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ON PARTIALLY SATURATED FORMATIONS OF FINITE GROUPS

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On partially saturated formations of finite groups

Tesis doctoral
presentada por
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dirigida por el doctor
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Don Adolfo Ballester Bolinches, profesor titular de universidad del Departament d'Àlgebra de la Universitat de València,

CERTIFICA:

Que la presente memoria *On partially saturated formations of finite groups* ha sido realizada bajo mi dirección por doña Clara Calvo López en el Departament d'Àlgebra de la Universitat de València y constituye su tesis para optar al grado de doctora.

Y para que así conste, autoriza la presentación de la referida tesis doctoral firmando el presente certificado.

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Contents

Introduction

The present work contains contributions to the theory of partially saturated formations of finite groups. This theory begins with the concept of local formation, introduced by Gaschütz in 1963 and it has been enriched by the results of Baer, Förster, and the school of Shemetkov in Gomel. This thesis is organised in four chapters, corresponding more or less to the contents of the papers we have written in the subject. After presenting the preliminary and basic results in Chapter 1, we study the relation between the known partially saturated formations in Chapter 2. The study of $\mathfrak{X}$-local products of formations is the main aim in Chapter 3. We bring the thesis into a close by giving the complete description of the factorisations of one-generated partially saturated formations in Chapter 4. Each chapter begins with a historical and mathematical introduction motivating the results included there. The main bibliographic references are the book of Doerk and Hawkes ([DH92]) and the one of Ballester and Ezquerro ([BBE06]).

Chapter 1

$\mathfrak{X}$-local formations

1.1 Preliminary concepts

The presentation of our work should begin with the concept of formation.

Definition 1.1.1. A *formation* is a class of groups satisfying the following two conditions:

1. If $G \in \mathfrak{F}$ and N is a normal subgroup of G, then $G/N \in \mathfrak{F}$.
2. If N and M are normal subgroups of G such that G/M, $G/N \in \mathfrak{F}$, then $G/(M \cap N) \in \mathfrak{F}$.

For a non-empty formation $\mathfrak{F}$ and a group G, the $\mathfrak{F}$-*residual* $G^{\mathfrak{F}}$ of G is the smallest normal subgroup N of G such that G/N belongs to $\mathfrak{F}$.

If $\mathfrak{Y}$ is a class of groups, the smallest formation containing $\mathfrak{Y}$ is denoted by form$(\mathfrak{Y})$. It is well-known that form$(\mathfrak{Y}) = \mathrm{Q}\,\mathrm{R}_0\,\mathfrak{Y}$ ([DH92; II, 2.2]).

Given two classes $\mathfrak{Y}$ and $\mathfrak{Z}$ of groups, a product class can be defined by setting

$$\mathfrak{Y}\mathfrak{Z} = (G \in \mathfrak{E} \mid \text{there is a normal subgroup } N \text{ of } G \text{ such that } N \in \mathfrak{Y} \text{ and } G/N \in \mathfrak{Z}),$$

Groups in $\mathfrak{Y}\mathfrak{Z}$ are sometimes called $\mathfrak{Y}$-by-$\mathfrak{Z}$-groups.

However this class is not in general a formation when $\mathfrak{Y}$ and $\mathfrak{Z}$ are formations. But there is a way of modifying the above definition to ensure that the class product of two formations is again a formation. If $\mathfrak{F}$ and $\mathfrak{G}$ are formations, the *formation product* or *Gaschütz product* of $\mathfrak{F}$ and $\mathfrak{G}$ is the class $\mathfrak{F} \circ \mathfrak{G}$ defined by

$$\mathfrak{F} \circ \mathfrak{G} := (G \in \mathfrak{E} \mid G^{\mathfrak{G}} \in \mathfrak{F}).$$

The class $\mathfrak{F} \circ \mathfrak{G}$ is again a formation and if $\mathfrak{F}$ is closed under taking subnormal subgroups, then $\mathfrak{F}\mathfrak{G} = \mathfrak{F} \circ \mathfrak{G}$ (see [DH92; IV, 1.7 and 1.8]).

Definition 1.1.2. A formation $\mathfrak{F}$ is said to be *saturated* when $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$, where $\Phi(G)$ denotes the Frattini subgroup of G.

Gaschütz [Gas63] introduced the concept of *local formation*, which enabled him to construct a rich family of saturated formations.

Definition 1.1.3.

- A *formation function* f assigns to every $p \in \mathbb{P}$ a (perhaps empty) formation $f(p)$.
- If f is a formation function, then the *local formation* $\mathrm{LF}(f)$ defined by f is the class of all groups G such that if H/K is a chief factor of G, then $G/\mathrm{C}_G(H/K) \in f(p)$ for all $p \in \pi(H/K)$.
- A formation $\mathfrak{F}$ is said to be *local* if there exists a formation function f such that $\mathfrak{F} = \mathrm{LF}(f)$.

The following remarkable theorem characterises local formations. It was proved by Gaschütz and Lubeseder in the soluble universe and later generalised by Schmid to the general finite one. It is now known as the Gaschütz-Lubeseder-Schmid theorem.

Theorem 1.1.4 (Gaschütz-Lubeseder-Schmid [DH92; IV, 4.6])**.** *A formation $\mathfrak{F}$ is saturated if and only if $\mathfrak{F}$ is local.*

Baer followed another approach to extend the theorem of Gaschütz and Lubeseder to the finite universe. He used a different notion of local formation, in which non-abelian chief factors were treated with more flexibility than abelian ones. This led him to find a new family of formations, the Baer-local formations, containing the local ones.

Definition 1.1.5.

- A *Baer function* assigns to every simple group J a class of groups $f(J)$ such that $f(C_p)$ is a formation for every $p \in \mathbb{P}$.
- If f is a Baer function, then the *Baer-local formation* or *Baer formation* $\mathrm{BLF}(f)$ defined by f is the class of all groups G such that if H/K is a chief factor of G, then $G/\mathrm{C}_G(H/K) \in f(J)$, where J is the composition factor of H/K.

- A formation $\mathfrak{F}$ is said to be *Baer-local* if there exists a Baer function f such that $\mathfrak{F} = \mathrm{BLF}(f)$.

Shemetkov introduced in [She75] the concept of *composition formation*. This notion is equivalent to the one of Baer-local formation.

Definition 1.1.6. A formation $\mathfrak{F}$ is said to be *solubly saturated* when, for every group G, the condition $G/\Phi(G_{\mathfrak{S}}) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$, where $G_{\mathfrak{S}}$ denotes the soluble radical of G.

Baer proved the following theorem:

Theorem 1.1.7 (Baer, [DH92; IV, 4.17])**.** *A formation $\mathfrak{F}$ is solubly saturated if and only if $\mathfrak{F}$ is Baer-local.*

The concepts of saturation and soluble saturation can be joined in a general definition. Other kinds of partially saturated formations appear naturally.

Definition 1.1.8. Let $\mathfrak{H}$ be a Fitting class and ω a fixed non-empty set of primes. We say that a formation $\mathfrak{F}$ is $(\omega, \mathfrak{H})$-*saturated* if $G/\mathrm{O}_{\omega}(\Phi(G_{\mathfrak{H}})) \in \mathfrak{F}$ always implies that $G \in \mathfrak{F}$.

The notion of saturated formation appears when $\mathfrak{H}$ coincides with the class $\mathfrak{E}$ of all finite groups and ω is the set $\mathbb{P}$ of all primes. When $\mathfrak{H}$ is the class of all soluble groups and ω is the set $\mathbb{P}$ of all primes, we find the concept of solubly saturated formation. Let us note other important special cases.

An $(\omega, \mathfrak{H})$-saturated formation $\mathfrak{F}$ is called

- *nilpotently saturated* when $\mathfrak{H} = \mathfrak{N}$, the class of all nilpotent groups, and $\omega = \mathbb{P}$.
- *p-solubly saturated* if $\mathfrak{H}$ coincides with the class of all p-soluble groups and $\omega = \mathbb{P}$.
- $\mathfrak{S}_p$*-saturated* if $\mathfrak{H} = \mathfrak{S}_p$, the class of all p-groups and $\omega = \mathbb{P}$.
- ω*-saturated* when $\mathfrak{H} = \mathfrak{E}$. We say that $\mathfrak{F}$ is *p-saturated* if $\omega = \{p\}$.

The concept of ω-saturation, where ω is a non-empty set of primes, has been widely studied, since it is a natural approach to the notion of saturation. The following question naturally arose:

How can ω-saturated formations be "locally" characterised?

The initial idea was to use formation functions for this purpose (in [She84] a formation $\mathfrak{H}$ was called *p-local* if the saturated formation generated by $\mathfrak{H}$ is contained in $\mathfrak{N}_{p'}\mathfrak{H}$). In the development of this idea, the following result was proved in [SS95]:

Theorem 1.1.9. *For a non-empty formation $\mathfrak{F}$, the following condtions are equivalent:*

1. *$\mathfrak{F}$ is ω-saturated.*
2. *$\mathfrak{S}_p(\mathfrak{F}/\operatorname{O}_{p',p}(\mathfrak{F})) \subseteq \mathfrak{F}$ for any $p \in \pi(\mathfrak{F}) \cap \omega$, where $\mathfrak{F}/O_{p',p}(\mathfrak{F}) = \operatorname{form}(G/\operatorname{O}_{p',p}(G) \mid G \in \mathfrak{F})$ if $p \in \pi(\mathfrak{F})$.*
3. *The saturated formation generated by $\mathfrak{F}$ is contained in $\mathfrak{N}_{\omega'}\mathfrak{F}$.*

The aim was to find an explicit function describing an ω-saturated formation. This problem was solved in [BBS97]. The main idea was to use the new concept of *small centraliser*. If G is a group, by $\mathcal{K}(G)$ we denote the class of all simple groups which are isomorphic to the composition factors of G.

Definition 1.1.10. Let H/K be a normal section of a group G. The *small centraliser* $\operatorname{c}_G(H/K)$ of H/K in G is the subgroup generated by all normal subgroups N of G such that $\mathcal{K}(NK/K) \cap \mathcal{K}(H/K) = \emptyset$.

In [BBS97], the definition of p-local satellite was introduced:

Definition 1.1.11. Let p be a prime number.

- A function f which associates with each group a formation is called a *p-local satellite* if it satisfies the following conditions:
  1. $f(S_1) = f(S_2)$ for any two characteristically simple groups S_1 and S_2 such that $\mathcal{K}(S_1) = \mathcal{K}(S_2)$.
  2. The value of f on any simple group whose order is divided by p coincides with the formation $f(p)$.
- If f is a p-local satellite, a chief factor H/K of a group G is called *f-central* if the following conditions hold:
  1. If the order of H/K is divided by p, then $G/\operatorname{C}_G(H/K) \in f(p)$.
  2. If H/K is a p'-group, then $G/\operatorname{c}_G(H/K) \in f(H/K)$.

It was shown [BBS97] that a non-empty formation $\mathfrak{F}$ is p-saturated if and only if it coincides with the class of all groups with f-central chief factors for some p-local satellite f. The following interesting fact was also obtained: every non-empty p-saturated formation has a p-local satellite whose values on non-identity p'-groups coincide. This fact was used as the foundation for the following definition which is used in works of different authors.

Definition 1.1.12. Let ω be a non-empty set of prime numbers.

- An *ω-local satellite* f assigns to every element of $\omega \cup \{\omega'\}$ a (perhaps empty) formation.
- The symbol $G_{\omega d}$ is used to denote the largest normal subgroup N of G such that $\omega \cap \pi(H/K) \neq \emptyset$ for every composition factor H/K of N (if $\omega \cap \pi\big(\mathrm{Soc}(G)\big) = \emptyset$, then we set $G_{\omega d} = 1$).
- If f is an ω-local satellite, then $\mathrm{LF}_\omega(f)$ denotes the class of groups G satisfying the following two conditions:
  1. if H/K is a chief factor of G, then $G/\mathrm{C}_G(H/K) \in f(p)$ for every $p \in \pi(H/K) \cap \omega$, and
  2. $G/G_{\omega d} \in f(\omega')$.
- A formation $\mathfrak{F}$ is *ω-local* when there exists an ω-local satellite f such that $\mathfrak{F} = \mathrm{LF}_\omega(f)$. In this case, f is called an *ω-local satellite* of $\mathfrak{F}$.

Remark 1.1.13. Note that Condition 1 in the above definition can be replaced by the following:

- $G/\mathrm{O}_{p',p}(G) \in f(p)$ for any $p \in \omega \cap \pi(G)$.

Definition 1.1.14. Let f be an ω-local satellite defining an ω-local formation $\mathfrak{F}$.

1. f is called *integrated* if $\mathfrak{F}$ contains $f(p)$ for every $p \in \omega$ and $f(\omega')$.
2. f is called *full* if $\mathfrak{S}_p f(p) = f(p)$ for every $p \in \omega$.

The next lemma shows that every ω-local formation has a full and integrated ω-local satellite.

Lemma 1.1.15 ([She03; 2.2])**.** *Let $\mathfrak{F}$ be an ω-local formation defined by an ω-local satellite f. Then $\mathfrak{F} = \mathrm{LF}_\omega(g)$, where*

$$\begin{aligned} g(p) &= \mathfrak{S}_p\big(f(p) \cap \mathfrak{F}\big) \qquad \text{if } p \in \omega, \\ g(\omega') &= f(\omega') \cap \mathfrak{F}. \end{aligned}$$

Moreover, g is a full and integrated ω-local satellite of $\mathfrak{F}$.

Theorem 1.1.16 ([SS00]). *A formation $\mathfrak{F}$ is ω-saturated if and only if $\mathfrak{F}$ is ω-local.*

A proof of this result is presented in [BBE06; Theorem 3.4.2].

1.2 $\mathfrak{X}$-local formations

With the aim of presenting a common generalisation of the Gaschütz-Lubeseder-Schmid and Baer theorems, Förster introduced in [För85] the concept of $\mathfrak{X}$-local formation, for a class $\mathfrak{X}$ of simple groups such that $\pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$, where

$$\pi(\mathfrak{X}) := \{p \in \mathbb{P} \mid \text{there exists } G \in \mathfrak{X} \text{ such that } p \in \pi(G)\}$$

and

$$\operatorname{char}\mathfrak{X} := \{p \in \mathbb{P} \mid C_p \in \mathfrak{X}\}.$$

This section is devoted to present the basic results of $\mathfrak{X}$-local formations, focusing our attention on some distinguished $\mathfrak{X}$-local formation functions defining them. Part of the results included here appear in Section 3.1 of [BBE06]. The rest can be found in [BBCSss].

In the rest of this thesis, $\mathfrak{X}$ will denote a class of simple groups satisfying the above condition.

Let $\mathfrak{J}$ denote the class of all simple groups. For any subclass $\mathfrak{Y}$ of $\mathfrak{J}$, we write $\mathfrak{Y}' := \mathfrak{J} \setminus \mathfrak{Y}$. Denote by $\mathrm{E}\,\mathfrak{Y}$ the class of groups whose composition factors belong to $\mathfrak{Y}$. It is clear that $\mathrm{E}\,\mathfrak{Y}$ is a Fitting class, and so each group G has a largest normal $\mathrm{E}\,\mathfrak{Y}$-subgroup, the $\mathrm{E}\,\mathfrak{Y}$-radical, denoted either by $G_{\mathrm{E}\,\mathfrak{Y}}$ or by $\mathrm{O}_{\mathfrak{Y}}(G)$. A chief factor which belongs to $\mathrm{E}\,\mathfrak{Y}$ is called a $\mathfrak{Y}$-*chief factor*. If p is a prime, we write $\mathfrak{Y}_p$ to denote the class of all simple groups $S \in \mathfrak{Y}$ such that $p \in \pi(S)$. Sometimes it will be convenient to identify the prime p with the cyclic group C_p of order p.

In the rest of this thesis, $\mathfrak{X}$ will denote a class of simple groups such that $\pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$.

Definition 1.2.1 ([För85]).

- An $\mathfrak{X}$-*formation function* f assigns to each $X \in (\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}'$ a (possibly empty) formation $f(X)$.
- If f is an $\mathfrak{X}$-formation function, then $\mathrm{LF}_{\mathfrak{X}}(f)$ is the class of all groups G satisfying the following two conditions:

    1. If H/K is an $\mathfrak{X}_p$-chief factor of G, then $G/\mathrm{C}_G(H/K) \in f(p)$.

2. If G/L is a monolithic quotient of G such that $\mathrm{Soc}(G/L)$ is an $\mathfrak{X}'$-chief factor of G, then $G/L \in f(E)$, where E is the composition factor of $\mathrm{Soc}(G/L)$.

Remark 1.2.2. By appealing to the classical theorem of Jordan-Hölder for operator groups, it is only necessary to consider the $\mathfrak{X}_p$-chief factors of a given chief series of a group G in order to check whether or not G satisfies Condition 1 of Definition 1.2.1.

Definition 1.2.3. Consider a prime $p \in \operatorname{char}\mathfrak{X}$ and a group G. Then the subgroup $\mathrm{C}^{\mathfrak{X}_p}(G)$ is defined to be the intersection of the centralisers of all $\mathfrak{X}_p$-chief factors of G, with $\mathrm{C}^{\mathfrak{X}_p}(G) = G$ if G has no $\mathfrak{X}_p$-chief factors.

Remark 1.2.4. Consider an $\mathfrak{X}$-formation function f and $p \in \operatorname{char}\mathfrak{X}$. If $f(p) \neq \emptyset$, a group G satisfies Condition 1 of Definition 1.2.1 for the prime p if and only if $G/\mathrm{C}^{\mathfrak{X}_p}(G) \in f(p)$.

Lemma 1.2.5. *Consider a prime $p \in \operatorname{char}\mathfrak{X}$, a group G and a normal subgroup N of G. Then*

$$\mathrm{C}^{\mathfrak{X}_p}(G) \cap N = \mathrm{C}^{\mathfrak{X}_p}(N)$$

Proof. Consider a chief series of G,

$$1 \leq G_0 \leq G_1 \leq \cdots \leq N = G_t \leq G_{t+1} \leq \cdots \leq G. \tag{1.1}$$

If N does not have $\mathfrak{X}_p$-chief factors, then either G does not have $\mathfrak{X}_p$-chief factors or all the $\mathfrak{X}_p$-chief factors of G are centralised by N. In both cases, $N = \mathrm{C}^{\mathfrak{X}_p}(N) = N \cap \mathrm{C}^{\mathfrak{X}_p}(G)$.

Assume that N has $\mathfrak{X}_p$-chief factors. Hence G also possesses $\mathfrak{X}_p$-chief factors. If H/K is an $\mathfrak{X}_p$-chief factor of G below N, by [DH92; A, 4.13] we have that

$$H/K = H_1/K \times \cdots \times H_r/K,$$

where H_i/K is a chief factor of N. Moreover,

$$\mathrm{C}_N(H/K) = \bigcap_{i=1}^{r} \mathrm{C}_N(H_i/K). \tag{1.2}$$

Since $\mathrm{C}^{\mathfrak{X}_p}(N)$ and $\mathrm{C}^{\mathfrak{X}_p}(G)$ do not depend on the considered chief series of N and G, respectively, and N centralises the chief factors of G above N, we have that

$$\begin{aligned}
N \cap \mathrm{C}^{\mathfrak{X}_p}(G) &= N \cap \bigcap \{\mathrm{C}_G(H/K) \mid H/K \ \mathfrak{X}_p\text{-chief factor of } G\} \\
&= N \cap \bigcap \{\mathrm{C}_G(H/K) \mid H/K \ \mathfrak{X}_p\text{-chief factor of } G,\ H \leq N\} \\
&= \bigcap \{N \cap \mathrm{C}_G(H/K) \mid H/K \ \mathfrak{X}_p\text{-chief factor of } G,\ H \leq N\} \\
&= \bigcap \{\mathrm{C}_N(H/K) \mid H/K \ \mathfrak{X}_p\text{-chief factor of } G,\ H \leq N\} \\
&= \bigcap \{\mathrm{C}_N(L/T) \mid L/T \ \mathfrak{X}_p\text{-chief factor of } N\} \qquad \text{by (1.2)} \\
&= \mathrm{C}^{\mathfrak{X}_p}(N)
\end{aligned}$$

□

Remark 1.2.6. If N is not a normal subgroup of G, Lemma 1.2.5 is not true in general. Take, for example, $\mathfrak{X} = (C_3)$, $p = 3$, $G \cong \Sigma_3$ and $N \cong C_2$. We have that $\mathrm{C}^{\mathfrak{X}_p}(N) = N$ and $\mathrm{C}^{\mathfrak{X}_p}(G) \cong C_3$. Therefore, $\mathrm{C}^{\mathfrak{X}_p}(G) \cap N \neq \mathrm{C}^{\mathfrak{X}_p}(N)$.

Lemma 1.2.7. *Consider a prime $p \in \operatorname{char} \mathfrak{X}$, a group G and a normal subgroup N of G such that $N \leq \mathrm{O}_{\mathfrak{X}}(G)$. Then*

$$\mathrm{C}^{\mathfrak{X}_p}\big(G/\Phi(N)\big) = \mathrm{C}^{\mathfrak{X}_p}(G)/\Phi(N).$$

Proof. Clearly, we can assume that $\Phi(N) \neq 1$.

Suppose that G does not have $\mathfrak{X}_p$-chief factors. Then $G/\Phi(N)$ does not have $\mathfrak{X}_p$-chief factors. In this case, $\mathrm{C}^{\mathfrak{X}_p}(G) = G$ and $\mathrm{C}^{\mathfrak{X}_p}\big(G/\Phi(N)\big) = G/\Phi(N)$ and the result holds.

Now assume that G has $\mathfrak{X}_p$-chief factors and $G/\Phi(N)$ does not have $\mathfrak{X}_p$-chief factors. Then there exists an $\mathfrak{X}_p$-chief factor H/K of G such that $H \leq \Phi(N)$. Since p divides the order of $\Phi(N)$, then we have that p also divides the order of $N/\Phi(N)$. It follows that $N/\Phi(N)$ has a chief factor A/B whose order is divisible by p. Since $N \leq \mathrm{O}_{\mathfrak{X}}(G)$, we can assume that A/B is an $\mathfrak{X}_p$-chief factor of $G/\Phi(N)$, contradicting the hypothesis. We have proved that if G has $\mathfrak{X}_p$-chief factors, then so does $G/\Phi(N)$.

Take a subgroup A such that

$$A/\Phi(N) = \mathrm{C}^{\mathfrak{X}_p}\big(G/\Phi(N)\big).$$

We have that

$$\Phi(N) \leq \Phi(G) \leq \mathrm{F}(G) \leq \mathrm{C}^{\mathfrak{X}_p}(G).$$

Therefore, we can consider $\mathrm{C}^{\mathfrak{X}_p}(G)/\Phi(N)$. Since

$$\mathrm{C}^{\mathfrak{X}_p}(G)/\Phi(N) \leq \mathrm{C}^{\mathfrak{X}_p}\big(G/\Phi(N)\big),$$

we have that $\mathrm{C}^{\mathfrak{X}_p}(G) \leq A$. We aim to prove that $A \leq \mathrm{C}^{\mathfrak{X}_p}(G)$.

Consider the formation function f defined by:

$$f(q) = \begin{cases} (1) & \text{if } q = p, \\ \mathfrak{E} & \text{if } q \neq p. \end{cases}$$

A can be considered as a group of operators of G by conjugation.

We are going to prove that A acts hypercentrally on N (see [DH92; IV, 6.1]). We will see that A acts hypercentrally on $N/\Phi(N)$ in order to apply [DH92; IV, 6.7]. The theorem of Jordan-Hölder will be used.

Consider an A-composition factor H/K of G between $N \cap A$ and N. Since $[N, A] \leq N \cap A$, we have that $\mathrm{C}_A(H/K) = A$. Take a chief factor H/K of G between $\Phi(N)$ and $N \cap A$ such that p divides $|H/K|$. It follows that H/K is an $\mathfrak{X}_p$-chief factor of G, since $N \leq \mathrm{O}_{\mathfrak{X}}(G)$. Therefore, H/K is centralised by A. Since H/K is a minimal normal subgroup of G/K and $H/K \leq A/K$, we have by [DH92; A, 4.13] that

$$H/K = L_1/K \times L_2/K \times \cdots \times L_s/K,$$

where L_i/K is an A-composition factor of G centralised by A. We can construct a part of an A-composition series of G between $\Phi(N)$ and N such that every A-composition factor whose order is divided by p is centralised by A.

Now we aim to see that A centralises every $\mathfrak{X}_p$-chief factor of G. Clearly, A centralises every $\mathfrak{X}_p$-chief factor of G above $\Phi(N)$. Consider an $\mathfrak{X}_p$-chief factor H/K of G below $\Phi(N)$. Since H/K is a minimal normal subgroup of G/K and $H/K \leq A/K$, we have that

$$H/K = L_1/K \times L_2/K \times \cdots \times L_r/K,$$

where L_i/K is an A-composition factor of G. Since A centralises every L_i/K, we have that A centralises H/K, as we wanted. Therefore, $A \leq \mathrm{C}^{\mathfrak{X}_p}(G)$. □

Förster proves in [För85] that the class $\mathrm{LF}_{\mathfrak{X}}(f)$ is a formation. An alternative proof of this result can be found in [BBE06; Theorem 3.1.4]. We present here a different approach to this proof.

Theorem 1.2.8. *Consider an $\mathfrak{X}$-formation function f. The class* $\mathrm{LF}_{\mathfrak{X}}(f)$ *is a formation.*

Proof. We can assume that $\mathrm{LF}_{\mathfrak{X}}(f) \neq \emptyset$.

First let us prove that $\mathrm{LF}_{\mathfrak{X}}(f)$ is Q-closed. Consider a group $G \in \mathrm{LF}_{\mathfrak{X}}(f)$ and a normal subgroup N of G. If $(H/N)/(K/N)$ is an $\mathfrak{X}_p$-chief factor of G/N, then H/K is an $\mathfrak{X}_p$-chief factor of G and $\mathrm{C}_{G/N}\big((H/N)/(K/N)\big) =$

$\mathrm{C}_G(H/K)/N$. Therefore, we have that $(G/N)/\mathrm{C}_{G/N}\big((H/N)/(K/N)\big) \cong G/\mathrm{C}_G(H/K) \in f(p)$ and, hence, G/N satisfies Condition 1 of Definition 1.2.1. Moreover, if $(G/N)/(L/N)$ is a monolithic group whose socle is an $\mathfrak{X}'$-chief factor of type E, it follows that $(G/N)/(L/N) \cong G/L \in f(E)$.

Now we aim to see that $\mathrm{LF}_{\mathfrak{X}}(f)$ is R_0-closed. By [DH92; II, 2.6], it is enough to prove that if N_1 and N_2 are minimal normal subgroups of G such that $G/N_1 \in \mathrm{LF}_{\mathfrak{X}}(f)$, $G/N_2 \in \mathrm{LF}_{\mathfrak{X}}(f)$ and $N_1 \cap N_2 = 1$, then $G \in \mathrm{LF}_{\mathfrak{X}}(f)$. Since $G/N_1 \in \mathrm{LF}_{\mathfrak{X}}(f)$, it is clear that Condition 1 of Definition 1.2.1 holds for $\mathfrak{X}$-chief factors H/K of G such that $N_1 \leq K$. If N_1 is not an $\mathfrak{X}$-chief factor of G, there is nothing to prove. Assume that N_1 is not an $\mathfrak{X}$-chief factor of G and let $p \in \pi(N_1)$. Since N_1N_2/N_2 is G-isomorphic to $N_1/N_1 \cap N_2$ and $N_1 \cap N_2 = 1$, we have that N_1N_2/N_2 is an $\mathfrak{X}_p$-chief factor of G above N_2 and $\mathrm{C}_G(N_1) = \mathrm{C}_G(N_1N_2/N_2)$. Since $G/N_2 \in \mathrm{LF}_{\mathfrak{X}}(f)$, it follows that $G/\mathrm{C}_G(N_1) = G/\mathrm{C}_G(N_1N_2/N_2) \in f(p)$.

Now consider $L \trianglelefteq G$ such that G/L is monolithic and the composition factor of $\mathrm{Soc}(G/L)$ is isomorphic to $S \in \mathfrak{X}'$. Now we distinguish two cases:

- If $f(S) = \emptyset$, then $\mathrm{LF}_{\mathfrak{X}}(f) \subseteq \mathrm{E}(S)'$. Assume that it is not true and consider a group H in $\mathrm{LF}_{\mathfrak{X}}(f) \setminus \mathrm{E}(S)'$ of minimal order. By [DH92; II, 2.5], we have that H is monolithic. Let T be the socle of H. If $T \in \mathrm{E}(S)$, we would have that $H \in f(S)$. Since $f(S) = \emptyset$, then $H \in \mathrm{E}(S)'$, contradicting the choice of G. Therefore, $\mathrm{LF}_{\mathfrak{X}}(f) \subseteq \mathrm{E}(S)'$.

  Since $G/N_1 \in \mathrm{LF}_{\mathfrak{X}}(f) \subseteq \mathrm{E}(S)'$, $G/N_2 \in \mathrm{LF}_{\mathfrak{X}}(f) \subseteq \mathrm{E}(S)'$ and $N_1 \cap N_2 = 1$, we have that $G \in \mathrm{E}(S)'$. This is a contradiction, since the composition factor of $\mathrm{Soc}(G/L)$ is isomorphic to S. This means that the case $f(S) = \emptyset$ is not possible.

- If $f(S) \neq \emptyset$, then $\mathrm{LF}_{\mathfrak{X}}(f) \subseteq \mathrm{E}(S)' \circ f(S)$.

  Assume that this is not true and consider a group H of minimal order in $\mathrm{LF}_{\mathfrak{X}}(f) \setminus \mathrm{E}(S)' \circ f(S)$. By [DH92; II, 2.5], H is a monolithic group. Consider $T := \mathrm{Soc}(G)$. If the composition factor of T is isomorphic to S, then $H \in f(S) \subseteq \mathrm{E}(S)' \circ f(S)$, which is a contradiction. Therefore, $T \in \mathrm{E}(S)'$. Since $H/T \in \mathrm{E}(S)' \circ f(S)$, there exists $M \trianglelefteq G$ such that $M/T \in \mathrm{E}(S)'$ and $G/T \in f(S)$. Hence $M \in \mathrm{E}(S)'$ and we obtain that $H \in \mathrm{E}(S)' \circ f(S)$. This contradiction shows that $\mathrm{LF}_{\mathfrak{X}}(f) \subseteq \mathrm{E}(S)' \circ f(S)$.

  Since $G/N_1 \in \mathrm{LF}_{\mathfrak{X}}(f) \subseteq \mathrm{E}(S)' \circ f(S)$, $G/N_2 \in \mathrm{LF}_{\mathfrak{X}}(f) \subseteq \mathrm{E}(S)' \circ f(S)$ and $N_1 \cap N_2 = 1$, it follows that $G \in \mathrm{E}(S)' \circ f(S)$. There exists $N \trianglelefteq G$ such that $N \in \mathrm{E}(S)'$ and $G/N \in f(S)$. Therefore $N \leq L$ (otherwise we would have that $NL/L \neq 1$ and $\mathrm{Soc}(G/L) \leq NL/L \in \mathrm{E}(S)'$). Therefore, $G/L \in \mathrm{Q}\, f(S) = f(S)$ and $G \in \mathrm{LF}_{\mathfrak{X}}(f)$. □

Definition 1.2.9. A formation $\mathfrak{F}$ is said to be $\mathfrak{X}$-*local* if there exists an $\mathfrak{X}$-formation function f such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$. In this case we say that f is an $\mathfrak{X}$-*local definition of* $\mathfrak{F}$ or that f *defines* $\mathfrak{F}$.

Remarks 1.2.10.

- Each formation $\mathfrak{F}$ is $\mathfrak{X}$-local for $\mathfrak{X} = \emptyset$ because $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$, where $f(S) = \mathfrak{F}$ for all $S \in \mathfrak{J}$.
- If $\mathfrak{X} = \mathfrak{J}$, then an $\mathfrak{X}$-formation function is simply a formation function and the $\mathfrak{X}$-local formations are exactly the local formations.
- If $\mathfrak{X} = \mathbb{P}$, then an $\mathfrak{X}$-formation function is a Baer function and the $\mathfrak{X}$-local formations are exactly the Baer-local ones.

Remark 1.2.11. When $\mathfrak{X} = \mathbb{P}$, the usage of the expression $\mathbb{P}$-*local formation* clashes with Definition 1.1.12 when $\omega = \mathbb{P}$. In this thesis, $\mathbb{P}$-local formations will be understood in the sense of Definition 1.2.9, that is, as a synonymous of Baer-local formations, unless otherwise stated.

The following lemma turns out to be very useful when we aim to show that a group belongs to an $\mathfrak{X}$-local formation.

Lemma 1.2.12. *Consider an $\mathfrak{X}$-local formation $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$. Let G be a group and N a normal subgroup of G such that $G/N \in \mathfrak{F}$. If $N \in \mathrm{E}\,\mathfrak{X}$ and $G/\mathrm{C}_G(N) \in f(p)$ for every $p \in \pi(N)$, then $G \in \mathfrak{F}$.*

Proof. If H/K is an $\mathfrak{X}_p$-chief factor of G above N, then $G/\mathrm{C}_G(H/K) \in f(p)$ because $G/N \in \mathrm{LF}_{\mathfrak{X}}(f)$. Let H/K be an $\mathfrak{X}_p$-chief factor of G below N. Then $\mathrm{C}_G(N) \leq \mathrm{C}_G(H/K)$ and so $G/\mathrm{C}_G(H/K) \in \mathrm{Q}\, f(p) = f(p)$. We have that G satisfies Condition 1 of Definition 1.2.1.

Let K be a normal subgroup of G such that G/K is a monolithic group with $\mathrm{Soc}(G/K) \in \mathrm{E}(S)$, $S \in \mathfrak{X}'$. Then, since $N \in \mathrm{E}\,\mathfrak{X}$, we have that $N \leq K$ (otherwise we would have that $NK/K \neq 1$. This would imply that $\mathrm{Soc}(G/K) \leq NK/K \cong N/N \cap K$, which is a contradiction). Therefore $G/K \in f(S)$ because $G/N \in \mathrm{LF}_{\mathfrak{X}}(f)$.

Consequently $G \in \mathrm{LF}_{\mathfrak{X}}(f)$. □

The intersection of a family of $\mathfrak{X}$-local formations is an $\mathfrak{X}$-local formation as the next result exhibits.

Lemma 1.2.13. *Consider a family $\{f_i \mid i \in \mathcal{I}\}$ of $\mathfrak{X}$-formation functions.*

Then $\bigcap_{i\in\mathcal{I}} \mathrm{LF}_{\mathfrak{X}}(f_i) = \mathrm{LF}_{\mathfrak{X}}(g)$, where $g(S) = \bigcap_{i\in\mathcal{I}} f_i(S)$ for every $S \in (\mathrm{char}\,\mathfrak{X}) \cup \mathfrak{X}'$.

Proof. Since for every $i \in \mathcal{I}$, we have that $g(S) \subseteq f_i(S)$ for every $S \in (\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}'$, it follows that $\mathrm{LF}_{\mathfrak{X}}(g) \subseteq \mathrm{LF}_{\mathfrak{X}}(f_i)$ for every $i \in \mathcal{I}$, that is, $\mathrm{LF}_{\mathfrak{X}}(g) \subseteq \bigcap_{i\in\mathcal{I}} \mathrm{LF}_{\mathfrak{X}}(f_i)$.

Now we prove that $\bigcap_{i\in\mathcal{I}} \mathrm{LF}_{\mathfrak{X}}(f_i) \subseteq \mathrm{LF}_{\mathfrak{X}}(g)$. If $G \in \bigcap_{i\in\mathcal{I}} \mathrm{LF}_{\mathfrak{X}}(f_i)$ and H/K is an $\mathfrak{X}_p$-chief factor of G, then $G/\mathrm{C}_G(H/K) \in f_i(p)$ for every $i \in \mathcal{I}$. Therefore, $G/\mathrm{C}_G(H/K) \in g(p)$. If $L \trianglelefteq G$, G/L is monolithic and $\mathrm{Soc}(G/L)$ is an $\mathfrak{X}'$-chief factor of G, then $G/L \in f_i(E)$ for every $i \in \mathcal{I}$, where E is the composition factor of $\mathrm{Soc}(G/L)$. Hence $G/L \in g(E)$ and, therefore, $G \in \mathrm{LF}_{\mathfrak{X}}(g)$. □

Let f_1 and f_2 be two $\mathfrak{X}$-formation functions. We write $f_1 \leq f_2$ if $f_1(S) \subseteq f_2(S)$ for all $S \in (\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}'$. Note that in this case $\mathrm{LF}_{\mathfrak{X}}(f_1) \subseteq \mathrm{LF}_{\mathfrak{X}}(f_2)$. The following corollary shows that each $\mathfrak{X}$-local formation $\mathfrak{F}$ has a unique $\mathfrak{X}$-formation function $\underline{f}$ defining $\mathfrak{F}$ such that $\underline{f} \leq f$ for each $\mathfrak{X}$-formation function f such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$. We say that $\underline{f}$ is the *minimal $\mathfrak{X}$-local definition* of $\mathfrak{F}$. This $\mathfrak{X}$-local formation function will always be denoted by the use of a "lower bar."

Corollary 1.2.14. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation and $\{f_i \mid i \in \mathcal{I}\}$ a family of $\mathfrak{X}$-formation functions defining $\mathfrak{F}$, that is, $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f_i)$ for every $i \in \mathcal{I}$. Then $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$, where $f(S) := \bigcap_{i\in\mathcal{I}} f_i(S)$ for every $S \in (\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}'$.*

Proof. It follows immediately from Lemma 1.2.13 □

Moreover if $\mathfrak{Y}$ is a class of groups, the intersection of all $\mathfrak{X}$-local formations containing $\mathfrak{Y}$ is the smallest $\mathfrak{X}$-local formation containing $\mathfrak{Y}$. Such $\mathfrak{X}$-local formation is denoted by $\mathrm{form}_{\mathfrak{X}}(\mathfrak{Y})$. If $\mathfrak{X} = \mathfrak{J}$, we also write $\mathrm{lform}(\mathfrak{Y}) = \mathrm{form}_{\mathfrak{J}}(\mathfrak{Y})$, and if $\mathfrak{X} = \mathbb{P}$, $\mathrm{form}_{\mathbb{P}}(\mathfrak{Y})$ is usually denoted by $\mathrm{bform}(\mathfrak{Y})$. Recall that the formation generated by $\mathfrak{Y}$ is denoted by $\mathrm{form}(\mathfrak{Y})$.

Theorem 1.2.15 ([BBE06; Theorem 3.1.11])**.** *Let $\mathfrak{Y}$ be a class of groups. Then $\mathfrak{F} = \mathrm{form}_{\mathfrak{X}}(\mathfrak{Y}) = \mathrm{LF}_{\mathfrak{X}}(\underline{f})$, where*

$$\underline{f}(p) = \mathrm{form}\big(G/\mathrm{C}_G(H/K) \mid G \in \mathfrak{Y} \text{ and } H/K \text{ is an } \mathfrak{X}_p\text{-chief factor of } G\big),$$

if $p \in \operatorname{char}\mathfrak{X}$, and

$$\underline{f}(S) = \mathrm{form}\big(G/L \mid G \in \mathfrak{Y},\ G/L \text{ is monolithic, and } \mathrm{Soc}(G/L) \in \mathrm{E}(S)\big),$$

if $S \in \mathfrak{X}'$. Moreover $\underline{f}(p) = \mathrm{form}\big(G/\mathrm{C}^{\mathfrak{X}_p}(G) \mid G \in \mathfrak{F}\big)$ for all $p \in \operatorname{char}\mathfrak{X}$ such that $\underline{f}(p) \neq \emptyset$.

Proof. Let g be an $\mathfrak{X}$-formation function such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(g)$. Since $\mathrm{LF}_{\mathfrak{X}}(\underline{f})$ is an $\mathfrak{X}$-local formation containing $\mathfrak{Y}$, we have $\mathfrak{F} \subseteq \mathrm{LF}_{\mathfrak{X}}(\underline{f})$. Assume that $\mathrm{LF}_{\mathfrak{X}}(\underline{f}) \neq \mathfrak{F}$. Then $\mathrm{LF}_{\mathfrak{X}}(\underline{f}) \setminus \mathfrak{F}$ contains a group G of minimal order. Such a G has a unique minimal normal subgroup N by [DH92; II, 2.5] and $G/N \in \mathfrak{F}$. If N is an $\mathfrak{X}'$-chief factor of G, then $G \in \underline{f}(S)$ for some $S \in \mathfrak{X}'$. This implies that $G \in \mathrm{Q}\,\mathrm{R}_0\,\mathfrak{Y} \subseteq \mathfrak{F}$, a contradiction. Therefore $N \in \mathrm{E}\,\mathfrak{X}$. Let p be a prime divisor of $|N|$. Then $G/\mathrm{C}_G(N) \in \underline{f}(p)$. Now if X is a group in $\mathfrak{Y}$ and H/K is an $\mathfrak{X}_p$-chief factor of X, then $X\big/\mathrm{C}_X(H/K) \in g(p)$ because $\mathfrak{Y} \subseteq \mathfrak{F}$. Therefore $\underline{f}(p) \subseteq g(p)$, and so $G/\mathrm{C}_G(N) \in g(p)$. Applying Lemma 1.2.12, $G \in \mathfrak{F}$, contrary to hypothesis. Consequently $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(\underline{f})$.

