, $H^d(\omega_X \otimes \alpha \otimes f^*\beta) \neq 0$, so $\alpha \in f^* \operatorname{Pic}^0 Y$.

□

This result was already well-known in the case of surfaces by the complete understanding of the positive dimensional components of $V^1(\omega_S)$ (see [B, Cor. 2.3]).

6.1 Characterization of theta-divisors among primitive varieties

In this section we will prove an improvement of Hacon-Pardini's cohomological characterization of theta-divisors [HP1, Prop. 4.2]. Indeed, we characterize the boundary cases of Theorem 5.4 with $\chi(\omega_X) = 1$ which are primitive varieties.

Proposition 6.4 below has also been proved independently, with a different proof, in [LP, Prop. 3.13]. An algebraic version, valid in any characteristic, is provided by Corollary 6.6 below.

Proposition 6.4 ([BLNP, Prop. 3.1]). *Let X be a d-dimensional compact Kähler manifold such that:*

(a) X is primitive;

(b) $d < q = q(X)$;

(c) $\chi(\omega_X) = 1$.

Then $\operatorname{Alb} X$ *is a principally polarized abelian variety and the Albanese map* $\operatorname{alb}\colon X \to \operatorname{Alb} X$ *maps X birationally onto a theta-divisor.*

As an immediate corollary we have the following result (see Remark 6.2 and Theorem 5.4)

Corollary 6.5. *Let X be a compact Kähler manifold such that:*

(a) X is primitive;

(b) $\operatorname{gv}(\omega_X) = \chi(\omega_X) = 1$.

Then $\operatorname{Alb} X$ *is a principally polarized abelian variety and the Albanese map* $\operatorname{alb}\colon X \to \operatorname{Alb} X$ *maps X birationally onto a theta-divisor.*

Proof of Proposition 6.4. By hypothesis *(a)* and *(b)* $\operatorname{gv}(\omega_X) \geqslant 1$. Therefore, by Corollary I.1.18, $\widehat{\mathcal{O}_X}$ is torsion-free. Since, by (I.1.4), $\operatorname{rk} \widehat{\mathcal{O}_X} = \chi(\omega_X) \overset{(c)}{=} 1$, we get that $\widehat{\mathcal{O}_X}$ is an ideal sheaf twisted by a line bundle of $\operatorname{Pic}^0 X$:

$$\widehat{\mathcal{O}_X} = \mathcal{I}_Z \otimes L.$$

By base change, the support of Z is contained in the union of the $V^i(\omega_X)$ for $i > 0$ which are assumed, by *(a)*, to be finite sets. Therefore $\mathcal{E}xt^i(\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 X}) = \mathcal{E}xt^{i+1}(\mathcal{O}_Z, \mathcal{O}_{\operatorname{Pic}^0 X}) = 0$ for $i + 1 \neq q$. On the other hand, by Proposition 3.2, and Corollary I.1.14 it follows that

$$\mathcal{E}xt^d(\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 X}) \cong (-1_{\operatorname{Pic}^0 X})^* R^d\Phi_P(\omega_X) \cong \mathbb{C}(\hat{0}). \tag{6.1}$$

This implies:

(a) $d = q - 1$.

(b) $\mathcal{O}_Z = \mathbb{C}(\hat{0})$. Indeed, $\mathcal{E}xt^i(\mathbb{C}(\hat{0}), \mathcal{O}_{\operatorname{Pic}^0 X})$ is zero for $i < q(X)$ and equal to $\mathbb{C}(\hat{0})$ for $i = q(X)$.

Since $\mathrm{R}\mathcal{H}om(\cdot, \mathcal{O}_{\operatorname{Pic}^0 X})$ is an involution, *(b)* follows from (6.1). In conclusion

$$\widehat{\mathcal{O}_X} = \mathcal{I}_{\hat{0}} \otimes L,$$

where L is a line bundle on $\operatorname{Pic}^0 X$ and $\mathcal{I}_{\hat{0}}$ is the ideal sheaf of the (reduced) point $\hat{0}$.

By Proposition I.1.1

$$\mathrm{R}\Phi_P(\mathcal{O}_X) = \mathrm{R}\Phi_{\mathcal{P}}(\mathrm{R}\operatorname{alb}_* \mathcal{O}_X) = \mathcal{I}_{\hat{0}} \otimes L[-q+1].$$

Therefore, by Mukai's Inversion Theorem I.2.1

$$\mathrm{R}\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) = (-1)^*_{\operatorname{Pic}^0 X} \mathrm{R}\operatorname{alb}_* \mathcal{O}_X[-1]. \tag{6.2}$$

In particular,

$$R^0\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) = 0 \qquad \text{and} \qquad R^1\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) \cong \operatorname{alb}_* \mathcal{O}_X \tag{6.3}$$

Applying $\psi_{\mathcal{P}}$ to the standard exact sequence

$$0 \to \mathcal{I}_{\hat{0}} \otimes L \to L \to \mathcal{O}_{\hat{0}} \otimes L \to 0, \tag{6.4}$$

and using (6.3) we get,

$$0 \to R^0\Psi_{\mathcal{P}}(L) \to \mathcal{O}_{\operatorname{Alb} X} \to \operatorname{alb}_* \mathcal{O}_X \tag{6.5}$$

whence $R^0\Psi_{\mathcal{P}}(L)$ is supported everywhere (since $\operatorname{alb}_* \mathcal{O}_X$ is supported on a divisor). It is well-known that this implies that L is *ample*. Therefore $R^i\Psi_{\mathcal{P}}(L) = 0$ for $i > 0$. Therefore, by sequence (6.4), $R^i\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) = 0$ for $i > 1$. By (6.2) and (6.3), this implies that $R^i \operatorname{alb}_*(\mathcal{O}_X) = 0$ for $i > 0$. Furthermore, (6.5) implies easily that $h^0(L) = 1$, i.e. L is a *principal* polarization. Therefore, via the identification $\operatorname{Alb} X \cong \operatorname{Pic}^0 X$ provided by ϕ_L, we have $R^0\Psi_{\mathcal{P}}(L) \cong L^{-1}$ (see Proposition I.2.2).

Since the arrow on the right in (6.5) is onto, it follows that $\mathrm{alb}_* \mathcal{O}_X = \mathcal{O}_D$, where D is a divisor in $|L|$. Since we already know that alb is generically finite (see Propostion 1.10), this implies that alb is a birational morphism onto D. □

Note that the proof is entirely algebraic, except for the use of Propostion 1.10, since all the other results used are in section I.§1 and we recall Remark I.2.3. Therefore, the following statement holds,

Corollary 6.6 ([BLNP, Cor. 3.2]). *let X be a smooth projective variety over any algebraically closed field such that:*

(a) X is a primitive variety, i.e. $\dim V^i(\omega_X) = 0$ for all $i > 0$;

(b) $d = \dim X < \dim \mathrm{Alb}\, X$ and the Albanese map of X is generically finite, and

(c) $\chi(\omega_X) = 1$.

Then $\mathrm{Alb}\, X$ is a principally polarized abelian variety and the Albanese map $\mathrm{alb}\colon X \to \mathrm{Alb}\, X$ maps X birationally onto a theta-divisor.

6.2 The bicanonical map of primitive varieties with $q(X) > \dim X$

As we have showed in section §5, for smooth projective varieties without “irregular” fibrations, the problem of classifying those with $\mathrm{gv}(\omega_X) = 1$ and non-birational bicanonical map seems related to the classification problem of those with $\chi(\omega_X) = \mathrm{gv}(\omega_X) = 1$. As we have announced, in this section we will show that for primitive varieties and $q(X) > \dim X$ or $\mathrm{gv}(\omega_X) \geqslant 1$, the problem of classifying varieties with non-birational bicanonical map is equivalent to the problem of classifying the boundary examples of Theorem 5.4 with $\chi(\omega_X) = 1$. As we have already done the classification of the boundary examples of Theorem 5.4 with $\chi(\omega_X) = 1$ in Corollary 6.5, we will be able to classify primitive varieties with $q(X) > \dim X$ (equivalently $\mathrm{gv}(\omega_X) \geqslant 1$) and non-birational bicanonical map.
The following Theorem that is the main result in [BLNP].

Theorem 6.7 ([BLNP, Thm. A]). *Let X be a primitive smooth complex projective variety such that $\dim X < q(X)$. The following are equivalent*

(a) the bicanonical map of X is non-birational,

(b) X is birationally equivalent to a theta-divisor of an indecomposable principally polarized abelian variety.

As an immediate corollary we have the following result (see Remark 6.2) that interprets the previous theorem as the classification of the primitive boundary examples of Theorem 5.1.

Corollary 6.8. *Let X be a primitive smooth complex projective variety such that the bicanonical map of X is non-birational. If $\mathrm{gv}(\omega_X) = 1$, then $\mathrm{Alb}\, X$ is a principally* *polarized abelian variety and the Albanese map* $\mathrm{alb}\colon X \to \mathrm{Alb}\, X$ *maps* X *birationally onto a theta-divisor.*

Proof of Theorem 6.7. Example 4.4 shows implication $(b) \Rightarrow (a)$.
Implication $(a) \Rightarrow (b)$ follows from Proposition 6.4 once we prove that $\chi(\omega_X) = 1$.

Claim $\chi(\omega_X) = 1$.
We will freely use Notation/Terminology 4.12 and Lemma 4.15. First, note that the hypothesis $\dim X < q(X)$, implies that $\mathrm{gv}(\omega_X) \geqslant 1$. On the other hand, Lemma 4.18 yields to the short exact sequence on $X \times \mathrm{Pic}^0 X$

$$0 \to ((g\circ \mathrm{alb})\times f)^*\mathcal{P}_C^{-1} \xrightarrow{\cdot \overline{\mathcal{Y}}} p^*\mathcal{O}_X(F)\otimes q^*\mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar{p}}) \to (p^*\mathcal{O}_X(F)\otimes q^*\mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar{p}}))|_{\overline{\mathcal{Y}}} \to 0,$$

where p, q are the projections of $X \times \mathrm{Pic}^0 X$.
Recall that $((g \circ \mathrm{alb}) \times f)^*\mathcal{P}_C = (\mathrm{alb} \times \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{O}_{B\times \mathrm{Pic}^0 B} \boxtimes \mathcal{P}_C)$. We apply the functor $R^d q_*(\cdot \otimes (\mathrm{alb} \times \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C\times \mathrm{Pic}^0 C}))$, that is, we tensor by the other "half" Poincaré line bundle (see (4.7)) and we take the top direct image. We get

$$\begin{aligned}
\cdots &\to R^d\Phi_{P^{-1}}(\mathcal{O}_X) \to \\
&\to R^d q_*\left(p^*\mathcal{O}_X(F) \otimes (\mathrm{alb} \times \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C\times \mathrm{Pic}^0 C})\right) \otimes \mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar{p}}) \to \\
&\to R^d q_*\left((p^*\mathcal{O}_X(F)\otimes q^*\mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar{p}}))|_{\overline{\mathcal{Y}}} \otimes (\mathrm{alb}\times \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1}\boxtimes \mathcal{O}_{C\times \mathrm{Pic}^0 C})\right) \to 0
\end{aligned}$$