Let $p \in \operatorname{char}\mathfrak{X}$ and $t(p) = \operatorname{form}\big(G/\mathrm{C}^{\mathfrak{X}_p}(G) \mid G \in \mathfrak{F}\big)$. If $G \in \mathfrak{F}$ and $\underline{f}(p) \neq \emptyset$, then $G/\mathrm{C}^{\mathfrak{X}_p}(G) \in \underline{f}(p)$. Therefore $t(p) \subseteq \underline{f}(p)$. On the other hand, if $X \in \mathfrak{Y}$, then $X/\mathrm{C}^{\mathfrak{X}_p}(X) \in t(p)$. Hence $X\big/\mathrm{C}_X(H/K) \in t(p)$ for every $\mathfrak{X}_p$-chief factor H/K of X. This means that $\underline{f}(p) \subseteq t(p)$ and the equality holds. This completes the proof of the theorem. □

Remark 1.2.16. If $\mathfrak{F}$ is a local formation and $\underline{f}$ is the smallest local definition of $\mathfrak{F}$, then $\underline{f}(p) = \mathrm{Q}\big(G/\mathrm{O}_{p',p}(G) \mid G \in \mathfrak{F}\big)$ for each $p \in \operatorname{char}\mathfrak{F}$ (cf. [DH92; IV, 3.10]). The equality $\underline{f}(p) = \mathrm{Q}\big(G/\mathrm{O}_{p',p}(G) \mid G \in \mathfrak{F}\big)$ does not hold for $\mathfrak{X}$-local formations in general: Let $\mathfrak{X} = (C_2)$ and consider $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$, where $f(2) = \mathfrak{S}$ and $f(S) = \mathfrak{E}$ for all $S \in \mathfrak{X}'$. Then $\mathrm{Alt}(5) \in \mathfrak{F}$ and so $\mathrm{Alt}(5) \in \mathrm{Q}\big(G/\mathrm{O}_{2',2}(G) \mid G \in \mathfrak{F}\big)$. Since $\underline{f}(2) \subseteq \mathfrak{S}$, it follows that $\mathrm{Alt}(5) \notin \underline{f}(2)$. Consequently $\underline{f}(2) \neq \mathrm{Q}\big(G/\mathrm{O}_{2',2}(G) \mid G \in \mathfrak{F}\big)$.

Corollary 1.2.17. *Let $\mathfrak{X}$ and $\bar{\mathfrak{X}}$ be classes of simple groups such that $\bar{\mathfrak{X}} \subseteq \mathfrak{X}$. Then every $\mathfrak{X}$-local formation is $\bar{\mathfrak{X}}$-local.*

Proof. Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(\underline{f})$ be an $\mathfrak{X}$-local formation. Since $\operatorname{char}\bar{\mathfrak{X}} \subseteq \operatorname{char}\mathfrak{X}$, we can consider the $\bar{\mathfrak{X}}$-formation function g defined by

$$\begin{aligned} g(p) &= \underline{f}(p) && \text{if } p \in \operatorname{char}\bar{\mathfrak{X}},\\ g(E) &= \mathfrak{F} && \text{if } E \in \bar{\mathfrak{X}}'. \end{aligned}$$

It is clear that $\mathfrak{F} \subseteq \mathrm{LF}_{\bar{\mathfrak{X}}}(g)$. Suppose that $\mathfrak{F} \neq \mathrm{LF}_{\bar{\mathfrak{X}}}(g)$, and choose a group Y of minimal order in $\mathrm{LF}_{\bar{\mathfrak{X}}}(g) \setminus \mathfrak{F}$. Then Y has a unique minimal normal subgroup N, and $G/N \in \mathfrak{F}$. If $N \in \mathrm{E}(\bar{\mathfrak{X}}')$, then $G \in \mathfrak{F}$, which contradicts the choice of G. Therefore $N \in \mathrm{E}\,\bar{\mathfrak{X}}$ and $G/\mathrm{C}_G(N) \in \underline{f}(p)$ for each prime p dividing $|N|$. Applying Lemma 1.2.12, we conclude that $G \in \mathfrak{F}$, contrary to supposition. Hence $\mathfrak{F} = \mathrm{LF}_{\bar{\mathfrak{X}}}(g)$ and $\mathfrak{F}$ is $\bar{\mathfrak{X}}$-local. □

Definition 1.2.18. Let f be an $\mathfrak{X}$-formation function defining an $\mathfrak{X}$-local formation $\mathfrak{F}$.

1. f is called *integrated* if $f(S) \subseteq \mathfrak{F}$ for all $S \in (\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}'$,

2. f is called *full* if $\mathfrak{S}_p f(p) = f(p)$ for all $p \in \operatorname{char}\mathfrak{X}$.

The following theorem describes a full and integrated $\mathfrak{X}$-formation function defining an $\mathfrak{X}$-local formation.

Theorem 1.2.19 ([BBE06; Theorem 3.1.14])**.** *Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$ be an $\mathfrak{X}$-local formation defined by the $\mathfrak{X}$-formation function f. Set*

$$\begin{aligned} f^*(p) &= \mathfrak{F} \cap \mathfrak{S}_p f(p) && \text{for all } p \in \operatorname{char}\mathfrak{X}, \\ f^*(S) &= \mathfrak{F} \cap f(S) && \text{for all } S \in \mathfrak{X}'. \end{aligned}$$

Then:

1. *f^* is an $\mathfrak{X}$-formation function such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f^*)$.*

2. *$\mathfrak{S}_p f^*(p) = f^*(p)$ for all $p \in \operatorname{char}\mathfrak{X}$.*

Proof. 1. It is clear that f^* is an $\mathfrak{X}$-formation function. Let $\mathfrak{F}^* = \mathrm{LF}_{\mathfrak{X}}(f^*)$ and let $G \in \mathfrak{F}^*$. If H/K is an $\mathfrak{X}_p$-chief factor of G, then

$$G/\mathrm{C}_G(H/K) \in \mathfrak{F} \cap \mathfrak{S}_p f(p).$$

Since, by [DH92; A, 13.6], $\mathrm{O}_p\big(G/\mathrm{C}_G(H/K)\big) = 1$, it follows that

$$G/\mathrm{C}_G(H/K) \in f(p).$$

Now if G/L is a monolithic quotient of G with $\mathrm{Soc}(G/L) \in \mathrm{E}(S)$ for some $S \in \mathfrak{X}'$, it follows that $G/L \in f(S)$. Therefore $G \in \mathfrak{F}$. If H/K is an $\mathfrak{X}_p$-chief factor of a group $A \in \mathfrak{F}$, then $A/\mathrm{C}_A(H/K) \in \mathrm{Q}\,\mathfrak{F} \cap f(p) \subseteq f^*(p)$. If A/L is a monolithic quotient of A with $\mathrm{Soc}(A/L) \in \mathrm{E}(S)$, $S \in \mathfrak{X}'$, then $A/L \in \mathrm{Q}\,\mathfrak{F} \cap f(S) \subseteq f^*(S)$. This implies that $A \in \mathfrak{F}^*$ and therefore $\mathfrak{F} = \mathfrak{F}^*$.

2. Let $G \in \mathfrak{S}_p f^*(p)$, $p \in \operatorname{char}\mathfrak{X}$. Then $G/\mathrm{O}_p(G) \in f^*(p)$ and so $G \in \mathfrak{S}_p f(p)$ because $\mathrm{O}_p\big(G/\mathrm{O}_p(G)\big) = 1$. Moreover $G/\mathrm{O}_p(G) \in \mathfrak{F}$. If H/K is an $\mathfrak{X}_p$-chief factor of G below $\mathrm{O}_p(G)$, then $\mathrm{O}_p(G) \le \mathrm{C}_G(H/K)$ by [DH92; B, 3.12 (b)] and so $G/\mathrm{C}_G(H/K) \in \mathrm{Q}\,f(p) = f(p)$. If G/L is a monolithic quotient of G such that $\mathrm{Soc}(G/L) \in \mathrm{E}(S)$, $S \in \mathfrak{X}'$, it follows that $\mathrm{O}_p(G) \le L$. Therefore $G/L \in \mathrm{Q}\,f^*(p) = f^*(p) \subseteq \mathfrak{F}$ and so $G/L \in f(S)$. This proves that $G \in \mathfrak{F}$. Consequently $G \in f^*(p)$ and $\mathfrak{S}_p f^*(p) = f^*(p)$. □

It is known (cf. [DH92; IV, 3.7]) that if $\mathfrak{X} = \mathfrak{J}$, then every $\mathfrak{X}$-local formation has a unique integrated and full $\mathfrak{X}$-local definition, the canonical one. This is not true in general. In fact, if $\emptyset \neq \mathfrak{X} \neq \mathfrak{J}$, we can find an $\mathfrak{X}$-local formation with several integrated and full $\mathfrak{X}$-local definitions.

Example 1.2.20. Let $\emptyset \neq \mathfrak{X} \neq \mathfrak{J}$. Then we can consider $X \in \mathfrak{J} \setminus \mathfrak{X}$ and a prime $p \in \operatorname{char} \mathfrak{X}$. The formation $\mathfrak{F} = \mathfrak{S}_p$ is an $\mathfrak{X}$-local formation which can be $\mathfrak{X}$-locally defined by the following integrated and full $\mathfrak{X}$-formation functions:

$$f_1(S) = \begin{cases} \mathfrak{S}_p & \text{if } S \cong C_p, \\ \emptyset & \text{if } S \ncong C_p, \end{cases}$$

and

$$f_2(S) = \begin{cases} \mathfrak{S}_p & \text{if } S \cong C_p, \\ \mathfrak{S}_p & \text{if } S \cong X, \\ \emptyset & \text{otherwise} \end{cases}$$

for all $S \in (\operatorname{char} \mathfrak{X}) \cup \mathfrak{X}'$.

When an $\mathfrak{X}$-local formation $\mathfrak{F}$ is considered, it is often interesting to work with a special full and integrated $\mathfrak{X}$-local definition of $\mathfrak{F}$, called the *canonical* $\mathfrak{X}$-local definition of $\mathfrak{F}$. This $\mathfrak{X}$-formation function is described in the following theorem. In order to present a general version of this result, we assume that $\mathfrak{F} = \operatorname{form}_{\mathfrak{X}}(\mathfrak{Y})$, where $\mathfrak{Y}$ is a class of groups.

Theorem 1.2.21. *Let $\mathfrak{Y}$ be a class of groups and consider $\mathfrak{F} = \operatorname{form}_{\mathfrak{X}}(\mathfrak{Y})$ with minimal $\mathfrak{X}$-local definition $\underline{f}$. Then $\mathfrak{F} = \operatorname{LF}_{\mathfrak{X}}(F)$, where F is the following $\mathfrak{X}$-formation function:*

$$F(p) = \mathfrak{S}_p \underline{f}(p) \qquad \text{if } p \in \operatorname{char} \mathfrak{X},$$
$$F(S) = \operatorname{form}(\mathfrak{Y}) \qquad \text{if } S \in \mathfrak{X}'.$$

Moreover, F is full and integrated.

Proof. Since $\underline{f} \leq F$, it follows that $\mathfrak{F} \subseteq \operatorname{LF}_{\mathfrak{X}}(F)$. Suppose, by way of contradiction, that the equality does not hold and let G be a group of minimal order in $\operatorname{LF}_{\mathfrak{X}}(F) \setminus \mathfrak{F}$. Then the group G has a unique minimal normal subgroup, N say, and $G/N \in \mathfrak{F}$. Furthermore $N \in \mathrm{E}\,\mathfrak{X}$ because otherwise $G \in F(S)$ for some $S \in \mathfrak{X}'$ and then $G \in \mathfrak{F}$, contrary to supposition. Let p be a prime dividing $|N|$. Then $G/\operatorname{C}_G(N) \in \mathfrak{S}_p \underline{f}(p)$ and so $G/\operatorname{C}_G(N) \in \underline{f}(p)$ because $\operatorname{O}_p\big(G/\operatorname{C}_G(N)\big) = 1$ by [DH92; A, 13.6 (b)]. Then Lemma 1.2.12 implies that $G \in \mathfrak{F}$. This contradiction yields $\operatorname{LF}_{\mathfrak{X}}(F) \subseteq \mathfrak{F}$ and then $\mathfrak{F} = \operatorname{LF}_{\mathfrak{X}}(F)$. It is clear that F is full. Let $p \in \operatorname{char} \mathfrak{X}$. If possible, choose a group

G of minimal order in $F(p) \setminus \mathfrak{F}$. We know that G has a unique minimal normal subgroup N and, since $\underline{f}(p) \subseteq \mathfrak{F}$, $\mathrm{O}_p(G) \neq 1$. Hence N is a p-group. Moreover $G/N \in \mathfrak{F}$ and $G/\mathrm{C}_G(N) \in \underline{f}(p)$ because $\mathrm{O}_p(G)$ centralises N. But then $G \in \mathfrak{F}$. This contradicts the choice of G, and so we conclude that $F(p) \subseteq \mathfrak{F}$. □

Corollary 1.2.22. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation with minimal $\mathfrak{X}$-local definition $\underline{f}$. Then $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$, where F is the following $\mathfrak{X}$-formation function:*

$$\begin{cases} F(p) = \mathfrak{S}_p\underline{f}(p) & \text{if } p \in \operatorname{char}\mathfrak{X} \\ F(S) = \mathfrak{F} & \text{if } S \in \mathfrak{X}' \end{cases}$$

Moreover, F is full and integrated.

The $\mathfrak{X}$-formation function described above will be identified by the use of an uppercase letter. Hence if we write $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$, we are assuming that F is the canonical $\mathfrak{X}$-local definition of $\mathfrak{F}$.

Theorem 1.2.23. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation and g an $\mathfrak{X}$-local definition of $\mathfrak{F}$. Then $F(p) = \mathfrak{F} \cap \mathfrak{S}_p g(p)$ for every $p \in \operatorname{char}\mathfrak{X}$.*

Proof. Since $\underline{f} \leq g$, it follows that $F(p) = \mathfrak{S}_p\underline{f}(p) \subseteq \mathfrak{F} \cap \mathfrak{S}_p g(p) = g^*(p)$ for all $p \in \operatorname{char}\mathfrak{X}$. Let X be a group in $g^*(p)$ and set $W = C_p \wr X$. As above, denote by $B = C_p^\natural$ the base group of W. Then $W/B \in g^*(p)$. Moreover $W/B \in \mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(g^*)$ by Theorem 1.2.19. Applying Lemma 1.2.12, we conclude that $W \in \mathfrak{F}$. Hence $X \in F(p)$ and $F(p) = g^*(p)$. □

The following corollary shows that the canonical definition of an $\mathfrak{X}$-local formation $\mathfrak{F}$ is the *maximal integrated* $\mathfrak{X}$-formation function defining $\mathfrak{F}$. This means that $g \leq f$ for each integrated $\mathfrak{X}$-formation function g such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(g)$.

Corollary 1.2.24. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation and g an integrated $\mathfrak{X}$-formation function defining $\mathfrak{F}$. Then $g \leq F$.*

Proof. Since g is integrated, we have that $g(p) \subseteq \mathfrak{F} \cap \mathfrak{S}_p g(p) = F(p)$ for all $p \in \operatorname{char}\mathfrak{X}$. Clearly, $g(S) \subseteq F(S)$ if $S \in \mathfrak{X}'$. Therefore, $g \leq F$. □

Corollary 1.2.25. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation and f and g two full and integrated $\mathfrak{X}$-formation functions defining $\mathfrak{F}$. Then $f(p) = g(p)$ for every $p \in \operatorname{char}\mathfrak{X}$.*

Proof. By Theorem 1.2.23, $f(p) = g(p) = F(p)$ for every $p \in \operatorname{char}\mathfrak{X}$. □

Theorem 1.2.26. *Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$ be an $\mathfrak{X}$-local formation. Then $F(p) = (G \mid C_p \wr G \in \mathfrak{F})$ for every $p \in \operatorname{char}\mathfrak{X}$.*

Proof. Let $p \in \operatorname{char}\mathfrak{X}$ and let $\bar{F}(p)$ denote the class $(G \mid C_p \wr G \in \mathfrak{F})$. If $G \in F(p)$, then $C_p \wr G \in \mathfrak{S}_p F(p) = F(p) \subseteq \mathfrak{F}$ by Theorem 1.2.21. Hence $G \in \bar{F}(p)$ and so $F(p) \subseteq \bar{F}(p)$. Now consider a group $G \in \bar{F}(p)$ and set $W = C_p \wr G$. Denote $B = C_p^{\natural}$ the base group of W and $A = \bigcap\{\mathrm{C}_W(H/K) \mid H \leq B \text{ and } H/K \text{ is a chief factor of } W\}$. Since $W \in \mathfrak{F}$, it follows that $W/A \in F(p)$. On the other hand, A acts as a group of operators for B by conjugation and A stabilises a chain of subgroups of B. Applying [DH92; A, 12.4], we have that $A/\mathrm{C}_A(B)$ is a p-group. Then A is itself a p-group because $\mathrm{C}_A(B) = B$ by [DH92; A, 18.8]. Consequently $W \in F(p)$ and $G \in \mathrm{Q}\,F(p) = F(p)$. This proves that $\bar{F}(p) = F(p)$. □

Corollary 1.2.27. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation and $\mathfrak{Y} \subseteq \mathfrak{X}$. Let F_1 and F_2 be the canonical $\mathfrak{Y}$-local and $\mathfrak{X}$-local definitions of $\mathfrak{F}$, respectively. Then $F_1(p) = F_2(p)$ for all $p \in \operatorname{char}\mathfrak{Y}$.*

Proof. Applying Corollary 1.2.17, we know that $\mathfrak{F}$ is $\mathfrak{Y}$-local. Let p be a prime in $\operatorname{char}\mathfrak{Y}$. Then $p \in \operatorname{char}\mathfrak{X}$ and by Theorem 1.2.26 we have that $F_1(p) = (G \mid C_p \wr G \in \mathfrak{F}) = F_2(p)$. □

Taking $\mathfrak{Y} = (C_p)$, $p \in \operatorname{char}\mathfrak{X}$ in Corollary 1.2.27 and, applying Theorem 1.2.15 and Theorem 1.2.26, we have:

Corollary 1.2.28. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation. If $p \in \operatorname{char}\mathfrak{X}$, then*

$$F(p) = \mathfrak{S}_p \operatorname{form}\big(G/\mathrm{C}_G(H/K) \mid G \in \mathfrak{F},\ H/K \text{ is an abelian } p\text{-chief factor of } G\big).$$

Corollary 1.2.29. *If $\mathfrak{F}$ is an $\mathfrak{X}$-local formation, p is a prime in $\operatorname{char}\mathfrak{X}$ and $C_p \in \mathcal{K}(\mathfrak{F})$, then $\mathfrak{S}_p \subseteq \mathfrak{F}$.*

Corollary 1.2.30. *Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(\underline{f}) = \mathrm{LF}_{\mathfrak{X}}(F)$ and $\mathfrak{G} = \mathrm{LF}_{\mathfrak{X}}(\underline{g}) = \mathrm{LF}_{\mathfrak{X}}(G)$ be $\mathfrak{X}$-local formations. Then any two of the following statements are equivalent:*

1. $\mathfrak{F} \subseteq \mathfrak{G}$
2. $F \leq G$
3. $\underline{f} \leq \underline{g}$

Corollary 1.2.31 ([BBCER05; Lemma 4.5]). *Let $\mathfrak{F}$ be a formation and let $\{\mathfrak{X}_i \mid i \in \mathcal{I}\}$ be a family of classes of simple groups such that $\pi(\mathfrak{X}_i) = \operatorname{char} \mathfrak{X}_i$ for all $i \in \mathcal{I}$. Put $\mathfrak{X} = \bigcup_{i\in\mathcal{I}} \mathfrak{X}_i$. If $\mathfrak{F}$ is $\mathfrak{X}_i$-local for all $i \in \mathcal{I}$, then $\mathfrak{F}$ is $\mathfrak{X}$-local.*

Proof. First of all, note that $\pi(\mathfrak{X}) = \operatorname{char} \mathfrak{X}$.

Applying Theorem 1.2.21, $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}_i}(F_i)$, where

$$F_i(S) = \begin{cases} (G \mid C_p \wr G \in \mathfrak{F}) & \text{if } S \cong C_p,\ p \in \operatorname{char} \mathfrak{X}_i, \\ \mathfrak{F} & \text{if } S \in \mathfrak{X}_i', \end{cases}$$

for all $i \in \mathcal{I}$.

Let f be the $\mathfrak{X}$-formation function defined by

$$f(S) = \begin{cases} (G \mid C_p \wr G \in \mathfrak{F}) & \text{if } S \cong C_p,\ p \in \operatorname{char} \mathfrak{X}, \\ \mathfrak{F} & \text{if } S \in \mathfrak{X}'. \end{cases}$$

It is clear that $\mathfrak{F} \subseteq \mathrm{LF}_{\mathfrak{X}}(f)$. Assume that the inclusion is proper and derive a contradiction. Let $G \in \mathrm{LF}_{\mathfrak{X}}(f) \setminus \mathfrak{F}$ of minimal order. Then G has a unique minimal normal subgroup N such that $G/N \in \mathfrak{F}$. It is clear that $N \in \mathrm{E}\,\mathfrak{X}$ because otherwise $G \in \mathfrak{F}$. Hence $N \in \mathrm{E}\,\mathfrak{X}_i$ for some $i \in \mathcal{I}$ and $G/\mathrm{C}_G(N) \in f(p) = F_i(p)$ for all $p \in \pi(N)$. Therefore $G \in \mathrm{LF}_{\mathfrak{X}_i}(F_i) = \mathfrak{F}$. This is a contradiction. Consequently $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$ and $\mathfrak{F}$ is an $\mathfrak{X}$-local formation. □

When $\mathfrak{X}$ is the class of all abelian simple groups, we have $\mathfrak{X} = \bigcup_{p\in\mathbb{P}}(C_p)$. Therefore

Corollary 1.2.32 ([BBCER05; Corollary 4.6]). *A formation $\mathfrak{F}$ is Baer-local if and only if $\mathfrak{F}$ is (C_p)-local for every prime p.*

Lemma 1.2.33. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation and let f be an $\mathfrak{X}$-formation function defining $\mathfrak{F}$. If f is integrated, then $\mathfrak{S}_p f(p) \subseteq \mathfrak{F}$ for every $p \in \operatorname{char} \mathfrak{X}$.*

Proof. Consider $p \in \operatorname{char} \mathfrak{X}$ and assume that $\mathfrak{S}_p f(p)$ is not contained in $\mathfrak{F}$. Let G be a group of minimal order in $\mathfrak{S}_p f(p) \setminus \mathfrak{F}$. Then G is a group with a unique minimal normal subgroup N. Clearly, G is a p-group and $G/N \in \mathfrak{F}$. Since $G/\mathrm{C}_G(N) \in f(p)$, we can apply Lemma 1.2.12 to deduce that $G \in \mathfrak{F}$, which is a contradiction. Therefore, $\mathfrak{S}_p f(p) \subseteq \mathfrak{F}$. □

The following lemma shows that if $\mathfrak{F}$ is an $\mathfrak{X}$-local formation and we consider an integrated $\mathfrak{X}$-formation function f defining $\mathfrak{F}$, a new $\mathfrak{X}$-local definition g of $\mathfrak{F}$ having the same values on the groups of $\mathfrak{X}'$ can be constructed.

Lemma 1.2.34. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation and f an integrated $\mathfrak{X}$-formation function defining $\mathfrak{F}$. Then $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f_*)$, where f_* is the following $\mathfrak{X}$-formation function:*

$$\begin{cases} f_*(p) = f(p) & \text{if } p \in \operatorname{char}\mathfrak{X} \\ f_*(S) = \operatorname{form}(\bigcup_{S\in\mathfrak{X}'} f(S)) & \text{if } S \in \mathfrak{X}' \end{cases}$$

Proof. Consider $\mathfrak{F}_* = \mathrm{LF}_{\mathfrak{X}}(f_*)$. Since $f \leq f_*$, it is clear that $\mathfrak{F}$ is contained in $\mathfrak{F}_*$. Now assume that $\mathfrak{F}_*$ is not contained in $\mathfrak{F}$ and let G be a group of minimal order in $\mathfrak{F}_* \setminus \mathfrak{F}$. Then G has a unique minimal normal subgroup N and $G/N \in \mathfrak{F}$. If N is an $\mathfrak{X}'$ chief factor of G, it follows that $G \in f_*(S)$, where $S \in \mathfrak{X}'$. Since $f_*(S)$ is contained in $\mathfrak{F}$, this is a contradiction. Assume that N is an $\mathfrak{X}$-chief factor of G and consider $p \in \pi(N)$. We have that $G/\mathrm{C}_G(N) \in f_*(p) = f(p)$. By Lemma 1.2.12, we can conclude that $G \in \mathfrak{F}$, which is a contradiction. Therefore, $\mathfrak{F}_*$ is contained in $\mathfrak{F}$. □

Therefore, in Definition 1.2.1 we can assume without loss of generality that an $\mathfrak{X}$-formation function has the same value on the groups of $\mathfrak{X}'$. Bearing this in mind, Definition 1.2.1 can be modified if $\mathfrak{X}' \neq \emptyset$.

Definition 1.2.35.

- An $\mathfrak{X}$-*formation function* f assigns to each $X \in \operatorname{char}\mathfrak{X} \cup \{\mathfrak{X}'\}$ a (possibly empty) formation $f(X)$.
- If f is an $\mathfrak{X}$-formation function, then $\mathrm{LF}_{\mathfrak{X}}(f)$ is the class of all groups G satisfying the following two conditions:
  1. If H/K is an $\mathfrak{X}_p$-chief factor of G, then $G/\mathrm{C}_G(H/K) \in f(p)$.
  2. If G/L is a monolithic quotient of G such that $\mathrm{Soc}(G/L)$ is an $\mathfrak{X}'$-chief factor of G, then $G/L \in f(\mathfrak{X}')$.
- A formation $\mathfrak{F}$ is said to be $\mathfrak{X}$-*local* if there exists an $\mathfrak{X}$-formation function f such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$. In this case we say that f is an $\mathfrak{X}$-*local definition of* $\mathfrak{F}$ or that f *defines* $\mathfrak{F}$.

Lemma 1.2.34 ensures us that the $\mathfrak{X}$-local formations of Definition 1.2.35 coincide with the ones presented in Definition 1.2.9.

Remark 1.2.36. The concepts of full and integrated $\mathfrak{X}$-formation function (see Definition 1.2.18) are defined analogously. When we work with this new definition we can also consider the canonical $\mathfrak{X}$-formation function F of an $\mathfrak{X}$-local formation $\mathfrak{F}$ (see Theorem 1.2.21), bearing in mind that, in this case, $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$, where $F(p) = \mathfrak{S}_p\underline{f}(p)$ if $p \in \operatorname{char}\mathfrak{X}$ and $F(\mathfrak{X}') = \mathfrak{F}$.

Now we aim to give an interpretation of $\mathfrak{X}$-local and ω-local formations as the classes of groups with generalised central chief factors. In [BBS97], the word *satellite* was introduced. The origin was the need of studying arbitrary formations by functional methods. The main idea, which was developed in [BBS97] and [She01], is the following:

Let $\mathfrak{M}$ be a class of finite groups. Consider a function f which assigns to every group in $\mathfrak{M}$ a formation. Suppose that there exists a rule (*an f-rule*) which decides whether a given chief factor of a group G is f-central or f-eccentric. We call a chief series f-central if all of its factors are f-central. We say that a normal subgroup H of G is f-hypercentral if all the G-chief factors of H are f-central in G. We can consider the class $F(f)$ of all the groups whose chief series are f-central. If $\mathfrak{F} = F(f)$, then the function f, which is considered together with the mentioned rule, is called a *satellite* of $\mathfrak{F}$. It is easy to note (see [BBS97] and [She01; Example 1]) that every non-empty formation has at least one satellite. Hence we can study arbitrary formations with the help of satellites.

Example 1.2.37. Let f be an $\mathfrak{X}$-formation function (Definition 1.2.35). We extend the definition domain of f in the following way. If $X \in \mathfrak{X}$ and $T \in \mathrm{E}(X)$, then we set

$$f(T) = f(X) = \bigcap_{p \in \pi(X)} f(p).$$

Now we define an f-rule. We say that a chief factor H/K of a group G is f-central if one of the following conditions holds:

- $H/K \in \mathrm{E}\,\mathfrak{X}$ and $G/\mathrm{C}_G(H/K) \in f(H/K)$.
- $H/K \in \mathrm{E}\,\mathfrak{X}'$ and $G/\mathrm{c}_G(H/K) \in f(\mathfrak{X}')$.

Example 1.2.38. Let f be an ω-local satellite. We extend the definition domain of f in the following way. If T is a group whose order is divided by a prime $p \in \omega$, we set

$$f(T) = \bigcap_{p \in \omega \cap \pi(T)} f(p).$$

Now we define an f-rule. We say that a chief factor H/K of a group G is f-central if one of the following conditions holds:

- The order of H/K is divided by a prime $p \in \omega$ and $G/\mathrm{C}_G(H/K) \in f(H/K)$.
- H/K is an ω'-group and $G/\mathrm{c}_G(H/K) \in f(\omega')$.

Our purpose is to prove that if $\mathfrak{F}$ is an $\mathfrak{X}$-local formation defined by an $\mathfrak{X}$-formation function f, then $\mathfrak{F}$ coincides with the class of all groups whose chief factors are f-central. We need some lemmas.

Lemma 1.2.12 can be also rewritten in the following way:

Lemma 1.2.39. *Consider an $\mathfrak{X}$-local formation $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$. Let G be a group and N a normal subgroup of G such that $G/N \in \mathfrak{F}$. If $N \in \mathrm{E}\,\mathfrak{X}$ and N is f-hypercentral, then $G \in \mathfrak{F}$.*

Lemma 1.2.40 ([She01; Lemma 4])**.** *Let $\mathfrak{F}$ be a formation and $\mathfrak{Y}$ a subclass of $\mathfrak{J}$. Let M_1 and M_2 be normal subgroups of a group G such that $M_1 \cap M_2 = 1$. If $G^{\mathfrak{F}}M_i/M_i$ is contained in the small centralizer of every $\mathfrak{Y}'$-chief factor of G/M_i ($i = 1, 2$), then $G^{\mathfrak{F}}$ is contained in the small centralizer of every $\mathfrak{Y}'$-chief factor of G.*

Lemma 1.2.41. *Consider the normal series*

$$1 \leq T \leq K \leq H \leq G$$

Then $\mathrm{c}_G(H/K)/T = \mathrm{c}_{G/T}\big((H/T)/(K/T)\big)$.

Proof. Set $R = \mathrm{c}_G(H/K)$ and $L/T = \mathrm{c}_{G/T}\big((H/T)/(K/T)\big)$. By Definition 1.1.10, $\mathcal{K}(R/K) \cap \mathcal{K}(H/K) = \emptyset$. But then

$$\mathcal{K}\big((R/T)/(K/T)\big) \cap \mathcal{K}\big((H/T)/(K/T)\big) = \emptyset.$$

From here it follows that $R/T \leq L/T$ and $R \leq L$. On the other hand, $K/T \leq L/T$ and

$$\mathcal{K}\big((L/T)/(K/T)\big) \cap \mathcal{K}\big((H/T)/(K/T)\big) = \emptyset.$$

Therefore, $\mathcal{K}(L/K) \cap \mathcal{K}(H/K) = \emptyset$ and $L \leq R$. □

Theorem 1.2.42 ([BBCSss])**.** *Let f be an $\mathfrak{X}$-formation function. Let $\mathfrak{H}$ be the class of all groups whose chief factors are f-central. Then $\mathfrak{H}$ is a formation.*

Proof. Let $G \in \mathfrak{H}$ and $K \trianglelefteq G$. Consider a chief series

$$1 \leq \cdots \leq K = G_t \leq \cdots \leq G_1 \leq G_0 = G$$

If G_{i-1}/G_i is an $\mathfrak{X}$-chief factor, it is clear that $(G_{i-1}/K)\big/(G_i/K)$ is f-central in G/K. Let G_{i-1}/G_i be an $\mathfrak{X}'$-chief factor. Then $G/\mathrm{c}_G(G_{i-1}/G_i)$ belongs to $f(\mathfrak{X}')$. Therefore, $L = \mathrm{c}_G(G_{i-1}/G_i)$ contains the $f(\mathfrak{X}')$-residual of G. By Lemma 1.2.41, $L/K = \mathrm{c}_{G/K}\big((G_{i-1}/K)\big/(G_i/K)\big)$. It follows that

$(G_{i-1}/K)\big/(G_i/K)$ is f-central in G/K. Hence we have proved that $\mathfrak{H}$ is Q-closed.

Assume that $G/M_i \in \mathfrak{H}$ $(i = 1, 2)$, $M_1 \cap M_2 = 1$. Clearly, every $\mathfrak{X}$-chief factor of G is f-central. Assume that $\mathfrak{X}' \neq \emptyset$ and $f(\mathfrak{X}') = \mathfrak{B}$. Applying Lemma 1.2.40, we see that every $\mathfrak{X}'$-chief factor of G is f-central. Therefore we have proved that $\mathfrak{H}$ is a formation. □

Theorem 1.2.43 ([BBCSss]). *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation and f an $\mathfrak{X}$-formation function defining $\mathfrak{F}$. Then $\mathfrak{F}$ coincides with the class of all groups whose chief factors are f-central.*

Proof. Let $\mathfrak{H}$ be the class of all groups whose chief factors are f-central. If $\mathfrak{H}$ is not contained in $\mathfrak{F}$, let G be a group of least order in $\mathfrak{H}\backslash\mathfrak{F}$. Then G is a monolithic group with socle $T = G^{\mathfrak{F}}$. Evidently, $\mathrm{c}_G(T) = 1$. Since $G \in \mathfrak{H}$, then T is f-central in G. If T is an $\mathrm{E}\,\mathfrak{X}$-group, since $G/T \in \mathfrak{F}$ by minimality of G, we have that $G \in \mathfrak{F}$, by Lemma 1.2.39. If T is a $\mathrm{E}\,\mathfrak{X}'$-group, then $G \in f(\mathfrak{X}')$. Now, applying Definition 1.2.35, we have that $G \in \mathfrak{F}$. Since this is a contradiction, we have that $\mathfrak{H}$ is contained in $\mathfrak{F}$.

Now assume that $\mathfrak{F}$ is not contained in $\mathfrak{H}$ and let G be a group of least order in $\mathfrak{F}\backslash\mathfrak{H}$. Then G is a monolithic group with socle $T = G^{\mathfrak{H}}$. Suppose that $T \in \mathrm{E}\,\mathfrak{X}'$. Then, by Definition 1.2.35, we have that $G \in f(\mathfrak{X}')$. It follows that all the $\mathfrak{X}'$-chief factors of G are f-central. Now assume that $T \in \mathrm{E}\,\mathfrak{X}$. It is clear that T and the other chief $\mathrm{E}\,\mathfrak{X}$-factors of G are f-central. Let H/K be a $\mathfrak{X}'$-chief factor of G. Then $T \leq K$ and, since $G/T \in \mathfrak{H}$, we have that $(H/T)\big/(K/T)$ is an f-central chief factor of G/T. If $R/T = \mathrm{c}_{G/T}\big((H/T)\big/(K/T)\big)$, then by Lemma 1.2.41 we have that $R = \mathrm{c}_G(H/K)$. Therefore, since $(G/T)\big/(R/T) \in f(\mathfrak{X}')$, it follows that $G/R \in f(\mathfrak{X}')$. Hence, H/K is f-central in G. We have proved that $G \in \mathfrak{H}$, which is a contradiction. Therefore, $\mathfrak{F}$ is contained in $\mathfrak{H}$. □

For ω-saturated formations the following analogous result is true.

Theorem 1.2.44 ([BBCSss]). *Let $\mathfrak{F}$ be an ω-local formation and f an ω-local satellite defining $\mathfrak{F}$. Then $\mathfrak{F}$ coincides with the class of all groups whose chief factors are f-central.*

Lemma 1.2.45 ([She01; Lemma 1]). *Let $\mathfrak{X}$ be a non-empty class of simple groups and $\{S_i \mid i \in \mathcal{I}\}$ the set of all $\mathfrak{X}'$-chief factors of a group G. Then*

$$\bigcap_{i\in\mathcal{I}} \mathrm{c}_G(S_i) = G_{\mathrm{E}\,\mathfrak{X}}.$$

Remark 1.2.46. We always suppose that the intersection of an empty set of subgroups of G coincides with G. In particular, in Lemma 1.2.45, the set $\mathcal{I}$ can be empty.

If $\mathfrak{Y}$ is a class of simple groups, let $\mathfrak{E}_{c\mathfrak{Y}}$ denote the class of all groups whose $\mathfrak{Y}$-chief factors are central. Clearly, $\mathfrak{E}_{c\mathfrak{Y}} = \mathrm{BLF}(f)$, where f is the Baer-function defined as $f(A) = (1)$ if $A \in \mathfrak{Y}$ and $f(A) = \mathfrak{E}$ if $A \in \mathfrak{Y}'$. Moreover, $\mathfrak{E}_{c\mathfrak{Y}}$ is a Fitting formation. We denote by $G_{c\mathfrak{Y}}$ the $\mathfrak{E}_{c\mathfrak{Y}}$-radical of a group G. If $\mathfrak{Y} = (C_p)$, then $\mathfrak{E}_{c\mathfrak{Y}}$ is denoted by $\mathfrak{E}_{cp}$ and $G_{c\mathfrak{Y}}$ is denoted by G_{cp}.

Lemma 1.2.47 ([She01; Lemma 2]). *Let $\mathfrak{Y}$ be a non-empty class of simple groups, and $\{S_i \mid i \in \mathcal{I}\}$ the set of all $\mathrm{E}\,\mathfrak{Y}$-chief factors of a group G. Then*

$$\bigcap_{i\in\mathcal{I}} \mathrm{C}_G(S_i) = G_{c\mathfrak{Y}}.$$

Bearing in mind Definition 1.2.3, we get this corollary.

Corollary 1.2.48. *For a group G, we have that $\mathrm{C}^{\mathfrak{X}_p}(G) = G_{c\mathfrak{X}_p}$.*

Combining Theorem 1.2.43 and Lemmas 1.2.45 and 1.2.47 we obtain the following results.

Corollary 1.2.49. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation and let f be an $\mathfrak{X}$-formation function defining $\mathfrak{F}$. Then $\mathfrak{F}$ coincides with the class of all the groups G which satisfy the following two conditions:*

1. *$G/G_{\mathrm{E}\,\mathfrak{X}} \in f(\mathfrak{X}')$ if $G \notin \mathrm{E}\,\mathfrak{X}$.*
2. *$G/G_{c\mathfrak{X}_p} \in f(p)$ for any prime p such that $\mathfrak{X}_p \cap \mathcal{K}(G) \neq \emptyset$.*

Corollary 1.2.50. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation, where $\mathfrak{X}$ is a class of abelian simple groups and $\omega = \mathrm{char}\,\mathfrak{X}$. Let f be an $\mathfrak{X}$-formation function defining $\mathfrak{F}$. Then $\mathfrak{F}$ coincides with the class of all the groups G which satisfy the following two conditions:*

1. *$G/G_{\mathfrak{S}_\omega} \in f(\mathfrak{Y}')$, if $G \notin \mathfrak{S}_\omega$.*
2. *$G/G_{cp} \in f(p)$ for any p in ω such that $C_p \in \mathcal{K}(G)$.*

When we work with the modified definition of $\mathfrak{X}$-local formation (see Definition 1.2.35), we can also consider the minimal $\mathfrak{X}$-local definition, since Lemma 1.2.13 and Corollary 1.2.14 can be adapted to this new definition.

Lemma 1.2.51. *Consider a family $\{f_i \mid i \in \mathcal{I}\}$ of $\mathfrak{X}$-formation functions.*

Then $\bigcap_{i\in\mathcal{I}} \mathrm{LF}_{\mathfrak{X}}(f_i) = \mathrm{LF}_{\mathfrak{X}}(g)$, where $g(S) = \bigcap_{i\in\mathcal{I}} f_i(S)$ for every $S \in (\operatorname{char}\mathfrak{X}) \cup \{\mathfrak{X}'\}$.

Corollary 1.2.52. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation and $\{f_i \mid i \in \mathcal{I}\}$ a family of $\mathfrak{X}$-formation functions defining $\mathfrak{F}$, that is, $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f_i)$ for every $i \in \mathcal{I}$. Then $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$, where $f(S) := \bigcap_{i\in\mathcal{I}} f_i(S)$ for every $S \in (\operatorname{char}\mathfrak{X}) \cup \{\mathfrak{X}'\}$.*

The minimal definition of an $\mathfrak{X}$-local formation is described in the following theorem.

Theorem 1.2.53 ([BBCSss])**.** *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation. Then $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(m)$, where*

$$\begin{aligned} m(p) &= \operatorname{form}(G/G_{c\mathfrak{X}_p} \mid G \in \mathfrak{F} \text{ and } \mathfrak{X}_p \cap \mathcal{K}(G) \neq \emptyset) && \text{if } p \in \operatorname{char}\mathfrak{X}, \\ m(\mathfrak{X}') &= \operatorname{form}(G/G_{\mathrm{E}\,\mathfrak{X}} \mid G \in \mathfrak{F} \text{ and } G \neq G_{\mathrm{E}\,\mathfrak{X}}) && \text{if } \mathfrak{X}' \neq \emptyset. \end{aligned}$$

Moreover, $m \leq f$ for every $\mathfrak{X}$-formation function f such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$.

Proof. Consider $\mathfrak{M} = \mathrm{LF}_{\mathfrak{X}}(m)$. Clearly, we have that $\mathfrak{F} \subseteq \mathfrak{M}$ applying Corollary 1.2.49. Consider an $\mathfrak{X}$-formation function f such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$. We aim to prove that $m \leq f$. If G is a group in $\mathfrak{F}$ such that $\mathfrak{X}_p \cap \mathcal{K}(G) \neq \emptyset$ for a prime $p \in \operatorname{char}\mathfrak{X}$, we have that $G/G_{c\mathfrak{X}_p} \in f(p)$ by Corollary 1.2.49. Therefore, $m(p) \subseteq f(p)$. Now if $G \in \mathfrak{F}$ and $G \neq G_{\mathrm{E}\,\mathfrak{X}}$, it follows by Corollary 1.2.49 that $G/G_{\mathrm{E}\,\mathfrak{X}} \in f(\mathfrak{X}')$. Consequently, $m(\mathfrak{X}') \leq f(\mathfrak{X}')$ and $\mathfrak{F} \subseteq \mathfrak{M}$. We have also proved that $m \leq f$ for every $\mathfrak{X}$-formation function f such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$. □

1.3 $\mathfrak{X}$-saturated formations

Besides the concept of $\mathfrak{X}$-local formation, Förster defined a Frattini-like subgroup $\Phi^*_{\mathfrak{X}}(G)$ for each group G, which enables him to introduce the concept of $\mathfrak{X}$-saturation. If our aim is to generalise the concepts of saturation and soluble saturation, we would expect the $\mathfrak{X}$-Frattini subgroup of a group G to be defined as $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$, since $\mathrm{O}_{\mathfrak{X}}(G) = G$ when $\mathfrak{X} = \mathfrak{J}$ and $\mathrm{O}_{\mathfrak{X}}(G) = G_{\mathfrak{S}}$ when $\mathfrak{X} = \mathbb{P}$. We will see that Förster's definition does not coincide with the natural one.