Recall that $\mathrm{R}\Phi_{P^{-1}} \cong (-1)^*_{\mathrm{Pic}^0 X}\mathrm{R}\Phi_P$ (see Corollary I.1.3). Then we have the following short exact sequence,

$$0 \to (-1)^*_{\mathrm{Pic}^0 X}\widehat{\mathcal{O}_X} \xrightarrow{\mu} \mathcal{E}(\mathcal{D}_{\bar{p}}) \to \mathcal{T} \to 0 \tag{6.6}$$

where:

(a) Since X is primitive $V^1(\omega_X)$ is a finite set of points. By Lemma 4.14 we know that $\dim\ker(\mathrm{id} - f) > 0$, so $f(V^1(\omega_X)) \neq \ker(\mathrm{id} - f)$ and by Lemma 4.15*(c)* $h^0(\mathcal{O}_X(M) \otimes \beta) = h^d(\mathcal{O}_X(F) \otimes \beta^{-1})$ is constant equal to $\chi(\omega_X)$ for all $\beta \in \ker f = \mathrm{Pic}^0 B$. Then, by base change, $\mathcal{E} = R^d q_*(p^*\mathcal{O}_X(F) \otimes (\mathrm{alb} \times \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C\times \mathrm{Pic}^0 C}))$ is a locally free sheaf of rank $\chi(\omega_X)$ on $\mathrm{Pic}^0 X$.

(b) The map μ is injective, since it is a generically surjective map of sheaves of the same rank (recall that $\mathrm{rk}\,\widehat{\mathcal{O}_X} = \chi(\omega_X)$) and, as $\mathrm{gv}(\omega_X) \geqslant 1$, the source $\widehat{\mathcal{O}_X}$ is torsion-free (Corollary I.1.18).

(c) $\mathcal{T} = R^d q_*\left((p^*\mathcal{O}_X(F)\otimes q^*\mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar{p}}))|_{\overline{\mathcal{Y}}} \otimes (\mathrm{alb}\times \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1}\boxtimes \mathcal{O}_{C\times \mathrm{Pic}^0 C})\right)$ is supported at the locus of the $\alpha \in \mathrm{Pic}^0 X$ such that the fiber of the projection $q\colon \overline{\mathcal{Y}} \to \mathrm{Pic}^0 X$ has dimension d, i.e. it coincides with X. Such locus is contained in $V^1(\omega_X)$ that, since X is primitive, is a finite set.

(d) μ is $R^d q_*(m_s)$, where m_s is the multiplication by the section defining $\overline{\mathcal{Y}}$. Hence by base change $R^d q_*(m_s) \otimes \mathbb{C}(\alpha) = H^d(m_s|_{q^{-1}\{\alpha\}})$ where q is the projection $q\colon \overline{\mathcal{Y}} \to \operatorname{Pic}^0 X$. When $q^{-1}\{\alpha\} = X$, $m_s|_{q^{-1}\{\alpha\}} = 0$, hence in these points $\dim_{\mathbb{C}}(\mathcal{T} \otimes \mathbb{C}(\alpha)) = \chi(\omega_X)$.

From (6.6), the fact that $\mathcal{E}$ is locally free and supp $\mathcal{T}$ is a finite scheme it follows that $\mathcal{E}xt^i(\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 X}) \cong \mathcal{E}xt^{i+1}(\mathcal{T}, \mathcal{O}_{\operatorname{Pic}^0 X}) = 0$ if $i \neq q(X) - 1$. On the other hand, by Corollary I.1.14 and Proposition 3.2, $\mathcal{E}xt^d(\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 X}) \cong \mathbb{C}(\hat{0})$. It follows that $d = q(X) - 1$ and that $\mathcal{T} = \mathbb{C}(\hat{0})$. The assertion follows since the dimension of $\mathcal{T} \otimes \mathbb{C}(0)$ is equal to $\chi(\omega_X)$. □

Corollary 6.9. *Let X be a primitive variety such that $\omega_X^2 \otimes \alpha$ is not birational for some $\alpha \in \operatorname{Pic}^0 X$ and that* $\dim X < q(X)$*. Then, the morphism f defined in Lemma 4.14 is injective. i.e.* $\operatorname{Pic}^0 X = \ker(\mathrm{id} - f)$, $f = \mathrm{id}_{\operatorname{Pic}^0 X}$ *and* $g = \mathrm{id}_{\operatorname{Alb} X}$.

6.3 The bicanonical map of primitive varieties with $q(X) = \dim X$

In this section we want to study primitive varieties with non-birational bicanonical map such that $\dim X = q(X)$. In this case $\operatorname{gv}(\omega_X) = 0$ because $V^d(\omega_X) = \{\hat{0}\}$ (see Remark 6.2).

It is remarkable that in this case, even in the case of surfaces, there are primitive varieties with $\chi(\omega_X) = 1$, $\operatorname{gv}(\omega_X) = 0$ and birational bicanonical map.

Example 6.10 ([CH2]). Chen and Hacon have constructed in [CH2] a minimal surface S with $\chi(\omega_S) = 1$, $q(S) = 2$ and $K_S^2 = 5$. This surface is birational to a triple cover of an abelian surface A. When A is a simple abelian surface, S does not have a pencil of curves of genus 2. Since a double cover $\tilde{S}$ branched along a reduced divisor in $|2\Theta|$ has $K_{\tilde{S}}^2 = 4$, the surface constructed by Chen and Hacon is a different example of variety with $\operatorname{gv}(\omega_X) = 0$ and $\chi(\omega_X) = 1$ and by Theorem 4.7*(c)*, its bicanonical map is birational.

This example shows that the problem of classifying primitive varieties with non-birational bicanonical map and $q(X) = \dim X$ is more subtle. Indeed, it is not equivalent to classify primitive varieties with $\chi(\omega_X) = 1$ and $q(X) = \dim X$.

We will focus and work out the case of Galois abelian covers with rational singularities.

Abelian covers

We will focus on the case where the Albanese map is essentially an abelian Galois cover (see [Pa] or [Mu2, II.§7]). More precisely, we will require that the Albanese map is generically finite onto $\mathrm{Alb}\, X$, and the finite part of its Stein factorization is an abelian Galois cover. We need to control the singularities of the Stein factorization.

First we recall the definition of an abelian Galois cover.

Definition 6.11 ([Pa, Def. 1.1]). Let Y be a reduced and irreducible scheme. An *abelian Galois cover* of Y with group G is a finite map $\pi : X \to Y$ together with a faithful action of G on X such that π exhibits Y as the quotient of X via G.

If we assume X to be normal and Y to be smooth, then π is flat and the action of G induces a splitting:

$$\pi_* \mathcal{O}_X = \bigoplus_{\rho \in G^*} L_\rho^{-1}, \tag{6.7}$$

where G^* is the group of characters of G and L_ρ^{-1} is the line bundle corresponding to the eigenspace of $\pi_* \mathcal{O}_X$ where the group acts via the character ρ.

First we will need the following technical lemma, consequence of the work of Pardini in [Pa].

Lemma 6.12. *Let X be a normal variety and let $\pi : X \to Y$ be an abelian Galois cover with group G. Suppose that*

$$L_\rho \cong \mathcal{O}_Y(\mathcal{D}) \qquad \textit{for all } \rho \in G^* - \{0\},$$

where L_ρ are those defined in (6.7) *and $\mathcal{D}$ is an irreducible divisor such that $h^0(\mathcal{D}) = 1$. Then $\pi : X \to Y$ is a double cover branched along a reduced divisor in $|2\mathcal{D}|$.*

Proof. The compatibility conditions for the abelian covers [Pa, Thm 2.1] tell us that

$$L_\rho \otimes L_{\rho'} \cong L_{\rho+\rho'} \otimes \bigotimes_{H \in \mathfrak{C}} \bigotimes_{\psi \in S_H} O_Y(\varepsilon^{H,\psi}_{\rho,\rho'} D_{H,\psi}),$$

where

(a) $D_{H,\psi}$ are effective divisors.

(b) $\mathfrak{C}$ is the set of cyclic quotients of G^*.

(c) S_H is the set of generators of H.

(d) Given $H \in \mathfrak{C}$ and a generator $\psi \in S_H$ we define by $i^{H,\psi}_\rho \in \{0, \dots, |H| - 1\}$ the integer (that depends on H and ψ) such that $\rho|_H = \psi^{i^{H,\psi}_\rho}$. Then

$$\varepsilon^{H,\psi}_{\rho,\rho'} = \begin{cases} 0, & i^{H,\psi}_\rho + i^{H,\psi}_{\rho'} < |H| \\ 1, & \text{otherwise.} \end{cases}$$

In our situation $L_\rho = \mathcal{O}_Y(\mathcal{D})$ for every $\rho \neq 0$. So, for every non-trivial ρ, ρ' such that $\rho + \rho' \neq 0$,

$$\mathcal{D} = \sum_{H \in \mathfrak{C}} \sum_{\psi \in S_H} \varepsilon^{H,\psi}_{\rho,\rho'} D_{H,\psi}.$$

Recall that we are assuming that $\mathcal{D}$ is an effective irreducible divisor with $h^0(\mathcal{D}) = 1$. Therefore, for every ρ, ρ' such that $\rho + \rho' \neq 0$, $\exists!(H,\psi)$ such that $\varepsilon^{H,\psi}_{\rho,\rho'} \neq 0$. Moreover, for this (H,ψ), $D_{H,\psi} = \mathcal{D}$. Since X is normal, by [Pa, Cor. 3.1], $\exists!(H,\psi)$ such that $D_{H,\psi} = \mathcal{D}$. Hence

$$\exists!(H,\psi) \text{ such that for every non-opposite and non-trivial } \rho, \rho', \quad \varepsilon^{H,\psi}_{\rho,\rho'} = 1 \quad (6.8)$$