Definition 1.3.1 (Förster)**.** Let G be a group.

- For a prime p, we define $\Phi^p_{\mathfrak{X}}(G)$:

– If $\mathrm{O}_{p'}(G) = 1$,

$$\Phi^p_{\mathfrak{X}}(G) := \begin{cases} \Phi(G) & \text{if } \mathrm{Soc}\big(G/\Phi(G)\big) \text{ and } \Phi(G) \text{ belong to } \mathrm{E}\,\mathfrak{X}, \\ \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) & \text{otherwise.} \end{cases}$$

– In general, $\Phi^p_{\mathfrak{X}}(G)$ is the subgroup of G such that

$$\Phi^p_{\mathfrak{X}}(G)/\mathrm{O}_{p'}(G) = \Phi^p_{\mathfrak{X}}\big(G/\mathrm{O}_{p'}(G)\big).$$

- Finally

$$\Phi^*_{\mathfrak{X}}(G) := \mathrm{O}_{\mathfrak{X}}(G) \cap \left(\bigcap_{p \in \mathrm{char}\,\mathfrak{X}} \Phi^p_{\mathfrak{X}}(G) \right).$$

A full account of the properties of this Frattini-like subgroup can be found in [BBE06; Proposition 3.2.2]. Nevertheless we give alternative proofs of some of them.

Definition 1.3.2. Let $\mathfrak{F}$ be a formation. We say that:

- $\mathfrak{F}$ is $\mathfrak{X}$-*saturated (N)* if $\mathfrak{F}$ contains a group G whenever it contains $G/\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$.
- $\mathfrak{F}$ is $\mathfrak{X}$-*saturated (F)* if $G \in \mathfrak{F}$ provided that $G/\Phi^*_{\mathfrak{X}}(G) \in \mathfrak{F}$.

The following definition can be deduced from [BBE06; Proposition 3.2.2 (2)].

Proposition 1.3.3. *If $\mathfrak{X} = \mathfrak{J}$, the class of all simple groups, then $\Phi^*_{\mathfrak{X}}(G) = \Phi(G)$ for every group G.*

Proof. We have that

$$\Phi(G)\,\mathrm{O}_{p'}(G)/\mathrm{O}_{p'}(G) \le \Phi\big(G/\mathrm{O}_{p'}(G)\big) = \Phi^p_{\mathfrak{X}}\big(G/\mathrm{O}_{p'}(G)\big) = \Phi^p_{\mathfrak{X}}(G)/\mathrm{O}_{p'}(G).$$

Hence $\Phi(G) \le \Phi^p_{\mathfrak{X}}(G)$ for every prime p. Since in this case

$$\Phi^*_{\mathfrak{X}}(G) = \bigcap_{p \in \mathbb{P}} \Phi^p_{\mathfrak{X}}(G),$$

then $\Phi(G) \le \Phi^*_{\mathfrak{X}}(G)$.

Now we aim to prove that $\Phi^*_{\mathfrak{X}}(G) \le \Phi(G)$. Note that $\Phi^*_{\mathfrak{X}}(G)$ is nilpotent, since for every prime p, we have that

$$\begin{aligned} \Phi^*_{\mathfrak{X}}(G)/\Phi^*_{\mathfrak{X}}(G) \cap \mathrm{O}_{p'}(G) &\cong \Phi^*_{\mathfrak{X}}(G)\,\mathrm{O}_{p'}(G)/\mathrm{O}_{p'}(G) \le \Phi^p_{\mathfrak{X}}(G)\,\mathrm{O}_{p'}(G)/\mathrm{O}_{p'}(G) \\ &= \Phi^p_{\mathfrak{X}}(G)/\mathrm{O}_{p'}(G) = \Phi\big(G/\mathrm{O}_{p'}(G)\big) \le \mathrm{F}\big(G/\mathrm{O}_{p'}(G)\big), \end{aligned}$$

where $\mathrm{F}\bigl(G/\mathrm{O}_{p'}(G)\bigr)$ is a p-group (otherwise, since $\mathrm{F}\bigl(G/\mathrm{O}_{p'}(G)\bigr)$ is p-nilpotent, it would have a non-trivial normal Hall p'-subgroup and so the factor group $\mathrm{O}_{p'}\bigl(G/\mathrm{O}_{p'}(G)\bigr)$ would be non-trivial). We have seen that $\Phi^*_{\mathfrak{X}}(G) \cap \mathrm{O}_{p'}(G)$ is a normal Hall p'-subgroup of $\Phi^*_{\mathfrak{X}}(G)$ and, therefore, $\Phi^*_{\mathfrak{X}}(G)$ is p-nilpotent.

Consider a prime p. Let us prove that $\mathrm{O}_p\bigl(\Phi^*_{\mathfrak{X}}(G)/\Phi(G)\bigr) = 1$. Assume that it is not true and consider a minimal normal subgroup $N/\Phi(G)$ of $G/\Phi(G)$ such that $N/\Phi(G) \leq \mathrm{O}_p\bigl(\Phi^*_{\mathfrak{X}}(G)/\Phi(G)\bigr)$. Let $M/\Phi(G)$ be a maximal subgroup of $G/\Phi(G)$ such that $G = MN$. Since $|G : M|$ is a power of p, it follows that $\mathrm{O}_{p'}(G) \leq M$ (otherwise, we would have that $G = \mathrm{O}_{p'}(G)M$ and $|G : M|$ would be a p'-number). Hence $M/\mathrm{O}_{p'}(G)$ is a maximal subgroup of $G/\mathrm{O}_{p'}(G)$ and, therefore,

$$\Phi^p_{\mathfrak{X}}(G)/\mathrm{O}_{p'}(G) = \Phi^p_{\mathfrak{X}}\bigl(G/\mathrm{O}_{p'}(G)\bigr) = \Phi\bigl(G/\mathrm{O}_{p'}(G)\bigr) \leq M/\mathrm{O}_{p'}(G).$$

Hence $\Phi^p_{\mathfrak{X}}(G) \leq M$ and

$$N/\Phi(G) \leq \Phi^*_{\mathfrak{X}}(G)/\Phi(G) \leq \Phi^p_{\mathfrak{X}}(G)/\Phi(G) \leq M/\Phi(G),$$

which is a contradiction. Therefore $\mathrm{O}_p\bigl(\Phi^*_{\mathfrak{X}}(G)/\Phi(G)\bigr) = 1$ for every prime p. We have proved that $\Phi^*_{\mathfrak{X}}(G) = \Phi(G)$ and the proof is complete. □

Remark 1.3.4. Let G be a group. Then

$$\Phi^*_{\mathfrak{X}}(G) = \mathrm{O}_{\mathfrak{X}}(G) \cap \left(\bigcap_{p\in\mathbb{P}} \Phi^p_{\mathfrak{X}}(G)\right).$$

Proof. Since $\Phi^*_{\mathfrak{X}}(G) \leq \mathrm{O}_{\mathfrak{X}}(G)$, it follows that $\Phi^*_{\mathfrak{X}}(G) \in \mathrm{E}\,\mathfrak{X}$. Therefore, if q is a prime such that $q \notin \operatorname{char}\mathfrak{X}$, q cannot divide the order of $\Phi^*_{\mathfrak{X}}(G)$. Hence $\Phi^*_{\mathfrak{X}}(G) \leq \mathrm{O}_{q'}(G) \leq \Phi^q_{\mathfrak{X}}(G)$ and, therefore

$$\Phi^*_{\mathfrak{X}}(G) \leq \mathrm{O}_{\mathfrak{X}}(G) \cap \left(\bigcap_{p\in\mathbb{P}} \Phi^p_{\mathfrak{X}}(G)\right).$$

The other inclusion is clear. □

Proposition 1.3.5.

1. *If $\mathfrak{X} \subseteq \overline{\mathfrak{X}}$, then $\Phi^*_{\mathfrak{X}}(G) \leq \Phi^*_{\overline{\mathfrak{X}}}(G)$ for every group G.*
2. *For every group G,*

$$\Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr) \leq \Phi^*_{\mathfrak{X}}(G) \leq \Phi(G)$$

Proof. 1. Clearly, if $p \in \mathbb{P}$, it follows that

$$\Phi^p_{\mathfrak{X}}\big(G/\operatorname{O}_{p'}(G)\big) \le \Phi^p_{\overline{\mathfrak{X}}}\big(G/\operatorname{O}_{p'}(G)\big),$$

since $\mathrm{E}\,\mathfrak{X} \subseteq \mathrm{E}\,\overline{\mathfrak{X}}$. Therefore, $\Phi^p_{\mathfrak{X}}(G) \le \Phi^p_{\overline{\mathfrak{X}}}(G)$. Bearing in mind Remark 1.3.4, we have that

$$\Phi^*_{\mathfrak{X}}(G) = \operatorname{O}_{\mathfrak{X}}(G) \cap \left(\bigcap_{p\in\mathbb{P}} \Phi^p_{\mathfrak{X}}(G)\right) \le \operatorname{O}_{\overline{\mathfrak{X}}}(G) \cap \left(\bigcap_{p\in\mathbb{P}} \Phi^p_{\overline{\mathfrak{X}}}(G)\right) = \Phi^*_{\overline{\mathfrak{X}}}(G)$$

2. Applying 1 and Proposition 1.3.3, we obtain that $\Phi^*_{\mathfrak{X}}(G) \le \Phi(G)$.

Now we aim to prove that $\Phi\big(\operatorname{O}_{\mathfrak{X}}(G)\big) \le \Phi^*_{\mathfrak{X}}(G)$. If $p \in \mathbb{P}$, it follows that

$$\begin{aligned}
&\Phi\big(\operatorname{O}_{\mathfrak{X}}(G)\big)\operatorname{O}_{p'}(G)/\operatorname{O}_{p'}(G) \le \Phi\big(\operatorname{O}_{\mathfrak{X}}(G)\operatorname{O}_{p'}(G)\big)\operatorname{O}_{p'}(G)/\operatorname{O}_{p'}(G)\\
&\le \Phi\big(\operatorname{O}_{\mathfrak{X}}(G)\operatorname{O}_{p'}(G)/\operatorname{O}_{p'}(G)\big) \le \Phi\Big(\operatorname{O}_{\mathfrak{X}}\big(G/\operatorname{O}_{p'}(G)\big)\Big)\\
&\le \Phi^p_{\mathfrak{X}}\big(G/\operatorname{O}_{p'}(G)\big) = \Phi^p_{\mathfrak{X}}(G)/\operatorname{O}_{p'}(G).
\end{aligned}$$

Hence $\Phi\big(\operatorname{O}_{\mathfrak{X}}(G)\big) \le \Phi^p_{\mathfrak{X}}(G)$ for every prime p and, therefore,

$$\Phi\big(\operatorname{O}_{\mathfrak{X}}(G)\big) \le \operatorname{O}_{\mathfrak{X}}(G) \cap \left(\bigcap_{p\in\mathbb{P}} \Phi^p_{\mathfrak{X}}(G)\right) = \Phi^*_{\mathfrak{X}}(G). \qquad \square$$

When $\mathfrak{X} = \mathbb{P}$, the class of all abelian simple groups, $\Phi^*_{\mathfrak{X}}(G)$ does not coincide in general with $\Phi\big(\operatorname{O}_{\mathfrak{X}}(G)\big)$, as the following example shows.

Example 1.3.6. Consider a non-abelian simple group E and a prime $p \in \pi(E)$. Let T denote the group algebra $\mathrm{GF}(p)E$. The structure of T as a T-module (regular module) leads to an action of E over T. Consider $G := [T]E$ and $A := T/\operatorname{Rad}(T)$, the head of T.

We aim to prove that $\operatorname{Rad}(T) = \Phi(G)$. In general, we have that $\operatorname{Rad}(M) = \Phi([M]H) \cap M$ for a group H and a $\mathrm{GF}(p)H$-module M (see [DH92; B, 3.14]). Therefore, $\operatorname{Rad}(T) \le \Phi(G)$.

On the other hand, since $\Phi([A]E)A/A \le \Phi([A]E/A) = 1$, we have that $\Phi([A]E) \le A$. Hence $\Phi([A]E) = \Phi([A]E) \cap A = \operatorname{Rad}(A) = 1$.

We have that $\Phi(G)\operatorname{Rad}(T)/\operatorname{Rad}(T) \le \Phi\big(G/\operatorname{Rad}(T)\big) \cong \Phi([A]E) = 1$. Hence $\Phi(G) \le \operatorname{Rad}(T)$ and, therefore, $\Phi(G) = \operatorname{Rad}(T)$.

Now let us prove that $\Phi^*_{\mathbb{P}}(G) = \Phi(G)$.

We know that $\Phi^p_{\mathbb{P}}(G) = \Phi(G)$, since $\operatorname{O}_{p'}(G) = 1$ and the groups $\Phi(G)$ and $\operatorname{Soc}\big(G/\Phi(G)\big)$ are soluble.

If $q \neq p$, it follows that $\Phi(G) \leq T \leq \mathrm{O}_{q'}(G) \leq \Phi^q_{\mathbb{P}}(G)$. Bearing in mind that $\Phi(G) \leq G_{\mathfrak{S}}$, we obtain that $\Phi^*_{\mathbb{P}}(G) = \Phi(G)$.

Since $\mathrm{char}\big(\mathrm{GF}(p)\big)$ divides the order of E, we have by Maschke's theorem that T is not completely reducible as a module and, therefore, $\mathrm{Rad}(T)$ is non-trivial. Hence $\Phi(G_{\mathfrak{S}}) = 1 \neq \Phi^*_{\mathbb{P}}(G)$.

Förster proved in [För85] the following theorem.

Theorem 1.3.7. *A formation is $\mathfrak{X}$-saturated (F) if and only if it is $\mathfrak{X}$-local.*

Remark 1.3.8. If $\mathfrak{X} = \mathfrak{J}$ in Theorem 1.3.7, then we obtain as a special case the Gaschütz-Lubeseder-Schmid theorem (see Theorem 1.1.4). However, Baer's theorem cannot be immediately deduced from this result.

Since $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$ is contained in $\Phi^*_{\mathfrak{X}}(G)$ for every group G, we can deduce from Förster's theorem that every $\mathfrak{X}$-local formation fulfils the following property:

$$\text{A group } G \text{ belongs to } \mathfrak{F} \text{ if and only if } G/\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) \text{ belongs to } \mathfrak{F}. \qquad (1.3)$$

Therefore from the very beginning the following question naturally arises:

Open question 1.3.9. *Let $\mathfrak{F}$ be a formation with the property* (1.3). *Is $\mathfrak{F}$ $\mathfrak{X}$-local?*

Now we introduce another $\mathfrak{X}$-Frattini subgroup which is smaller than Förster's one. We will prove that $\mathfrak{X}$-local formations are exactly the $\mathfrak{X}$-saturated ones defined using our $\mathfrak{X}$-Frattini subgroup. This fact will draw near the solution of Question 1.3.9.

Definition 1.3.10.

- Let p be a prime number. We say that a group G belongs to the class $\mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ provided there exists an elementary abelian normal p-subgroup N of G such that
  1. $N \leq \Phi(G)$ and G/N is a primitive group with a unique non-abelian minimal normal subgroup, i. e., G/N is a primitive group of type 2,
  2. $\mathrm{Soc}(G/N) \in \mathrm{E}\,\mathfrak{X} \setminus \mathfrak{E}_{p'}$, and
  3. $\mathrm{C}^h_G(N) \leq N$, where

$$\mathrm{C}^h_G(N) := \bigcap \{\mathrm{C}_G(H/K) \mid H/K \text{ is a chief factor of } G \text{ below } N\}.$$

- The $\mathfrak{X}$-Frattini subgroup of a group G is the subgroup $\Phi_{\mathfrak{X}}(G)$ defined as

$$\Phi_{\mathfrak{X}}(G) := \begin{cases} \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) & \text{if } G \notin \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2) \text{ for all } p \in \operatorname{char}\mathfrak{X}, \\ \Phi(G) & \text{otherwise.} \end{cases}$$

- A formation $\mathfrak{F}$ is said to be $\mathfrak{X}$-*saturated* if $G \in \mathfrak{F}$ for every group G such that $G/\Phi_{\mathfrak{X}}(G) \in \mathfrak{F}$.

Remark 1.3.11. $\Phi_{\mathfrak{X}}(G)$ is a π-group, where $\pi = \pi(\mathfrak{X})$.

Proposition 1.3.12.

1. *If $\mathfrak{X} = \mathfrak{J}$, the class of all simple groups, $\Phi_{\mathfrak{X}}(G) = \Phi(G)$ for every group G. In this case, the concept of $\mathfrak{X}$-saturation coincides with the one of saturation.*

2. *If $\mathfrak{X} \subseteq \mathbb{P}$, that is, if $\mathfrak{X}$ is a class of abelian simple groups, then $\Phi_{\mathfrak{X}}(G) = \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$ for every group G. In particular, if $\mathfrak{X} = \mathbb{P}$, then $\Phi_{\mathfrak{X}}(G) = \Phi(G_{\mathfrak{S}})$ for every group G and, in this case, the $\mathfrak{X}$-saturated formations are exactly the solubly saturated ones.*

Proof.

1. It is clear, since $\mathrm{O}_{\mathfrak{X}}(G) = G$ for every group G if $\mathfrak{X} = \mathfrak{J}$.

2. If $\mathfrak{X} \subseteq \mathbb{P}$, we have that $\mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2) = \emptyset$ for every prime p, because if $G \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$, it follows that $G/\Phi(G)$ is a primitive group of type 2, that is, $\operatorname{Soc}\big(G/\Phi(G)\big)$ is non-abelian. On the other hand, $\operatorname{Soc}\big(G/\Phi(G)\big) \in \mathrm{E}\,\mathfrak{X} \subseteq \mathrm{E}\,\mathbb{P} = \mathfrak{S}$, implying that it is abelian. Therefore, $\Phi_{\mathfrak{X}}(G) = \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$.
   If $\mathfrak{X} = \mathbb{P}$, then $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) = \Phi(G_{\mathfrak{S}})$. □

Definition 1.3.13. Let $\mathfrak{Y}$ be a class of groups. We define the following class:

$$\mathrm{E}_{\Phi_{\mathfrak{X}}}(\mathfrak{Y}) := (G \in \mathfrak{E} \mid \text{there exists } N \trianglelefteq G \text{ such that } N \leq \Phi_{\mathfrak{X}}(G) \text{ and } G/N \in \mathfrak{Y}).$$

Remark 1.3.14. A formation $\mathfrak{F}$ is $\mathfrak{X}$-saturated if and only if $\mathrm{E}_{\Phi_{\mathfrak{X}}}(\mathfrak{F}) = \mathfrak{F}$.

Lemma 1.3.15. *If $\mathfrak{X} \subseteq \mathbb{P}$, that is, if $\mathfrak{X}$ only contains abelian simple groups, and $\mathfrak{Y}$ is a Q-closed class of groups, then $\mathrm{E}_{\Phi_{\mathfrak{X}}}(\mathfrak{Y})$ is also Q-closed.*

Proof. Consider a group $G \in \mathrm{E}_{\Phi_{\mathfrak{X}}}(\mathfrak{Y})$ and a normal subgroup M of G. We aim to prove that $G/M \in \mathrm{E}_{\Phi_{\mathfrak{X}}}(\mathfrak{Y})$. There exists a normal subgroup N of G such that $N \leq \Phi_{\mathfrak{X}}(G)$ and $G/N \in \mathfrak{Y}$. Consider the normal subgroup NM/M of G/M, which satisfies that

$$\begin{aligned} NM/M &\leq \Phi_{\mathfrak{X}}(G)M/M = \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)M/M \leq \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)M\big)M/M \\ &\leq \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)M/M\big) \leq \Phi\big(\mathrm{O}_{\mathfrak{X}}(G/M)\big) = \Phi_{\mathfrak{X}}(G/M) \end{aligned}$$

Moreover, $(G/M)\big/(NM/M) \cong G/NM$. Since $G/N \in \mathfrak{Y}$ and $\mathfrak{Y}$ is Q-closed, It follows that $(G/M)\big/(NM/M) \in \mathfrak{Y}$. Therefore, $G/M \in \mathrm{E}_{\Phi_{\mathfrak{X}}}(\mathfrak{Y})$. □

Lemma 1.3.16. *If $\mathfrak{X}$ and $\overline{\mathfrak{X}}$ two classes of simple groups such that $\mathfrak{X} \subseteq \overline{\mathfrak{X}}$, then $\Phi_{\mathfrak{X}}(G) \leq \Phi_{\overline{\mathfrak{X}}}(G)$ for every group G. Therefore, if a formation $\mathfrak{F}$ is $\overline{\mathfrak{X}}$-saturated, then $\mathfrak{F}$ is $\mathfrak{X}$-saturated.*

Proof. If $\Phi_{\mathfrak{X}}(G) = \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$, it follows that $\Phi_{\mathfrak{X}}(G) \leq \Phi\big(\mathrm{O}_{\overline{\mathfrak{X}}}(G)\big) \leq \Phi_{\overline{\mathfrak{X}}}(G)$. If $G \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ for a prime $p \in \operatorname{char}\mathfrak{X}$, then it is clear that $G \in \mathrm{A}_{\overline{\mathfrak{X}}_p}(\mathfrak{P}_2)$ and, therefore, $\Phi_{\mathfrak{X}}(G) = \Phi(G) = \Phi_{\overline{\mathfrak{X}}}(G)$. □

The following proposition shows that the new $\mathfrak{X}$-Frattini subgroup is smaller than the one defined by Förster.

Proposition 1.3.17. *Let G be a group. Then $\Phi_{\mathfrak{X}}(G)$ is contained in $\Phi^*_{\mathfrak{X}}(G)$.*

Proof. By Proposition 1.3.5, we have that $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) \leq \Phi^*_{\mathfrak{X}}(G)$. Therefore, we only need to prove that if $G \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ for a prime $p \in \operatorname{char}\mathfrak{X}$, then $\Phi(G) \leq \Phi^*_{\mathfrak{X}}(G)$. In this case, $\Phi(G)$ is a p-subgroup of G. We have that $\mathrm{O}_{p'}(G) = 1$, since $\mathrm{O}_{p'}(G) \leq \mathrm{C}^h_G\big(\Phi(G)\big)$. Bearing in mind the definition of the class $\mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$, we have that $\mathrm{C}^h_G\big(\Phi(G)\big) \leq \Phi(G)$. Since $\Phi(G)$ and $\operatorname{Soc}\big(G/\Phi(G)\big)$ belong to $\mathrm{E}\,\mathfrak{X}$, it follows that $\Phi^p_{\mathfrak{X}}(G) = \Phi(G)$. If $q \in \operatorname{char}\mathfrak{X}$ and $q \neq p$, then $\Phi(G) \leq \mathrm{O}_{q'}(G) \leq \Phi^q_{\mathfrak{X}}(G)$. Therefore, $\Phi(G) \leq \mathrm{O}_{\mathfrak{X}}(G) \cap \Big(\bigcap_{p\in\operatorname{char}\mathfrak{X}} \Phi^p_{\mathfrak{X}}(G)\Big) = \Phi^*_{\mathfrak{X}}(G)$. □

In general the subgroups $\Phi_{\mathfrak{X}}(G)$ and $\Phi^*_{\mathfrak{X}}(G)$ do not coincide (see Example 1.3.6). Moreover, the following example shows that for every class $\mathfrak{X}$ of simple groups such that $\mathfrak{X} \neq \emptyset$ and $\mathfrak{X} \neq \mathfrak{J}$, there exists a group G such that $\Phi_{\mathfrak{X}}(G) < \Phi^*_{\mathfrak{X}}(G)$.

Example 1.3.18. Let $\mathfrak{X}$ be a class of simple groups different from $\emptyset$ and $\mathfrak{J}$. First note that there exists a non-abelian simple group E and a prime $p \in \pi(E)$ such that $E \in \mathfrak{X}'$ and $p \in \operatorname{char}\mathfrak{X}$.

- If $\operatorname{char}\mathfrak{X} = \mathbb{P}$, take $E \in \mathfrak{X}'$, which is non-abelian. Moreover, if $p \in \pi(E)$, we have that $p \in \operatorname{char}\mathfrak{X}$.

- If $\operatorname{char}\mathfrak{X} \neq \mathbb{P}$, take a prime $p \in \operatorname{char}\mathfrak{X}$ and a prime q such that $q \notin \operatorname{char}\mathfrak{X}$. Consider $E = \operatorname{Alt}(p+q)$, the alternating group of degree $p+q$, which is a non-abelian simple group. We have that $p \in (\operatorname{char}\mathfrak{X}) \cap \pi(E)$. Moreover, $E \in \mathfrak{X}'$, because otherwise, since $q \in \pi(E)$, we would have that $q \in \operatorname{char}\mathfrak{X}$.

Now consider $G = [T]E$ (see Example 1.3.6). We know that $\Phi(G) = \operatorname{Rad}(T)$. In this case, $\Phi^*_{\mathfrak{X}}(G) = \Phi(G)$, bearing in mind that $\Phi(G)$ and $\operatorname{Soc}\bigl(G/\Phi(G)\bigr)$ belong to the class $\mathrm{E}\,\mathfrak{X}$. Note that $G \notin \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$, since $G/\Phi(G) \cong [A]E$, which is not a primitive group of type 2. Clearly, given a prime $q \in \operatorname{char}\mathfrak{X}$ different from p, G does not belong to $\mathrm{A}_{\mathfrak{X}_q}(\mathfrak{P}_2)$. Hence $\Phi_{\mathfrak{X}}(G) = \Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr) = \Phi(T) = 1$. Therefore, $\Phi_{\mathfrak{X}}(G) < \Phi^*_{\mathfrak{X}}(G)$.

There exist groups G satisfying that $\Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr)$ is a proper subgroup of $\Phi_{\mathfrak{X}}(G)$, since Example 1.3.20 (suggested by John Cossey [Cos04]) shows. We need the following lemma.

Lemma 1.3.19. *Let G be a group and let X be a primitive group of type 2, that is, a monolithic primitive group whose socle is non-abelian. Then $X \wr G$, the regular wreath product of X with G, is a primitive group of type 2.*

Proof. Consider $W = X \wr G$ and let S denote the direct product of $|G|$ copies of $\operatorname{Soc}(X)$, considered as a subgroup of the base group B of W. By [DH92; A, 18.5(a)], we have that S is a minimal normal subgroup of W.

Next we prove that $\mathrm{C}_W(S) = 1$. Take $w = tg \in \mathrm{C}_W(S)$, where $t \in B$ and $g \in G$. Consider $S_i = \operatorname{Soc}(X_i)$, where X_i is the copy number i of X in B. It follows that $S_i = S_i^w = S_i^{tg} = S_i^g = S_{ig}$, implying that $g = 1$. Therefore, $t \in \mathrm{C}_B(S) = 1$ and we have that $w = 1$. Consequently, $\mathrm{C}_W(S) = 1$. Hence W is a monolithic group and $\operatorname{Soc}(W) = S$. Since $S \not\leq \Phi(W)$, there exist a maximal subgroup M of W not containing S. It follows that $\operatorname{Core}_W(M) = 1$. By [DH92; A, 15.2], W is a primitive group of type 2. □

Example 1.3.20. Consider $\mathfrak{X} = \bigl(\operatorname{Alt}(5), C_2, C_3, C_5\bigr)$.

Consider the alternating group $\operatorname{Alt}(5)$ of degree 5. There exist a group X having an elementary abelian minimal normal subgroup F of order 5^3 such that $F = \Phi(X)$, $\mathrm{C}_X(F) = F$ and $X/F \cong \operatorname{Alt}(5)$ (see [GS78; Example 1 and Proposition 2]). Since 5 divides $11 - 1$, C_5 possesses a faithful irreducible module V of dimension 1 over $\mathrm{GF}(11)$. Let S denote the corresponding semidirect product, which is a non-abelian group of order 55. Consider $Y := X \wr S$. Let B be the base group of Y. It is clear that $A = BV$ is a normal subgroup of Y of index 5. Now consider a cyclic group C of order 25 and D its subgroup of index 5. Now we construct a subdirect product

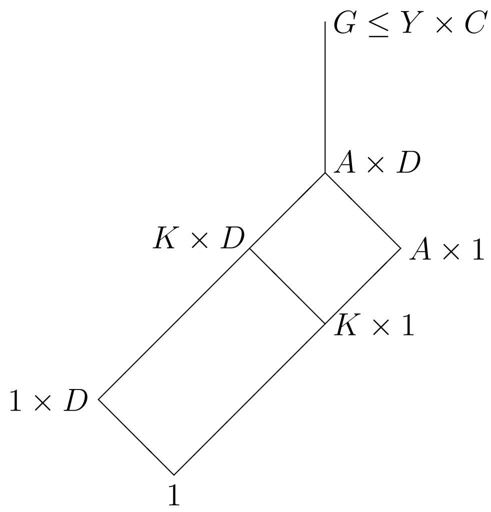

Figure 1.1: Group considered in Example 1.3.20

of Y and C with amalgamated factor group $Y/A \cong C/D \cong C_5$ (see [DH92; A, 19.2]). G can be seen as a subgroup of the direct product $Y \times C$ (see diagram 1.1). By [DH92; A, 19.1], $G/(1 \times D) \cong Y$ and $G/(A \times 1) \cong C$.

Next we study $\Phi(G)$. By [DH92; A, 18.8], we have that $A \cong X^5 \wr V$, where X^5 the direct product of 5 copies of X. Let K denote the direct product of all the copies of F in B. We shall prove that $K \times 1 \leq \Phi(G)$. Note that $F \times 1 \leq \Phi(X)$. Hence $K \times 1 \leq \Phi(B)$ by [DH92; A, 9.4]. Consequently, $K \times 1 \leq \Phi(G)$ by [DH92; A, 9.1].

The next step is to prove that $1 \times D \leq \Phi(G)$. Assume that it is not true. There exist a maximal subgroup M of G such that $(1 \times D)M = G$. It follows that

$$M \cong M/\big(M \cap (1 \times D)\big) \cong M(1 \times D)/(1 \times D) = G/(1 \times D) \cong Y.$$

Take quotients by $A \times 1$. We have that

$$\big((1 \times D)(A \times 1)/(A \times 1)\big)\big(M(A \times 1)/(A \times 1)\big) = G/(A \times 1).$$

Since $(1 \times D)(A \times 1)/(A \times 1) \cong C_5$ y $G/(A \times 1) \cong C_{25}$, it follows that $M(A \times 1)/(A \times 1) \cong C_{25}$. We conclude that $M/\big((A \times 1) \cap M\big) \cong C_{25}$, that is, $M \cong Y$ has a quotient which is isomorphic to C_{25}. We shall see that this is not possible. Bearing in mind that X is perfect, by [DH92; A, 18.4] it follows that $Y' = BS' = BV = A$. Hence $Y/Y' \cong C_5$ and Y cannot have a quotient which is isomorphic to C_{25}. This contradiction shows that $1 \times D \leq \Phi(G)$.

Next we prove that $\Phi(G) = K \times D$. We have seen that $K \times D \leq \Phi(G)$. Now we shall prove that $\Phi\big(G/(K \times D)\big) = 1$. We have that $G/(K \times D) \cong Y/K$ and, by [DH92; A, 18.2], $Y/K \cong \mathrm{Alt}(5) \wr S$. Applying Lemma 1.3.19, $\mathrm{Alt}(5) \wr S$ is a primitive group of type 2. In particular, $\Phi\big(\mathrm{Alt}(5) \wr S\big) = 1$ and, therefore, $\Phi\big(G/(K \times D)\big) = 1$. It follows that $\Phi(G) = K \times D$.

Now we analyse $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$. It is clear that $B \times D \leq \mathrm{O}_{\mathfrak{X}}(G)$. Moreover, $\mathrm{O}_{\mathfrak{X}}\big(G/(B \times D)\big) \cong \mathrm{O}_{\mathfrak{X}}(S) = 1$. Hence $\mathrm{O}_{\mathfrak{X}}(G) = B \times D$ and $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) = K$.

We shall prove that $G \in \mathrm{A}_{\mathfrak{X}_5}(\mathfrak{P}_2)$. On one hand, $\Phi(G) = K \times D$ is 5-elementary abelian. On the other hand, we have that $G/\Phi(G) \cong \mathrm{Alt}(5) \wr S$ is a primitive group of type 2. Moreover, $\mathrm{Soc}\big(G/\Phi(G)\big)$ is a direct product of copies of Alt(5). Hence it belongs to $\mathrm{E}_{\mathfrak{X}} \setminus \mathfrak{E}_{p'}$. Next we prove that $\mathrm{C}^h_G\big(\Phi(G)\big) \leq \Phi(G)$. Since F is a minimal normal subgroup of X and F is not central in X, by [DH92; A, 18.5] it follows that K is a minimal normal subgroup of Y. If $K \times 1$ were not a minimal normal subgroup of G, there would exist a minimal subgroup $T \times 1$ of G strictly contained in $K \times 1$. Hence $(T \times D)/(1 \times D)$ would be a minimal normal subgroup of $G/(1 \times D)$ strictly contained in $(K \times D)/(1 \times D)$. It would follows that K is not a minimal normal subgroup of Y, which is a contradiction. Therefore, $K \times 1$ is a minimal normal subgroup of G. Moreover, $\mathrm{C}_G(K \times 1) = K \times D$, since $\mathrm{C}_Y(K) = K$ because $\mathrm{C}_X(F) = F$. It follows that $\mathrm{C}^h\big(\Phi(G)\big) = \Phi(G)$. Hence $G \in \mathrm{A}_{\mathfrak{X}_5}(\mathfrak{P}_2)$ and, consequently, $\Phi_{\mathfrak{X}}(G) = \Phi(G)$ and $\Phi_{\mathfrak{X}}(G)$ is a proper group of $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$.

1.4 A generalisation of the Gaschütz-Lubeseder-Schmid-Baer theorem

We have introduced the concepts of $\mathfrak{X}$-saturation (N), $\mathfrak{X}$-saturation (F) and $\mathfrak{X}$-saturation. In this section the relation between these types of saturation is studied. Since $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) \leq \Phi_{\mathfrak{X}}(G) \leq \Phi^*_{\mathfrak{X}}(G)$ (see Propositions 1.3.5 and 1.3.17), it is clear that the family of $\mathfrak{X}$-saturated formations contains the family of $\mathfrak{X}$-saturated(F) ones and it is contained in the family of $\mathfrak{X}$-saturated(N) ones. Since there exist groups G such that $\Phi_{\mathfrak{X}}(G) < \Phi^*_{\mathfrak{X}}(G)$, the following question arises.

Open question 1.4.1. *If $\mathfrak{F}$ is an $\mathfrak{X}$-saturated formation, is $\mathfrak{F}$ an $\mathfrak{X}$-local formation?*

The main goal of this section is to prove that the answer is positive.

We can deduce from Theorem 1.3.7 that every $\mathfrak{X}$-local formation is $\mathfrak{X}$-saturated. Here a direct proof is given.

Theorem 1.4.2. *If $\mathfrak{F}$ is an $\mathfrak{X}$-local formation, then $\mathfrak{F}$ is $\mathfrak{X}$-saturated.*

Proof. Consider an $\mathfrak{X}$-formation function f such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$ and a group G. Assume that $G/\Phi_{\mathfrak{X}}(G) \in \mathfrak{F}$. We distinguish two cases:

1. If $G \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$, where $p \in \operatorname{char} \mathfrak{X}$, it follows that $\Phi_{\mathfrak{X}}(G) = \Phi(G)$. Hence $G/\Phi(G) \in \mathfrak{F}$. Next we prove that $G \in \mathfrak{F}$. We have that $\Phi(G)$ is p-elementary abelian, $G/\Phi(G)$ is a primitive group of type 2 and $\mathrm{Soc}(G/\Phi(G))$ is an $\mathfrak{X}_p$-chief factor of G. Therefore,
$$G/\mathrm{C}_G\Big(\mathrm{Soc}\big(G/\Phi(G)\big)\Big) = G/\Phi(G) \in f(p)$$
and hence $G/\mathrm{C}_G\big(\Phi(G)\big) \in \mathrm{Q}\, f(p) = f(p)$. Applying Lemma 1.2.12, we can conclude that $G \in \mathfrak{F}$.

2. If there does not exist $p \in \operatorname{char} \mathfrak{X}$ such that $G \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$, we have that $G/\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) \in \mathfrak{F}$. Next we prove that $G \in \mathfrak{F}$.

   If H/K is an $\mathfrak{X}_p$-chief factor of G below $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$, then p divides $\big|\mathrm{O}_{\mathfrak{X}}(G)/\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)\big|$, since p divides $\big|\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)\big|$. Therefore, G has an $\mathfrak{X}_p$-chief factor above $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$ and $f(p) \neq \emptyset$. By Lemma 1.2.7, it follows that
$$\begin{aligned} G/\mathrm{C}^{\mathfrak{X}_p}(G) &\cong G/\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)/\mathrm{C}^{\mathfrak{X}_p}(G)/\big(\mathrm{O}_{\mathfrak{X}}(G)\big) \\ &= G/\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)/\mathrm{C}^{\mathfrak{X}_p}\Big(G/\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)\Big) \in f(p). \end{aligned}$$

   Therefore, $G\big/\mathrm{C}_G(H/K) \in f(p)$.

   If $L \trianglelefteq G$, G/L is monolithic and $\mathrm{Soc}(G/L)$ is an $\mathfrak{X}'$-chief factor of G, it follows that $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) \leq L$ and, therefore, $G/L \in \mathfrak{F}$. We can conclude that $G \in \mathfrak{F}$. □

The following series of lemmas on $\mathfrak{X}$-saturated formations is needed to prove the main result of this section. It can be found in [BBCER05].

Lemma 1.4.3. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-saturated formation, X a group, and p a prime in* $\operatorname{char} \mathfrak{X}$. *If there exists a faithful X-module M over* $\mathrm{GF}(p)$ *such that $[M]X \in \mathfrak{F}$, then $[N]X \in \mathfrak{F}$ for every irreducible* $\mathrm{GF}(p)X$*-module N.*

Proof. We can argue as in [DH92; IV, 4.1], bearing in mind that the Hartley group used in the proof is a p-group and hence it belongs to $\mathrm{E}\,\mathfrak{X}$. □

Lemma 1.4.4. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-saturated formation, G a group and let p be a prime in* $\operatorname{char} \mathfrak{X}$. *If $C_p \in \mathfrak{F}$ and N is a normal elementary abelian p-subgroup of G such that and $[N](G/N) \in \mathfrak{F}$, then $G \in \mathfrak{F}$.*

Proof. Analogous to [DH92; IV, 4.15], noting that the Hartley group is a p-group as in the previous lemma. □

Lemma 1.4.5. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-saturated formation and p a prime in* $\operatorname{char}\mathfrak{X}$*. If $X \in \mathrm{R}_0\big(G/\operatorname{C}_G(H/K) \mid G \in \mathfrak{F}$ and H/K is an $\mathfrak{X}_p$-chief factor of $G\big)$, then $[N]X \in \mathfrak{F}$ for every irreducible* $\mathrm{GF}(p)X$*-module N.*

Proof. By Lemma 1.4.3, it is enough to find a faithful X-module M over $\mathrm{GF}(p)$ such that $[M]X \in \mathfrak{F}$.

Since

$$X \in \mathrm{R}_0\big(G/\operatorname{C}_G(H/K) \mid G \in \mathfrak{F} \text{ and } H/K \text{ is an } \mathfrak{X}_p\text{-chief factor of } G\big),$$

there exist a natural number n and normal subgroups X_i of X, for $i = 1, 2, \dots, n$, such that $\bigcap_{i=1}^{n} X_i = 1$ and $X/X_i \cong G_i/\operatorname{C}_{G_i}(H_i/K_i)$, where $G_i \in \mathfrak{F}$ and H_i/K_i is an $\mathfrak{X}_p$-chief factor of G_i.

Assume that H_i/K_i is non-abelian for some $i = 1, 2, \dots, n$. Then $G := G_i/\operatorname{C}_{G_i}(H_i/K_i)$ is a primitive group of type 2. Consider the maximal Frattini extension E of G corresponding to the prime p ([DH92; B, 11.8]). Then E has a elementary abelian normal p-subgroup $\mathrm{A}_p(G)$, the Frattini p-module of G, such that $E/\mathrm{A}_p(G) \cong G$. $\mathrm{A}_p(G)$ can be regarded as a $\mathrm{GF}(p)G$-module and so viewed we have that $\operatorname{Ker}\Big(G \text{ on } \operatorname{Soc}\big(\mathrm{A}_p(G)\big)\Big) = \mathrm{O}_{p',p}(G)$ (cf. [DH92; Appendix β]). In this case $\mathrm{O}_{p',p}(G) = 1$ and, therefore, there exists an irreducible $\mathrm{GF}(p)G$-submodule of $\mathrm{A}_p(G)$, say T, such that $\operatorname{C}_G(T) = 1$.

Note that $E \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$. This means that $\Phi_{\mathfrak{X}}(E) = \Phi(E) = \mathrm{A}_p(G)$ and, therefore, $E/\Phi_{\mathfrak{X}}(E) \cong G \in \mathfrak{F}$. Since $\mathfrak{F}$ is $\mathfrak{X}$-saturated, it follows that $E \in \mathfrak{F}$. We have that T is an abelian $\mathfrak{X}_p$-chief factor of E such that $E/\operatorname{C}_E(T) \cong G$ and $E \in \mathfrak{F}$. Hence we can assume that H_i/K_i is abelian for all i.

By [DH92; IV, 1.5], it follows that $[H_i/K_i]\big(G_i/\operatorname{C}_{G_i}(H_i/K_i)\big) \in \mathfrak{F}$. Consequently, $[H_i/K_i](X/X_i) \in \mathfrak{F}$.

Consider now $W := [H_i/K_i]X$. We have that $W/(H_i/K_i) \in \mathfrak{F}$ and $W/X_i \in \mathfrak{F}$. Therefore $W \in \mathrm{R}_0\,\mathfrak{F} = \mathfrak{F}$.

Write $M := (H_1/K_1) \times (H_2/K_2) \times \cdots \times (H_n/K_n)$. We have that H_i/K_i is a faithful $G_i/\operatorname{C}_{G_i}(H_i/K_i)$-module over $GF(p)$. Since $G_i/\operatorname{C}_{G_i}(H_i/K_i) \cong X/X_i$, we obtain that H_i/K_i is a $GF(p)X$-module and $\operatorname{C}_X(H_i/K_i) = X_i$. Since $\bigcap_{i=1}^{n} X_i = 1$, it follows that M is a faithful X-module over $GF(p)$. Moreover, $[M]X \in \mathrm{R}_0\,\mathfrak{F} = \mathfrak{F}$, as desired. □

Theorem 1.4.6 ([BBCER05; Theorem 3.4])**.** *If $\mathfrak{F}$ is an $\mathfrak{X}$-saturated formation, then $\mathfrak{F}$ is $\mathfrak{X}$-local.*

Proof. Let f be the $\mathfrak{X}$-formation function defined as

$$f(X) = \begin{cases} \mathrm{Q}\,\mathrm{R}_0\big(G/\operatorname{C}_G(H/K) \mid G \in \mathfrak{F} \text{ and } H/K \text{ is an } \mathfrak{X}_p\text{-chief factor of } G\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } X = p \in \operatorname{char}\mathfrak{X}, \\ \mathfrak{F} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } X \in \mathfrak{X}'. \end{cases}$$

It is clear that $\mathfrak{F} \subseteq \mathrm{LF}_{\mathfrak{X}}(f)$. Suppose that $\mathfrak{F} \neq \mathrm{LF}_{\mathfrak{X}}(f)$ and take a minimal group in $\mathrm{LF}_{\mathfrak{X}}(f) \setminus \mathfrak{F}$. Clearly, G is a monolithic group and $N := \operatorname{Soc}(G)$ is an abelian $\mathfrak{X}$-chief factor of G. Let p be the prime dividing the order of N. Thus, $f(p) \neq \emptyset$ and we can take a group

$$Y \in \mathrm{R}_0\big(G/\operatorname{C}_G(H/K) \mid G \in \mathfrak{F} \text{ and } H/K \text{ is an } \mathfrak{X}_p\text{-chief factor of } G\big).$$

Consider the trivial $\mathrm{GF}(p)Y$-module V. By Lemma 1.4.5, we have that $[V]Y \cong V \times Y \in \mathfrak{F}$ and thus $C_p \cong V \in \mathfrak{F}$.

Now we distinguish two cases:

- If $\operatorname{C}_G(N) = N$, we have that $G/N \in f(p)$. Therefore there exist $X \in \mathrm{R}_0\big(G/\operatorname{C}_G(H/K) \mid G \in \mathfrak{F}$ and H/K is an $\mathfrak{X}_p$-chief factor of $G\big)$ and $T \trianglelefteq X$ such that $G/N \cong X/T$.

  N can be regarded as an irreducible $\mathrm{GF}(p)G$-module and as an irreducible $\mathrm{GF}(p)X$-module. By Lemma 1.4.5, we have that $[N]X \in \mathfrak{F}$. This means that $[N](X/T)$ and $[N](G/N)$ also belong to $\mathfrak{F}$. We can now apply Lemma 1.4.4 to obtain that $G \in \mathfrak{F}$, a contradicction.

- Assume now that $N < \operatorname{C}_G(N)$. By [DH92; IV, 1.5], the group $B := [N](G/N)$ belongs to $\mathrm{LF}_{\mathfrak{X}}(f)$.

  Consider $M := \operatorname{C}_B(N) \cap (G/N)$. Since $M \neq 1$, the minimality of G implies that $B/M \in \mathfrak{F}$. Moreover, since $B/N \in \mathfrak{F}$ and $M \cap N = 1$, we have that $B \in \mathrm{R}_0\,\mathfrak{F} = \mathfrak{F}$. By Lemma 1.4.4, we obtain that $G \in \mathfrak{F}$, a contradiction.

  We have proved that $\mathrm{LF}_{\mathfrak{X}}(f) \subseteq \mathfrak{F}$ and, thus, $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$. □

Förster's main result ([För85]) and Theorem 1.4.6 can be combined to get the following theorem:

Theorem 1.4.7. *Let $\mathfrak{F}$ be a formation and $\mathfrak{X}$ a class of simple groups. The following statements are pairwise equivalent:*

1. *$\mathfrak{F}$ is $\mathfrak{X}$-local.*
2. *$\mathfrak{F}$ is $\mathfrak{X}$-saturated (F).*

3. $\mathfrak{F}$ *is* $\mathfrak{X}$*-saturated.*

The equivalence shown in Theorem 1.4.7 will be used in the rest of the thesis without further reference.

Corollary 1.4.8. *A formation* $\mathfrak{F}$ *is* $\mathfrak{X}$*-saturated if and only if* $\mathfrak{F}$ *is* $\mathfrak{X}_p$*-saturated for every* $p \in \operatorname{char}\mathfrak{X}$*. In particular, a formation* $\mathfrak{F}$ *is solubly saturated if and only if* $\mathfrak{F}$ *is* (C_p)*-saturated for every prime* p*.*

Proof. It follows from Theorem 1.4.6 and Corollary 1.2.31. □

Corollary 1.4.9. *Let* F *be a formation. The following statements are equivalent:*

- $\mathfrak{F}$ *is solubly saturated*
- $\mathfrak{F}$ *is nilpotently saturated*

Proof. If $\mathfrak{F}$ is solubly saturated, it is clear that $\mathfrak{F}$ is nilpotently saturated, since $\Phi\big(\mathrm{F}(G)\big) \leq \Phi(G_{\mathfrak{S}})$ for every group G.

Assume that $\mathfrak{F}$ is nilpotently saturated. Then for every prime number p and every group G, it follows that $G \in \mathfrak{F}$ whenever $G/\Phi\big(\mathrm{O}_p(G)\big) \in \mathfrak{F}$, since $\Phi\big(\mathrm{O}_p(G)\big) \leq \Phi\big(\mathrm{F}(G)\big)$. Therefore, $\mathfrak{F}$ is (C_p)-saturated for every prime p. By Corollary 1.4.8, $\mathfrak{F}$ is solubly saturated. □

Lemma 1.4.10. *Let* G *be a group and* M *a minimal normal subgroup of* G*. Then there exists a normal subgroup* N *of* G *such that* $N \cap M = 1$ *and* G/N *is monolithic with socle* MN/N*.*

Proof. Consider the family

$$\mathcal{T}_M := \{T \trianglelefteq G \mid T \cap M = 1\} \neq \emptyset$$

Take an element N of $\mathcal{T}_M$ of maximal order. Next we prove that G/N is monolithic. Clearly, MN/N is a minimal normal subgroup of G/N. Now assume that T/N is a minimal normal subgroup of G/N and $T \neq MN$. If $T \cap M \neq 1$, we would have that $M \leq T$ and then $MN/N \leq T/N$ and $T = MN$, which is a contradiction. Therefore, $T \in \mathcal{T}_M$, contradicting the choice of N. □

Remark 1.4.11. Note that the group found in Example 1.3.20 is not monolithic. Bearing this in mind, it would be a nice idea to modify the definition of the subgroup $\Phi_{\mathfrak{X}}(G)$ of a group G in the following way:

$$\Phi_{\mathfrak{X}}(G) := \begin{cases} \Phi(G) & \text{if } G \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2) \text{ for a prime } p \in \operatorname{char}\mathfrak{X} \\ & \qquad\qquad \text{and } G \text{ is monolithic,} \\ \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) & \text{otherwise.} \end{cases}$$

This new $\mathfrak{X}$-Frattini subgroup would be different from $\Phi_{\mathfrak{X}}(G)$ and closer to $\Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr)$, as Example 1.3.20 proves. This remark is devoted to show that the $\mathfrak{X}$-saturated formations (with the new definition) are also $\mathfrak{X}$-local formations. This is the approach presented in [BBE06; Section 3.2]. We only need to confirm that Lemma 1.4.5 still holds, since there the definition of $\Phi_{\mathfrak{X}}(G)$ was used.

In the proof of Lemma 1.4.5 we found a group E in $\mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ having a minimal normal subgroup T such that $T \leq \mathrm{A}_p(G)$ and $\mathrm{C}_G(T) = 1$. If E is not monolithic, by Lemma 1.4.10 there exists a normal subgroup N of E such that $N \cap T = 1$ and $\overline{E} := E/N$ is monolithic with socle NT/N. Moreover, $N \leq \mathrm{C}_E(T) = \mathrm{A}_p(G) = \Phi(E)$. Therefore, $\Phi(\overline{E}) = \Phi(E)/N$ and hence $\Phi(\overline{E})$ is p-elementary abelian. Moreover, $\overline{E}/\Phi(\overline{E}) \cong E/\Phi(E)$ is a primitive group of type 2. Since $\mathrm{C}_G(T) = 1$, we have that $\mathrm{C}_G(NT/N) = 1$ and, therefore, $\mathrm{C}_{\overline{E}}(NT/N) = \Phi(\overline{E})$. Consequently, $\overline{E}$ is a monolithic group in the class $\mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ and the proof can be continued by using $\overline{E}$ instead of E.

Considering the modified definition of $\Phi_{\mathfrak{X}}(G)$, we have that $\Phi_{\mathfrak{X}}(G) = \Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr)$ if $G \notin \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ for $p \in \operatorname{char}\mathfrak{X}$ or G is not monolithic. However, we do not know whether in monolithic groups belonging to $\mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ for some $p \in \operatorname{char}\mathfrak{X}$ the above equality holds. This raises the following question:

Open question 1.4.12. *Let $\mathfrak{X}$ be a class of simple groups such that* $\operatorname{char}\mathfrak{X} = \pi(\mathfrak{X})$ *and let* $p \in \operatorname{char}\mathfrak{X}$. *If* $G \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ *and* G *is monolithic, is it true that* $\Phi(G) = \Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr)$*?*

Chapter 2

Relation between $\mathfrak{X}$-saturated and ω-saturated formations

2.1 General results

The concepts of $\mathfrak{X}$-local formation and ω-local formation have been defined in Chapter 1. They are both an approach to the notion of local formation. The main aim of this chapter is to give a detailed account of the relation between these two kinds of partial saturation. The first results that we present can be found in [BBCER03].

Let us start with this lemma:

Lemma 2.1.1. *Let ω be a set of primes. If $\mathfrak{F}$ is an ω-saturated formation, then $\mathfrak{F}$ is $\mathfrak{X}_\omega$-saturated, where $\mathfrak{X}_\omega$ is the class of all simple ω-groups.*

Proof. Consider a group G such that $G/\Phi_{\mathfrak{X}_\omega}(G) \in \mathfrak{F}$. Clearly, $\Phi_{\mathfrak{X}_\omega}(G) \leq \Phi(G)$. Moreover, $\Phi_{\mathfrak{X}_\omega}(G) \leq \mathrm{O}_\omega(G)$, since $\mathrm{char}(\mathfrak{X}_\omega) = \omega$. This implies that $\Phi_{\mathfrak{X}_\omega}(G) \leq \Phi(G) \cap \mathrm{O}_\omega(G)$ and, hence, $G/\Phi(G) \cap \mathrm{O}_\omega(G) \in \mathrm{Q}\big(G/\Phi_{\mathfrak{X}_\omega}(G)\big) \subseteq \mathrm{Q}(\mathfrak{F}) = \mathfrak{F}$. Since $\mathfrak{F}$ is ω-saturated, we conclude that $G \in \mathfrak{F}$. Therefore, $\mathfrak{F}$ is $\mathfrak{X}_\omega$-saturated. $\square$

However, the family of $\mathfrak{X}_\omega$-saturated formations does not coincide with the one of ω-saturated formations in general. This follows from the fact that there exist Baer formations which are not ω-saturated for any $\omega \subseteq \mathbb{P}$, as Example 2.1.2 shows.

Example 2.1.2. Let us consider the formation $\mathfrak{F} = \mathfrak{E}_{\mathfrak{Y}}$, where $\mathfrak{Y} = \big(\mathrm{Alt}(n) \mid n \geq 5\big)$, i. e., the formation of all finite groups whose composition factors are isomorphic to an alternating group of degree $n \geq 5$. Next we prove that $\mathfrak{F}$ is a Baer formation, that is, a solubly saturated formation (see Theorem 1.1.7).

If G is a group such that $G/\Phi(G_{\mathfrak{S}}) \in \mathfrak{F}$, it follows that $G_{\mathfrak{S}} = \Phi(G_{\mathfrak{S}})$ (otherwise, $G/\Phi(G_{\mathfrak{S}})$ would have abelian chief factors, contradicting the fact that $G/\Phi(G_{\mathfrak{S}}) \in \mathfrak{F}$). Therefore, $G_{\mathfrak{S}} = 1$ and $G \in \mathfrak{F}$. Hence $\mathfrak{F}$ is a Baer formation. In particular, $\mathfrak{F}$ is $\mathfrak{X}$-saturated for every $\mathfrak{X} \subseteq \mathbb{P}$.

Assume that $\mathfrak{F}$ is p-saturated for a prime p. If $p \geq 5$, set $k = p$; otherwise, set $k = 5$. As p divides $|\mathrm{Alt}(k)|$, by [DH92; B, 11.8] there exists a group E with a normal elementary abelian p-subgroup $A \neq 1$ such that $A \leq \Phi(E)$ and $E/A \cong \mathrm{Alt}(k)$. We have that $E/\big(\mathrm{O}_p(E) \cap \Phi(E)\big) = E/\big(\mathrm{O}_p(E) \cap A\big) = E/A \in \mathfrak{F}$. Therefore $E \in \mathfrak{F}$, a contradiction.

Therefore $\mathfrak{F}$ is not ω-saturated for any set ω of primes. Moreover, by setting $\mathfrak{X} = (C_2)$ and $\omega = \{2\}$, we have that $\mathfrak{F}$ is $\mathfrak{X}$-saturated, but not 2-saturated.

From the above discussion, the following question naturally arises:

> *Let $\omega \subseteq \mathbb{P}$. Is it possible to ensure the existence of a class $\mathfrak{X}(\omega)$ of simple groups such that $\omega \subseteq \operatorname{char} \mathfrak{X}(\omega) = \pi\big(\mathfrak{X}(\omega)\big)$ satisfying that a formation is ω-saturated if and only if it is $\mathfrak{X}(\omega)$-saturated?*

The following example shows that the answer is negative.

Example 2.1.3. Consider the formation

$$\mathfrak{F} = (G \mid \text{all abelian composition factors of } G \text{ are isomorphic to } C_2).$$

Next we prove that $\mathfrak{F}$ is $\mathfrak{X}$-saturated exactly for the classes $\mathfrak{X}$ such that $\mathfrak{X} \subseteq \mathbb{P}$.

1. *$\mathfrak{F}$ is solubly saturated. Therefore, $\mathfrak{F}$ is $\mathfrak{X}$-saturated for the classes $\mathfrak{X}$ such that $\mathfrak{X} \subseteq \mathbb{P}$.*

   If $G/\Phi(G_{\mathfrak{S}}) \in \mathfrak{F}$, we have that $G_{\mathfrak{S}}/\Phi(G_{\mathfrak{S}})$ is a 2-group, since its composition factors are abelian. Therefore, $G_{\mathfrak{S}}$ is a 2-group and hence every abelian composition factor of G is isomorphic to C_2.

2. *If $\mathfrak{X}$ is a class of simple groups such that $\mathfrak{F}$ is $\mathfrak{X}$-saturated, then $\mathfrak{X}$ only contains abelian simple groups.*

   Suppose that $\mathfrak{F}$ is $\mathfrak{X}$-saturated for a class $\mathfrak{X}$ containing a non-abelian simple group E and $\pi(\mathfrak{X}) = \operatorname{char} \mathfrak{X}$. There exists a prime $p \neq 2$ dividing the order of E. Hence $p \in \mathfrak{X}$. Since $E \in \mathfrak{F}$, it follows that $f(p) \neq \emptyset$. Therefore $C_p \in \mathfrak{F}$, a contradiction.

Since $\mathfrak{F}$ is clearly 2-saturated, if we assume the existence of a class $\mathfrak{X}(2)$ fulfilling the property, it follows that $\mathfrak{X}(2) \subseteq \mathbb{P}$. This implies that the formation considered in Example 2.1.2 is 2-saturated, but we have proved that this is not true.

However, inside the ω-separable universe, the situation is clearer.

Lemma 2.1.4. *Consider a set of primes ω and a formation $\mathfrak{F}$. If G is a group of minimal order satisfying that $G/\big(\Phi(G)\cap \mathrm{O}_\omega(G)\big)\in\mathfrak{F}$ and $G\notin\mathfrak{F}$, then G is monolithic.*

Proof. If A is a minimal normal subgroup of G, we have that $G/\big(\mathrm{O}_\omega(G)A\cap \Phi(G)A\big)\in \mathrm{Q}\,\mathfrak{F}=\mathfrak{F}$. Therefore, $(G/A)/\Big(\big(\mathrm{O}_\omega(G)A/A\big)\cap\big(\Phi(G)A/A\big)\Big)\in\mathfrak{F}$. Since $\mathrm{O}_\omega(G)A/A\le \mathrm{O}_\omega(G/A)$ and $\Phi(G)A/A\le\Phi(G/A)$, it follows that $(G/A)/\big(\mathrm{O}_\omega(G/A)\cap\Phi(G/A)\big)\in\mathrm{Q}\,\mathfrak{F}=\mathfrak{F}$. By the choice of G, we have that $G/A\in\mathfrak{F}$. If B is a minimal normal subgroup of G different from A, we can argue the same way to obtain that $G/B\in\mathfrak{F}$. Therefore, $G\cong G/(A\cap B)\in \mathrm{R}_0\,\mathfrak{F}=\mathfrak{F}$, contradicting the minimality of G. □

Theorem 2.1.5. *Let ω be a set of primes and let $\mathfrak{X}_\omega$ be the class of all simple ω-groups. If $\mathfrak{F}$ is an $\mathfrak{X}_\omega$-saturated formation composed of ω-separable groups, then $\mathfrak{F}$ is ω-saturated.*

Proof. Consider an $\mathfrak{X}_\omega$-formation function f such that $\mathfrak{F}=\mathrm{LF}_{\mathfrak{X}_\omega}(f)$.

If $\mathfrak{F}$ is not ω-saturated, there exists a group G such that $G/\big(\Phi(G)\cap \mathrm{O}_p(G)\big)\in\mathfrak{F}$ but $G\notin\mathfrak{F}$ for a prime $p\in\omega$. We choose one of minimal order. By Lemma 2.1.4, G has a unique minimal normal subgroup N. It follows that $G/N\in\mathfrak{F}$ and $N\le\Phi(G)\cap\mathrm{O}_p(G)$. Note that if p divides the order of a chief factor H/K of G, then H/K is an $(\mathfrak{X}_\omega)_p$-chief factor of G, since G is ω-separable. Therefore, the intersection of the centralisers of the $(\mathfrak{X}_\omega)_p$-chief factors of G is, in this case, $\mathrm{O}_{p',p}(G)$ (see [DH92; A, 13.8]).

Since $G/N\in\mathfrak{F}$, we obtain that $G/N/\,\mathrm{O}_{p',p}(G/N)\in f(p)$ (note that $p\in\pi(G/N)$ and hence $f(p)\neq\emptyset$). Since $N\le\Phi(G)$, it follows by [Hup67; VI, 6.3] that $\mathrm{O}_{p',p}(G/N)=\mathrm{O}_{p',p}(G)/N$ and, therefore, $G/\,\mathrm{O}_{p',p}(G)\in f(p)$. In particular, we have that $G/\,\mathrm{C}_G(N)\in f(p)$. By Lemma 1.2.12, this implies that $G\in\mathfrak{F}$, contradicting the choice of G. □

Corollary 2.1.6. *Let $\mathfrak{F}$ be a formation composed of ω-separable groups. Then $\mathfrak{F}$ is ω-saturated if and only if $\mathfrak{F}$ is $\mathfrak{X}_\omega$-saturated, where $\mathfrak{X}_\omega$ is the class of all simple ω-groups.*

The following theorem shows that an $\mathfrak{X}$-local formation always contains a largest ω-local formation for $\omega=\operatorname{char}\mathfrak{X}$. It appears in [BBE06; Chapter 3] and [BBCSss].

Theorem 2.1.7. *Let $\mathfrak{X}$ be a class of simple groups such that $\omega=\operatorname{char}\mathfrak{X}=\pi(\mathfrak{X})$. Let $\mathfrak{F}=\mathrm{LF}_{\mathfrak{X}}(F)$ be an $\mathfrak{X}$-local formation. Then the ω-local formation $\mathfrak{F}_\omega=\mathrm{LF}_\omega(f)$, where $f(p)=F(p)$ for every $p\in\omega$ and $f(\omega')=\mathfrak{F}$, is the largest ω-local formation contained in $\mathfrak{F}$.*

Proof. Suppose, for a contradiction, that $\mathfrak{F}_\omega$ is not contained in $\mathfrak{F}$. Let G be a group of minimal order in $\mathfrak{F}_\omega \setminus \mathfrak{F}$. Then, as usual, G has a unique minimal normal subgroup N, and $G/N \in \mathfrak{F}$. If $G_{\omega d} = 1$, we would have that $G \in f(\omega') = \mathfrak{F}$, contradicting the choice of G. Assume that $G_{\omega d} \neq 1$. Then N is contained in $G_{\omega d}$. This means that there exists a prime $p \in \omega$ dividing $|N|$. Hence $G/\operatorname{C}_G(N) \in f(p) = F(p)$. If N is a p-group, it follows that N is an $\mathfrak{X}$-chief factor of G. By Lemma 1.2.12, we conclude that $G \in \operatorname{LF}_{\mathfrak{X}}(F) = \mathfrak{F}$, against the choice of G. Hence N is non-abelian and so $\operatorname{C}_G(N) = 1$ and $G \in F(p)$. Since $F(p) = \mathfrak{S}_p \underline{f}(p)$ and $\operatorname{O}_p(G) = 1$, it follows that $G \in \underline{f}(p) \subseteq \mathfrak{F}$. This contradiction proves that $\mathfrak{F}_\omega \subseteq \mathfrak{F}$.

Now let $\mathfrak{G} = \operatorname{LF}_\omega(g)$ be an ω-local formation contained in $\mathfrak{F}$. Suppose, if possible, that $\mathfrak{G}$ is not contained in $\mathfrak{F}_\omega$ and let A be a group of minimal order in the supposed non-empty class $\mathfrak{G} \setminus \mathfrak{F}_\omega$. Then A has a unique minimal normal subgroup B, and $A/B \in \mathfrak{F}_\omega$. Since $A \in \mathfrak{G} \subseteq \mathfrak{F}$, we have that $A/A_{\omega d} \in \mathfrak{F} = f(\omega')$. Suppose that $p \in \omega \cap \pi(B)$. If B is an $\mathfrak{X}$-chief factor of A, it follows that $A/\operatorname{C}_A(B) \in F(p) = f(p)$. If B is an $\mathfrak{X}'$-chief factor of A, then B is non-abelian and $A \cong A/\operatorname{C}_A(B) \in g(p)$. Then $\operatorname{O}_p(A) = 1$ and so, by [DH92; B, 10.9], A has a faithful irreducible representation over $\operatorname{GF}(p)$. Let M be the corresponding module and $G = [M]A$ the corresponding semidirect product. Let us see that $G \in \mathfrak{G}$. Since M is contained in $G_{\omega d}$, it follows that $G/G_{\omega d} \in g(\omega')$ because $A/A_{\omega d} \in g(\omega')$. Moreover, we have that $G/\operatorname{C}_G(M) \cong A \in g(p)$. We conclude that $G \in \mathfrak{G}$ and, consequently, $G = [M]A \in \mathfrak{F}$. This implies that $A \cong G/\operatorname{C}_G(M) \in f(p)$. Now we can state that $A \in \mathfrak{F}_\omega$, contradicting the choice of A. Therefore $\mathfrak{G}$ is contained in $\mathfrak{F}_\omega$. □

As an immediate application of Theorem 2.1.7, we find an alternative proof of Theorem 2.1.5.

Corollary 2.1.8. *Let ω be a set of primes and let $\mathfrak{X}_\omega$ be the class of all simple ω-groups. If $\mathfrak{F}$ is an $\mathfrak{X}_\omega$-local formation composed of ω-separable groups, then $\mathfrak{F}$ is ω-local.*

Proof. Suppose that $\mathfrak{F}$ is an $\mathfrak{X}_\omega$-local formation. According to Theorem 2.1.7, $\mathfrak{F} = \operatorname{LF}_{\mathfrak{X}_\omega}(F)$ contains a largest ω-local formation $\mathfrak{F}_\omega = \operatorname{LF}_\omega(f)$, where $f(p) = F(p)$ for every $p \in \omega$ and $f(\omega') = \mathfrak{F}$. Suppose that the inclusion is proper, and let G be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{F}_\omega$. Then G has a unique minimal normal subgroup N, and $G/N \in \mathfrak{F}_\omega$. It is clear that $G/G_{\omega d} \in f(\omega') = \mathfrak{F}$. If $p \in \pi(N) \cap \omega$, it follows that N is an ω-group, since G is ω-separable. Hence, N is an $\mathfrak{X}_\omega$-chief factor of G and, therefore, $G/\operatorname{C}_G(N) \in F(p) = f(p)$. Taking into account that $G/N \in \mathfrak{F}_\omega$, we conclude that $G \in \mathfrak{F}_\omega$ by Lemma 1.2.12. This contradiction proves that $\mathfrak{F} = \mathfrak{F}_\omega$ is ω-local. □

The following consequence of Theorem 2.1.7 is of interest.

Corollary 2.1.9 ([Sal83; Satz 5.6])**.** *Every Baer-local formation contains a maximal local formation with respect to inclusion.*

There exist formations not containing a maximal local formation as Example 2.1.10 shows:

Example 2.1.10. [Sal83] Let $\mathfrak{F}$ be the class of all soluble groups G such that the Sylow subgroups of G corresponding to different primes permute. By [Hup67; VI, 3.2], $\mathfrak{F}$ is a formation. Let q be a prime and consider the formation function f_q given by $f_q(p) = \mathfrak{S}_{\{p,q\}}$ for every $p \in \mathbb{P}$. Then the local formation $\mathfrak{F}_q = \mathrm{LF}(f_q)$ is contained in $\mathfrak{F}$ by [Hup67; VI, 3.1]. Let q_1 and q_2 be two different primes and let $\mathfrak{F}_{q_1,q_2}$ be the smallest local formation containing $\mathfrak{F}_{q_1}$ and $\mathfrak{F}_{q_2}$. Then $C_{q_1} \times C_{q_2} \in F(p)$ for every $p \in \mathbb{P}$, where F is the canonical local definition of $\mathfrak{F}_{q_1,q_2}$. This is due to the fact that $C_{q_1} \in F_{q_1}(p)$ and $C_{q_2} \in F_{q_2}(p)$, where F_{q_1} and F_{q_2} are the canonical local definitions of $\mathfrak{F}_{q_1}$ and $\mathfrak{F}_{q_2}$, respectively. Let q_3 be a prime, $q_3 \neq q_1$, q_2. By [DH92; B, 10.9], $C_{q_1} \times C_{q_2}$ has an irreducible and faithful module M over $\mathrm{GF}(q_3)$. Let $G = [M](C_{q_1} \times C_{q_2})$ be the corresponding semidirect product. Then $G \in \mathfrak{F}_{q_1,q_2}$, but $G \notin \mathfrak{F}$. This shows that $\mathfrak{F}$ does not contain a maximal local formation with respect to the inclusion.

The following example shows that the converse of Corollary 2.1.9 does not hold, that is, Baer-local formations are not characterised as the formations containing a maximal local formation.

Example 2.1.11. Consider the formation $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ is the formation $\mathrm{D}_0\big(1, \mathrm{Alt}(5)\big)$ composed of all groups that are direct products of copies of $\mathrm{Alt}(5)$ together with the trivial group and $\mathfrak{G} = \mathfrak{S}_2$. It is shown in [Sal83; 6.1] that $\mathfrak{H}$ is not a Baer-local formation. However $\mathfrak{S}_2$ is the maximal saturated formation contained in $\mathfrak{H}$.

A natural question arising from the above results is the following:

> *What are the precise conditions to ensure that an $\mathfrak{X}$-local formation is ω-local for $\omega = \mathrm{char}\,\mathfrak{X}$?*

The next result, which appears in [BBCSss], gives the answer.

Theorem 2.1.12. *Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$ be an $\mathfrak{X}$-local formation and $\omega = \mathrm{char}\,\mathfrak{X}$. The following conditions are pairwise equivalent:*

1. *$\mathfrak{F}$ is ω-local.*

2. *$G/\operatorname{c}_G(H/K) \in \underline{f}(p)$ for every $G \in \mathfrak{F}$ and every $\mathfrak{X}'$-chief factor H/K of G such that $p \in \pi(H/K) \cap \omega$.*

3. *$\underline{f}(S) \subseteq \underline{f}(p)$ for every $S \in \mathfrak{X}'$ and $p \in \pi(S) \cap \omega$.*

4. *$\mathfrak{S}_p\underline{f}(S) \subseteq \mathfrak{F}$ for every $S \in \mathfrak{X}'$ and $p \in \pi(S) \cap \omega$.*

Proof. 1 implies 2. Suppose that $\mathfrak{F}$ is ω-local. Then, by Theorem 2.1.7, $\mathfrak{F} = \mathrm{LF}_\omega(f)$, where

$$\begin{cases} f(p) = \mathfrak{S}_p\underline{f}(p) & \text{if } p \in \omega, \\ f(\omega') = \mathfrak{F}. \end{cases}$$

Assume that a group $G \in \mathfrak{F}$ has a chief factor H/K such that $H/K \cong S \times \cdots \times S$, where $S \in \mathfrak{X}'$ and the order of S is divisible by a prime $p \in \omega$. Evidently, S is non-abelian. Set $C = \operatorname{c}_G(H/K)$. It is clear that $HC/C \cong H/K$. Since $\operatorname{c}_G(HC/C) = C$, we have by Lemma 1.2.41 that $\operatorname{c}_{G/C}(HC/C) = 1$. Hence the socle of $\overline{G} = G/C$ is a direct product of non-abelian minimal normal subgroups:

$$\operatorname{Soc}(\overline{G}) = \overline{H}_0 \times \overline{H}_1 \times \cdots \times \overline{H}_t,$$

where $\overline{H}_i \in \mathrm{E}(S)$ for any i (the case $t = 0$ is possible). Since all $\overline{H}_i$ are f-central in G, then

$$\overline{G}/\operatorname{C}_{\overline{G}}(\overline{H}_i) \in f(p) = \mathfrak{S}_p\underline{f}(p).$$

Since $\operatorname{O}_p(\overline{H}_i) = 1$, it follows that actually $\overline{G}/\operatorname{C}_{\overline{G}}(\overline{H}_i) \in \underline{f}(p)$. But then $\overline{G}/\overline{D} \in \underline{f}(p)$, where $\overline{D} = \bigcap_i \operatorname{C}_{\overline{G}}(\overline{H}_i)$. The subgroup $\overline{D}$ centralizes $\operatorname{Soc}(\overline{G})$, and since $\operatorname{F}(\overline{G}) = 1$, we have that $\overline{D} = 1$ by the well-known Schmid's theorem (see [Sch72]). We have that $\overline{G} = G/C$ belongs to $\underline{f}(p)$.

2 implies 3. Consider $S \in \mathfrak{X}'$ and $p \in \pi(S) \cap \omega$. By Theorem 1.2.15,

$$\underline{f}(S) = \operatorname{form}\big(G/L \mid G \in \mathfrak{F},\, G/L \text{ is monolithic, and } \operatorname{Soc}(G/L) \in \mathrm{E}(S)\big).$$

Let G be a group in $\mathfrak{F}$ and let L be a normal subgroup of G such that G/L is monolithic and $T/L := \operatorname{Soc}(G/L) \in \mathrm{E}(S)$. We have that $G/\operatorname{c}_G(T/L) \in \underline{f}(p)$. Clearly, $\operatorname{c}_G(T/L) = L$ and, therefore, $G/L \in \underline{f}(p)$. We have proved that $\underline{f}(S) \subseteq \underline{f}(p)$.

3 implies 4. Let $S \in \mathfrak{X}'$ and $p \in \pi(S) \cap \omega$. Then $\mathfrak{S}_p\underline{f}(S) \subseteq \mathfrak{S}_p\underline{f}(p) = F(p) \subseteq \mathfrak{F}$.

4 implies 1. Applying Theorems 2.1.7 and 1.2.21, it is known that $\mathfrak{F}_\omega = \mathrm{LF}_\omega(f)$, where

$$\begin{cases} f(p) = \mathfrak{S}_p \underline{f}(p) & \text{if } p \in \omega, \\ f(\omega') = \mathfrak{F}. \end{cases}$$

is the largest ω-local formation contained in $\mathfrak{F}$. Suppose, by way of contradiction, that $\mathfrak{F}$ is not ω-local. Then $\mathfrak{F}_\omega \neq \mathfrak{F}$. Let G be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{F}_\omega$. By a familiar argument, G has a unique minimal normal subgroup N, and $G/N \in \mathfrak{F}_\omega$. It is clear that $G/G_{\omega d} \in \mathfrak{F}$. If $\pi(N) \cap \omega = \emptyset$, we would have that $G \in \mathfrak{F}_\omega$, since $G/N \in \mathfrak{F}_\omega$. Therefore $\pi(N) \cap \omega \neq \emptyset$. Let p be a prime in $\pi(N) \cap \omega$. If N is an $\mathfrak{X}_p$-chief factor of G, $G/\mathrm{C}_G(N) \in \underline{f}(p) \subseteq f(p)$. Assume that N is an $\mathfrak{X}'$-chief factor of G and $N \in \mathrm{E}(S)$. Then S is non-abelian and so $\mathrm{O}_p(G) = 1$. By [DH92; B, 10.9], G has an irreducible and faithful module M over $\mathrm{GF}(p)$. Let $Z = [M]G$ be the corresponding semidirect product. Since $G \in \underline{f}(S)$, it follows that $Z \in \mathfrak{S}_p \underline{f}(S) \subseteq \mathfrak{F}$. This implies that $G \cong Z/\mathrm{C}_Z(M) \in \underline{f}(p) \subseteq f(p)$. Consequently $G/\mathrm{C}_G(N) \in f(p)$ for every $p \in \pi(N) \cap \omega$ and $G \in \mathfrak{F}_\omega$. This contradicts our initial supposition. Therefore $\mathfrak{F} = \mathfrak{F}_\omega$ and $\mathfrak{F}$ is ω-local. □

As an application of Theorem 2.1.12, we prove that the formation considered in Example 2.1.2 is not p-local for any prime p.

Example 2.1.13. Consider the formation $\mathfrak{F} = \mathfrak{E}_{\mathfrak{Y}}$, where $\mathfrak{Y} = \big(\mathrm{Alt}(n) \mid n \geq 5\big)$, i. e., the formation of all finite groups whose composition factors are isomorphic to an alternating group of degree $n \geq 5$. It is clear that $\mathfrak{F}$ is a Baer formation, since it is closed under taking extensions by the Frattini subgroup of the soluble radical. In particular, $\mathfrak{F}$ is (C_p)-local for every prime p. Let us see that $\mathfrak{F}$ is not p-local for any prime p. By Corollary 1.2.28,

$$F(p) = \mathrm{form}\big(G/\mathrm{C}_G(H/K) \mid G \in \mathfrak{F} \text{ and } H/K \text{ is an abelian } p\text{-chief factor of } G\big).$$

In this case, $F(p) = \emptyset$, since the groups in $\mathfrak{F}$ do not have abelian chief factors. We have that $\underline{f}(p) \subseteq \mathfrak{S}_p \underline{f}(p) = F(p)$ and, therefore, $\underline{f}(p) = \emptyset$. Consider $n \geq 5$ such that p divides $|\mathrm{Alt}(n)|$. We have that $\mathrm{Alt}(n) \in \mathfrak{X}'$ and

$$\mathrm{Alt}(n) \in \underline{f}\big(\mathrm{Alt}(n)\big) = \mathrm{form}\Big(G/L \mid G \in \mathfrak{F},\ G/L \text{ is monolithic, and } \mathrm{Soc}(G/L) \in \mathrm{E}\big(\mathrm{Alt}(n)\big)\Big).$$

Therefore, $\underline{f}\big(\mathrm{Alt}(n)\big) \neq \emptyset$. Since $\underline{f}\big(\mathrm{Alt}(n)\big) \not\subseteq \underline{f}(p)$, we can apply Theorem 2.1.12 to conclude that $\mathfrak{F}$ is not a p-local formation.

Now additional information about $\mathfrak{X}$-saturated formations will be obtained and some results from [She97] will be deduced. The following results are included in [BBCER05].

In the rest of this chapter, π will denote $\pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$.

Theorem 2.1.14. *Let $\mathfrak{F}$ be a formation.*

1. *If $\operatorname{form}_{\mathfrak{X}}(\mathfrak{F}) \subseteq \mathfrak{E}_{\pi'}\mathfrak{F}$, then $\mathfrak{F}$ is $\mathfrak{X}$-saturated.*
2. *If $\mathfrak{X}$ is a class of abelian simple groups and $\mathfrak{F}$ is $\mathfrak{X}$-saturated, then $\mathfrak{N}_{\pi'}\mathfrak{F}$ is $\mathfrak{X}$-saturated.*

Proof. 1. Assume that $G/\Phi_{\mathfrak{X}}(G) \in \mathfrak{F}$. Since $G \in \operatorname{form}_{\mathfrak{X}}(\mathfrak{F}) \subseteq \mathfrak{E}_{\pi'}\mathfrak{F}$, there exists a normal π'-subgroup M of G such that $G/M \in \mathfrak{F}$. Since $\Phi_{\mathfrak{X}}(G)$ is a π-group, it follows that $M \cap \Phi_{\mathfrak{X}}(G) = 1$ and so $G \in \mathfrak{F}$. Hence $\mathfrak{F}$ is $\mathfrak{X}$-saturated.

2. Assume that $\mathfrak{N}_{\pi'}\mathfrak{F}$ is not $\mathfrak{X}$-saturated and let G be a group of minimal order satisfying $G/\Phi_{\mathfrak{X}}(G) \in \mathfrak{N}_{\pi'}\mathfrak{F}$ and $G \notin \mathfrak{N}_{\pi'}\mathfrak{F}$. If M is a normal subgroup of G, we have that $(G/M)/\Phi_{\mathfrak{X}}(G/M) \in \mathfrak{N}_{\pi'}\mathfrak{F}$, since $\Phi_{\mathfrak{X}}(G)M/M \leq \Phi_{\mathfrak{X}}(G/M)$. This means that G is a monolithic group. Since $N := \operatorname{Soc}(G) \leq \Phi_{\mathfrak{X}}(G)$, we have that N is a π-group.

   Let M be a normal subgroup of G such that $M/\Phi_{\mathfrak{X}}(G) \in \mathfrak{N}_{\pi'}$ and

$$\big(G/\Phi_{\mathfrak{X}}(G)\big)/\big(M/\Phi_{\mathfrak{X}}(G)\big) \cong G/M \in \mathfrak{F}.$$

   Since M is nilpotent by [Hup67; VI, 6.3], we have that $M = \Phi_{\mathfrak{X}}(G) \times \overline{M}$, where $\overline{M}$ is a normal Hall π'-subgroup of M. Since $\mathrm{O}_{\pi'}(G) = 1$, it follows that $M = \Phi_{\mathfrak{X}}(G)$ and $G/\Phi_{\mathfrak{X}}(G) \in \mathfrak{F}$. Therefore $G \in \mathfrak{F} \subseteq \mathfrak{N}_{\pi'}\mathfrak{F}$, a contradiction. □

Corollary 2.1.15. *Assume that $\mathfrak{X}$ is a class of abelian simple groups and let $\mathfrak{F}$ be a formation. The following statements are pairwise equivalent:*

1. *$\mathfrak{F}$ is $\mathfrak{X}$-saturated,*
2. *$\mathfrak{N}_{\pi'}\mathfrak{F}$ is $\mathfrak{X}$-saturated, and*
3. *$\operatorname{form}_{\mathfrak{X}}(\mathfrak{F}) \subseteq \mathfrak{N}_{\pi'}\mathfrak{F}$*

Theorem 2.1.16. *Let $\mathfrak{F}$ be a formation and let σ be a set of primes. Denote by $\mathfrak{Y} := \mathfrak{J} \cap \mathfrak{E}_{\sigma}$, the class of all simple σ-groups. The formation $\mathfrak{N}_{\sigma}\mathfrak{F}$ is $\mathfrak{Y}$-saturated and it can be defined by the following $\mathfrak{Y}$-formation function:*

$$f(p) = \mathfrak{F} \textit{ if } p \in \sigma,$$
$$f(E) = \mathfrak{N}_{\sigma}\mathfrak{F} \textit{ if } E \in \mathfrak{Y}'.$$

Proof. Clearly, $\mathfrak{N}_\sigma\mathfrak{F} \subseteq \mathrm{LF}_{\mathfrak{Y}}(f)$.