Suppose that there exist, $H \in \mathfrak{C}$ of order $m = |H| > 2$. First observe that $|S_H| \geqslant 2$, since fixed an isomorphism $H \cong \mathbb{Z}/m\mathbb{Z}$, 1 and $m-1$ are two different generators. Then, for any generator $\psi \in S_H$ consider $\rho \in G^*$ such that $i^{H,\psi}_\rho = m-1$. Then it is clear, that $\rho \neq 0$, $i^{H,\psi}_{2\rho} = m-2 \neq 0$ so $2\rho \neq 0$ and $2(m-1) > |H|$, so $\varepsilon^{H,\psi}_{\rho,\rho} = 1$. Since $|S_H| \geqslant 2$, we get a contradiction with (6.8).

Therefore all the cyclic quotients of G^* are of order two. So $G^* \cong \mathbb{Z}/2\mathbb{Z} \times \overset{n}{\ldots} \times \mathbb{Z}/2\mathbb{Z}$. And every cyclic quotient $H \in \mathfrak{C}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and has a unique generator, so we omit the reference to $\psi \in S_H$ in the notation. In fact, since $G^* \to H$ is a morphism of $\mathbb{Z}/2\mathbb{Z}$-vector spaces it splits and we have as many non-trivial cyclic quotients as non-trivial elements in G^*, that is, $2^n - 1$. Moreover, for each cyclic quotient H half of the elements of G^* are sent to 0 in H and the other half is sent to the generator of H. So if $n > 1$ we have more than one non-trivial cyclic quotient H and for each H we have two different non-trivial elements $\rho, \rho' \in G^*$ such that $i^H_\rho = i^H_{\rho'} = 1$, so $\varepsilon^H_{\rho,\rho'} = 1$ contradicting (6.8). So $|G^*| = 2$.

Hence $\pi : X \to Y$ is a double cover and, being X normal, its branch locus is a reduced divisor $B \in |L^{\otimes 2}_\rho| = |2\mathcal{D}|$ (see again [Pa, Cor 3.1]). □

Now we are ready to prove the following theorem.

Theorem 6.13. *Let X be a primitive smooth complex variety of general type, $q(X) = \dim X$ and suppose $\operatorname{Alb} X$ is simple. Then, the following are equivalent,*

(a) the bicanonical map of X is non-birational and the finite part of the Stein factorization of $\mathrm{alb}\colon X \to \operatorname{Alb} X$ is an abelian Galois cover with rational singularities,

(b) X is birationally equivalent to a double cover of an indecomposable principally polarized abelian variety (A, Θ), branched along a reduced divisor $|2\Theta|$.

Recall that by Proposition 3.9, since X is of general type, $\chi(\omega_X) > 0$.

Proof. Example 4.5 shows the implication $(b) \Rightarrow (a)$. Hence we have only to prove $(a) \Rightarrow (b)$.
Lemma 4.18 yields to the standard short exact sequence on $X \times \operatorname{Pic}^0 X$

$$0 \to ((g \circ \mathrm{alb}) \times f)^* \mathcal{P}_C^{-1} \xrightarrow{\cdot \mathcal{Y}} p^* \mathcal{O}_X(F) \otimes q^* \mathcal{O}_{\operatorname{Pic}^0 X}(\mathcal{D}_{\bar{p}}) \to (p^* \mathcal{O}_X(F) \otimes q^* \mathcal{O}_{\operatorname{Pic}^0 X}(\mathcal{D}_{\bar{p}}))|_{\overline{\mathcal{Y}}} \to 0,$$

where p, q are the projections of $X \times \operatorname{Pic}^0 X$.
Recall that $((g \circ \mathrm{alb}) \times f)^* \mathcal{P}_C = (\mathrm{alb} \times \mathrm{id}_{\operatorname{Pic}^0 X})^* (\mathcal{O}_{B \times \operatorname{Pic}^0 B} \boxtimes \mathcal{P}_C)$. We apply the functor

$$R^d q_* (\,\cdot\, \otimes (\mathrm{alb} \times \mathrm{id}_{\operatorname{Pic}^0 X})^* (\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C \times \operatorname{Pic}^0 C})),$$

that is, we tensor by the other "half" Poincaré line bundle (see (4.7)) and we take the top direct image. We get

$$\begin{aligned} \cdots &\to R^d \Phi_{P^{-1}}(\mathcal{O}_X) \to \\ &\to R^d q_* \left(p^* \mathcal{O}_X(F) \otimes (\mathrm{alb} \times \mathrm{id}_{\operatorname{Pic}^0 X})^* (\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C \times \operatorname{Pic}^0 C})\right) \otimes \mathcal{O}_{\operatorname{Pic}^0 X}(\mathcal{D}_{\bar{p}}) \to \\ &\to R^d q_* \Big((p^* \mathcal{O}_X(F) \otimes q^* \mathcal{O}_{\operatorname{Pic}^0 X}(\mathcal{D}_{\bar{p}}) \otimes (\mathrm{alb} \times \mathrm{id}_{\operatorname{Pic}^0 X})^* (\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C \times \operatorname{Pic}^0 C}))|_{\overline{\mathcal{Y}}} \Big) \to 0 \end{aligned}$$