Assume that $\mathrm{LF}_{\mathfrak{Y}}(f) \setminus \mathfrak{N}_\sigma\mathfrak{F} \neq \emptyset$ and consider a group G of least order in $\mathrm{LF}_{\mathfrak{Y}}(f) \setminus \mathfrak{N}_\sigma\mathfrak{F}$. Then G is a monolithic group whose socle N is a p-group for a prime $p \in \sigma$. We have that $G/\mathrm{C}_G(N) \in f(p) = \mathfrak{F}$. Moreover, since $G/N \in \mathfrak{N}_\sigma\mathfrak{F}$, there exists a normal subgroup M of G such that $N \leq M$, $M/N \in \mathfrak{N}_\sigma$ and $G/M \in \mathfrak{F}$. We observe that M is a σ-group. If $N \leq \Phi(G)$, we would obtain that M is nilpotent and, therefore, $G \in \mathfrak{N}_\sigma\mathfrak{F}$. This could not be possible. Therefore N is complemented in G and $\mathrm{C}_G(N) = N$. In particular, G belongs to $\mathfrak{N}_\sigma\mathfrak{F}$, a contradiction. Consequently $\mathrm{LF}_{\mathfrak{Y}}(f) \subseteq \mathfrak{N}_\sigma\mathfrak{F}$, as desired. □

Corollary 2.1.17. *Let $\mathfrak{F}$ be a formation. If $\mathfrak{F}$ is $\mathfrak{X}$-saturated, then $\mathfrak{N}_{\pi'}\mathfrak{F}$ is $\overline{\mathfrak{X}}$-local, where $\overline{\mathfrak{X}} := (\mathfrak{X} \cap \mathbb{P}) \cup (\mathfrak{J} \cap \mathfrak{E}_{\pi'})$. In particular, $\mathfrak{N}_{\pi'}\mathfrak{F}$ is a Baer-local formation.*

Proof. Since $\mathfrak{F}$ is $\mathfrak{X}$-saturated, we have that $\mathfrak{F}$ is also $(\mathfrak{X} \cap \mathbb{P})$-local. By Theorem 2.1.14, it follows that $\mathfrak{N}_{\pi'}\mathfrak{F}$ is $(\mathfrak{X} \cap \mathbb{P})$-local, since $\mathrm{char}(\mathfrak{X} \cap \mathbb{P}) = \mathrm{char}\,\mathfrak{X} = \pi$. Moreover, by Theorem 2.1.16, we have that $\mathfrak{N}_{\pi'}\mathfrak{F}$ is $(\mathfrak{J} \cap \mathfrak{E}_{\pi'})$-local. Now, by Corollary 1.2.31, we obtain that $\mathfrak{N}_{\pi'}\mathfrak{F}$ is $\overline{\mathfrak{X}}$-local. Since $\mathrm{char}(\overline{\mathfrak{X}}) = \mathbb{P}$, we have that $\mathfrak{N}_{\pi'}\mathfrak{F}$ is, in particular, a Baer-local formation. □

Corollary 2.1.18. *Let $\mathfrak{F}$ be a formation and $\mathfrak{X}$ a class of abelian simple groups. Then* $\mathrm{bform}(\mathfrak{F}) \subseteq \mathfrak{N}_{\pi'}\mathfrak{F}$ *if and only if $\mathfrak{F}$ is $\mathfrak{X}$-saturated.*

If we take $\mathfrak{X} = (C_p)$ in Corollary 2.1.18, we obtain Shemetkov's main result in [She97] (Theorem 3.2).

Lemma 2.1.19. *Let ω be a set of primes and $\mathfrak{F}$ an ω-saturated formation. Then $\mathfrak{N}_{\omega'}\mathfrak{F}$ is also ω-saturated.*

Proof. Assume that the result is not true and consider a group G of minimal order satisfying $G/\mathrm{O}_\omega(G) \cap \Phi(G) \in \mathfrak{N}_{\omega'}\mathfrak{F}$ and $G \notin \mathfrak{N}_{\omega'}\mathfrak{F}$. Note that G is a monolithic group whose socle is an ω-group. There exists a normal subgroup N_1 of G such that $G/N_1 \in \mathfrak{F}$ and $N_1/\mathrm{O}_\omega(G) \cap \Phi(G) \in \mathfrak{N}_{\omega'}$. Since N_1 is nilpotent by [Hup67; VI, 6.3], we have that $N_1 = \big(\mathrm{O}_\omega(G) \cap \Phi(G)\big) \times N_2$, where N_2 is a normal Hall ω'-subgroup of N_1. Since $\mathrm{O}_{\omega'}(G) = 1$, it follows that $N_2 = 1$ and, hence, $G/\mathrm{O}_\omega(G) \cap \Phi(G) \in \mathfrak{F}$. This implies that $G \in \mathfrak{F} \subseteq \mathfrak{N}_{\omega'}\mathfrak{F}$. This contradiction completes the proof. □

Corollary 2.1.20. *Let ω be a set of primes and let $\mathfrak{F}$ be an ω-saturated formation. Then* $\mathrm{bform}(\mathfrak{F})$ *is ω-saturated.*

Proof. Assume that the result is false and consider a group G of least order such that $G/\operatorname{O}_\omega(G) \cap \Phi(G) \in \operatorname{bform}(\mathfrak{F})$ and $G \notin \operatorname{bform}(\mathfrak{F})$. We have that G is a monolithic group whose socle is an ω-group. We know that $\mathfrak{F}$ is $\mathfrak{X}_\omega$-saturated by Lemma 2.1.1, where $\mathfrak{X}_\omega$ is the class of all simple ω-groups. By Corollary 2.1.17, $\mathfrak{N}_{\omega'}\mathfrak{F}$ is a Baer-local formation. Therefore $G/\operatorname{O}_\omega(G) \cap \Phi(G) \in \mathfrak{N}_{\omega'}\mathfrak{F}$, since $\operatorname{bform}(\mathfrak{F}) \subseteq \mathfrak{N}_{\omega'}\mathfrak{F}$. By Lemma 2.1.19, it follows that $\mathfrak{N}_{\omega'}\mathfrak{F}$ is ω-saturated and, hence, $G \in \mathfrak{N}_{\omega'}\mathfrak{F}$. Bearing in mind that $\operatorname{O}_{\omega'}(G) = 1$, we conclude $G \in \operatorname{bform}(\mathfrak{F})$, a contradiction. □

As a particular case of Corollary 2.1.20, taking $\omega = \{p\}$, we get a result proved by Shemetkov [She97; Theorem 3.1].

2.2 Some remarks on a result of Shemetkov

In this section, the contents of which appear in [BBCSss], we give an extension of a result of Shemetkov on p-local formations (the main result of [She03]). In that paper, Shemetkov studies direct decompositions of the finite-dimensional FG-module, where F is a field of characteristic $p > 0$ and G is a group. He introduces the concept of f-centrality in the following way:

Definition 2.2.1. Let $\mathfrak{F}$ be a non-empty p-local formation and let f be a p-local satellite of $\mathfrak{F}$. Let V be a finite-dimensional FG-module, where G is a finite group and F is a field of characteristic $p > 0$. A composition factor R/S of V is called

- *f-central* if $G/\operatorname{Ker}(G \,\mathrm{on}\, R/S) \in f(p)$.
- *f-eccentric* if R/S is not f-central.

Shemetkov proves the following theorem:

Theorem 2.2.2 ([She03; Theorem 4.1])**.** *Let V be a finite-dimensional FG-module, where F is a field of characteristic $p > 0$ and G is a finite group in $\mathfrak{F} = \operatorname{LF}_p(f)$. Then V can be represented as the direct sum $V = V_1 \oplus V_2$, where the modules V_1 and V_2 posses the following properties:*

1. *Every factor of a composition series of V_1 is f-central.*
2. *Every factor of a composition series of V_2 is f-eccentric.*

Theorem 2.2.2 is a consequence of a more general result:

Theorem 2.2.3. *Let V be a finite-dimensional FG-module, where F is a field of characteristic $p > 0$ and G is a finite group. Let N be a normal p'-subgroup of G. Then V can be represented as the direct sum $V = V_1 \oplus V_2$, where the modules V_1 and V_2 possess the following properties:*

1. $\operatorname{Ker}(N \operatorname{on} V_1) = N$.
2. *Every FG-composition factor R/S of V_2 satisfies that*

$$\operatorname{Ker}(N \operatorname{on} R/S) \neq N.$$

Proof. Regard V as an FN-module. By Maschke's theorem (see [DH92; B, 4.5]), V_N is a completely reducible module. Consider the following sums:

$$V_1 = \sum\{U \mid U \text{ is an irreducible } FN\text{-submodule of } V_N \text{ and } \operatorname{Ker}(N \operatorname{on} U) = N\}$$

and

$$V_2 = \sum\{U \mid U \text{ is an irreducible } FN\text{-submodule of } V_N \text{ and } \operatorname{Ker}(N \operatorname{on} U) \neq N\}.$$

Let us see that V_1 and V_2 are FG-submodules of V. Assume that $V_1, V_2 \neq 0$. If U is an irreducible FN-submodule of V_N such that $\operatorname{Ker}(N \operatorname{on} U) = N$ and $g \in G$, it follows that Ug is an irreducible FN-module of V_N and it is isomorphic to the conjugate FN-module $U^{g^{-1}}$ by [DH92; B, 7.2]. As it is observed in [DH92; B, 7.6], we have that $\operatorname{Ker}(N \operatorname{on} Ug) = \operatorname{Ker}(N \operatorname{on} U^{g^{-1}}) = \big(\operatorname{Ker}(N \operatorname{on} U)\big)^g = N^g = N$. Therefore, we have that $Ug \leq V_1$ and V_1 is a FG-submodule of V. Now consider an irreducible FN-submodule U of V_N such that $\operatorname{Ker}(N \operatorname{on} U) \neq N$ and $g \in G$. It follows that $\operatorname{Ker}(N \operatorname{on} Ug) \neq N$, because otherwise we would have by [DH92; B, 7.2 and 7.6] that

$$\operatorname{Ker}(N \operatorname{on} U) = \big(\operatorname{Ker}(N \operatorname{on} U^{g^{-1}})\big)^{g^{-1}} = \big(\operatorname{Ker}(N \operatorname{on} Ug)\big)^{g^{-1}} = N^{g^{-1}} = N,$$

contradicting the choice of U. We have proved that $Ug \leq V_2$ and, hence, V_2 is a FG-submodule of V.

Let R/S be an FG-composition factor of V_1. By [DH92; B, 7.1], it follows that $R/S = A_1 \oplus A_2 \oplus \cdots \oplus A_n$, where A_i, $1 \leq i \leq n$, is an irreducible FN-module. Moreover, by virtue of the Jordan-Hölder theorem, A_i is isomorphic to an irreducible FN-submodule of V_1. This implies that $\operatorname{Ker}(N \operatorname{on} A_i) = N$ for every $i \in \{1, 2, \ldots, n\}$ and, therefore, $\operatorname{Ker}(N \operatorname{on} R/S) = N$. Since V_1 is a direct sum of irreducible FN-submodules, it follows that $\operatorname{Ker}(N \operatorname{on} V_1) = N$.

Now assume that R/S is an FG-composition factor of V_2. We know that $R/S = A_1 \oplus A_2 \oplus \cdots \oplus A_n$, where A_i, $1 \leq i \leq n$, is an irreducible FN-module such that $\operatorname{Ker}(N \operatorname{on} A_i) \neq N$. Therefore, $\operatorname{Ker}(N \operatorname{on} R/S) \neq N$.

□

Proof of Theorem 2.2.2. If $f(p) = \emptyset$, we can take $V_1 = 0$ and $V_2 = V$. Suppose that $f(p) \neq \emptyset$.

First assume that f is a full and integrated p-local satellite of $\mathfrak{F}$. Since $G/\operatorname{O}_{p',p}(G) \in f(p)$, we have that $N := G^{f(p)}$ is a p'-group. Consider the decomposition $V = V_1 \oplus V_2$ given in Theorem 2.2.3. If R/S is a composition factor of V_1, it follows that $\operatorname{Ker}(N \operatorname{on} R/S) = N$. Therefore, $N \leq \operatorname{Ker}(G \operatorname{on} R/S)$ and $G/\operatorname{Ker}(G \text{ on } R/S) \in f(p)$. If R/S is a composition factor of V_2, R/S is f-eccentric, because otherwise we would have that $N \leq \operatorname{Ker}(G \operatorname{on} R/S)$. This would imply that $N = \operatorname{Ker}(N \operatorname{on} R/S)$ and this is not possible.

Now let f be any p-local satellite of $\mathfrak{F}$. If R/S is a composition factor of V_1, we have proved that $G/\operatorname{Ker}(G \text{ on } R/S) \in \mathfrak{S}_p(f(p) \cap \mathfrak{F})$, bearing in mind Lemma 1.1.15. Since $\operatorname{O}_p(G) \leq \operatorname{Ker}(G \text{ on } R/S)$, it follows that $G/\operatorname{Ker}(G \text{ on } R/S) \in f(p) \cap \mathfrak{F} \subseteq f(p)$. If R/S is a composition factor of V_2, we have that $G/\operatorname{Ker}(G \text{ on } R/S) \notin \mathfrak{S}_p(f(p) \cap \mathfrak{F})$. This implies that $G/\operatorname{Ker}(G \text{ on } R/S) \notin f(p) \cap \mathfrak{F}$. Since $G \in \mathfrak{F}$, we can conclude that R/S is an f-eccentric composition factor of V_2. □

Chapter 3

Products of formations of finite groups

It is well known that the formation product of two local formations is again a local formation (see [DH92; IV, 3.13 and 4.8]). However, the formation product of two $\mathfrak{X}$-local formations is not in general an $\mathfrak{X}$-local formation, as it is shown in Example 3.1.3. Taking this into account, the following question arises:

> *Which are the precise conditions on two $\mathfrak{X}$-local formations $\mathfrak{F}$ and $\mathfrak{G}$ to ensure that $\mathfrak{F} \circ \mathfrak{G}$ is an $\mathfrak{X}$-local formation?*

This question was studied by Salomon in [Sal83] for Baer-local formations. We present a complete answer in Section 3.1. We prove that the formation product of a local formation and an $\mathfrak{X}$-local one is $\mathfrak{X}$-local. In particular, [DH92; IV, 3.13 and 4.8] follow from our results. In Section 3.2, which is independent of Section 3.1, we study when the product of two arbitrary formations $\mathfrak{F}$ and $\mathfrak{G}$ is $\mathfrak{X}$-local.

On the other hand, Shemetkov posed the following question in *The Kourovka Notebook* ([MK90]):

> Question 10.72 (Shemetkov). *To prove indecomposability of $\mathfrak{S}_p$, p a prime, into a product of two non-trivial subformations.*

This question was solved positively by Shemetkov and Skiba in [SS89]. In Section 3.3 we deal with ω-saturated formations and we prove a general version of this conjecture as a corollary of a more general result. The contents of this chapter are the ones of [BBCER06].

3.1 Products of $\mathfrak{X}$-local formations

We begin with the following definition.

Definition 3.1.1. If $\mathfrak{K}$ is a class of groups and $p \in \operatorname{char}\mathfrak{X}$, denote

$$K_{\mathfrak{X}}(p) := \mathfrak{S}_p \operatorname{form}\bigl(G/\operatorname{C}_G(H/K) \mid G \in \mathfrak{K} \text{ and } H/K \text{ is an } \mathfrak{X}_p\text{-chief factor of } G\bigr),$$

taking into account that $K_{\mathfrak{X}}(p) = \emptyset$ if there does not exist any group $G \in \mathfrak{K}$ with an $\mathfrak{X}_p$-chief factor.

Lemma 3.1.2. *Consider $p \in \operatorname{char}\mathfrak{X}$. If $\mathfrak{K}$ is a quotient-closed class of groups, then*

$$K_{\mathfrak{X}}(p) := \mathfrak{S}_p \operatorname{form}\bigl(G/\operatorname{C}_G(N) \mid G \textit{ is monolithic, } G \in \mathfrak{K} \textit{ and } N = \operatorname{Soc}(G) \in \mathrm{E}\,\mathfrak{X}_p\bigr).$$

Proof. Consider a group $L \in \mathfrak{K}$ and let M/N be an $\mathfrak{X}_p$-chief factor of L. Take R maximal among the normal subgroups of L such that $M \cap R = N$. Then the quotient group $G = L/R \in \mathfrak{K}$ is monolithic and its minimal normal subgroup is $V = MR/R \cong M/M \cap R = M/N$ (see Lemma 1.4.10). Moreover, $R \leq \operatorname{C}_L(M/N)$ and $L/\operatorname{C}_L(M/N) \cong G/\operatorname{C}_G(V)$. □

The following example shows that the formation product of two $\mathfrak{X}$-local formations is not in general an $\mathfrak{X}$-local formation.

Example 3.1.3 ([Sal83])**.** Consider $\mathfrak{F} = \mathrm{D}_0\bigl(1, \operatorname{Alt}(5)\bigr)$, the formation composed of all groups that are direct products of copies of $\operatorname{Alt}(5)$ together with the trivial group, and $\mathfrak{G} = \mathfrak{S}_2$. It is clear that $\mathfrak{F}$ and $\mathfrak{G}$ are Baer formations, that is, $\mathfrak{X}$-local where $\mathfrak{X} = \mathbb{P}$. Assume that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is a $\mathbb{P}$-local formation. By Theorem 1.2.21, we have that $\mathfrak{H} = \operatorname{LF}_{\mathbb{P}}(H)$, where

$$\begin{cases} H(p) = H_{\mathbb{P}}(p) & \text{if } p \in \mathbb{P}, \\ H(E) = \mathfrak{H} & \text{if } E \in \mathbb{P}'. \end{cases}$$

Since $\mathfrak{G} \subseteq \mathfrak{H}$, it follows that $H(2) \neq \emptyset$. Consider $G = \operatorname{SL}(2,5)$. Then $G/\operatorname{Z}(G) \in \mathfrak{H}$ and $G/\operatorname{C}_G\bigl(\operatorname{Z}(G)\bigr) \in H(2)$. Applying Lemma 1.2.12, we have that $G \in \mathfrak{H}$. This is not true. Hence $\mathfrak{H}$ is not a Baer-local formation.

Taking the above example into account, it is natural to study conditions on two non-empty $\mathfrak{X}$-local formations $\mathfrak{F}$ and $\mathfrak{G}$ to ensure that $\mathfrak{F} \circ \mathfrak{G}$ is an $\mathfrak{X}$-local formation.

In the following $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations and $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$.

The next theorem provides an $\mathfrak{X}$-local definition of $\mathrm{form}_{\mathfrak{X}}(\mathfrak{H})$. We use the notation introduced in 3.1.1.

Theorem 3.1.4. *Assume that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations, and $\mathfrak{F}$ is an $\mathfrak{X}$-local formation. Then the smallest $\mathfrak{X}$-local formation* $\mathrm{form}_{\mathfrak{X}}(\mathfrak{H})$ *containing $\mathfrak{H}$ is $\mathfrak{X}$-locally defined by the $\mathfrak{X}$-formation function h given by*

$$h(p) = \begin{cases} F_{\mathfrak{X}}(p) \circ \mathfrak{G} & \text{if } \mathfrak{S}_p \subseteq \mathfrak{F}, \\ G_{\mathfrak{X}}(p) & \text{if } \mathfrak{S}_p \not\subseteq \mathfrak{F}, \end{cases} \qquad \text{when } p \in \mathrm{char}\,\mathfrak{X};$$

$$h(S) = \mathfrak{H} \qquad \text{when } S \in \mathfrak{X}'.$$

Proof. We know by Theorem 1.2.21 that $\mathrm{form}_{\mathfrak{X}}(\mathfrak{H}) = \mathrm{LF}_{\mathfrak{X}}(H)$, where H is the $\mathfrak{X}$-formation function defined by

$$\begin{cases} H(p) = H_{\mathfrak{X}}(p) & \text{if } p \in \mathrm{char}\,\mathfrak{X}, \\ H(S) = \mathfrak{H} & \text{if } S \in \mathfrak{X}'. \end{cases}$$

If we prove that $H(p) = h(p)$ for every $p \in \mathrm{char}\,\mathfrak{X}$, then the result is clear. By Lemma 3.1.2, we have that

$$H(p) := \mathfrak{S}_p \,\mathrm{Q}\,\mathrm{R}_0\big(G/\,\mathrm{C}_G(N) \mid G \text{ is monolithic, } G \in \mathfrak{H}, \\ \text{and } N = \mathrm{Soc}(G) \in \mathrm{E}\,\mathfrak{X}_p\big).$$

Assume that G is a monolithic group in $\mathfrak{H}$, where $N = \mathrm{Soc}(G) \in \mathrm{E}\,\mathfrak{X}_p$, and consider $A = G^{\mathfrak{G}}$. If $A = 1$, it follows that $G \in \mathfrak{G}$. If $\mathfrak{S}_p \subseteq \mathfrak{F}$, then $h(p) = F_{\mathfrak{X}}(p) \circ \mathfrak{G}$ and, therefore, $G \in h(p)$. If $\mathfrak{S}_p \not\subseteq \mathfrak{F}$, then $G/\,\mathrm{C}_G(N) \in G_{\mathfrak{X}}(p) = h(p)$. Now suppose that $A \neq 1$. Since $N \leq A$, applying [DH92; A, 4.13], it follows that $N = V_1 \times \cdots \times V_n$, where V_i is a minimal normal subgroup of A, $1 \leq i \leq n$. Since $A \in \mathfrak{F}$, it follows that $A/\,\mathrm{C}_A(V_i) \in F_{\mathfrak{X}}(p)$, for all $i \in \{1, \ldots, n\}$, and p divides $|N|$. Consequently $\big(G/\,\mathrm{C}_G(N)\big)^{\mathfrak{G}} \cong A/\,\mathrm{C}_A(N) \in \mathrm{R}_0\, F_{\mathfrak{X}}(p) = F_{\mathfrak{X}}(p)$ and so $G/\,\mathrm{C}_G(N) \in F_{\mathfrak{X}}(p) \circ \mathfrak{G} = h(p)$ for all p divides $|N|$. It follows that $H(p) = H_{\mathfrak{X}}(p) \subseteq \mathfrak{S}_p \,\mathrm{Q}\,\mathrm{R}_0\, h(p) = h(p)$.

Now we prove that $h(p) \subseteq H(p) = H_{\mathfrak{X}}(p)$. If $\mathfrak{S}_p \not\subseteq \mathfrak{F}$, then clearly $h(p) = G_{\mathfrak{X}}(p) \subseteq H_{\mathfrak{X}}(p)$. Suppose that $\mathfrak{S}_p \subseteq \mathfrak{F}$, that is, $h(p) = F_{\mathfrak{X}}(p) \circ \mathfrak{G}$. Consider a group $G \in F_{\mathfrak{X}}(p) \circ \mathfrak{G}$. Then the wreath product $C_p \wr G \in \mathfrak{S}_p(F_{\mathfrak{X}}(p) \circ \mathfrak{G}) \subseteq \mathfrak{S}_p F_{\mathfrak{X}}(p) \circ \mathfrak{G}$. By Theorem 1.2.21 and Lemma 1.2.33, we know that $\mathfrak{S}_p F_{\mathfrak{X}}(p) \subseteq \mathfrak{F}$ and, hence, $C_p \wr G \in \mathfrak{F} \circ \mathfrak{G} = \mathfrak{H} \subseteq \mathrm{form}_{\mathfrak{X}}(\mathfrak{H})$. Then Theorem 1.2.26 shows that $G \in H_{\mathfrak{X}}(p)$. This proves that $h(p) \subseteq H(p)$. □

The following definition was introduced in [Sal83] for Baer-local formations.

Definition 3.1.5. Consider $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations. We say that the boundary $\mathrm{b}(\mathfrak{H})$ of $\mathfrak{H}$ is $\mathfrak{X}\mathfrak{G}$-*free* if every group $G \in \mathrm{b}(\mathfrak{H})$ such that $\mathrm{Soc}(G)$ is a p-group for some prime $p \in \operatorname{char} \mathfrak{X}$ satisfies that $G/\mathrm{C}_G\big(\mathrm{Soc}(G)\big) \notin G_{\mathfrak{X}}(p)$.

Remark 3.1.6. Note that in Example 3.1.3, $\mathrm{b}(\mathfrak{H})$ is not $\mathbb{P}\mathfrak{G}$-free.

Lemma 3.1.7. *If $\mathfrak{K}$ is a formation, $G \in \mathrm{b}(\mathfrak{K}) \cap \mathrm{form}_{\mathfrak{X}}(\mathfrak{K})$, and $N = \mathrm{Soc}(G)$, then N is an abelian p-group for a prime $p \in \operatorname{char} \mathfrak{X}$.*

Proof. According to Theorem 1.2.21, $\mathrm{form}_{\mathfrak{X}}(\mathfrak{K}) = \mathrm{LF}_{\mathfrak{X}}(K)$, where K is the following $\mathfrak{X}$-formation function:

$$\begin{cases} K(p) = K_{\mathfrak{X}}(p) & \text{if } p \in \operatorname{char} \mathfrak{X}, \\ K(E) = \mathfrak{K} & \text{if } E \in \mathfrak{X}'. \end{cases}$$

Clearly, N is a minimal normal subgroup of G. If N were an $\mathfrak{X}'$-group, we would have that $G \in K(E)$ for some $E \in \mathfrak{X}'$. This would imply that $G \in \mathfrak{K}$, contrary to supposition. Hence N is an $\mathfrak{X}$-chief factor of G. Let p be a prime dividing $|N|$. Since $p \in \operatorname{char} \mathfrak{X}$, it follows that $G/\mathrm{C}_G(N) \in K(p)$. Since $K(p) = K_{\mathfrak{X}}(p) \subseteq \mathfrak{S}_p\mathfrak{K}$ and $\mathrm{O}_p\big(G/\mathrm{C}_G(N)\big) = 1$, it follows that $G/\mathrm{C}_G(N) \in \mathfrak{K}$. Therefore, $\mathrm{C}_G(N) \neq 1$ and so N is an abelian p-group. □

The next result provides a test for $\mathfrak{X}$-locality of $\mathfrak{H}$ in terms of its boundary.

Theorem 3.1.8. *Assume that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations, and $\mathfrak{F}$ is $\mathfrak{X}$-local. Then $\mathfrak{H}$ is an $\mathfrak{X}$-local formation if and only if $\mathrm{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free.*

Proof. Suppose that $\mathfrak{H}$ is $\mathfrak{X}$-local. Then $\mathfrak{H} = \mathrm{LF}_{\mathfrak{X}}(H)$, where H is the canonical $\mathfrak{X}$-local definition of $\mathfrak{H}$. Let G be a group in $\mathrm{b}(\mathfrak{H})$ such that $\mathrm{Soc}(G)$ is a p-group for some $p \in \operatorname{char} \mathfrak{X}$. If $G/\mathrm{C}_G\big(\mathrm{Soc}(G)\big)$ were in $G_{\mathfrak{X}}(p)$, then we would have that $G/\mathrm{C}_G\big(\mathrm{Soc}(G)\big) \in H_{\mathfrak{X}}(p) = H(p)$, since $\mathfrak{G} \subseteq \mathfrak{H}$. By Lemma 1.2.12, it would imply that $G \in \mathfrak{H}$. This would be a contradiction. Therefore $G/\mathrm{C}_G\big(\mathrm{Soc}(G)\big) \notin G_{\mathfrak{X}}(p)$ and $\mathrm{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free.

Conversely, suppose that $\mathrm{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free. By Theorem 3.1.4, $\mathrm{form}_{\mathfrak{X}}(\mathfrak{H}) = \mathrm{LF}_{\mathfrak{X}}(h)$, where

$$\begin{aligned} h(p) &= \begin{cases} F_{\mathfrak{X}}(p) \circ \mathfrak{G} & \text{if } \mathfrak{S}_p \subseteq \mathfrak{F}, \\ G_{\mathfrak{X}}(p) & \text{if } \mathfrak{S}_p \not\subseteq \mathfrak{F}, \end{cases} && \text{for } p \in \operatorname{char} \mathfrak{X}; \\ h(S) &= \mathfrak{H} && \text{for } S \in \mathfrak{X}'. \end{aligned}$$

We shall prove that $\mathfrak{H} = \mathrm{form}_{\mathfrak{X}}(\mathfrak{H})$. Assume that this is not the case and choose a group G of minimal order in $\mathrm{form}_{\mathfrak{X}}(\mathfrak{H}) \setminus \mathfrak{H}$. Then $G \in \mathrm{b}(\mathfrak{H})$ and, according to Lemma 3.1.7, $N = \mathrm{Soc}(G)$ is an abelian p-group for a prime $p \in \mathrm{char}\,\mathfrak{X}$. Therefore, $G/\mathrm{C}_G(N) \in h(p)$. Assume that $\mathfrak{S}_p$ is not contained in $\mathfrak{F}$. Then $h(p) = G_{\mathfrak{X}}(p)$. We conclude that $\mathrm{b}(\mathfrak{H})$ is not $\mathfrak{X}\mathfrak{G}$-free. This contradiction shows that $\mathfrak{S}_p$ is contained in $\mathfrak{F}$. Then $G/\mathrm{C}_G(N) \in F_{\mathfrak{X}}(p) \circ \mathfrak{G}$. It follows that $G^{\mathfrak{G}}/\mathrm{C}_{G^{\mathfrak{G}}}(N) \in F_{\mathfrak{X}}(p)$. Since $G^{\mathfrak{G}}/N \in \mathfrak{F}$, we can apply Lemma 1.2.12 to conclude that $G^{\mathfrak{G}} \in \mathfrak{F}$, that is, $G \in \mathfrak{H}$. This contradiction shows that $\mathrm{form}_{\mathfrak{X}}(\mathfrak{H})$ is contained in $\mathfrak{H}$ and, therefore, $\mathfrak{H}$ is $\mathfrak{X}$-local. □

Example 3.1.9. Let S be a non-abelian simple group with trivial Schur multiplier. Consider $\mathfrak{F} = \mathrm{D}_0(1, S)$, the formation of all groups which are a direct product of copies of S together with the trivial group. Let $\mathfrak{X}$ be a class of simple groups such that $\pi(\mathfrak{X}) = \mathrm{char}\,\mathfrak{X}$ and $S \notin \mathfrak{X}$. Note that $\mathfrak{F}$ is $\mathfrak{X}$-local. Let $\mathfrak{G}$ be any formation. Suppose that $G \in \mathrm{b}(\mathfrak{H})$, $N = \mathrm{Soc}(G)$ is the minimal normal subgroup of G, and N is a p-group for some $p \in \mathrm{char}\,\mathfrak{X}$. If $G/\mathrm{C}_G(N) \in G_{\mathfrak{X}}(p)$, then $N \leq \mathrm{Z}(G^{\mathfrak{G}})$ because $1 \neq G^{\mathfrak{G}} \leq \mathrm{C}_G(N)$. Since $G/N \in \mathfrak{H}$, it follows that $G^{\mathfrak{G}}/N \in \mathfrak{F}$. Assume that $G^{\mathfrak{G}} \neq N$. This implies that $G^{\mathfrak{G}}/N$, a direct product of copies of S, has non-trivial Schur multiplier, contrary to [Suz82; Exercise 4 (c), page 265]. Thus $G^{\mathfrak{G}} = N$ and then $G \in \mathrm{form}_{\mathfrak{X}}(\mathfrak{G})$ by Lemma 1.2.12 and Theorem 1.2.21. Assume, in addition, that $\mathrm{form}_{\mathfrak{X}}(\mathfrak{G}) \subseteq \mathfrak{N}_{p'}\mathfrak{G}$ for all primes $p \in \mathrm{char}\,\mathfrak{X}$. It follows then that $G \in \mathfrak{G}$ and this contradicts our choice of G. Hence, $\mathrm{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free and $\mathfrak{H}$ is $\mathfrak{X}$-local by Theorem 3.1.8. Consequently, $\mathfrak{H}$ is $\mathfrak{X}$-local for all formations $\mathfrak{G}$ satisfying that $\mathrm{form}_{\mathfrak{X}}(\mathfrak{G}) \subseteq \mathfrak{N}_{p'}\mathfrak{G}$ for all primes $p \in \mathrm{char}\,\mathfrak{X}$.

We bring this section to a close with an application of Theorem 3.1.8 and some consequences.

Theorem 3.1.10. *Consider $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations, $\mathfrak{F}$ is $\mathfrak{X}$-local and one of the following two conditions is satisfied:*

1. *$\mathfrak{G}$ is $\mathfrak{X}$-local.*

2. *$\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ when $p \in \mathrm{char}\,\mathfrak{X}$ and $F_{\mathfrak{X}}(p) = \emptyset$.*

Assume also that

$$\text{if } p \in \mathrm{char}\,\mathfrak{X},\ F_{\mathfrak{X}}(p) = \emptyset, \text{ and } \mathfrak{S}_p \subseteq \mathfrak{G}, \text{ then } \mathfrak{F} \subseteq \mathfrak{E}_{p'}. \tag{3.1}$$

Then $\mathfrak{H}$ is $\mathfrak{X}$-local.

Proof. By Theorem 3.1.8, it suffices to prove that $\mathrm{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free. Consider $G \in \mathrm{b}(\mathfrak{H})$ such that $N = \mathrm{Soc}(G)$ is a p-group for a prime $p \in \mathrm{char}\,\mathfrak{X}$ and assume that $G/\mathrm{C}_G(N) \in G_{\mathfrak{X}}(p)$. Consider $K = G^{\mathfrak{G}}$. Since $G_{\mathfrak{X}}(p) \subseteq \mathfrak{S}_p\mathfrak{G}$ and $\mathrm{O}_p\big(G/\mathrm{C}_G(N)\big) = 1$, it follows that $G/\mathrm{C}_G(N) \in \mathfrak{G}$, which implies that $K \leq \mathrm{C}_G(N)$. If $K = 1$, then $G \in \mathfrak{G} \subseteq \mathfrak{H}$, which contradicts the fact that $G \in \mathrm{b}(\mathfrak{H})$. Therefore $K \neq 1$ and, hence, $N \leq K$.

Now we aim to verify the hypotheses of (3.1). Since $G/N \in \mathfrak{H}$, it follows that $K/N \in \mathfrak{F}$. If $F_{\mathfrak{X}}(p) \neq \emptyset$, then $K/\mathrm{C}_K(N) = 1 \in F_{\mathfrak{X}}(p)$ and Lemma 1.2.12 implies that $K \in \mathfrak{F}$, which means that $G \in \mathfrak{H}$, contradicting the choice of G. Therefore $F_{\mathfrak{X}}(p) = \emptyset$. Since $G/\mathrm{C}_G(N) \in G_{\mathfrak{X}}(p)$, it is clear that $G_{\mathfrak{X}}(p) \neq \emptyset$ so, if Condition 1 holds, it follows that $\mathfrak{S}_p \subseteq \mathfrak{G}$. On the other hand, if Condition 2 holds, then $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ and hence $\mathfrak{S}_p \subseteq \mathfrak{G}$. Now we can deduce that $\mathfrak{F} \subseteq \mathfrak{E}_{p'}$.

We know that $K/N \in \mathfrak{F}$, so K/N is a p'-group and it follows from the Schur-Zassenhaus theorem ([DH92; A, 11.3]) that K has a subgroup Y which complements N. Since $K \leq \mathrm{C}_G(N)$, this means that $K = N \times Y$ and $Y = \mathrm{O}_{p'}(K) \trianglelefteq G$. Moreover, G is monolithic, so we deduce that $Y = 1$ and $K = N$. This means that $G/N \in \mathfrak{G}$. Since $G/\mathrm{C}_G(N) \in G_{\mathfrak{X}}(p)$, if Condition 1 holds, then it follows from Lemma 1.2.12 that $G \in \mathfrak{G}$. If Condition 2 holds, then $G \in \mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$. Thus in both cases $G \in \mathfrak{G} \subseteq \mathfrak{H}$, which gives the final contradiction. □

Since local formations are $\mathfrak{X}$-local for every class of simple groups $\mathfrak{X}$ (see Theorem 1.2.17), we obtain as a special case of Theorem 3.1.10 the following results:

Corollary 3.1.11. *Suppose that either of the following conditions is fulfilled:*

1. *$\mathfrak{F}$ is local and $\mathfrak{G}$ is $\mathfrak{X}$-local.*
2. *$\mathfrak{F}$ is local and $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ for all $p \in \operatorname{char}\mathfrak{X}$ such that $F_{\mathfrak{X}}(p) = \emptyset$.*

Then $\mathfrak{H}$ is an $\mathfrak{X}$-local formation.

Proof. If $\mathfrak{F}$ is local and $p \in \pi(\mathfrak{F})$, then $F_{\mathfrak{X}}(p) \neq \emptyset$. Therefore, Condition (3.1) in Theorem 3.1.10 is satisfied. □

Corollary 3.1.12 ([DH92; IV, 3.13 and 4.8])**.** *The formation $\mathfrak{H}$ is local if either of the following conditions is satisfied:*

1. *$\mathfrak{F}$ and $\mathfrak{G}$ are both local.*
2. *$\mathfrak{F}$ is local and $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ for all primes p such that $F_{\mathfrak{J}}(p) = \emptyset$.*

3.2 $\mathfrak{X}$-local products of formations

Example 3.1.9 shows that there are many cases in which a product of an $\mathfrak{X}$-local formation and a non $\mathfrak{X}$-local formation is $\mathfrak{X}$-local. This observation leads to the following question:

Are there $\mathfrak{X}$-local products of non $\mathfrak{X}$-local formations?

The local version of the above question is the one appearing in *The Kourovka Notebook* ([MK90]) as Question 9.58. It was posed by Shemetkov and Skiba and answered affirmatively in several papers (see [BBPR98, Ved88, Vor93]).

The above question has a affirmative answer when $|\mathrm{char}\,\mathfrak{X}| \geq 2$, as the next example shows.

Example 3.2.1 ([BBPR98]). Assume that p and q are different primes in $\mathrm{char}\,\mathfrak{X}$. Consider the formations $\mathfrak{F} = \mathfrak{S}_p\mathfrak{A}_q \cap \mathfrak{A}_q\mathfrak{S}_p$ and $\mathfrak{G} = \mathfrak{S}_q\mathfrak{A}_p$, where $\mathfrak{A}_r$ denotes the formation of all abelian r-groups for a prime r. $\mathfrak{F}$ is not (C_q)-local and $\mathfrak{G}$ is not (C_p)-local. Therefore, by Corollary 1.2.17, $\mathfrak{F}$ and $\mathfrak{G}$ are not $\mathfrak{X}$-local. However $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is local and so it is $\mathfrak{X}$-local.

Bearing in mind Example 3.2.1, the following question naturally arises:

Which are the precise conditions on two non-empty formations $\mathfrak{F}$ and $\mathfrak{G}$ to ensure that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is $\mathfrak{X}$-local?

Our next results answer this question.

Theorem 3.2.2. *Let $\mathfrak{K}$ be a non-empty formation. Then $\mathfrak{K}$ is $\mathfrak{X}$-local if and only if the following conditions hold:*

1. $K_{\mathfrak{X}}(p) \subseteq \mathfrak{K}$ *for all* $p \in \mathrm{char}\,\mathfrak{X}$.

2. *If* $G \in \mathrm{b}(\mathfrak{K})$, $N = \mathrm{Soc}(G) \in \mathfrak{S}_p$ *with* $p \in \mathrm{char}\,\mathfrak{X}$, *and K is the natural semidirect product* $[N]\big(G/\,\mathrm{C}_G(N)\big)$, *then* $K \in \mathrm{b}(\mathfrak{K})$.

Proof. Assume that $\mathfrak{K}$ is an $\mathfrak{X}$-local formation. Then $\mathfrak{K} = \mathrm{LF}_{\mathfrak{X}}(K)$, where K is the $\mathfrak{X}$-formation function defined in Theorem 1.2.21. Consider a prime $p \in \mathrm{char}\,\mathfrak{X}$. By Theorem 1.2.21, it follows that $K_{\mathfrak{X}}(p) \subseteq \mathfrak{K}$ so Condition 1 holds. Suppose that $G \in \mathrm{b}(\mathfrak{K})$, where $N = \mathrm{Soc}(G)$ is a p-group with $p \in \mathrm{char}\,\mathfrak{X}$ and consider $K = [N]\big(G/\,\mathrm{C}_G(N)\big)$. If $K \in \mathfrak{K}$, we would have that $K/\,\mathrm{C}_K(N) \in K_{\mathfrak{X}}(p)$ and, therefore, $G/\,\mathrm{C}_G(N) \in K_{\mathfrak{X}}(p)$. Since $G/N \in \mathfrak{K}$, this would imply by Lemma 1.2.12 that $G \in \mathrm{LF}_{\mathfrak{X}}(K) = \mathfrak{K}$. This contradiction proves that $K \notin \mathfrak{K}$. On the other hand, since $N \leq \mathrm{C}_G(N)$, it is clear that $G/\,\mathrm{C}_G(N) \in \mathfrak{K}$. Therefore, $K/N \in \mathfrak{K}$. Since K is monolithic, it follows that $K \in \mathrm{b}(\mathfrak{K})$ and Condition 2 holds.

To prove the sufficiency, assume that $\mathfrak{K}$ satisfies Conditions 1 and 2. We will obtain a contradiction by supposing that $\mathrm{form}_{\mathfrak{X}}(\mathfrak{K}) \setminus \mathfrak{K}$ contains a group G of minimal order. Such a G has a unique minimal normal subgroup N, and $G/N \in \mathfrak{K}$. This is to say that $G \in \mathrm{b}(\mathfrak{K})$. According to Lemma 3.1.7, N is an abelian p-group for a prime $p \in \mathrm{char}\,\mathfrak{X}$. We have that $K = [N]\big(G/\mathrm{C}_G(N)\big) \in \mathrm{b}(\mathfrak{K})$. Since $G \in \mathrm{form}_{\mathfrak{X}}(\mathfrak{K})$, $G/\mathrm{C}_G(N) \in K_{\mathfrak{X}}(p)$. Consequently, $K/N \in K_{\mathfrak{X}}(p)$ and $K \in \mathfrak{S}_p K_{\mathfrak{X}}(p) = K_{\mathfrak{X}}(p)$. Since Condition 1 states that $K_{\mathfrak{X}}(p) \subseteq \mathfrak{K}$, it follows that $K \in \mathfrak{K}$, which is a contradiction. □

Remark 3.2.3. If $\mathfrak{X} = \mathfrak{J}$, then Condition 1 implies Condition 2 in the above theorem.