Recall that $\mathrm{R}\Phi_{P^{-1}} \cong (-1)^*_{\operatorname{Pic}^0 X} \mathrm{R}\Phi_P$ (see Corollary I.1.3). Then we have the following short exact sequence,

$$0 \to \ker \mu \to (-1)^*_{\operatorname{Pic}^0 X} \widehat{\mathcal{O}_X} \xrightarrow{\mu} \mathcal{E}(\mathcal{D}_{\bar{p}}) \to \mathcal{T} \to 0 \tag{6.9}$$

where:

(a) $\mathcal{E} = R^d q_* (p^* \mathcal{O}_X(F) \otimes (\mathrm{alb} \times \mathrm{id}_{\operatorname{Pic}^0 X})^* (\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C \times \operatorname{Pic}^0 C}))$. Since X is primitive $V^1(\omega_X)$ is a finite set of points. By Lemma 4.14 $\dim \ker(\mathrm{id} - f) > 0$, so $f(V^1(\omega_X)) \neq \ker(\mathrm{id} - f)$ and by Lemma 4.15*(c)* $h^0(\mathcal{O}_X(M) \otimes \beta) = h^d(\mathcal{O}_X(F) \otimes \beta^{-1})$ is constant equal to $\chi(\omega_X)$ for all $\beta \in \ker f = \operatorname{Pic}^0 B$. Then, by base change, $\mathcal{E}$ is a locally free sheaf of rank $\chi(\omega_X)$ on $\operatorname{Pic}^0 X$.

(b) Since the map μ is a generically surjective map of sheaves of the same rank (recall that $\operatorname{rk} \widehat{\mathcal{O}_X} = \chi(\omega_X)$), $\ker \mu$ is a torsion sheaf. So $\ker \mu \subseteq \tau((-1)^*_{\operatorname{Pic}^0 X} \widehat{\mathcal{O}_X})$ that by Corollary 3.6 it is $\mathbb{C}(\hat{0})$. Since $\ker \mu \neq 0$ because then $\widehat{\mathcal{O}_X}$ would be torsion-free, which is impossible by Corollary I.1.18, we have $\ker \mu \cong \mathbb{C}(\hat{0})$.

(c) $\mathcal{T} = R^d q_* \Big((p^* \mathcal{O}_X(F) \otimes q^* \mathcal{O}_{\operatorname{Pic}^0 X}(\mathcal{D}_{\bar{p}}) \otimes (\mathrm{alb} \times \mathrm{id}_{\operatorname{Pic}^0 X})^* (\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_{C \times \operatorname{Pic}^0 C}))|_{\overline{\mathcal{Y}}} \Big)$ is set-theoretically supported at the locus of the $\alpha \in \operatorname{Pic}^0 X$ such that the fiber of the projection $q \colon \overline{\mathcal{Y}} \to \operatorname{Pic}^0 X$ has dimension d, i.e. it coincides with X. Such locus is contained in $V^1(\omega_X)$ that, since X is primitive, is a finite set.

(d) μ is $R^d q_*(m_s)$, where m_s is the multiplication by the section defining $\overline{\mathcal{Y}}$. Hence by base change $R^d q_*(m_s) \otimes k(\alpha) = H^d(m_s|_{q^{-1}\{\alpha\}})$ where q is the projection $q \colon \overline{\mathcal{Y}} \to \operatorname{Pic}^0 X$. When $q^{-1}\{\alpha\} = X$, $m_s|_{q^{-1}\{\alpha\}} = 0$, hence in these points

$$\dim_{\mathbb{C}}(\mathcal{T} \otimes \mathbb{C}(\alpha)) = \chi(\omega_X).$$

Hence we have the following exact sequence,

$$0 \to \mathbb{C}(\hat{0}) \to (-1)^*_{\mathrm{Pic}^0 X}\widehat{\mathcal{O}_X} \xrightarrow{\mu} \mathcal{E}(\mathcal{D}_{\bar{p}}) \to \mathcal{T} \to 0. \tag{6.10}$$

Until now we have only used that X is a primitive variety and $X \to \mathrm{Alb}$ is generically finite. Consider, now the Stein factorization of $X \to \mathrm{Alb}\, X$, that is

$$\begin{tikzcd} X \arrow[rr, "\mathrm{alb}"] \arrow[dr, "u"'] & & \mathrm{Alb}\, X \\ & \tilde{X} \arrow[ur, "\widetilde{\mathrm{alb}}"'] & \end{tikzcd}$$

where $\tilde{X}$ is a normal variety u is a birational map (a desingularization of $\tilde{X}$).

By assumption $\widetilde{\mathrm{alb}}$ is an abelian Galois cover with group G and $\mathrm{R}u_*\mathcal{O}_X = \mathcal{O}_{\tilde{X}}$ since $\tilde{X}$ has rational singularities. Hence, by (6.7), we have that

$$\mathrm{R}\,\mathrm{alb}_*\,\mathcal{O}_X = \widetilde{\mathrm{alb}}_*\mathcal{O}_{\tilde{X}} = \bigoplus_{\rho \in G^*} L_\rho^{-1}, \tag{6.11}$$

where G^* is the group of characters of G and L_ρ^{-1} is the line bundle corresponding to the eigenspace of $\widetilde{\mathrm{alb}}_*\mathcal{O}_{\tilde{X}}$ where the group acts via the character ρ.