Proof. Assume that $\mathfrak{K}$ satisfies Condition 1. Consider $G \in \mathrm{b}(\mathfrak{K})$, where $N = \mathrm{Soc}(G)$ is a p-group with $p \in \mathrm{char}\,\mathfrak{X}$ and $K = [N]\big(G/\mathrm{C}_G(N)\big)$.

Suppose that $\Phi(G) = 1$. Then G is a primitive group, $\mathrm{C}_G(N) = N$, and G is isomorphic to $K = [N](G/N)$. Therefore, $K \in \mathrm{b}(\mathfrak{K})$. Now assume that $\Phi(G) \neq 1$. Consider $T/N := \mathrm{O}_{p'}(G/N)$. Since T/N is p-nilpotent and $N \leq \Phi(G)$, we have by [Hup67; VI, 6.3] that T is p-nilpotent. This implies that $T = N$ because otherwise we would find a non-trivial normal p'-subgroup of G. Hence, $\mathrm{O}_{p'}(G/N) = 1$. Consequently, $G/N \in K_{\mathfrak{X}}(p)$ by [DH92; IV, 3.7] and, hence, $G \in \mathfrak{S}_p K_{\mathfrak{X}}(p) = K_{\mathfrak{X}}(p)$. By Condition 1 we conclude that $G \in \mathfrak{K}$, which contradicts our supposition. □

Here we recall a well-known result, due to Bryant, Bryce and Hartley.

Theorem 3.2.4 ([DH92; IV, 1.14])**.** *Let U be a subgroup of a group G such that $G = UN$ for some nilpotent normal subgroup N of G. If G belongs to a formation $\mathfrak{F}$, then U belongs to $\mathfrak{F}$.*

The following proposition will be useful for later applications.

Proposition 3.2.5. *Let $\mathfrak{F}$ be a non-empty formation and let G be a group. If N is normal subgroup of G, then we have:*

1. *$(G/N)^{\mathfrak{F}} = G^{\mathfrak{F}}N/N$.*
2. *If U is a subgroup of $G = UN$, then $U^{\mathfrak{F}}N = G^{\mathfrak{F}}N$.*
3. *If N is nilpotent and $G = UN$, then $U^{\mathfrak{F}}$ is contained in $G^{\mathfrak{F}}$.*

Proof. 1. Denote $R/N = (G/N)^{\mathfrak{F}}$. It is clear that $G/R \in \mathfrak{F}$. Hence $G^{\mathfrak{F}}N$ is contained in R. Moreover $G/G^{\mathfrak{F}}N \in \mathfrak{F}$. It implies that the factor group $(G/N)\big/(G^{\mathfrak{F}}N/N) \in \mathfrak{F}$ and so $R/N \leq G^{\mathfrak{F}}N/N$. Therefore $R = G^{\mathfrak{F}}N$.

2. Let θ denote the canonical isomorphism from $G/N = UN/N$ to $U/(U \cap N)$. Then $\big((G/N)^{\mathfrak{F}}\big)^{\theta} = \big(U/(U \cap N)\big)^{\mathfrak{F}}$, which is equal to $U^{\mathfrak{F}}(U \cap N)/(U \cap N)$ by Statement 1. Hence $U^{\mathfrak{F}}N/N = (G/N)^{\mathfrak{F}} = G^{\mathfrak{F}}N/N$ and $U^{\mathfrak{F}}N = G^{\mathfrak{F}}N$.

3. We have $G/G^{\mathfrak{F}} = (UG^{\mathfrak{F}}/G^{\mathfrak{F}})(NG^{\mathfrak{F}}/G^{\mathfrak{F}}) \in \mathfrak{F}$. Applying Theorem 3.2.4, it follows that $UG^{\mathfrak{F}}/G^{\mathfrak{F}} \in \mathfrak{F}$. Therefore $U^{\mathfrak{F}}$ is contained in $U \cap G^{\mathfrak{F}}$. □

If $\mathfrak{Y}$ is a class of groups, denote $\mathfrak{Y}^{\mathfrak{G}} = (Y^{\mathfrak{G}} \mid Y \in \mathfrak{Y})$. The following lemma can be deduced from the proof of [BBPR98; Theorem A].

Lemma 3.2.6. *Consider two non-empty formations $\mathfrak{F}$ and $\mathfrak{G}$ and $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$. If T is a group such that $T \notin \mathfrak{G}$ and $\mathfrak{S}_p(T) \subseteq \mathfrak{H}$ for some prime p, then $\mathfrak{S}_p(T^{\mathfrak{G}}) \subseteq \mathfrak{F}$.*

Proof. Let Z be a group in $\mathfrak{S}_p(T^{\mathfrak{G}})$. Then Z has a normal subgroup P such that P is a p-group and Z/P is isomorphic to $T^{\mathfrak{G}} \neq 1$. Assume that p^s is the exponent of the abelian p-group P/P'. Consider $Q = P \wr_{\mathrm{nat}} H$, where $H = \langle (1, 2, \ldots, p^s) \rangle$ is a cyclic group of order p^s regarded as a subgroup of the symmetric group of degree p^s. Here the wreath product is taken with respect to the natural permutation representation of H of degree p^s. Set $D = \{(a, \ldots, a) \mid a \in P\}$ the diagonal subgroup of $P^{\natural}$, the base group of Q. Since $a^{p^s} \in P'$, we have that D is contained in $[P^{\natural}, H]$ by [DH92; A, 18.4]. In particular D is contained in Q'. Let $Y = Q \wr T$ be the regular wreath product of Q with T. Since $Y \in \mathfrak{S}_p(T) \subseteq \mathfrak{H}$, it follows that $Y \in \mathfrak{H}$. Therefore $Y^{\mathfrak{G}} \in \mathfrak{F}$. Now, by Proposition 3.2.5, we know that $Y^{\mathfrak{G}} = (B \cap Y^{\mathfrak{G}})T^{\mathfrak{G}}$, where $B = Q^{\natural}$ is the base group of Y. Now, by [DH92; A, 18.8], $BT^{\mathfrak{G}}$ is isomorphic to $(Q^n) \wr T^{\mathfrak{G}}$, where $n = |T : T^{\mathfrak{G}}|$ and $C' \leq [C, T^{\mathfrak{G}}]$, for $C = (Q^n)^{\natural}$, by virtue of [DH92; A, 18.4]. This implies that $B' = [B, T^{\mathfrak{G}}] \leq [B, Y^{\mathfrak{G}}] \leq B \cap Y^{\mathfrak{G}}$. Hence $B'T^{\mathfrak{G}}$ is contained in $Y^{\mathfrak{G}}$. Applying Theorem 3.2.4, $B'T^{\mathfrak{G}} \in \mathfrak{F}$. Therefore $\big((Q')^n\big) \wr T^{\mathfrak{G}} \in \mathfrak{F}$. Since P is isomorphic to a subgroup of Q', it follows that $P^n \wr T^{\mathfrak{G}} \in \mathfrak{F}$ by Theorem 3.2.4. Since P can be regarded as a subgroup of P^n, we have that $P \wr T^{\mathfrak{G}}$ is a subgroup of $P^n \wr T^{\mathfrak{G}}$ supplementing the Fitting subgroup of $P^n \wr T^{\mathfrak{G}}$. Applying again Theorem 3.2.4, we have that $P \wr T^{\mathfrak{G}} \in \mathfrak{F}$. By [DH92; A, 18.9], Z is isomorphic to a subgroup of $P \wr T^{\mathfrak{G}}$ supplementing the Fitting subgroup of $P \wr T^{\mathfrak{G}}$. Therefore $Z \in \mathfrak{F}$ by virtue of Theorem 3.2.4. This completes the proof of the lemma. □

Lemma 3.2.7. *Suppose that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations. Consider $p \in \operatorname{char} \mathfrak{X}$. Then the following conditions are equivalent:*

1. $H_{\mathfrak{X}}(p) \subseteq \mathfrak{H}$.
2. $H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$.

3. *$H_{\mathfrak{X}}(p) \subseteq \mathfrak{G}$ or $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$.*

Proof. It is clear that Conditions 1 and 2 are equivalent. Now assume that Condition 1 is satisfied. If $H_{\mathfrak{X}}(p) \not\subseteq \mathfrak{G}$, take a group $T \in H_{\mathfrak{X}}(p) \setminus \mathfrak{G}$. We have that $\mathfrak{S}_p(T) \subseteq \mathfrak{S}_p H_{\mathfrak{X}}(p) = H_{\mathfrak{X}}(p) \subseteq \mathfrak{H}$. Hence, by Lemma 3.2.6, we have that $\mathfrak{S}_p(T^{\mathfrak{G}}) \subseteq \mathfrak{F}$. Now consider a group G in $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}}$. Then G has a normal p-subgroup N such that $G/N \cong R^{\mathfrak{G}}$, where $R \in H_{\mathfrak{X}}(p)$. If $R^{\mathfrak{G}} \neq 1$, we have just proved that $\mathfrak{S}_p(R^{\mathfrak{G}}) \subseteq \mathfrak{F}$ and, therefore, $G \in \mathfrak{F}$. If $R^{\mathfrak{G}} = 1$, then $G \in \mathfrak{S}_p$. Consider the group $K := G \times T^{\mathfrak{G}}$. We have that $K \in \mathfrak{S}_p(T^{\mathfrak{G}}) \subseteq \mathfrak{F}$ and, therefore, $G \in \mathrm{Q}(\mathfrak{F}) = \mathfrak{F}$. We conclude that $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$ and Condition 3 holds.

Now suppose that Condition 3 is satisfied. If $H_{\mathfrak{X}}(p) \subseteq \mathfrak{G}$, it is clear that $H_{\mathfrak{X}}(p) \subseteq \mathfrak{H}$ and Condition 1 holds. If $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$, then $H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$ and Condition 2 is satisfied. □

The following theorem follows from Theorem 3.2.2 and Lemma 3.2.7.

Theorem 3.2.8. *Suppose that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations. Then $\mathfrak{H}$ is $\mathfrak{X}$-local if and only if the following conditions hold:*

1. *If $p \in \operatorname{char} \mathfrak{X}$, then $H_{\mathfrak{X}}(p) \subseteq \mathfrak{G}$ or $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$.*
2. *If $G \in \mathrm{b}(\mathfrak{H})$, $N = \mathrm{Soc}(G) \in \mathfrak{S}_p$ with $p \in \operatorname{char} \mathfrak{X}$, and K is the natural semidirect product $[N]\big(G/\mathrm{C}_G(N)\big)$, then $K \in \mathrm{b}(\mathfrak{H})$.*

Note that if Conditions 1 and 2 hold for $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, then it is easy to deduce that $\mathrm{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free. However it seems harder to prove directly that if $\mathfrak{F}$ is $\mathfrak{X}$-local and $\mathrm{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free, then Conditions 1 and 2 are satisfied.

Corollary 3.2.9 ([BBPR98; Theorem A])**.** *A formation product $\mathfrak{H}$ of two non-empty formations $\mathfrak{F}$ and $\mathfrak{G}$ is local if and only if $\mathfrak{H}$ satisfies the following condition:*

If p is a prime, then $H_{\mathfrak{X}}(p) \subseteq \mathfrak{G}$ or $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$.

When a product $\mathfrak{H}$ of two non-empty formations $\mathfrak{F}$ and $\mathfrak{G}$ is $\mathfrak{X}$-local, the formation $\mathfrak{G}$ has a very nice property.

Corollary 3.2.10. *If $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is $\mathfrak{X}$-local, then $\mathrm{form}_{\mathfrak{X}}(\mathfrak{G}) \subseteq \mathfrak{N}_{p'}\mathfrak{G}$ for all primes $p \in (\operatorname{char} \mathfrak{X}) \setminus \pi(\mathfrak{F})$.*

Proof. Let $p \in (\operatorname{char} \mathfrak{X}) \setminus \pi(\mathfrak{F})$. By Theorem 3.2.8, we have that $H_{\mathfrak{X}}(p) \subseteq \mathfrak{G}$. Consider the canonical $\mathfrak{X}$-formation function G defining $\mathrm{form}_{\mathfrak{X}}(\mathfrak{G})$ (see Theorem 1.2.21). Suppose that $\mathrm{form}_{\mathfrak{X}}(\mathfrak{G})$ is not contained in $\mathfrak{N}_{p'}\mathfrak{G}$, and let $G \in \mathrm{form}_{\mathfrak{X}}(\mathfrak{G}) \setminus \mathfrak{N}_{p'}\mathfrak{G}$ be a group of minimal order. Then G has a unique minimal normal subgroup, N say. Since $G \in \mathrm{form}_{\mathfrak{X}}(\mathfrak{N}_{p'}\mathfrak{G}) \cap \mathrm{b}(\mathfrak{N}_{p'}\mathfrak{G})$, it follows form Lemma 3.1.7 that $N \in \mathfrak{S}_q$ for a prime $q \in \mathrm{char}\,\mathfrak{X}$. Assume that $\Phi(G) = 1$. Then G is a primitive group and $N = \mathrm{C}_G(N)$. Therefore $G \in G(q)$. If $p \neq q$, then $G \in \mathfrak{N}_{p'}\mathfrak{G}$ because $G(q) \subseteq \mathfrak{S}_q\mathfrak{G}$ and if $p = q$, then $G \in \mathfrak{S}_p H_{\mathfrak{X}}(p) = H_{\mathfrak{X}}(p) \subseteq \mathfrak{G}$. In both cases, we reach a contradiction. Hence N is contained in $\Phi(G)$. If $p \neq q$, then the Fitting subgroup $\mathrm{F}(G)$ of G is a p'-group and $G/\mathrm{F}(G) \cong (G/N)\big/\mathrm{F}(G/N) \in \mathfrak{G}$. Hence, $G \in \mathfrak{N}_{p'}\mathfrak{G}$, contrary to supposition. Assume that $p = q$. Then, since $G/N \in \mathfrak{N}_{p'}\mathfrak{G}$, it follows that $(G/N)^{\mathfrak{G}} = G^{\mathfrak{G}}/N$ is a p'-group. Thus $G^{\mathfrak{G}}/N$ is contained in $\mathrm{O}_{p'}(G/N) = 1$ by [Hup67; VI, 6.3]. Hence $N = G^{\mathfrak{G}}$. Since $G \in \mathfrak{H}$, we have that $G^{\mathfrak{G}} = N \in \mathfrak{F}$ and $p \in \pi(\mathfrak{F})$. This final contradiction proves that $\mathrm{form}_{\mathfrak{X}}(\mathfrak{G}) \subseteq \mathfrak{N}_{p'}\mathfrak{G}$. □

If $\mathfrak{X} = \mathfrak{J}$, we have:

Corollary 3.2.11 ([She84])**.** *If* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ *is local, then* $\mathrm{lform}(\mathfrak{G})$ *is contained in* $\mathfrak{N}_{p'}\mathfrak{G}$ *for all primes* $p \notin \pi(\mathfrak{F})$ *(in other words,* $\mathfrak{G}$ *is* p*-local for all primes* $p \notin \pi(\mathfrak{F})$*).*

3.3 p-saturated products of formations

Here we study when the formation product of two formations is a p-saturated formation.

The following lemma gives a characterization of p-saturated formations.

Lemma 3.3.1. *Let* $\mathfrak{K}$ *be a non-empty formation. Then* $\mathfrak{K}$ *is* p*-saturated if and only if* $K_{\mathfrak{J}}(p) \subseteq \mathfrak{K}$*.*

Proof. Suppose that $\mathfrak{K}$ is a p-saturated formation, where p is a prime. Then $\mathrm{lform}(\mathfrak{K}) \subseteq \mathfrak{N}_{p'}\mathfrak{K}$ by Theorems 1.1.9 and 1.1.16. By Theorem 1.2.21, we know that $K_{\mathfrak{J}}(p) \subseteq \mathrm{lform}(\mathfrak{K})$ and, therefore, $K_{\mathfrak{J}}(p) \subseteq \mathfrak{N}_{p'}\mathfrak{K}$. This implies that $K_{\mathfrak{J}}(p) \subseteq \mathfrak{K}$.

Now suppose that $\mathfrak{K}$ is not p-saturated and $K_{\mathfrak{J}}(p) \subseteq \mathfrak{K}$. Let G be a group of minimal order satisfying $G/\big(\Phi(G) \cap \mathrm{O}_p(G)\big) \in \mathfrak{K}$ and $G \notin \mathfrak{K}$. Then G is a monolithic group and $N := \mathrm{Soc}(G) \leq \Phi(G) \cap \mathrm{O}_p(G)$. We have that $\mathrm{O}_{p',p}(G/N) = \mathrm{O}_{p',p}(G)/N$, since $N \leq \Phi(G)$. Moreover, $G/N \in \mathfrak{K}$ and, therefore, $G/\mathrm{O}_{p',p}(G) \in K_{\mathfrak{J}}(p)$, bearing in mind that $p \in \pi(G/N)$. Since $\mathrm{O}_{p',p}(G) = \mathrm{O}_p(G)$, $G \in K_{\mathfrak{J}}(p) \subseteq \mathfrak{K}$. This is not possible. □

Theorem 3.3.2. *Suppose that* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$*, where* $\mathfrak{F}$ *and* $\mathfrak{G}$ *are non-empty formations, and let* p *be a prime. Then the following statements are equivalent:*

1. *$\mathfrak{H}$ is a p-saturated formation.*

2. *Either $H_{\mathfrak{J}}(p) \subseteq \mathfrak{G}$ or $\mathfrak{S}_p H_{\mathfrak{J}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$.*

Proof. It is clear applying Lemma 3.3.1 and Lemma 3.2.7 for $\mathfrak{X} = \mathfrak{J}$. □

Theorem 3.3.2 also confirms a more general version of the abovementioned conjecture of Shemetkov concerning the non-decomposability of the formation of all p-groups (p a prime) as a formation product of two nontrivial subformations.

Corollary 3.3.3. *Suppose that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations, and $\mathfrak{H}$ is p-saturated. If $\mathfrak{F} \subseteq \mathfrak{S}_p$ and $\mathfrak{F} \neq \mathfrak{S}_p$, then $\mathfrak{H} = \mathfrak{G}$.*

Proof. If $H_{\mathfrak{J}}(p) = \emptyset$, it follows that $\mathfrak{H} \subseteq \mathfrak{E}_{p'}$. In this case, we have that $\mathfrak{H} \subseteq \mathfrak{E}_{p'} \cap (\mathfrak{S}_p\mathfrak{G})$. Therefore, $\mathfrak{H} \subseteq \mathfrak{G}$. If $H_{\mathfrak{J}}(p) \neq \emptyset$, we have that $\mathfrak{H} \subseteq \mathfrak{E}_{p'} H_{\mathfrak{J}}(p)$ by Remark 1.2.16. If $H_{\mathfrak{J}}(p)$ is contained in $\mathfrak{G}$, then $\mathfrak{H} \subseteq \big(\mathfrak{E}_{p'} H_{\mathfrak{J}}(p)\big) \cap (\mathfrak{S}_p\mathfrak{G}) \subseteq (\mathfrak{E}_{p'}\mathfrak{G}) \cap (\mathfrak{S}_p\mathfrak{G}) = \mathfrak{G}$ and the result holds. Suppose that $H_{\mathfrak{J}}(p)$ is not contained in $\mathfrak{G}$. Then $\mathfrak{S}_p H_{\mathfrak{J}}(p)^{\mathfrak{G}}$ is contained in $\mathfrak{F}$ by Theorem 3.3.2. In particular, $\mathfrak{S}_p \subseteq \mathfrak{F}$, and we have a contradiction. □

Chapter 4

Factorisations of one-generated formations

4.1 A question of Skiba

The formation product of two formations does not normally yield a saturated formation (see [BBPR98]). In fact, if $\mathfrak{F}$ and $\mathfrak{G}$ are formations such that $\mathfrak{F} \circ \mathfrak{G}$ is saturated, then $\mathfrak{G}$ is p-saturated for all primes p not dividing the order of any group in $\mathfrak{F}$ (see [She84]).

In [MK92], Skiba asked the following question:

> *If $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is a one-generated Baer formation, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-trivial formations, is $\mathfrak{F}$ a Baer formation?*

In the 1999 edition of the same book [MK99], it is announced that Skiba has answered the question negatively. The reader is referred to [Guo00; page 224] for a detailed example.

We note that in the known examples of that situation, the equalities $\mathfrak{H} = \mathfrak{G}$ and $\mathfrak{H} = \mathfrak{S}_p\mathfrak{H}$ for a prime p hold, where $\mathfrak{S}_p$ denotes the class of all p-groups. Consequently the following question still remains open:

> *Assume that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is a Baer formation generated by a group G, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-trivial formations. Is $\mathfrak{F}$ a Baer formation provided that $\mathfrak{H} \neq \mathfrak{G}$ or $\mathfrak{H} \neq \mathfrak{S}_p\mathfrak{H}$ for every prime p?*

In this section an affirmative answer to this question is given. The result has been published in [BBCER03].

Note that an analogous result was proved by Vishnevskaya in [Vis00] for p-saturated formations. She shows that the p-saturated formation $\mathfrak{H}$ generated by a finite group cannot be the Gaschütz product $\mathfrak{F} \circ \mathfrak{G}$ of two non-p-saturated formations provided $\mathfrak{H} \neq \mathfrak{G}$. This motivates us to present the most general version of our result by using $\mathfrak{X}$-saturated formations. Although in general there does not exist a class of simple groups $\mathfrak{X}$ such that the $\mathfrak{X}$-saturated formations are exactly the ω-saturated formations (see Example 2.1.3), the arguments used in the proof of our result still hold for ω-saturated formations. It leads to an alternative proof of Vishnevskaya's result.

Definition 4.1.1. A formation $\mathfrak{F}$ is said to be a *one-generated $\mathfrak{X}$-local formation* if there exists a group G such that $\mathfrak{F}$ is the smallest $\mathfrak{X}$-local formation containing G.

Lemma 4.1.2. *Let G be a group and p a prime. The class $\mathfrak{S}_p$ of all p-groups is not contained in the class $\mathrm{S}\big(\mathrm{form}(G)\big)$.*

Proof. Consider the class

$$\mathfrak{E}(e) := \big(X \in \mathfrak{E} \mid \mathrm{Exp}(X) \leq e\big),$$

where $e = \mathrm{Exp}(G)$, the exponent of G. It is clear that $\mathfrak{E}(e)$ is closed under taking direct products, quotients, and subgroups. In particular, $\mathfrak{E}(e)$ is a subgroup-closed formation. Since $G \in \mathfrak{E}(e)$, it follows that $\mathrm{form}(G) \subseteq \mathfrak{E}(e)$ and so $\mathrm{S}\big(\mathrm{form}(G)\big) \subseteq \mathfrak{E}(e)$. On the other hand, we know by [Hup67; III, 15.3] how to construct a p-group P such that $\mathrm{Exp}(P) > e$. The result is then proved. □

The following result of Skiba [Ski92; Lemma 1] will be applied several times in this chapter.

Lemma 4.1.3. *Let $G = A \wr B = [K]B$, where $K = A^\natural = \prod_{b\in B} A_1^b$ is the base group of G and let A_1 be the first copy of A in K. If L is a normal subgroup of G contained in K and M is the projection of L into A_1, then $(A_1/M) \wr B \in \mathrm{Q}(G/L)$.*

Theorem 4.1.4 ([BBCER03; Theorem 1])**.** *Let $\mathfrak{X}$ be a class of simple groups such that $\pi(\mathfrak{X}) = \mathrm{char}\,\mathfrak{X}$. Let $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ be an $\mathfrak{X}$-local formation generated by a group G. If $\mathfrak{F}$ and $\mathfrak{G}$ are non-trivial and $\mathfrak{H} \neq \mathfrak{G}$ or $\mathfrak{S}_p\mathfrak{H} \neq \mathfrak{H}$ for all primes $p \in \mathrm{char}\,\mathfrak{X}$, then $\mathfrak{F}$ is $\mathfrak{X}$-saturated.*

Proof. It is known that $\mathfrak{N}\,\mathrm{form}(G)$, where $\mathfrak{N}$ is the class of all nilpotent groups, is a saturated formation ([DH92; IV, 3.4(b)]). Therefore $\mathfrak{N}\,\mathrm{form}(G)$ is $\mathfrak{X}$-saturated and so $\mathfrak{H}$ is contained in $\mathfrak{N}\,\mathrm{form}(G)$.

Assume, arguing by contradiction, that $\mathfrak{F}$ is not $\mathfrak{X}$-saturated. Then there exists a group $A \notin \mathfrak{F}$ and a normal subgroup N of A such that $E = A/N \in \mathfrak{F}$ and $N \leq \Phi_{\mathfrak{X}}(A) \leq \Phi(A) \cap \mathrm{O}_{\mathfrak{X}}(A)$.

Step 1: For any group $1 \neq U \in \mathfrak{G}$, the group $(E \wr U)^{\mathfrak{G}}$ is not subdirectly contained in the base group of $E \wr U$, the regular wreath product of E with U.

Let $1 \neq U \in \mathfrak{G}$ and denote $D = E \wr U$. Then $D^{\mathfrak{G}}$ is contained in $K = E^{\natural}$, the base group of D. Suppose that $D^{\mathfrak{G}}$ is subdirect in K. Then $D^{\mathfrak{G}} \in \mathfrak{F}$ because $K \in \mathfrak{F}$.

Consider now $D_1 = A \wr U$, and if $A_0 \leq A$, we write $A_0^{\natural} = \{(a_1, \dots, a_{|U|}) \mid a_i \in A_0, \quad 1 \leq i \leq |U|\} \leq A^{\natural} = K_1$. Denote $R = N^{\natural}$. By [DH92; A, 18.2], we have that D_1/R is isomorphic to D. Assume that $\Phi_{\mathfrak{X}}(A) = \Phi\big(\mathrm{O}_{\mathfrak{X}}(A)\big)$. In this case, $\mathrm{O}_{\mathfrak{X}}(K_1) = \big(\mathrm{O}_{\mathfrak{X}}(A)\big)^{\natural}$ and $\Phi\big(\mathrm{O}_{\mathfrak{X}}(K_1)\big) = \Big(\Phi\big(\mathrm{O}_{\mathfrak{X}}(A)\big)\Big)^{\natural}$. This means that R is contained in $\Phi_{\mathfrak{X}}(D_1)$.

Suppose now that $\Phi_{\mathfrak{X}}(A) = \Phi(A)$. Then there exists a prime p such that $A \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$, that is, A has an elementary abelian normal p-subgroup T such that $T \leq \Phi(A)$, A/T is primitive of type 2, $\mathrm{Soc}(A/T) \in \mathrm{E}\,\mathfrak{X} \setminus \mathfrak{E}_{p'}$ and $\mathrm{C}_A^h(T) \leq T$. Since A/T is a primitive group of type 2, it follows that $(A/T) \wr U \cong (A \wr U)/T^{\natural} = D_1/T^{\natural}$ is a primitive group of type 2 by [DH92; A, 18.5]. It is clear that $T^{\natural}$ is an elementary abelian normal p-subgroup of D_1 contained in $\Phi(D_1)$. Moreover $\mathrm{Soc}(D_1/T^{\natural}) \cong \mathrm{Soc}(A/T)^{\natural}$. Therefore $\mathrm{Soc}(D_1/T^{\natural}) \in \mathrm{E}\,\mathfrak{X} \setminus \mathfrak{E}_{p'}$. Assume that there exists $x \in D_1$ such that x centralises all chief factors of D_1 below $T^{\natural}$. Let H/K be a chief factor of A below T such that $H/K \not\leq \mathrm{Z}(A/K)$. Then $(H/K)^{\natural}$ is a chief factor of D_1 below $T^{\natural}$ and so x centralises $(H/K)^{\natural}$. In particular, each component of x centralises the corresponding component of $(H/K)^{\natural}$. This implies that $x \in T^{\natural}$. Consequently $\mathrm{C}_{D_1}^h(T^{\natural}) \leq T^{\natural}$. Therefore we have proved that $D_1 \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$. In this case, $\Phi_{\mathfrak{X}}(D_1) = \Phi(D_1)$ and so $R \leq \Phi(A)^{\natural} \leq \Phi(D_1)$. Since $\mathfrak{H}$ is $\mathfrak{X}$-saturated, it follows that $D_1 \in \mathfrak{H}$. This implies that $D_1^{\mathfrak{G}} \in \mathfrak{F}$. Since $D_1/R \cong D$, we have that $D^{\mathfrak{G}}$ is isomorphic to $D_1^{\mathfrak{G}}R/R$. It follows that $D_1^{\mathfrak{G}}$ is subdirect in K_1. But hence A is an epimorphic image of $D_1^{\mathfrak{G}}$. Therefore $A \in \mathfrak{F}$, a contradiction.

Step 2: If C_q were the unique simple group in $\mathfrak{F}$, we would have $\mathfrak{S}_q\mathfrak{G} = \mathfrak{G}$ and $\mathfrak{H} = \mathfrak{G}$. This would be a contradiction if $q \in \operatorname{char}\mathfrak{X}$. Therefore there does not exist any prime $q \in \operatorname{char}\mathfrak{X}$ such that $\mathfrak{S}_q$ is contained in $\mathfrak{G}$.

First of all, we prove that $E^{\mathfrak{G}} \neq E$. Assume that $E^{\mathfrak{G}} = E$. Since $E \in \mathfrak{F}$, it follows that $E \in \mathfrak{H}$ and, therefore, $A \in \mathfrak{H}$. We have that $E^{\mathfrak{G}} = A^{\mathfrak{G}}N/N = A/N$. Consequently $A^{\mathfrak{G}}N = A$ and $A^{\mathfrak{G}} = A$, since $N \leq \Phi(A)$. Therefore we obtain that $A \in \mathfrak{F}$, which is a contradiction.

Now assume that C_q is the only simple group in $\mathfrak{F}$ and let F be a maximal normal subgroup of E such that $E^{\mathfrak{G}} \leq F$. We have that $E/F \cong C_q$. Since $E/F \in \mathrm{Q}(E/E^{\mathfrak{G}})$, it follows that $C_q \in \mathfrak{G}$.

Now let B be a group of minimal order in $\mathfrak{S}_q\mathfrak{G} \setminus \mathfrak{G}$. Hence $C = \mathrm{Soc}(B)$ is a minimal normal q-subgroup of B and $1 \neq B/C \in \mathfrak{G}$.

Consider the group $D = E \wr (B/C)$. From Step 1, we know that $D^{\mathfrak{G}}$ is not subdirect in the base group K of D. Let E_1 be the first copy of E in K and let F be the projection of $D^{\mathfrak{G}}$ in E_1. Then F is a proper normal subgroup of E_1. Let E_0 be a maximal normal subgroup of E_1 such that $F \leq E_0$. Then E_1/E_0 is a simple group in $\mathfrak{F}$. Consequently $E_1/E_0 \cong C_q$. It is clear that $K_0 = E_0^{\natural}$ is a normal subgroup of D and $D/K_0 \cong C_q \wr (B/C)$ by [DH92; A, 18.2]. On the other hand, by Lemma 4.1.3, we have that $(E_1/F) \wr (B/C)$ is a quotient of $D/D^{\mathfrak{G}}$. It follows then that $C_q \wr (B/C) \in \mathfrak{G}$. Therefore $C \wr (B/C) \in \mathrm{R}_0\big(C_q \wr (B/C)\big) \subseteq \mathfrak{G}$. Applying [DH92; A, 18.9 and IV, 1.14], B belongs to $\mathrm{Q}\,\mathrm{R}_0\big(C_q \wr (B/C)\big) \subseteq \mathfrak{G}$, a contradiction. Therefore $\mathfrak{S}_q\mathfrak{G} = \mathfrak{G}$. Let $G \in \mathfrak{H}$, and assume that $T = G^{\mathfrak{G}} \neq 1$. Hence $\mathrm{O}^q(T) < T$. Therefore $G/\mathrm{O}^q(T) \in \mathfrak{S}_q\mathfrak{G} = \mathfrak{G}$, a contradiction. It follows that $T = 1$ and so $G \in \mathfrak{G}$. Consequently $\mathfrak{H} = \mathfrak{G}$ and so $\mathfrak{S}_q\mathfrak{H} = \mathfrak{S}_q\mathfrak{G} = \mathfrak{G} = \mathfrak{H}$, a contradiction if $q \in \operatorname{char}\mathfrak{X}$.

Assume now that there exists a prime $q \in \operatorname{char}\mathfrak{X}$ such that $\mathfrak{S}_q \subseteq \mathfrak{G}$. Since $\mathfrak{F} \neq (1)$, there exists a simple group $S \in \mathfrak{F}$. Assume $S \not\cong C_q$. Let G_1 be an arbitrary q-group. Then $S \wr G_1 \in \mathfrak{H} \subseteq \mathfrak{N}\operatorname{form}(G)$. If S is not abelian, then $S \wr G_1 \in \operatorname{form}(G)$, and so $G_1 \in \operatorname{form}(G)$. Assume now that S is abelian, then S is isomorphic to C_r for a prime $r \neq q$. Then $G_1 \in \operatorname{form}(G)$. In both cases, $\mathfrak{S}_q \subseteq \operatorname{form}(G)$, a contradiction by Lemma 4.1.2. Therefore C_q is the only simple group in $\mathfrak{F}$ and the conclusion holds.

Step 3: G has a composition factor in $\mathfrak{X}$.

Denote by $\mathfrak{K}$ the class of composition factors of G and assume that $\mathfrak{K} \cap \mathfrak{X} = \emptyset$. Consider the class $\mathrm{E}\,\mathfrak{K}$ of finite groups whose composition factors belong to $\mathfrak{K}$. Let $\mathfrak{L}$ be a formation contained in $\mathrm{E}\,\mathfrak{K}$. Then it is rather easy to see that $\mathfrak{L}$ is $\mathfrak{X}$-saturated. Since $\operatorname{form}(G) \subseteq \mathrm{E}\,\mathfrak{K}$ we have that $\operatorname{form}(G)$ is $\mathfrak{X}$-saturated. Therefore $\mathfrak{F} \circ \mathfrak{G} = \operatorname{form}(G)$. By [Guo00; 4.5.8], it follows that $\mathfrak{F}$ consists of nilpotent groups. Since $\mathfrak{F}$ is subgroup-closed by [DH92; IV, 1.16], we have that $\mathfrak{F} \subseteq \mathfrak{H} \subseteq \mathrm{E}\,\mathfrak{K}$. It follows that $\mathfrak{F}$ is $\mathfrak{X}$-saturated, a contradiction.

Step 4: Final contradiction.

Let q be a prime dividing the order of a composition factor of G in $\mathfrak{X}$. It follows that $\mathfrak{S}_q \subseteq \mathfrak{H}$ because $\mathfrak{H}$ is $\mathfrak{X}$-saturated by Corollary 1.2.29. By Step 2, we have that $\mathfrak{S}_q$ is not contained in $\mathfrak{G}$. By [BBPR98; Corollary] and Step 2, we have that $\mathfrak{S}_q \subseteq \mathfrak{F}$. Moreover, by Step 1 and [Ski92; Lemma 3], there exists a prime p such that $\mathfrak{S}_p \subseteq \mathrm{S}\,\mathfrak{G}$, that is, given a group $P \in \mathfrak{S}_p$, there exists a group $G(P)$ in $\mathfrak{G}$ such that $P \leq G(P)$.

Assume that $p \neq q$. Consider $X_P = C_q \wr G(P) \in \mathfrak{H} \subseteq \mathfrak{N}\operatorname{form}(G)$. We have that $T_P = X_P/\mathrm{F}(X_P) \in \operatorname{form}(G)$. But $\mathrm{F}(X_P)$ is a q-group. It follows that $\operatorname{form}(G)$ contains a group T_P with a subgroup isomorphic to P. This is a contradiction by Lemma 4.1.2. Therefore $p = q$.

On the other hand, by Step 2 we know that there exists a simple group S in $\mathfrak{F}$ such that $S \not\cong C_q$. We have that $S \wr G(P) \in \mathfrak{F} \circ \mathfrak{G} = \mathfrak{H} \subseteq \mathfrak{N}\,\mathrm{form}(G)$. If S is not abelian, then $G(P) \in \mathrm{form}(G)$, that is, $\mathfrak{S}_p \subseteq \mathrm{S}\big(\mathrm{form}(G)\big)$. This is a contradiction by Lemma 4.1.2. Therefore $S \cong C_r$ for a prime $r \neq q$. Let $Y = C_r \wr G(P)$. Then $Y \in \mathfrak{H} \subseteq \mathfrak{N}\,\mathrm{form}(G)$. Moreover, $\mathrm{F}(Y)$ is an r-group and $Y/\mathrm{F}(Y) \in \mathrm{form}(G)$. It follows that $\mathrm{form}(G)$ contains a subgroup isomorphic to P. It is not possible by Lemma 4.1.2. □

When $\mathfrak{X} = \mathbb{P}$, the class of all abelian simple groups, the $\mathfrak{X}$-saturated formations are exactly the Baer formations. Therefore we have:

Corollary 4.1.5. *Let $\mathfrak{H}$ be the Baer formation generated by a group G. If $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ for non-trivial formations $\mathfrak{F}$ and $\mathfrak{G}$, and $\mathfrak{H} \neq \mathfrak{G}$ or $\mathfrak{S}_p\mathfrak{H} \neq \mathfrak{H}$ for all primes p, then $\mathfrak{F}$ is a Baer formation.*

The above result also seems to be the answer of the problem 18 of Guo's book [Guo00]. Note that there is an obvious misprint in the statement of that problem, because the example in page 224 answers that question negatively.

The same arguments to those used in the proof of Theorem 4.1.4 replacing $\Phi_{\mathfrak{X}}(G)$ by $\Phi(G) \cap \mathrm{O}_p(G)$ give an alternative proof of the result of Vishnevskaya for p-saturated formations. Moreover a second condition could be added to that result: $\mathfrak{G} \neq \mathfrak{S}_p\mathfrak{G}$. Taking into account that the ω-saturated formations are exactly the p-saturated ones for all $p \in \omega$, the main result of [Vis00] can be improved in the following way:

Theorem 4.1.6. *Let ω be set of primes. Let $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ be an ω-saturated formation generated by a group G. If $\mathfrak{F}$ and $\mathfrak{G}$ are non-trivial and $\mathfrak{H} \neq \mathfrak{G}$ or $\mathfrak{S}_p\mathfrak{H} \neq \mathfrak{H}$ for all primes $p \in \omega$, then $\mathfrak{F}$ is ω-saturated.*

4.2 A characterisation of factorised one-generated $\mathfrak{X}$-local formations

Now we aim to give a complete description of the factorisations of a one-generated $\mathfrak{X}$-local formation. These results appear in [BBC]. We will need the following lemmas:

Lemma 4.2.1 ([GS03; Lemma 3.11]). *Let $\mathfrak{F}$ be a Fitting formation such that $\mathfrak{F} = \mathfrak{F}\mathfrak{F}$ and $\mathfrak{K}$ a non-empty class of groups. If G is a group such that $G \in \mathrm{form}(\mathfrak{K})$ and $G_{\mathfrak{F}} = 1$, then $G \in \mathrm{form}(A/A_{\mathfrak{F}} \mid A \in \mathfrak{K})$.*

Lemma 4.2.2. *Consider a prime* $p \in \operatorname{char}\mathfrak{X}$ *and a group* G *such that* $\mathrm{O}_p(G) = 1$. *Let* W *denote the wreath product* $C_p \wr G$ *and* $B = C_p^\natural$ *the base group of* W. *Then* $\mathrm{C}^{\mathfrak{X}_p}(W) = B$.

Proof. $\mathrm{C}^{\mathfrak{X}_p}(W)$ can be regarded as a group of operators for B, acting by conjugation. Consider a chief series of W passing through B:

$$1 = T_0 \trianglelefteq T_1 \trianglelefteq \cdots \trianglelefteq T_n = B \trianglelefteq T_{n+1} \trianglelefteq \cdots \trianglelefteq T_m = W.$$

Then $\mathrm{C}^{\mathfrak{X}_p}(W)$ stabilises the following series of B:

$$1 = T_0 \trianglelefteq T_1 \trianglelefteq \cdots \trianglelefteq T_n = B.$$

Since B is a p-group, we can apply [DH92; A, 12.4] in order to deduce that $\mathrm{C}^{\mathfrak{X}_p}(W)/\,\mathrm{C}_{\mathrm{C}^{\mathfrak{X}_p}(W)}(B)$ is a p-group. We can conclude that $\mathrm{C}^{\mathfrak{X}_p}(W)$ is a p-group, since $\mathrm{C}_{\mathrm{C}^{\mathfrak{X}_p}(W)}(B) = \mathrm{C}_W(B) \cap \mathrm{C}^{\mathfrak{X}_p}(W) = B \cap \mathrm{C}^{\mathfrak{X}_p}(W)$ is a p-group. The result now follows, bearing in mind that $B \leq \mathrm{C}^{\mathfrak{X}_p}(W)$ and $\mathrm{O}_p(G) = 1$. □

The following lemma is a generalisation of [Ski92; Lemma 3].

Lemma 4.2.3. *Consider two non-empty formations* $\mathfrak{F}$ *and* $\mathfrak{G}$, $\mathfrak{G} \neq (1)$, *and the formation product* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$. *Let* n *be a natural number. Assume that there exists a group* $A \in \mathfrak{F}$ *such that for every* $B \in \mathfrak{G}$ *satisfying that* $|B| > n$, $(A \wr B)^{\mathfrak{G}}$ *is not subdirectly contained in the base group of* $A \wr B$. *Then there exists a prime* $p \in \pi(\mathfrak{H})$ *such that each* p*-group is isomorphic to a subgroup of a* $\mathfrak{G}$*-group.*

Proof. Note that $\mathfrak{G}$ contains groups of order greater than n. If $B \in \mathfrak{G}$ and $|B| > n$, consider the wreath product $B_1 = A \wr B$ and the base group $B_2 = A^\natural$ of B_1. Let A^B be the first copy of A in B_2 and A_B the projection of $B_1^{\mathfrak{G}}$ in A^B. Since $B_1^{\mathfrak{G}}$ is not subdirectly contained in B_2, it follows that A_B is strictly contained in A^B.

Now let D be a group in $\mathfrak{G}$ such that $|D| > n$. Denote $T_1 = (A^D/A_D) \wr D$. By Lemma 4.1.3, we have that $T_1 \in \mathfrak{G}$. Moreover, $|T_1| > n$. Now consider $T_2 = (A^{T_1}/A_{T_1}) \wr T_1$. Following the sequence, denote $T_t = (A^{T_{t-1}}/A_{T_{t-1}}) \wr T_{t-1}$. Clearly, $T_i \in \mathfrak{G}$ and $|T_i| > n$ for every index i. Since $\pi(A)$ is finite, there exists a prime p such that for every natural number i, there exists an index $\alpha_i > i$ such that $p \in \pi(A^{T_{\alpha_i}}/A_{T_{\alpha_i}})$.

Consider the sequence $G_0 = C_p$, $G_1 = C_p \wr C_p$, …, $G_i = C_p \wr G_{i-1}$. We prove that, for every natural number i, there exists a natural number k such that G_i is isomorphic to a subgroup of T_k. By induction, assume that G_{i-1} is isomorphic to a subgroup of T_j. There exists $t > j$ such that $p \in \pi(A^{T_t}/A_{T_t})$.

Since T_j is isomorphic to a subgroup of T_t, it follows that G_{i-1} is isomorphic to a subgroup S of T_t. By [DH92; A, 18.2], we have that $C_p \wr T_t$ is isomorphic to a subgroup of $(A^{T_t}/A_{T_t}) \wr T_t = T_{t+1}$. Let T be the base group of $C_p \wr T_t$. We have that TS is isomorphic to a subgroup of T_{t+1}. By [DH92; A, 18.8], $TS \cong C_p^m \wr G_{i-1}$, where $m = |T_t : S|$. Therefore, $C_p^m \wr G_{i-1}$ is isomorphic to a subgroup of T_{t+1}. Applying [DH92; A, 18.2], we have that $G_i = C_p \wr G_{i-1}$ is isomorphic to a subgroup of T_{t+1}.

Now let P be a p-group of order p^n. Let L be a normal subgroup of P of order p. Then $|P/L| = p^{n-1}$. Assume inductively that P/L is isomorphic to a subgroup T of G_i. Consider $G_{i+1} = C_p \wr G_i$ and let M be the base group of G_{i+1}. By [DH92; A, 18.8], $MT \cong C_p^m \wr T$, where $m = |G_i : T|$. Therefore, $C_p^m \wr T$ is isomorphic to a subgroup of G_{i+1}. Applying [DH92; A, 18.2], we have that $L \wr (P/L)$ is isomorphic to a subgroup of G_{i+1}. By [DH92; A, 18.9], $L \wr (P/L)$ contains a subgroup isomorphic with P. Therefore, P is isomorphic to a subgroup of G_{i+1}. □

In the sequel, π denotes the characteristic of $\mathfrak{X}$.

Definition 4.2.4. We say that a group G is *π-nilpotent* when it is p-nilpotent for all primes $p \in \pi$.

Definition 4.2.5. A formation $\mathfrak{F}$ is said to be nilpotent (respectively π-nilpotent, abelian, ...) whenever $\mathfrak{F}$ is contained in the class of all nilpotent (respectively π-nilpotent, abelian, ...) groups.

Theorem 4.2.6. *Let $\mathfrak{H}$ be a one-generated formation. Suppose that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ for some non-trivial formations $\mathfrak{F}$ and $\mathfrak{G}$. Assume also that $\mathfrak{H} \neq \mathfrak{G}$ or $\mathfrak{S}_p\mathfrak{H} \neq \mathfrak{H}$ for all primes $p \in \pi(\mathfrak{F})$. Put $\pi = \operatorname{char} \mathfrak{X}$.*

Then:

1. *Every simple group in $\mathfrak{F}$ is abelian.*
2. *Every primitive monolithic group in $\mathfrak{F}$ is abelian-by-nilpotent.*
3. *$\mathfrak{F}$ is metanilpotent and π-local.*
4. *$\mathcal{K}_{\mathfrak{A}}(\mathfrak{G}) \cap \mathfrak{X} \subseteq \mathfrak{F}$, where $\mathcal{K}_{\mathfrak{A}}(\mathfrak{G})$ is the class of all abelian composition factors of groups in $\mathfrak{G}$.*
5. *$H_{\mathfrak{X}}(p) = F_{\mathfrak{X}}(p) \circ \mathfrak{G}$ for every prime $p \in \pi$.*
6. *If $\mathfrak{F}$ is not π-nilpotent, then $\mathfrak{G}$ is abelian and one-generated.*
7. *If $A \in \mathfrak{F}$ and $B \in \mathfrak{G}$, then $\pi\big(A/\operatorname{C}^{\mathfrak{X}_p}(A)\big) \cap \pi(B) = \emptyset$ for every $p \in \pi$.*

8. *If $\mathfrak{F} = \mathfrak{S}_p$, the class of all p-groups, $p \in \pi$, then*

$$\mathfrak{G}/\operatorname{O}_p(\mathfrak{G}) := \operatorname{form}(X/\operatorname{O}_p(X) \mid X \in \mathfrak{G})$$

is a one-generated formation.

9. *If $\mathfrak{F}$ is π-nilpotent, $|\pi(\mathfrak{F})| > 1$ and $\pi(\mathfrak{F}) \subseteq \pi$, then $\mathfrak{G}$ is one-generated.*

Proof. To fix ideas, let us suppose that $\mathfrak{H}$ is the $\mathfrak{X}$-local formation generated by a group G.

1. Assume that it is not true and consider a non-abelian simple group $S \in \mathfrak{F}$. Let B be a group in $\mathfrak{G}$ such that $|B| > |G|$ and consider the wreath product $X := S \wr B$. Then $X \in \mathfrak{H}$. It is known that $\mathfrak{N} \operatorname{form}(G)$, where $\mathfrak{N}$ is the class of all nilpotent groups, is a local formation ([DH92; IV, 3.4(b)]). Therefore, $\mathfrak{N} \operatorname{form}(G)$ is $\mathfrak{X}$-local and so $\mathfrak{H}$ is contained in $\mathfrak{N} \operatorname{form}(G)$. It follows that $X \in \mathfrak{N} \operatorname{form}(G)$. Since $\operatorname{F}(X) = 1$, $X \in \operatorname{form}(G)$. By [DH92; A, 18.5], the base group $S^\natural$ of X is a minimal normal subgroup of X. Therefore, $S^\natural$ is isomorphic to a chief factor of G, which is a contradiction.

2. Suppose that A is a monolithic primitive group in $\mathfrak{F}$. We prove that $\operatorname{Soc}(A)$ is abelian and $A/\operatorname{Soc}(A)$ is nilpotent. Assume that the assertion is false. Then either $\operatorname{Soc}(A)$ is not abelian or $\operatorname{Soc}(A)$ is abelian and $A/\operatorname{Soc}(A)$ is not nilpotent. Let B be a group in $\mathfrak{G}$ such that $|B| > |G|$ and consider the wreath product $D := A \wr B$. Suppose that $\operatorname{Soc}(A)$ is not abelian. In this case, A is a primitive group of type 2. By Lemma 1.3.19, D is also a primitive group of type 2 and $\operatorname{Soc}(D) = \operatorname{Soc}(A)^\natural$. Since $\operatorname{Soc}(D)$ is not contained in $\operatorname{F}(D)$, it follows that $\operatorname{F}(D) = 1$. Now suppose that $\operatorname{Soc}(A)$ is abelian but $A/\operatorname{Soc}(A)$ is not nilpotent. Since $\operatorname{C}_A\big(\operatorname{Soc}(A)\big) = \operatorname{Soc}(A)$ and $A/\operatorname{Soc}(A) \neq 1$, we have that $\operatorname{Soc}(A)$ is not contained in $\operatorname{Z}(A)$. By [DH92; A, 18.5], $\operatorname{Soc}(A)^\natural$ is a minimal normal subgroup of D. Clearly, $\operatorname{Soc}(A)^\natural$ is contained in $\operatorname{F}(D)$ and, since $\operatorname{C}_D\big(\operatorname{Soc}(A)^\natural\big) = \operatorname{Soc}(A)^\natural$, it follows that $\operatorname{F}(D) = \operatorname{Soc}(A)^\natural$.

   Assume that $D \in \mathfrak{H}$. Then $D/\operatorname{F}(D) \in \operatorname{form}(G)$, because $\mathfrak{H}$ is contained in $\mathfrak{N} \operatorname{form}(G)$. If $\operatorname{F}(D) = 1$, we would have that $D \in \operatorname{form}(G)$. Then $\operatorname{Soc}(D) = \operatorname{Soc}(A)^\natural$ would be a chief factor of D, which is isomorphic to a chief factor of G. This is a contradiction. If $\operatorname{Soc}(A)$ is abelian and $A/\operatorname{Soc}(A)$ is not nilpotent, it follows that $D/\operatorname{Soc}(A)^\natural \in \operatorname{form}(G)$. Consider a chief factor H/K of A such that $\operatorname{Soc}(A) \leq K$ and $\operatorname{Z}(H/K) \neq A/K$. By [DH92; A, 18.5], it follows that $(H/K)^\natural$ is a minimal normal subgroup of $(A/K) \wr B$ isomorphic to $(A \wr B)/K^\natural$. Therefore $(H/K)^\natural$ is a chief factor of $D/\operatorname{Soc}(A)^\natural$. This means that $(H/K)^\natural$ is isomorphic to a chief factor of G, which is not possible.

We have proved that for every group in B in $\mathfrak{G}$ such that $|B| > |G|$, the wreath product $D := A \wr B$ does not belong to $\mathfrak{H}$, which means that $D^{\mathfrak{G}}$ is not subdirectly contained in $A^\natural$, the base group of D. By Lemma 4.2.3, there exists a prime q such that $\mathfrak{S}_q$, the class of all q-groups, is contained in $\mathrm{S}\,\mathfrak{G}$, the class of all subgroups of groups in $\mathfrak{G}$.

On the other hand, consider a group C in $\mathfrak{G}$ such that $|C| > |G|$ and the wreath product $E := A \wr C$. We know that $E^{\mathfrak{G}}$ is not subdirect in the base group $A^\natural$ of E. Let A_1 be the first copy of A in $A^\natural$ and let M be the projection of $E^{\mathfrak{G}}$ in A_1. Then M is a proper normal subgroup of A_1. By 4.1.3 , we have that $(A_1/M) \wr C$ is a quotient of $E/E^{\mathfrak{G}}$. Let A_0 be a maximal normal subgroup of A_1 containing M. Then A_1/A_0 is isomorphic to C_p for a prime p by Statement 1. By [DH92; A, 18.2], $(A_1/A_0) \wr C$ is a quotient of $(A_1/M) \wr C \in \mathfrak{G}$. Therefore $C_p \wr C \in \mathfrak{G}$. Let N be the diagonal subgroup of the base group $C_p^\natural$ of $C_p \wr C$. Then N is a central minimal normal subgroup of $C_p \wr C$ isomorphic to C_p. We can apply [DH92; IV, 1.5] to deduce that $C_p \in \mathfrak{G}$.

Now suppose that $p \neq q$ and consider a q-group Q. We know that there exists a group $G(Q)$ in $\mathfrak{G}$ such that Q is contained in $G(Q)$. Let $X = C_p \wr G(Q)$ be the corresponding wreath product. Since $X \in \mathfrak{H} \subseteq \mathfrak{N}\operatorname{form}(G)$, it follows that $X/\operatorname{F}(X) \in \operatorname{form}(G)$. Since $\operatorname{F}(X)$ is a p-group, we have that $X/\operatorname{F}(X)$ has a subgroup isomorphic to Q. This is a contradiction by Lemma 4.1.2. Therefore, $p = q$.

Now let us prove that $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$. If it is not true, we can consider a group $U \in \mathfrak{S}_p\mathfrak{G} \setminus \mathfrak{G}$ of minimal order. Then $V = \operatorname{Soc}(U)$ is a p-group and it is a minimal normal subgroup of U. Since $C_p \in \mathfrak{G}$, it follows that $U/V \neq 1$. Consider the group $W = A \wr (U^m/V^m)$, where $m = |G|$. We know that $(U^m/V^m)^{\mathfrak{G}}$ is not subdirect in the base group $A^\natural$ of W. Arguing as above, we have that $C_p \wr (U^m/V^m) \in \mathfrak{G}$. Thus $V^m \wr (U^m/V^m) \in \mathrm{Q}\,\mathrm{R}_0\big(C_p \wr (U^m/V^m)\big) \subseteq \mathfrak{G}$. Applying [DH92; A, 18.9 and IV, 1.14], U^m belongs to $\mathrm{Q}\,\mathrm{R}_0\big(V^m \wr (U^m/V^m)\big) \subseteq \mathfrak{G}$. This contradiction shows that $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$.

Suppose, arguing by contradiction, that $\mathfrak{H}$ is not equal to $\mathfrak{G}$ and take a group $X \in \mathfrak{H} \setminus \mathfrak{G}$ of minimal order. Then $X^{\mathfrak{G}}$ is a minimal normal subgroup of X and $X^{\mathfrak{G}} \in \mathfrak{F}$. By Statement 1, $X^{\mathfrak{G}}$ is an r-group for some prime r. Since $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$, it follows that r is different from p. Considering the wreath product $C_r \wr G(Q)$ of C_r with $G(Q)$, where $G(Q)$

is a group in $\mathfrak{G}$ containing the p-group Q, we have that every p-group is isomorphic to a subgroup of a group in the formation generated by G. This is a contradiction by Lemma 4.1.2. Consequently $\mathfrak{H} = \mathfrak{G}$ and so $\mathfrak{S}_p\mathfrak{H} = \mathfrak{S}_p\mathfrak{G} = \mathfrak{G} = \mathfrak{H}$. This final contradiction shows that A is abelian-by-nilpotent.

3. Suppose that $\mathfrak{F}$ is not metanilpotent and among the $\mathfrak{F}$-groups Z which are not metanilpotent, we take a group U of minimal order. It follows that U is a monolithic primitive group as the class of all metanilpotent groups is a saturated formation. By Statement 2, U is abelian by nilpotent. This contradicts the choice of U. Hence $\mathfrak{F}$ is a metanilpotent formation.

   By Theorem 4.1.4, we have that $\mathfrak{F}$ is an $\mathfrak{X}$-local formation. Since $\mathfrak{F}$ is composed by soluble groups, $\mathfrak{F}$ is a π-local formation.

4. Consider $C_p \in \mathcal{K}_{\mathfrak{A}}(\mathfrak{G}) \cap \mathfrak{X}$. Since $C_p \in \mathcal{K}_{\mathfrak{A}}(\mathfrak{H})$ and $\mathfrak{H}$ is $\mathfrak{X}$-local, we have that $\mathfrak{S}_p \subseteq \mathfrak{H}$ by Corollary 1.2.29. If $\mathfrak{S}_p \subseteq \mathfrak{F}$, then $C_p \in \mathfrak{F}$, as we wanted. Suppose that $\mathfrak{S}_p \not\subseteq \mathfrak{F}$. Since $\mathfrak{F}$ is $\mathfrak{X}$-local, this implies that there does not exist any p-group in $\mathfrak{F}$ by Corollary 1.2.29 again. Take a p-group P. Since $P \in \mathfrak{H}$, we obtain that $P^{\mathfrak{G}} \in \mathfrak{F}$. Thus $P^{\mathfrak{G}} = 1$. Therefore $P \in \mathfrak{G}$. It follows that $\mathfrak{S}_p \subseteq \mathfrak{G}$. Consider a prime $q \neq p$ such that $C_q \in \mathfrak{F}$ and $W = C_q \wr P$, where P is a p-group. Clearly, $W \in \mathfrak{H}$ and, hence, $W/\mathrm{F}(W) \in \mathrm{form}(G)$. Since $\mathrm{F}(W)$ is a q-group, $P \in \mathrm{form}(G)$. We have proved that $\mathfrak{S}_p \subseteq \mathrm{form}(G)$, which is not possible by Lemma 4.1.2. Therefore, $\mathcal{K}_{\mathfrak{A}}(\mathfrak{G}) \cap \mathfrak{X} \subseteq \mathfrak{F}$.

5. Consider a prime $p \in \pi$. If $\mathfrak{S}_p$ is not contained in $\mathfrak{H}$, then $\mathfrak{S}_p$ is not contained in $\mathfrak{F}$ either, and we have that $H_{\mathfrak{X}}(p) = F_{\mathfrak{X}}(p) = \emptyset$. Now suppose that $\mathfrak{S}_p$ is contained in $\mathfrak{H}$. If $\mathfrak{S}_p \subseteq \mathfrak{F}$, we can apply Theorem 3.1.4 to obtain that $H_{\mathfrak{X}}(p) = F_{\mathfrak{X}}(p) \circ \mathfrak{G}$. If $\mathfrak{S}_p \not\subseteq \mathfrak{F}$, it follows that $C_p \in \mathcal{K}_{\mathfrak{A}}(\mathfrak{G})$. Since $\mathcal{K}_{\mathfrak{A}}(\mathfrak{G}) \cap \mathfrak{X} \subseteq \mathfrak{F}$, we have that $C_p \in \mathfrak{F}$, a contradiction.

6. Assume that there exists a prime $p \in \pi$ such that $\mathfrak{F}$ is not p-nilpotent. If H/K is a p-chief factor of a group $U \in \mathfrak{F}$, then $U/\mathrm{C}_U(H/K)$ is a nilpotent p'-group by Statement 3. This means that $F_{\mathfrak{X}}(p) \subseteq \mathfrak{S}_p\mathfrak{N}_{p'}$, where $\mathfrak{N}_{p'}$ is the class of all nilpotent p'-groups. If $F_{\mathfrak{X}}(p) = \mathfrak{S}_p$, we would have that every p-chief factor of $\mathfrak{F}$ is central. Since this is not possible, we have that $F_{\mathfrak{X}}(p) \neq \mathfrak{S}_p$ and we may consider a prime $q \neq p$ such that $C_q \in F_{\mathfrak{X}}(p)$.

   Suppose that $\mathfrak{G}$ is not abelian and take a non-abelian $\mathfrak{G}$-group, B say. Assume that $q \in \pi(B)$ and let L be a subgroup of B of order

q. Let m be a natural number with $m > |G|$ and $W = C_q \wr B^m$. Clearly, $W \in F_{\mathfrak{X}}(p) \circ \mathfrak{G} = H_{\mathfrak{X}}(p)$. Since $\mathfrak{H} \subseteq \mathfrak{N}\,\mathrm{form}(G)$, it follows that $H_{\mathfrak{X}}(p) \subseteq \mathfrak{S}_p\,\mathrm{form}(G)$ and, hence, $W \in \mathrm{form}(G)$. Consider $(C_q)^{\natural}L^m \leq W$. By [DH92; A, 18.8], $(C_q)^{\natural}L^m \cong ((C_q)^{|B^m:L^m|} \wr L^m)$. Therefore, $C_q \wr L^m \in \mathrm{Q}\big((C_q)^{\natural}L^m\big)$ and hence the nilpotent class, $\mathrm{cl}(C_q \wr L^m)$, of $C_q \wr L^m$ is less or equal than nilpotent class of the Sylow q-subgroups of G. But $\mathrm{cl}(C_q \wr L^m) \geq m+1$, so we have reached a contradiction. We have proved that $q \notin \pi(B)$.

Consider the algebraic closure F of the field of q-elements and the regular module FB. By Maschke's theorem, $FB = V_1 \oplus V_2 \oplus \cdots \oplus V_r$, where V_i is an irreducible FB-module for $i = 1, 2, \ldots, r$. If $\dim V_i = 1$ for every $i \in \{1, 2, \ldots, r\}$, then $B/\mathrm{C}_B(V_i)$ is abelian and, hence, $B' \leq \bigcap \mathrm{C}_B(V_i) = 1$. This is not possible, since B is not abelian. Therefore, there exists an irreducible FB-module V such that $\dim V \geq 2$. By [DH92; B, 5.23], $V \otimes \overset{(m)}{\cdots} \otimes V$ is an irreducible FB^m-module. Applying [DH92; B, 5.14], there exists an irreducible $GF(q)B^m$-module W such that W_F contains $V \otimes \overset{(m)}{\cdots} \otimes V$ as a submodule. Then the corresponding semidirect product $R = [W]B^m$ belongs to $F_{\mathfrak{X}}(p) \circ \mathfrak{G}$. It follows that $R \in \mathrm{form}(G)$ and, hence, W is isomorphic to a chief factor of G. Since $\dim_F(W) \geq 2^m$, this is a contradiction. Therefore $\mathfrak{G}$ is an abelian formation.

Now let us see that $\mathfrak{G}$ is contained in $\mathrm{form}(G)$. If this is not true, consider a group $U \in \mathfrak{G} \setminus \mathrm{form}(G)$ of minimal order. It follows that U is an abelian monolithic group and so U is an r-group, where r is a prime. Consider the wreath product $W = C_q \wr U$. Then $W \in F_{\mathfrak{X}}(p) \circ \mathfrak{G} = H_{\mathfrak{X}}(p)$. It follows that $W \in \mathrm{form}(G)$ and, hence, $U \in \mathrm{form}(G)$, which contradicts the choice of U.

Since $\mathfrak{F}$ is soluble and $\mathfrak{G}$ is abelian, it follows that $\mathfrak{H}$ is soluble. Hence G is a soluble group. By [DH92; VII, 1.6], $\mathfrak{G}$ contains finitely many subformations. Consider the series $(1) = \mathfrak{G}_0 \subseteq \mathfrak{G}_1 \subseteq \cdots \subseteq \mathfrak{G}_{n-1} \subseteq \mathfrak{G}_n = \mathfrak{G}$, where $\mathfrak{G}_i$ is a maximal subformation of $\mathfrak{G}_{i+1}$. Clearly, $\mathfrak{G} = \mathrm{form}(\mathfrak{G}_{n-1}, G_{n-1})$, where G_{n-1} is a group in $\mathfrak{G} \setminus \mathfrak{G}_{n-1}$. Moreover, $\mathfrak{G}_{n-1} = \mathrm{form}(\mathfrak{G}_{n-2}, G_{n-2})$, where $G_{n-2} \in \mathfrak{G}_{n-1} \setminus \mathfrak{G}_{n-2}$. Repeating this process, we obtain that $\mathfrak{G} = \mathrm{form}(G_0, G_1, \ldots, G_{n-1})$, where $G_i \in \mathfrak{G}_{i+1} \setminus \mathfrak{G}_i$. Therefore, $\mathfrak{G} = \mathrm{form}(G_0 \times G_1 \times \cdots \times G_{n-1})$.

7. Consider a prime $p \in \pi$. Let q be a prime in $\pi\big(A/\mathrm{C}^{\mathfrak{X}_p}(A)\big)$. Since A is metanilpotent and $\mathrm{F}(A)$ is contained in $\mathrm{C}^{\mathfrak{X}_p}(A)$, we have that $A/\mathrm{C}^{\mathfrak{X}_p}(A)$ is nilpotent. Moreover, $\mathrm{O}_p\big(A/\mathrm{C}^{\mathfrak{X}_p}(A)\big) = 1$ and, therefore,

$p \notin \pi\big(A/\mathrm{C}^{\mathfrak{X}_p}(A)\big)$. This means that $p \neq q$. Since $A/\mathrm{C}^{\mathfrak{X}_p}(A) \in F_{\mathfrak{X}}(p)$ and $A/\mathrm{C}^{\mathfrak{X}_p}(A) \in \mathfrak{N}$, it follows that $C_q \in F_{\mathfrak{X}}(p)$. Arguing as in 6, we can prove that $q \notin \pi(B)$. Consequently, $\pi\big(A/\mathrm{C}^{\mathfrak{X}_p}(A)\big) \cap \pi(B) = \emptyset$.

8. Let A be a group in $\mathfrak{G}$. Then $T := C_p \wr \big(A/\mathrm{O}_p(A)\big) \in \mathfrak{F} \circ \mathfrak{G} = \mathfrak{H}$. By Lemma 4.2.2, it follows that $\mathrm{C}^{\mathfrak{X}_p}(T) = C_p^{\natural}$, the base group of T. Therefore, we have that $A/\mathrm{O}_p(A) \cong T/C_p^{\natural} \in \mathrm{form}\big(G/\mathrm{C}^{\mathfrak{X}_p}(G)\big)$ by Theorem 1.2.15. Hence $\mathfrak{G}/\mathrm{O}_p(\mathfrak{G})$ is contained in $\mathrm{form}(G/\mathrm{C}^{\mathfrak{X}_p}(G)\big)$. On the other hand, since $G \in \mathfrak{H} = \mathfrak{S}_p\mathfrak{G}$, we have that $G^{\mathfrak{G}} \in \mathfrak{S}_p$. Therefore $G/\mathrm{O}_p(G) \in \mathfrak{G}$. Since $\mathrm{O}_p(G) \leq \mathrm{C}^{\mathfrak{X}_p}(G)$ and $G/\mathrm{O}_p(G) \in \mathfrak{G}/\mathrm{O}_p(\mathfrak{G})$, we obtain that $G/\mathrm{C}^{\mathfrak{X}_p}(G) \in \mathfrak{G}/\mathrm{O}_p(\mathfrak{G})$. Consequently, $\mathrm{form}\big(G/\mathrm{C}^{\mathfrak{X}_p}(G)\big)$ is contained in $\mathfrak{G}/\mathrm{O}_p(\mathfrak{G})$ and so $\mathfrak{G}/\mathrm{O}_p(\mathfrak{G})$ is a one-generated formation.

9. Consider two primes p and q such that $p \neq q$ and $C_p, C_q \in \mathfrak{F}$. By Theorem 1.2.15, $\underline{h}(p)$ and $\underline{h}(q)$ are one-generated formations. Take two groups A and B such that $\underline{h}(p) = \mathrm{form}(A)$ and $\underline{h}(q) = \mathrm{form}(B)$. Since $\mathfrak{F}$ is p-nilpotent for every prime $p \in \pi$ and $\pi(\mathfrak{F}) \subseteq \pi$, it follows that $\mathfrak{F}$ is nilpotent. Therefore, $\underline{h}(p)$ and $\underline{h}(q)$ are contained in $\mathfrak{G}$. This implies that A and B belong to $\mathfrak{G}$ and, hence, $\mathrm{form}(A \times B) \subseteq \mathfrak{G}$. Assume that $\mathfrak{G}$ is not contained in $\mathrm{form}(A \times B)$ and consider a group D of minimal order in $\mathfrak{G} \setminus \mathrm{form}(A \times B)$. It follows that D is a monolithic group. Consider $R = \mathrm{Soc}(D)$. It is clear that either $\mathrm{O}_p(D) = 1$ or $\mathrm{O}_q(D) = 1$. Suppose that $\mathrm{O}_p(D) = 1$ and consider the wreath product $W = C_p \wr D$. Since $W \in \mathfrak{F} \circ \mathfrak{G} = \mathfrak{H}$ and $\mathrm{C}^{\mathfrak{X}_p}(W) = C_p^{\natural}$ by Lemma 4.2.2, we have that $D \cong W/C_p^{\natural} \in \underline{h}(p) = \mathrm{form}(A) \subseteq \mathrm{form}(A \times B)$. If $\mathrm{O}_q(D) = 1$, we obtain that $D \in \underline{h}(q) = \mathrm{form}(B) \subseteq \mathrm{form}(A \times B)$. This contradiction shows that $\mathfrak{G} = \mathrm{form}(A \times B)$. □

Our next aim is to prove that the converse of the above theorem holds provided that $\pi(\mathfrak{F}) \subseteq \pi$. We need the following lemma:

Lemma 4.2.7. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation. Assume that $\mathfrak{F}$ has an integrated $\mathfrak{X}$-formation function f such that:*

- *$f(p) = \emptyset$ except for a finite number of primes $p \in \mathrm{char}\,\mathfrak{X}$.*
- *If $f(p) \neq \emptyset$, there exists a group G_p such that $f(p) = \mathrm{form}(G_p)$.*
- *There exists a group H such that for every $E \in \mathfrak{X}'$, $f(E) = \mathrm{form}(H)$.*

Then $\mathfrak{F}$ is a one-generated $\mathfrak{X}$-local formation.

Proof. Let $\{p_1, p_2, \dots, p_n\}$ be the set of primes p such that $f(p) \neq \emptyset$. Consider the group

$$X = H \times \Big(C_{p_1} \wr \big(G_{p_1}/\operatorname{O}_{p_1}(G_{p_1})\big)\Big) \times \cdots \times \Big(C_{p_n} \wr \big(G_{p_n}/\operatorname{O}_{p_n}(G_{p_n})\big)\Big).$$

We will prove that $\mathfrak{F} = \operatorname{form}_{\mathfrak{X}}(X)$. Consider an $\mathfrak{X}$-formation function g such that $\operatorname{form}_{\mathfrak{X}}(X) = \operatorname{LF}_{\mathfrak{X}}(g)$. If E is a group in $\mathfrak{X}'$, we have that $H \in f(E) \subseteq \mathfrak{F}$. On the other hand, by Lemma 1.2.33, $C_{p_i} \wr \big(G_{p_i}/\operatorname{O}_{p_i}(G_{p_i})\big) \in \mathfrak{S}_{p_i} \operatorname{form}(G_{p_i}) \subseteq \mathfrak{F}$ for every $i \in \{1, \dots, n\}$. Therefore, we have that $X \in \mathfrak{F}$. Since $\mathfrak{F}$ is $\mathfrak{X}$-local, we obtain that $\operatorname{form}_{\mathfrak{X}}(X) \subseteq \mathfrak{F}$.

If $\mathfrak{F}$ is not contained in $\operatorname{form}_{\mathfrak{X}}(X)$, we may consider a group G of minimal order in $\mathfrak{F} \setminus \operatorname{form}_{\mathfrak{X}}(X)$. It follows that G is monolithic and $G/N \in \operatorname{form}_{\mathfrak{X}}(X)$, where $N = \operatorname{Soc}(G)$. If N is an $\mathfrak{X}'$-chief factor of G, then $G \in \operatorname{form}(H) \subseteq \operatorname{form}(X) \subseteq \operatorname{form}_{\mathfrak{X}}(X)$, which is a contradiction. Now assume that N is an $\mathfrak{X}$-chief factor of G. If $p \in \pi(N)$, we have that $G/\operatorname{C}_G(N) \in f(p) = \operatorname{form}(G_p)$. Since $\operatorname{O}_p\big(G/\operatorname{C}_G(N)\big) = 1$, it follows that $G/\operatorname{C}_G(N) \in \operatorname{form}\big(G_p/\operatorname{O}_p(G_p)\big)$, by Lemma 4.2.1. Let $T_p = C_p \wr \big(G_p/\operatorname{O}_p(G_p)\big)$. We have that $T_p \in \operatorname{form}(X) \subseteq \operatorname{form}_{\mathfrak{X}}(X)$ and $\operatorname{C}^{\mathfrak{X}_p}(T_p) = C_p^{\natural}$ by Lemma 4.2.2. Therefore, $G_p/\operatorname{O}_p(G_p) \cong T_p/C_p^{\natural} \in g(p)$. We obtain that $\operatorname{form}\big(G_p/\operatorname{O}_p(G_p)\big) \subseteq g(p)$ and, hence, $G/\operatorname{C}_G(N) \in g(p)$. By Lemma 1.2.12, it follows that $G \in \operatorname{LF}_{\mathfrak{X}}(g)$, contradicting the choice of G. Consequently, $\mathfrak{F} = \operatorname{form}_{\mathfrak{X}}(X)$. □

Theorem 4.2.8. *Let $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ be an $\mathfrak{X}$-local formation which is the product of the non-trivial formations $\mathfrak{F}$ and $\mathfrak{G}$. Put $\pi = \operatorname{char} \mathfrak{X}$. Assume that the following conditions hold:*

1. *$\mathfrak{F}$ is metanilpotent and a one-generated π-local formation.*

2. *$\mathcal{K}_{\mathfrak{A}}(\mathfrak{G}) \cap \mathfrak{X} \subseteq \mathfrak{F}$, where $\mathcal{K}_{\mathfrak{A}}(\mathfrak{G})$ is the class of all abelian composition factors of groups in $\mathfrak{G}$.*

3. *$H_{\mathfrak{X}}(p) = F_{\mathfrak{X}}(p) \circ \mathfrak{G}$ for every prime $p \in \pi$.*

4. *If $A \in \mathfrak{F}$ and $B \in \mathfrak{G}$, then $\pi\big(A/\operatorname{C}^{\mathfrak{X}_p}(A)\big) \cap \pi(B) = \emptyset$ for every $p \in \pi$.*

5. *If $\mathfrak{F}$ is not π-nilpotent, then $\mathfrak{G}$ is abelian and one-generated.*

6. *If $\mathfrak{F}$ is π-nilpotent, then $\mathfrak{G}$ is one-generated or*

$$\mathfrak{G}/\operatorname{O}_p(\mathfrak{G}) := \operatorname{form}(G/\operatorname{O}_p(G) \mid G \in \mathfrak{G})$$

   is one-generated for every $p \in \pi$.

If $\pi(\mathfrak{F}) \subseteq \pi$, then $\mathfrak{H}$ is a one-generated $\mathfrak{X}$-local formation.

Proof. First of all, applying Lemmas 1.2.34 and 1.2.53, we have that $\mathfrak{H} = \mathrm{LF}_{\mathfrak{X}}(h)$, where

$$\begin{aligned} h(p) &= \underline{h}(p) && \text{if } p \in \pi, \\ h(E) &= \mathrm{form}(H/H_{\mathrm{E}\,\mathfrak{X}} \mid H \in \mathfrak{H}) && \text{for every } E \in \mathfrak{X}'. \end{aligned}$$

On the other hand, since $\mathfrak{F}$ is a one-generated π-local formation and $\mathfrak{G}$ is one-generated or $\mathfrak{G}/\mathrm{O}_p(\mathfrak{G}) := \mathrm{form}(G/\mathrm{O}_p(G) \mid G \in \mathfrak{G})$ is one-generated for every $p \in \pi$, it follows that $\pi(\mathfrak{H})$ is finite. Therefore $h(p) = \emptyset$ except for a finite number of primes of π. Let $p \in \pi(\mathfrak{H}) \cap \pi$. Since $\mathfrak{H}$ is $\mathfrak{X}$-local, it follows that $C_p \in \mathfrak{H}$ and $h(p) = \underline{h}(p) \neq \emptyset$. Consequently, $C_p \in \mathfrak{F}$ or $C_p \in \mathfrak{G}$. In both cases, we have that $C_p \in \mathfrak{F}$, since $\mathcal{K}_{\mathfrak{A}}(\mathfrak{G}) \cap \mathfrak{X} \subseteq \mathfrak{F}$. It implies that $\mathfrak{G}/\mathrm{O}_p(\mathfrak{G})$ is contained in $\underline{h}(p)$ as the wreath product $C_p \wr X$, for each $X \in \mathfrak{G}$ such that $\mathrm{O}_p(X) = 1$, belongs to $\mathfrak{H}$ by Lemma 4.2.2.

Assume that $\mathfrak{F}$ is not π-nilpotent. In this case, $\mathfrak{G}$ is abelian and one-generated. We shall prove that that $h(p)$, where $p \in \pi$ and $h(p) \neq \emptyset$, and $h(E)$, where $E \in \mathfrak{X}'$, contain finitely many subformations.

Consider a prime $p \in \pi$ such that $h(p) \neq \emptyset$. Bearing in mind that $H_{\mathfrak{X}}(p) = F_{\mathfrak{X}}(p) \circ \mathfrak{G}$ for every prime $p \in \pi$, we deduce that $\underline{h}(p) \subseteq \underline{f}(p) \circ \mathfrak{G}$. In addition, $\underline{f}(p)$ is one-generated by Theorem 1.2.15. Since $\mathfrak{F}$ is metanilpotent, it follows that $\underline{f}(p)$ is nilpotent. On the other hand, $\pi(\underline{f}(p)) \cap \pi(\mathfrak{G}) = \emptyset$ by Condition 4. By [Ski83], $\underline{f}(p) \circ \mathfrak{G}$ is a one-generated formation. Since $\underline{f}(p) \circ \mathfrak{G}$ is soluble, it follows by [DH92; VII, 1.6] that it contains only finitely many subformations and so does $\underline{h}(p)$. Now let $H \in \mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$. Since $H^{\mathfrak{G}} \in \mathfrak{F}$ and $\mathfrak{F} \subseteq \mathrm{E}\,\mathfrak{X}$ because $\mathfrak{F}$ is soluble and $\pi(\mathfrak{F}) \subseteq \pi$, it follows that $H^{\mathfrak{G}} \leq H_{\mathrm{E}\,\mathfrak{X}}$. Therefore, $H/H_{\mathrm{E}\,\mathfrak{X}} \in \mathfrak{G}$. Consequently, $h(E) \subseteq \mathfrak{G}$ for every $E \in \mathfrak{X}'$. Applying [DH92; VII, 1.6], $\mathfrak{G}$ contains finitely many subformations and so does $h(E)$.

If $\mathfrak{M}$ is an $\mathfrak{X}$-local subformation of $\mathfrak{H}$, we know that $\mathfrak{M} = \mathrm{LF}_{\mathfrak{X}}(m)$, where

$$\begin{aligned} m(p) &= \underline{m}(p) && \text{if } p \in \pi, \\ m(E) &= \mathrm{form}(M/M_{\mathrm{E}\,\mathfrak{X}} \mid M \in \mathfrak{M}) && \text{for every } E \in \mathfrak{X}'. \end{aligned}$$

Observe that $m(p) \subseteq h(p)$ for every $p \in \pi$ and $m(E) \subseteq h(E)$ for every $E \in \mathfrak{X}'$. Therefore, there are only finitely many $\mathfrak{X}$-local subformations $\mathfrak{M}$ of $\mathfrak{H}$. Consider the series $(1) = \mathfrak{H}_0 \subseteq \mathfrak{H}_1 \subseteq \cdots \subseteq \mathfrak{H}_{n-1} \subseteq \mathfrak{H}_n = \mathfrak{H}$, where $\mathfrak{H}_i$ is a maximal $\mathfrak{X}$-local subformation of $\mathfrak{H}_{i+1}$. Clearly, $\mathfrak{H} = \mathrm{form}_{\mathfrak{X}}(\mathfrak{H}_{n-1}, G_{n-1})$, where G_{n-1} is a group in $\mathfrak{H} \setminus \mathfrak{H}_{n-1}$. Moreover, $\mathfrak{H}_{n-1} = \mathrm{form}_{\mathfrak{X}}(\mathfrak{H}_{n-2}, G_{n-2})$, where $G_{n-2} \in \mathfrak{H}_{n-1} \setminus \mathfrak{H}_{n-2}$. Repeating this process, we obtain that $\mathfrak{H} = \mathrm{form}_{\mathfrak{X}}(G_0, G_1, \ldots, G_{n-1})$, where $G_i \in \mathfrak{H}_{i+1} \setminus \mathfrak{H}_i$. Therefore, $\mathfrak{H} = \mathrm{form}_{\mathfrak{X}}(G_0 \times G_1 \times \cdots \times G_{n-1})$ and $\mathfrak{H}$ is a one-generated $\mathfrak{X}$-formation.

Now suppose that $\mathfrak{F}$ is π-nilpotent. Since $\pi(\mathfrak{F}) \subseteq \pi$, $\mathfrak{F}$ is actually a nilpotent formation.

Let p be a prime in π such that $h(p) \neq \emptyset$. Then

$$h(p) = \underline{h}(p) = \operatorname{form}(H/\operatorname{C}^{\mathfrak{X}_p}(H) \mid H \in \mathfrak{H}) \subseteq \mathfrak{G}.$$

Hence $h(p) = \mathfrak{G}/\operatorname{O}_p(\mathfrak{G})$, as $\operatorname{O}_p\big(H/\operatorname{C}^{\mathfrak{X}_p}(H)\big) = 1$ for every group $H \in \mathfrak{H}$. Now, if $\mathfrak{G}$ is one-generated, then so is $\mathfrak{G}/\operatorname{O}_p(\mathfrak{G})$ by Lemma 4.2.1. Consequently, $h(p)$ is one-generated for all $p \in \pi$.

Next we prove that $h(E)$ is a one-generated formation for $E \in \mathfrak{X}'$. We have that $h(E) \subseteq \mathfrak{G}$, because if $H \in \mathfrak{H}$, it follows that $H^{\mathfrak{G}} \in \mathfrak{F} \subseteq \operatorname{E}\mathfrak{X}$ and, hence $H^{\mathfrak{G}} \leq H_{\operatorname{E}\mathfrak{X}}$.

Assume that $\mathfrak{G}$ is a one-generated formation and consider a group T such that $\mathfrak{G} = \operatorname{form}(T)$. Let us prove that $h(E) = \operatorname{form}(T/T_{\operatorname{E}\mathfrak{X}})$. Consider a group $H \in \mathfrak{H}$. We know that $H/H_{\operatorname{E}\mathfrak{X}} \in \mathfrak{G}$. This implies, by Lemma 4.2.1, that $H/H_{\operatorname{E}\mathfrak{X}} \in \operatorname{form}(T/T_{\operatorname{E}\mathfrak{X}})$. Therefore, $h(E) \subseteq \operatorname{form}(T/T_{\operatorname{E}\mathfrak{X}})$. On the other hand, $T \in \mathfrak{G} \subseteq \mathfrak{H}$. Hence $T/T_{\operatorname{E}\mathfrak{X}} \in h(E)$ and $\operatorname{form}(T/T_{\operatorname{E}\mathfrak{X}}) \subseteq h(E)$.