Since $\mathrm{gv}(\omega_X) \geqslant 0$, $\mathrm{R}\,\mathrm{alb}_*\,\mathcal{O}_X = \bigoplus_{\rho\in G^*} L_\rho^{-1}$ is a WIT_d sheaf. Hence, the line bundles L_ρ are ample or topologically trivial. But since $\tau(\widehat{\mathcal{O}_X}) = \mathbb{C}(\hat{0})$, only $L_0 \cong \mathcal{O}_X$ is topologically trivial and for $\rho \in G^* - \{0\}$, L_ρ is ample. In particular, since any ample line bundle in an abelian variety has sections, $h^0(L_\rho) \geqslant 1$ for all $\rho \in G^*$. Moreover, it is clear that the inclusion $\mathbb{C}(\hat{0}) \hookrightarrow (-1)^*_{\mathrm{Pic}^0 X}\widehat{\mathcal{O}_X}$ in exact sequence (6.10), splits and we have,

$$0 \to \bigoplus_{\rho \in G^* - \{0\}} R^d\Phi_{\mathcal{P}}\, L_\rho^{-1} \to \mathcal{E}(\mathcal{D}_{\bar{p}}) \to \mathcal{T} \to 0. \tag{6.12}$$

Dualizing, we get that $\mathcal{E}xt^d(\mathcal{T}, \mathcal{O}_{\mathrm{Pic}^0 X}) = 0$, so $\mathcal{T} = 0$. Hence, the exact sequence (6.10) becomes a short exact sequence that splits,

$$0 \to \mathbb{C}(\hat{0}) \to (-1)^*_{\mathrm{Pic}^0 X}\widehat{\mathcal{O}_X} \xrightarrow{\mu} \mathcal{E}(\mathcal{D}_{\bar{p}}) \to 0. \tag{6.13}$$

Dualizing we get that

$$\mathcal{E}^{\vee}(-\mathcal{D}_{\bar{p}}) \cong R^0\Phi_P\,\omega_X \cong \bigoplus_{\rho\in G^*-\{0\}} R^0\Phi_{\mathcal{P}}\,L_\rho, \tag{6.14}$$

and comparing ranks we have that,

$$\chi(\omega_X) = \sum_{\rho\in G^*-\{0\}} h^0(L_\rho). \tag{6.15}$$

Recall now that $\operatorname{Alb} X$ is a simple abelian variety, so $\ker f$ in Lemma 4.14 is trivial and

$$\mathcal{E} = R^d q_*(p^*\mathcal{O}_X(F)) = \mathcal{O}_{\operatorname{Pic}^0 X}^{\oplus\chi(\omega_X)}. \tag{6.16}$$

Now we go back to exact sequence (6.13) with (6.16),

$$0 \to \mathbb{C}(\hat{0}) \to (-1)^*_{\operatorname{Pic}^0 X}\widehat{\mathcal{O}_X} \xrightarrow{\mu} \bigoplus_{\chi(\omega_X)} \mathcal{O}_{\operatorname{Pic}^0 X}(\mathcal{D}_{\bar{p}}) \to 0.$$

By (6.12), we have $\bigoplus_{\chi(\omega_X)} \mathcal{O}_{\operatorname{Pic}^0 X}(\mathcal{D}_{\bar{p}}) \cong \bigoplus_{\rho\in G^*-\{0\}} R^d\Phi_{\mathcal{P}}\,L_\rho^{-1}$, so taking cohomologies we get

$$\chi(\omega_X)\cdot h^0(\mathcal{D}_{\bar{p}}) = \deg \operatorname{alb} -1 = |G|-1, \tag{6.17}$$

since L_ρ is ample and $h^0(\operatorname{Pic}^0 X, R^d\Phi_{\mathcal{P}}\,L_\rho^{-1}) = \operatorname{rk} L_\rho^{-1} = 1$ (see Theorem I.2.1, Corollary I.1.14 and Proposition I.2.2).

Thus,

$$\begin{aligned} |G|-1 &= \chi(\omega_X)\cdot h^0(\mathcal{D}_{\bar{p}}) && \text{by (6.17)}\\ &= \sum_{\rho\in G^*-\{0\}} h^0(L_\rho)\cdot h^0(\mathcal{D}_{\bar{p}}) && \text{by (6.15)}, \end{aligned}$$

so $h^0(\mathcal{D}_{\bar{p}}) = h^0(L_\rho) = 1$ for all $\rho \in G$. $\mathcal{D}_{\bar{p}}$ is a *principal* polarization, so via the identification $\operatorname{Alb} X \cong \operatorname{Pic}^0 X$ provided by $\phi_{\mathcal{D}_{\bar{p}}}$, we have $R^d\Psi_{\mathcal{P}}(-\mathcal{D}_{\bar{p}}) \cong \mathcal{D}_{\bar{p}}$ (see Proposition I.2.2). Hence

$$\bigoplus_{\chi(\omega_X)} \mathcal{O}(\mathcal{D}_{\bar{p}}) \cong \bigoplus_{\rho\in G^*-\{0\}} L_\rho.$$

Tensoring by $\mathcal{O}(-\mathcal{D}_{\bar{p}})$ we get $\bigoplus_{\chi(\omega_X)} \mathcal{O}_{\operatorname{Alb} X} \cong \bigoplus_{\rho\in G^*-\{0\}} L_\rho \otimes \mathcal{O}(-\mathcal{D}_{\bar{p}})$, so

$$\sum_{\rho\in G^*-\{0\}} h^0(L_\rho\otimes\mathcal{O}(-\mathcal{D}_{\bar{p}})) = \chi(\omega_X).$$

Since $|G|-1 = \chi(\omega_X)$ and $h^0(L_\rho\otimes\mathcal{O}(-\mathcal{D}_{\bar{p}})) \leqslant h^0(L_\rho) = 1$ for all $\rho \in G$, we have

$h^0(L_\rho \otimes \mathcal{O}(-\mathcal{D}_{\bar{p}})) = 1$ for all $\rho \in G^* - \{0\}$. Therefore,

$$L_\rho \cong \mathcal{O}_{\mathrm{Alb}\,X}(\mathcal{D}_{\bar{p}}) \quad \text{for all } \rho \in G^* - \{0\}\,.$$

Since we are assuming that $\mathrm{Alb}\,X$ is simple abelian variety and $h^0(\mathcal{D}_{\bar{p}}) = 1$, $\mathcal{D}_{\bar{p}}$ has to be an irreducible divisor by the Decomposition Theorem of abelian varieties (e.g. [BL, Thm. 4.3.1]).

By the previous Lemma 6.12, $\deg \mathrm{alb} = 2$ and X is birationally equivalent to a double cover of a principally polarized abelian variety branched along a reduced divisor in $|2\mathcal{D}_{\bar{p}}|$. □

Remark 6.14. *As Example 4.5 shows, it is clear that the implication (b) ⇒ (a) does not need to assume that* $\mathrm{Alb}\,X$ *is simple.*

From the previous proposition, we have the following immediate corollary

Corollary 6.15. *Let X be a primitive smooth complex variety of general type with $q(X) = \dim X$ and suppose $\mathrm{Alb}\,X$ is simple. If the finite part of the Stein factorization of $\mathrm{alb}\colon X \to \mathrm{Alb}\,X$ is an abelian Galois cover with rational singularities and the bicanonical map of X is non-birational, then $\chi(\omega_X) = 1$.*

Proof. Let $h\colon X \to A$ be a double cover branched along a reduced divisor $B \in |2\Theta|$. Then,

$$\chi(\omega_X) = \chi(\omega_X \otimes \alpha) = h^0(\omega_X \otimes \alpha) = h^0(h^*\mathcal{O}(\Theta) \otimes \alpha) = h^0(\mathcal{O}(\Theta) \otimes \alpha) + h^0(\alpha) = 1,$$

for a general $\alpha \in \mathrm{Pic}^0 X = \mathrm{Pic}^0 A$. □

In fact we have also the following Proposition.

Proposition 6.16. *Let X be a primitive smooth complex variety of general type and $q(X) = \dim X$. Then, the following are equivalent,*

- *(a) the finite part of the Stein factorization of $\mathrm{alb}\colon X \to \mathrm{Alb}\,X$ is an abelian Galois cover, has rational singularities and $\chi(\omega_X) = 1$,*
- *(b) X is birationally equivalent to a double cover of an indecomposable principally polarized abelian variety (A, Θ), branched along a reduced divisor in $|2\Theta|$.*

Proof. If $\chi(\omega_X) = 1$ the consequences of the birationality criterion, Corollary 4.11, are trivially satisfied. The rest of section §4, from Corollary 4.11 and beyond, follows from the consequences of Corollary 4.11. Hence, following the proof of Theorem 6.13 we get again the short exact sequence (6.13)

$$0 \to \mathbb{C}(\hat{0}) \to (-1)^*_{\mathrm{Pic}^0 X}\widehat{\mathcal{O}_X} \xrightarrow{\mu} \mathcal{E}(\mathcal{D}_{\bar{p}}) \to 0.$$

Now, we proceed like in the proof of Theorem 6.13 until equality (6.15) that directly tells us

$$1 = \sum_{\rho \in G^* - \{0\}} h^0(L_\rho),$$

so $|G| = 2$ and $h^0(L_\rho) = 1$. Observe that, until equality (6.15) we have not used that $\mathrm{Alb}\, X$ is simple so we can avoid this assumption. Moreover, this way we take an easy shortcut that allows us to avoid Lemma 6.12.

The other implication follows from the same calculation as Corollary 6.15. □

These two results, show that we have to find the difficulties and interesting examples like Chen-Hacon's surface (Example 6.10) in the non-abelian case. However, we expect that the key point to study the primitive varieties of general type and $q(X) = \dim X$ with non-birational bicanonical map, is the type of singularities that are allowed in the Stein factorization. So, we end with the following conjecture,

Conjecture 6.17. *Let X be a smooth primitive variety of general type and $q(X) = \dim X$. Then, the following are equivalent,*

(a) the bicanonical map of X is non-birational,
(b) the finite part of the Stein factorization of $\mathrm{alb}\colon X \to \mathrm{Alb}\, X$ *has canonical singularities and $\chi(\omega_X) = 1$,*
(c) X is birationally equivalent to a double cover of an indecomposable principally polarized abelian variety (A, Θ), branched along a reduced divisor in $|2\Theta|$.

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