Suppose that $\mathfrak{G}/\operatorname{O}_p(\mathfrak{G}) = \operatorname{form}(T)$. We aim to prove that $h(E) = \operatorname{form}(T/T_{\operatorname{E}\mathfrak{X}})$. If $H \in \mathfrak{H}$, then $H/H_{\operatorname{E}\mathfrak{X}} \in \mathfrak{G}$. Since $(H/H_{\operatorname{E}\mathfrak{X}})_{\operatorname{E}\mathfrak{X}} = 1$, we have by Lemma 4.2.1 that $H/H_{\operatorname{E}\mathfrak{X}} \in \operatorname{form}(T/T_{\operatorname{E}\mathfrak{X}})$. On the other hand, since $T \in \mathfrak{G} \subseteq \mathfrak{H}$, it follows that $T/T_{\operatorname{E}\mathfrak{X}} \in h(E)$ and $\operatorname{form}(T/T_{\operatorname{E}\mathfrak{X}}) \subseteq h(E)$.

Now we can apply Lemma 4.2.7 to conclude that $\mathfrak{H}$ is a one-generated $\mathfrak{X}$-local formation. □

The following result arises as a combination of Theorems 4.2.6 and 4.2.8.

Theorem 4.2.9. *Let $\mathfrak{X}$ be a class of simple groups such that $\pi = \pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$. Let $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ be an $\mathfrak{X}$-local formation such that $\pi(\mathfrak{F}) \subseteq \pi$. Assume that $\mathfrak{F}$ and $\mathfrak{G}$ are non-trivial and $\mathfrak{H} \neq \mathfrak{G}$ or $\mathfrak{S}_p\mathfrak{H} \neq \mathfrak{H}$ for all primes $p \in \pi$.*

Then $\mathfrak{H}$ is a one-generated $\mathfrak{X}$-local formation if and only if the following conditions hold:

1. *$\mathfrak{F}$ is metanilpotent and a one-generated local formation.*
2. *$\mathcal{K}_{\mathfrak{A}}(\mathfrak{G}) \cap \mathfrak{X} \subseteq \mathfrak{F}$, where $\mathcal{K}_{\mathfrak{A}}(\mathfrak{G})$ is the class of all abelian composition factors of groups in $\mathfrak{G}$.*
3. *$H_{\mathfrak{X}}(p) = F_{\mathfrak{X}}(p) \circ \mathfrak{G}$ for every prime $p \in \pi$.*
4. *If $A \in \mathfrak{F}$ and $B \in \mathfrak{G}$, then $\pi\big(A/\operatorname{C}^{\mathfrak{X}_p}(A)\big) \cap \pi(B) = \emptyset$ for every $p \in \pi$.*
5. *If $\mathfrak{F}$ is not π-nilpotent, then $\mathfrak{G}$ is abelian and one-generated.*

6. *If $\mathfrak{F}$ is π-nilpotent, then $\mathfrak{G}$ is one-generated or*

$$\mathfrak{G}/\operatorname{O}_p(\mathfrak{G}) := \operatorname{form}(G/\operatorname{O}_p(G) \mid G \in \mathfrak{G})$$

is one-generated for every $p \in \pi$.

Proof. Assume that $\mathfrak{H}$ is the $\mathfrak{X}$-local formation generated by a group G. By Theorem 4.2.6, $\mathfrak{F}$ is π-local. Since $\pi(\mathfrak{F}) \subseteq \pi$, it is clear that $\mathfrak{F}$ is a local formation. Since $\mathfrak{F}$ is metanilpotent, we obtain that $\underline{f}(p)$ is nilpotent. Applying [DH92; IV, 1.16], it follows that $\underline{f}(p)$ is S-closed. Now, by [DH92; IV, 3.14], we have that $\mathfrak{F}$ is an S-closed formation. Therefore, $\mathfrak{F} \subseteq \mathfrak{H} = \operatorname{form}_{\mathfrak{X}}(G) \subseteq \operatorname{lform}(G)$. By [BBS97], $\operatorname{lform}(G)$ has only many finitely S-closed subformations and so does $\mathfrak{F}$. Consider the series $(1) = \mathfrak{F}_0 \subseteq \mathfrak{F}_1 \subseteq \cdots \subseteq \mathfrak{F}_{n-1} \subseteq \mathfrak{F}_n = \mathfrak{F}$, where $\mathfrak{F}_i$ is a maximal S-closed local subformation of $\mathfrak{F}_{i+1}$. Since every local subformation of $\mathfrak{F}$ is S-closed, we have that $\mathfrak{F}_i$ is a maximal local subformation of $\mathfrak{F}_{i+1}$. Clearly, $\mathfrak{F} = \operatorname{lform}(\mathfrak{F}_{n-1}, G_{n-1})$, where G_{n-1} is a group in $\mathfrak{F} \setminus \mathfrak{F}_{n-1}$. Moreover, $\mathfrak{F}_{n-1} = \operatorname{lform}(\mathfrak{F}_{n-2}, G_{n-2})$, where $G_{n-2} \in \mathfrak{F}_{n-1} \setminus \mathfrak{F}_{n-2}$. Repeating this process, we obtain that $\mathfrak{F} = \operatorname{lform}(G_0, G_1, \ldots, G_{n-1})$, where $G_i \in \mathfrak{F}_{i+1} \setminus \mathfrak{F}_i$. Therefore, $\mathfrak{F} = \operatorname{lform}(G_0 \times G_1 \times \cdots \times G_{n-1})$.

The converse is clear by Theorem 4.2.8, bearing in mind that if $\mathfrak{F}$ is a one-generated local formation, it can be also seen as a one-generated π-local formation, since $\pi(\mathfrak{F}) \subseteq \pi$. □

Remarks 4.2.10.

1. The main result of [Ski83] is Theorem 4.2.9 for the class $\mathfrak{X}$ of all simple groups, and the main result of [GS01] is our Theorem 4.2.9 for the class $\mathfrak{X}$ of all abelian simple groups.

2. A representation of a non-trivial formation $\mathfrak{H}$ as the product $\mathfrak{H} = \mathfrak{H}_1 \circ \cdots \circ \mathfrak{H}_t$, where $\mathfrak{H}_1, \ldots, \mathfrak{H}_t$ are formations, is called *irreducible* if $\mathfrak{H} \neq \mathfrak{H}_1 \circ \cdots \circ \mathfrak{H}_{i-1} \mathfrak{H}_{i+1} \circ \cdots \circ \mathfrak{H}_t$ for all $i = 1, 2, \ldots, t$. Assume that $\mathfrak{H}$ is a one-generated $\mathfrak{X}$-local formation. If $\mathfrak{H} = \mathfrak{H}_1 \circ \cdots \circ \mathfrak{H}_t$ is an irreducible representation of $\mathfrak{H}$, $t \geq 4$, and $\pi(\mathfrak{H}_i) \subseteq \pi$ for all $i = 1, 2, 3$, it follows that $\mathfrak{H}_1 \circ \mathfrak{H}_2 \circ \mathfrak{H}_3$ is a metanilpotent one-generated local formation. Applying the main result of [Ski92], $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are nilpotent, $\mathfrak{H}_3$ is abelian and $\mathfrak{H}_2 \cap \mathfrak{H}_3 = (1)$. Moreover $|\pi(\mathfrak{H}_1)| > 1$ by Theorem 4.2.9. This is impossible by considering suitable non-trivial groups $A \in \mathfrak{H}_1$, $B \in \mathfrak{H}_2$, $C \in \mathfrak{H}_3$ and the wreath product $A \wr (B \wr C)$. Therefore, $t \leq 3$. Assume that $t = 3$ and $\mathfrak{H} = \mathfrak{H}_1 \circ \mathfrak{H}_2 \circ \mathfrak{H}_3$. Since $\mathfrak{H}_3 \neq \mathfrak{H}$, it follows that $\mathfrak{H}_1 \circ \mathfrak{H}_2$ is a metanilpotent one-generated formation. If $\mathfrak{H}_1 \circ \mathfrak{H}_2$ were nilpotent, we would have that either $\mathfrak{H}_1 \circ \mathfrak{H}_2 = \mathfrak{H}_2$ or $\mathfrak{H}_1 \circ \mathfrak{H}_2 \subseteq \mathfrak{H}_1$ by [BBPR98; Corollary A.2]. Clearly $\mathfrak{H}_1 \circ \mathfrak{H}_2 = \mathfrak{H}_2$ contradicts the irreducibility of the factorisation. Hence $\mathfrak{H}_1 \circ \mathfrak{H}_2 \subseteq \mathfrak{H}_1$. On the other hand, if $\mathfrak{H}_1$ is not nilpotent, then $\mathfrak{H}_2 \circ \mathfrak{H}_3$ is abelian by Theorem 4.2.9. It implies that

$\mathfrak{H}_2 \circ \mathfrak{H}_3 = \mathfrak{H}_3$ or $\mathfrak{H}_2 \circ \mathfrak{H}_3 \subseteq \mathfrak{H}_2 \subseteq \mathfrak{H}_2 \circ \mathfrak{H}_3$ since $\mathfrak{H}_2$ is nilpotent. This is not possible as the factorisation is irreducible. Hence $\mathfrak{H}_1$ is nilpotent and so $\mathfrak{H}_1 = \mathfrak{H}_1 \circ \mathfrak{H}_2$, another contradiction. Consequently, $\mathfrak{H}_1 \circ \mathfrak{H}_2$ is not nilpotent. Applying Theorem 4.2.9, $\mathfrak{H}_3$ is abelian. Suppose that $\mathfrak{H}_1$ is not nilpotent. Then $\mathfrak{H}_2 \circ \mathfrak{H}_3$ is abelian. This is not possible, since we can consider a prime p such that $C_p \in \mathfrak{H}_2$, take a group $X \in \mathfrak{H}_3$, $X \neq 1$, and construct the wreath product $C_p \wr X$, which belongs to $\mathfrak{H}_2 \circ \mathfrak{H}_3$ and is not abelian. Therefore $\mathfrak{H}_1$ is not nilpotent. Assume finally that $\mathfrak{H}_2$ is not nilpotent and let X be a non-nilpotent group in $\mathfrak{H}_2$ of minimal order. Then $\mathrm{Soc}(X) = \mathrm{F}(X)$ is a q-group. Let $p \neq q$ be a prime dividing the order of $X/\mathrm{F}(X)$ and consider $T = C_p \wr X$. By Theorem 4.2.9, $C_p \in \mathfrak{H}_1$ and so $T \in \mathfrak{H}_1 \circ \mathfrak{H}_2$. It is clear that T is not metanilpotent. Consequently $\mathfrak{H}_2$ is nilpotent. Now assume that $q \in \pi(\mathfrak{H}_2)$. Let $p \neq q$ be a prime such that $C_p \in \mathfrak{H}_1$. Then if $A = C_p \wr C_q \in \mathfrak{H}_1\mathfrak{H}_2$, it follows that $A/\mathrm{C}^{\mathfrak{X}_p}(A) \cong C_q$. By Theorem 4.2.9, $q \notin \pi(\mathfrak{H}_3)$.
Therefore we have:
Theorem 4.2.11. *If $\mathfrak{H} = \mathfrak{H}_1 \circ \cdots \circ \mathfrak{H}_t$ is an irreducible factorisation of a one-generated $\mathfrak{X}$-local formation and $\pi(\mathfrak{H}_i) \subseteq \pi$ for all $i = 1, 2, 3$, then $t \leq 3$, and if $t = 3$, then $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are nilpotent, $\mathfrak{H}_3$ is abelian and $\pi(\mathfrak{H}_2) \cap \pi(\mathfrak{H}_3) = \emptyset$.*
Consequently Theorems 4.2.9 and 4.2.11 extend the main results of [Ski92] and [GS03].

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Apèndix A

Resum

Tots els grups considerats al llarg del treball són finits. Les principals referències bibliogràfiques són el llibre de Doerk i Hawkes ([DH92]) i el de Ballester i Ezquerro ([BBE06]).

La tesi s'emmarca dins de la teoria de formacions de grups. Una *formació* és una classe de grups que compleix les dues següents propietats:

- Si $G \in \mathfrak{F}$ i $N \trianglelefteq G$, aleshores $G/N \in \mathfrak{F}$.
- Si G/N, $G/M \in \mathfrak{F}$, aleshores $G/(M \cap N) \in \mathfrak{F}$.

Diem que una formació $\mathfrak{F}$ és *saturada* si, per a cada grup G, la condició $G/\Phi(G) \in \mathfrak{F}$ implica que $G \in \mathfrak{F}$, on $\Phi(G)$ és el subgrup de Frattini de G.

Una *funció formació* f associa a cada nombre primer p una formació $f(p)$, que pot ser buida. Si f és una funció formació, aleshores la *formació local* $\mathrm{LF}(f)$ definida per f és la clase de tots els grups G que satisfan la següent propietat:

- Si H/K és un factor principal de G, aleshores $G/\mathrm{C}_G(H/K) \in f(p)$ per a tot $p \in \pi(H/K)$.

Una formació $\mathfrak{F}$ és *local* si existeix una funció formació f tal que $\mathfrak{F} = \mathrm{LF}(f)$. Gaschütz va introduir aquest concepte a [Gas63], la qual cosa li permetia construir una gran família de formacions saturades. De fet, el teorema de Gaschütz-Lubeseder-Schmid, que és un dels resultats més rellevants en teoria de formacions, afirma que amb el mètode anterior es poden obtenir totes les formacions saturades. Aquest teorema va ser provat per Gaschütz i Lubeseder en l'univers resoluble. Posteriorment, Schmid va provar que aquesta restricció és innecessària.

Baer va generalitzar de manera diferent el teorema anterior en l'univers finit. Va fer servir un concepte diferent de formació local, tractant els factors principals no abelians amb més flexibilitat que els abelians. Açò el va portar a trobar una família de formacions, la de les formacions Baer-locals, que conté les locals. Aquestes dues famílies coincideixen en l'univers resoluble. Una *funció de Baer* associa a cada grup simple J una classe de grups $f(J)$, de forma que $f(C_p)$ és una formació per a tot nombre primer p. Si f és una funció de Baer, aleshores la *formació Baer-local* o *formació de Baer* $\mathrm{BLF}(f)$ definida per f és la classe de tots els grups G que satisfan la següent propietat:

- Si H/K és un factor principal de G, aleshores $G/\mathrm{C}_G(H/K) \in f(J)$, on J és el factor de composició de H/K.

Les formacions Baer-locals també van ser estudiades per Shemetkov (vegeu [She75]), sota el nom de *formacions de composició* (*composition formations*). Diem que una formació $\mathfrak{F}$ és *resolublement saturada* si, per a cada grup G, la condició $G/\Phi(G_{\mathfrak{S}}) \in \mathfrak{F}$ implica que $G \in \mathfrak{F}$, on $G_{\mathfrak{S}}$ és el radical resoluble de G. Baer va provar que una formació és resolublement saturada si, i només si, és Baer-local.

Amb el propòsit de presentar una generalització comú dels teoremes de Gaschütz-Lubeseder-Schmid i de Baer, Förster va introduir en [För85] el concepte de formació $\mathfrak{X}$-local, on $\mathfrak{X}$ és una classe de grups simples que compleix que $\pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$, on

$$\pi(\mathfrak{X}) := \{p \in \mathbb{P} \mid \text{existeix } G \in \mathfrak{X} \text{ tal que } p \in \pi(G)\}$$

i

$$\operatorname{char}\mathfrak{X} := \{p \in \mathbb{P} \mid C_p \in \mathfrak{X}\}.$$

Anomenem $\mathfrak{J}$ a la classe de tots els grups simples. Si $\mathfrak{Y}$ és una subclasse de $\mathfrak{J}$, $\mathfrak{Y}'$ denota la classe $\mathfrak{J} \setminus \mathfrak{Y}$. Fem servir $\mathrm{E}\,\mathfrak{Y}$ per denotar la classe de tots els grups els factors de composició dels quals es troben en $\mathfrak{Y}$. Tot grup G posseeix un subgrup normal que pertany a $\mathrm{E}\,\mathfrak{Y}$ i que conté tots els subgrups normals de G en $\mathrm{E}\,\mathfrak{Y}$, és a dir, un $\mathrm{E}\,\mathfrak{Y}$-radical. Per referir-nos-hi escrivim $\mathrm{O}_{\mathfrak{Y}}(G)$ o bé $G_{\mathrm{E}\,\mathfrak{Y}}$. Si H/K és un factor principal d'un grup G i $H/K \in \mathrm{E}\,\mathfrak{Y}$, diem que H/K és un $\mathfrak{Y}$*-factor principal* de G. Escrivim $\mathfrak{Y}_p$ per denotar la classe dels grups simples de $\mathfrak{Y}$ l'ordre del qual és divisible per p, on p és un nombre primer. Si $\mathfrak{Y}$ és una classe de grups, la menor formació que conté $\mathfrak{Y}$ es denota per $\mathrm{form}(\mathfrak{Y})$.

Una $\mathfrak{X}$*-funció formació* f és una aplicació que associa una formació, possiblemente buida, a cada grup simple del conjunt $(\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}'$. Si f és una $\mathfrak{X}$-funció formació, $\mathrm{LF}_{\mathfrak{X}}(f)$ es defineix com la classe de tots els grups G que satisfan les dues següents propietats:

1. Si $p \in \operatorname{char}(\mathfrak{X})$ i H/K és un $\mathfrak{X}_p$-factor principal de G, aleshores $G/\operatorname{C}_G(H/K) \in f(p)$.
2. Si $L \trianglelefteq G$, G/L és monolític i $\operatorname{Soc}(G/L)$ és un $\mathfrak{X}'$-factor principal de G, aleshores $G/L \in f(E)$, on E és el factor de composició de $\operatorname{Soc}(G/L)$.

Förster demostra en [För85] que la classe $\operatorname{LF}_{\mathfrak{X}}(f)$ és una formació. Una formació $\mathfrak{F}$ de grups finits és $\mathfrak{X}$-*local* si existeix una $\mathfrak{X}$-funció formació f tal que $\mathfrak{F} = \operatorname{LF}_{\mathfrak{X}}(f)$. Si $\mathfrak{Y}$ és una classe de grups, la menor formació $\mathfrak{X}$-local que conté $\mathfrak{Y}$ es denota per $\operatorname{form}_{\mathfrak{X}}(\mathfrak{Y})$.

Si $\mathfrak{X}$ és la classe de tots els grups simples, les formacions $\mathfrak{X}$-locals són exactament les formacions locals. Si $\mathfrak{X}$ és la classe dels grups simples abelians, es demostra que les formacions $\mathfrak{X}$-locals coincideixen amb les Baer-locals. Al capítol 1 de la tesi, presentem resultats bàsics de formacions $\mathfrak{X}$-locals i caracteritzem aquestes formacions com les classes de grups amb factors principals centrals generalitzats. Si f és una $\mathfrak{X}$-funció formació, definim el concepte de factor principal f-central i provem el següent teorema:

Teorema ([BBCSss])**.** *Considerem una formació $\mathfrak{X}$-local $\mathfrak{F}$ i una $\mathfrak{X}$-funció formació f que definisca $\mathfrak{F}$. Aleshores $\mathfrak{F}$ coincideix amb la classe dels grups els factors principals dels quals són f-centrals.*

Förster també va introduir a [För85] un subgrup $\mathfrak{X}$-Frattini $\Phi^*_{\mathfrak{X}}(G)$ per a cada grup G. Va definir el concepte de $\mathfrak{X}$-saturació de la manera natural i va provar que les formacions $\mathfrak{X}$-saturadas són exactament les $\mathfrak{X}$-locals. Tot i que els teoremes de Gaschütz-Lubeseder-Schmid i Baer es dedueixen del teorema de Förster, la definició que fa servir de $\mathfrak{X}$-saturació no és la més natural i la demostració del teorema de Baer no es dedueix fàcilment del seu resultat. A la segona secció del capítol 1, introduïm un nou subgrup de tipus Frattini $\Phi_{\mathfrak{X}}(G)$ per a cada grup G associat a la classe $\mathfrak{X}$. Es defineix de la següent manera:

- Considerem un nombre primer p. Diem que un grup G pertany a la classe $\operatorname{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ si es compleix:
  1. $\Phi(G)$ és p-elemental abelià.
  2. $G/\Phi(G)$ és un grup primitiu amb un únic subgrup normal minimal no abelià, és a dir, $G/\Phi(G)$ és un grup primitiu de tipus 2.
  3. $\operatorname{Soc}\big(G/\Phi(G)\big) \in \mathrm{E}\,\mathfrak{X} \setminus \mathfrak{E}_{p'}$
  4. $\operatorname{C}^h_G\big(\Phi(G)\big) \leq \Phi(G)$, on

$$\operatorname{C}^h_G\big(\Phi(G)\big) := \bigcap\{\operatorname{C}_G(H/K) \mid H/K \text{ és un factor principal de } G \text{ tal que } H \leq \Phi(G)\}$$

- El subgrup $\mathfrak{X}$-Frattini de G, $\Phi_{\mathfrak{X}}(G)$, es defineix com

$$\Phi_{\mathfrak{X}}(G) := \begin{cases} \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) & \text{si } G \notin \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2) \text{ per a tot } p \in \mathrm{char}(\mathfrak{X}), \\ \Phi(G) & \text{en un altre cas.} \end{cases}$$

- Una formación $\mathfrak{F}$ es diu *$\mathfrak{X}$-saturada* si $G/\Phi_{\mathfrak{X}}(G) \in \mathfrak{F}$ sempre implica que $G \in \mathfrak{F}$.

Aquest subgrup $\mathfrak{X}$-Frattini proporciona una definició de $\mathfrak{X}$-saturació més senzilla que la de Förster i més propera a la definició natural. Al final del capítol provem que les formacions $\mathfrak{X}$-saturades, amb aquesta nova definició de $\mathfrak{X}$-saturació, també coincideixen amb les formacions $\mathfrak{X}$-locals de Förster ([BBCER05]).

Shemetkov i Skiba van introduir el concepte de formació ω-local, on ω és un conjunt de nombres primers. Un *ω-satèl·lit local* f associa a cada element de $\omega \cup \{\omega'\}$ una formació, que pot ser buida. El símbol $G_{\omega d}$ denota el subgrup normal més gran N de G tal que $\omega \cap \pi(H/K) \neq \emptyset$ per a tot factor de composició H/K de N (si $\omega \cap \pi\big(\mathrm{Soc}(G)\big) = \emptyset$, definim $G_{\omega d} = 1$). Si f és un ω-satèl·lit local, aleshores $\mathrm{LF}_{\omega}(f)$ denota la classe de grups G que satisfan les dues següents condicions:

1. Si H/K és un factor principal de G i $p \in \pi(H/K) \cap \omega$, aleshores $G/\mathrm{C}_G(H/K) \in f(p)$.
2. $G/G_{\omega d} \in f(\omega')$.

Hom diu que una formació $\mathfrak{F}$ és *ω-local* si existeix un ω-satèl·lit local f tal que $\mathfrak{F} = \mathrm{LF}_{\omega}(f)$. En aquest cas, es diu que f és un ω-satèl·lit local de $\mathfrak{F}$. Si p és un nombre primer, es diu que una formació $\mathfrak{F}$ és *p-saturada* si $G \in \mathfrak{F}$ sempre que $G/\mathrm{O}_p(G) \cap \Phi(G) \in \mathfrak{F}$. Si ω és un conjunt de primers, diem que $\mathfrak{F}$ és *ω-saturada* si $\mathfrak{F}$ és p-saturada per a cada primer $p \in \omega$.

A [SS00] es prova el següent teorema:

Teorema. *Una formación $\mathfrak{F}$ és ω-saturada si, i només si, $\mathfrak{F}$ és ω-local.*

Si ω és el conjunt de tots els nombres primers, les formacions ω-locals són exactament les locals, per la qual cosa la idea de formació ω-local és un altre apropament al concepte de formació local. Al segon capítol estudiem la relació entre les formacions $\mathfrak{X}$-locals i les formacions ω-locals. Els primers resultats del capítol apareixen en [BBCER03]. Si ω és un conjunt de nombres primers i $\mathfrak{F}$ una formació ω-saturada, se segueix que $\mathfrak{F}$ és $\mathfrak{X}_{\omega}$-saturada, on $\mathfrak{X}_{\omega}$ és la classe dels ω-grups simples. Tanmateix, la família de les formacions $\mathfrak{X}_{\omega}$-saturades no coincideix en general amb la de les formacions ω-saturades. Açò se segueix del fet que existeixen formacions de Baer que no són ω-saturades per a cap conjunt no buit ω de primers. La següent pregunta sorgeix de manera natural:

> *Considerem un conjunt de nombres primers ω. Es pot assegurar l'existència d'una classe $\mathfrak{X}(\omega)$ de grups simples tal que $\operatorname{char}\big(\mathfrak{X}(\omega)\big) = \pi\big(\mathfrak{X}(\omega)\big)$ que complisca que una formació és ω-saturada si, i només si, és $\mathfrak{X}(\omega)$-saturada?*

Provem mitjançant un exemple que la resposta és negativa. En el següent resultat, que apareix en [BBE06; Chapter 3] i [BBCSss], provem que una formació $\mathfrak{X}$-local $\mathfrak{F}$ sempre conté una formació ω-local, on $\omega = \operatorname{char}\mathfrak{X}$, que té la propietat de ser la formación ω-local més gran continguda en $\mathfrak{F}$.

Teorema. *Considerem una classe de grups simples $\mathfrak{X}$ tal que $\omega = \pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$. Considerem una formació $\mathfrak{X}$-local $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$. Aleshores la formació ω-local $\mathfrak{F}_\omega = \mathrm{LF}_\omega(f)$, on $f(p) = F(p)$ per a tot $p \in \omega$ i $f(\omega') = \mathfrak{F}$, és la formación ω-local més gran continguda en $\mathfrak{F}$.*

Com a corol·lari, obtenim els següents resultats:

Corol·lari. *Considerem un conjunt de primers ω i una formació $\mathfrak{F}$ composta de grups ω-separables. Aleshores $\mathfrak{F}$ és ω-local si, i només si, $\mathfrak{F}$ és $\mathfrak{X}(\omega)$-local.*

Corol·lari. *Tota formació Baer-local conté una formació local maximal respecte de la inclusió.*

A més a més, donem condicions per assegurar que una formació $\mathfrak{X}$-local és ω-local per a $\omega = \operatorname{char}\mathfrak{X}$, com mostra el següent resultat. En l'enunciat, $\underline{f}$ denota la $\mathfrak{X}$-función formación més petita que defineix $\mathfrak{F}$.

Teorema ([BBCSss])**.** *Siga $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$ una formació $\mathfrak{X}$-local i $\omega = \operatorname{char}\mathfrak{X}$. Les següents condicions són equivalents:*

1. *$\mathfrak{F}$ és ω-local.*
2. *$G/\mathrm{c}_G(H/K) \in \underline{f}(p)$ per a cada $G \in \mathfrak{F}$ i cada $\mathfrak{X}'$-factor principal H/K de G tal que $p \in \pi(H/K) \cap \omega$.*
3. *$\underline{f}(S) \subseteq \underline{f}(p)$ per a cada $S \in \mathfrak{X}'$ i $p \in \pi(S) \cap \omega$.*
4. *$\mathfrak{S}_p\underline{f}(S) \subseteq \mathfrak{F}$ per a cada $S \in \mathfrak{X}'$ i $p \in \pi(S) \cap \omega$.*

Donades dues classes de grups $\mathfrak{Y}$ i $\mathfrak{Z}$, es pot definir la classe producte de la següent manera:

$$\mathfrak{Y}\mathfrak{Z} = (G \in \mathfrak{E} \mid \text{existeix un subgrup normal } N \text{ de } G \text{ tal que } N \in \mathfrak{Y} \text{ i } G/N \in \mathfrak{Z}).$$

Tanmateix, aquesta classe no és en general una formació quan $\mathfrak{Y}$ i $\mathfrak{Z}$ són formacions. Es pot modificar la definició anterior per assegurar que açò sí ocòrrega, és a dir, que la classe producte de dues formacions és de nou una formació. Si $\mathfrak{F}$ i $\mathfrak{G}$ són formacions, el *producte formació* o *producte de Gaschütz* de $\mathfrak{F}$ i $\mathfrak{G}$ és la classe $\mathfrak{F} \circ \mathfrak{G}$ així definida:

$$\mathfrak{F} \circ \mathfrak{G} := (G \in \mathfrak{E} \mid G^{\mathfrak{G}} \in \mathfrak{F}).$$

És conegut el fet que el producte formació de dues formacions locals és de nou una formació local (vegeu [DH92; IV, 3.13 i 4.8]). Tanmateix, el producte formació de dues formacions $\mathfrak{X}$-locals no és en general una formació $\mathfrak{X}$-local. Per tant, és natural preguntar-se el següent:

> *Quines condicions podem exigir a dues formacions $\mathfrak{F}$ i $\mathfrak{G}$ per assegurar que $\mathfrak{F} \circ \mathfrak{G}$ és una formació $\mathfrak{X}$-local?*

Aquesta qüestió, que ja va ser estudiada per Salomon en [Sal83] per a formacions Baer-locals, és el centre del capítol 3 de la tesi. Els nostres resultats han sigut publicats en [BBCER06].

Si $\mathfrak{K}$ és una classe de grups i $p \in \operatorname{char} \mathfrak{X}$, definim la classe

$$K_{\mathfrak{X}}(p) := \mathfrak{S}_p \operatorname{form}\bigl(G/\operatorname{C}_G(H/K) \mid G \in \mathfrak{K} \text{ i } H/K \text{ és un } \mathfrak{X}_p\text{-factor principal de } G\bigr).$$

Tinguem en compte que $K_{\mathfrak{X}}(p) = \emptyset$ si no existeix cap grup $G \in \mathfrak{K}$ que tinga algun $\mathfrak{X}_p$-factor principal.

El primer resultat del capítol afirma que si $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, on $\mathfrak{F}$ i $\mathfrak{G}$ són formacions no buides i $\mathfrak{F}$ és una formació $\mathfrak{X}$-local, aleshores la formació $\mathfrak{X}$-local més petita $\operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$ que conté $\mathfrak{H}$ es pot definir mitjançant la $\mathfrak{X}$-funció formació h donada per:

$$h(p) = \begin{cases} F_{\mathfrak{X}}(p) \circ \mathfrak{G} & \text{if } \mathfrak{S}_p \subseteq \mathfrak{F}, \\ G_{\mathfrak{X}}(p) & \text{if } \mathfrak{S}_p \nsubseteq \mathfrak{F}, \end{cases} \qquad \text{si } p \in \operatorname{char} \mathfrak{X};$$
$$h(S) = \mathfrak{H} \qquad \text{si } S \in \mathfrak{X}'.$$

Considerem $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, on $\mathfrak{F}$ i $\mathfrak{G}$ són formacions no buides. Diem que la frontera $\operatorname{b}(\mathfrak{H})$ de $\mathfrak{H}$ és $\mathfrak{X}\mathfrak{G}$-*lliure* si cada grup $G \in \operatorname{b}(\mathfrak{H})$ tal que $\operatorname{Soc}(G)$ és un p-grup per a algun primer $p \in \operatorname{char} \mathfrak{X}$ comple que $G/\operatorname{C}_G\bigl(\operatorname{Soc}(G)\bigr) \notin G_{\mathfrak{X}}(p)$.

El següent teorema és un dels resultats centrals del capítol.

Teorema. *Considerem* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$*, on* $\mathfrak{F}$ *i* $\mathfrak{G}$ *són formacions no buides i* $\mathfrak{F}$ *és* $\mathfrak{X}$*-local. Aleshores* $\mathfrak{H}$ *és una formació* $\mathfrak{X}$*-local si, i només si,* $\mathrm{b}(\mathfrak{H})$ *és* $\mathfrak{X}\mathfrak{G}$*-lliure.*

Com a conseqüència d'aquest resultat obtenim condicions sota les quals la formació producte $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ és $\mathfrak{X}$-local.

Teorema. *Considerem* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$*, on* $\mathfrak{F}$ *i* $\mathfrak{G}$ *són formacions no buides,* $\mathfrak{F}$ *és* $\mathfrak{X}$*-local i es compleix una de las dues condicions següents:*

1. $\mathfrak{G}$ *és* $\mathfrak{X}$*-local.*
2. $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ *si* $p \in \operatorname{char}\mathfrak{X}$ *i* $F_{\mathfrak{X}}(p) = \emptyset$*.*

Suposem, a més a més, que si $p \in \operatorname{char}\mathfrak{X}$*,* $F_{\mathfrak{X}}(p) = \emptyset$ *i* $\mathfrak{S}_p \subseteq \mathfrak{G}$*, aleshores* $\mathfrak{F} \subseteq \mathfrak{E}_{p'}$*. Aleshores* $\mathfrak{H}$ *és* $\mathfrak{X}$*-local.*

Com que les formacions locals són les formacions $\mathfrak{X}$-locals quan $\mathfrak{X}$ és la classe de tots els grups simples, obtenim els següents resultats com a corol·laris al teorema anterior:

Corol·lari. *Suposem que se satisfà alguna de les condicions següents:*

1. $\mathfrak{F}$ *és local i* $\mathfrak{G}$ *és* $\mathfrak{X}$*-local.*
2. $\mathfrak{F}$ *és local i* $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ *per a tot* $p \in \operatorname{char}\mathfrak{X}$ *tal que* $F_{\mathfrak{X}}(p) = \emptyset$*.*

Aleshores $\mathfrak{H}$ *és una formació* $\mathfrak{X}$*-local.*

Corol·lari ([DH92; IV, 3.13 y 4.8])**.** *La formació* $\mathfrak{H}$ *és local si se satisfà alguna de les següents condicions*

1. $\mathfrak{F}$ *i* $\mathfrak{G}$ *són locals.*
2. $\mathfrak{F}$ *és local i* $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ *per a tot primer* p *tal que* $F_{\mathfrak{J}}(p) = \emptyset$*.*

A la secció següent ens plantegem la següent pregunta:

> *Quines condicions cal imposar a les formacions* $\mathfrak{F}$ *i* $\mathfrak{G}$ *per poder assegurar que* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ *és* $\mathfrak{X}$*-local?*

Amb el següent resultat responem a la qüestió:

Teorema. *Considerem una formació no buida* $\mathfrak{K}$*. Hom té que* $\mathfrak{K}$ *és* $\mathfrak{X}$*-local si, i només si, se satisfan les condicions següents:*

1. $K_{\mathfrak{X}}(p) \subseteq \mathfrak{K}$ *per a tot* $p \in \operatorname{char}\mathfrak{X}$*.*

2. *Si* $G \in \mathrm{b}(\mathfrak{K})$, $N = \mathrm{Soc}(G) \in \mathfrak{S}_p$ *on* $p \in \mathrm{char}\,\mathfrak{X}$ *i* K *es el producte semidirecte natural* $[N]\big(G/\mathrm{C}_G(N)\big)$, *aleshores* $K \in \mathrm{b}(\mathfrak{K})$.

Per acabar, estudiem quan el producte de dues formacions és una formació p-local. Obtenim el següent teorema:

Teorema. *Considerem* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, *on* $\mathfrak{F}$ *i* $\mathfrak{G}$ *són dues formacions no buides, i un primer* p. *Les següents afirmacions són equivalents:*

1. $\mathfrak{H}$ *és una formació* p*-local.*

2. $H_{\mathfrak{J}}(p) \subseteq \mathfrak{G}$ *o* $\mathfrak{S}_p H_{\mathfrak{J}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$.

Shemetkov va proposar la següent qüestió en *The Kourovka Notebook* ([MK90]):

> Qüestió 10.72 (Shemetkov). *Proveu la indescomponibilitat de la formació dels* p*-grups,* $\mathfrak{S}_p$, *on* p *és un primer, com a producte de dues subformacions no trivials.*

Shemetkov i Skiba van provar la conjectura a [SS89]. Como a conseqüència del teorema anterior, en demostrem a la fi del capítol 3 una versió més general.

Diem que una formació $\mathfrak{F}$ és una *formació* $\mathfrak{X}$*-local 1-generada* si existeix un grup G tal que $\mathfrak{F}$ és la formació $\mathfrak{X}$-local més petita que conté G. El punt de partida del capítol 4 és la pregunta següent, plantejada per Skiba a [MK92].

> *Si* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ *és una formació de Baer 1-generada, on* $\mathfrak{F}$ *i* $\mathfrak{G}$ *són formacions no trivials, és* $\mathfrak{F}$ *una formació de Baer?*

El mateix Skiba va donar una resposta negativa a la pregunta, però la següent qüestió encara quedava oberta:

> *Suposem que* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ *és la formació de Baer generada per un grup* G, *on* $\mathfrak{F}$ *i* $\mathfrak{G}$ *són dues formacions no trivials. És* $\mathfrak{F}$ *una formació de Baer si es compleix que* $\mathfrak{H} \neq \mathfrak{G}$ *o* $\mathfrak{H} \neq \mathfrak{S}_p\mathfrak{H}$ *per a tot primer* p*?*

En aquest capítol donem resposta afirmativa a una pregunta més general, ja que plantegem el problema en termes de formacions $\mathfrak{X}$-locals. El primer resultat és el següent:

Teorema ([BBCER03; Teorema 1]). *Suposem que* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ *és una formació* $\mathfrak{X}$*-local generada per un grup* G*. Si* $\mathfrak{F}$ *i* $\mathfrak{G}$ *són no trivials i* $\mathfrak{H} \neq \mathfrak{G}$ *o* $\mathfrak{S}_p\mathfrak{H} \neq \mathfrak{H}$ *per a tot primer* $p \in \operatorname{char} \mathfrak{X}$*, aleshores* $\mathfrak{F}$ *és* $\mathfrak{X}$*-saturada.*

Tot seguit donem una descripció completa de les factoritzacions d'una formació $\mathfrak{X}$-local 1-generada. Els resultats apareixen a [BBC].

Teorema. *Considerem una formació* $\mathfrak{X}$*-local* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ *tal que* $\pi(\mathfrak{F}) \subseteq \pi$*. Suposem que* $\mathfrak{F}$ *i* $\mathfrak{G}$ *són formacions no trivials i* $\mathfrak{H} \neq \mathfrak{G}$ *o* $\mathfrak{S}_p\mathfrak{H} \neq \mathfrak{H}$ *per a tot primer* $p \in \pi$*.*

Aleshores $\mathfrak{H}$ *és una formació* $\mathfrak{X}$*-local 1-generada si, i només si, se satisfan les propietats següents:*

1. $\mathfrak{F}$ *és i una formació local 1-generada metanilpotent.*
2. $\mathcal{K}_{\mathfrak{A}}(\mathfrak{G}) \cap \mathfrak{X} \subseteq \mathfrak{F}$*, on* $\mathcal{K}_{\mathfrak{A}}(\mathfrak{G})$ *és la classe de tots els factors de composició dels grups de* $\mathfrak{G}$*.*
3. $H_{\mathfrak{X}}(p) = F_{\mathfrak{X}}(p) \circ \mathfrak{G}$ *per a tot primer* $p \in \pi$*.*
4. *Si* $A \in \mathfrak{F}$ *i* $B \in \mathfrak{G}$*, aleshores* $\pi\bigl(A/\operatorname{C}^{\mathfrak{X}_p}(A)\bigr) \cap \pi(B) = \emptyset$ *per a tot* $p \in \pi$*.*
5. *Si* $\mathfrak{F}$ *no és* π*-nilpotent, aleshores* $\mathfrak{G}$ *és abeliana i 1-generada.*
6. *Si* $\mathfrak{F}$ *és* π*-nilpotent, aleshores* $\mathfrak{G}$ *és 1-generada o*

$$\mathfrak{G}/\operatorname{O}_p(\mathfrak{G}) := \operatorname{form}(G/\operatorname{O}_p(G) \mid G \in \mathfrak{G})$$

   és 1-generada per a tot $p \in \pi$*.*

Si $\mathfrak{X}$ és la classe de tots els grups simples, obtenim el principal resultat de [Ski83] com a corol·lari al nostre teorema. Per a la classe $\mathfrak{X}$ de tots els grups simples abelians, es dedueix el teorema central de [GS01].

Una factorització d'una formació no trivial $\mathfrak{H}$ com a producte $\mathfrak{H} = \mathfrak{H}_1 \circ \cdots \circ \mathfrak{H}_t$, donde $\mathfrak{H}_1, \ldots, \mathfrak{H}_t$ són formacions, s'anomena *irreductible* si $\mathfrak{H} \neq \mathfrak{H}_1 \circ \cdots \circ \mathfrak{H}_{i-1}\mathfrak{H}_{i+1} \circ \cdots \circ \mathfrak{H}_t$ per a tot $i = 1, 2, \ldots, t$. Com a culminació al capítol, estudiem com són les factoritzacions irreductibles d'una formació $\mathfrak{X}$-local 1-generada. Obtenim el següent teorema i estenem els principals resultats de [Ski92] i [GS03].

Teorema. *Si* $\mathfrak{H} = \mathfrak{H}_1 \circ \cdots \circ \mathfrak{H}_t$ *és una factorización irreductible d'una formació* $\mathfrak{X}$*-local 1-generada i* $\pi(\mathfrak{H}_i) \subseteq \pi$ *per a tot* $i = 1, 2, 3$*, aleshores* $t \leq 3$*, i si* $t = 3$*, aleshores* $\mathfrak{H}_1$ *i* $\mathfrak{H}_2$ *són nilpotents,* $\mathfrak{H}_3$ *és abeliana i* $\pi(\mathfrak{H}_2) \cap \pi(\mathfrak{H}_3) = \emptyset$*.*

Apéndice B

Resumen

Todos los grupos considerados a lo largo del trabajo son finitos. Las principales referencias bibliográficas utilizadas son el libro de Doerk y Hawkes ([DH92]) y el de Ballester y Ezquerro ([BBE06]).

La tesis se enmarca dentro de la teoría de formaciones de grupos. Una *formación* es una clase de grupos que cumple las dos siguientes propiedades:

- Si $G \in \mathfrak{F}$ y $N \trianglelefteq G$, entonces $G/N \in \mathfrak{F}$.
- Si G/N, $G/M \in \mathfrak{F}$, entonces $G/(M \cap N) \in \mathfrak{F}$.

Decimos que una formación $\mathfrak{F}$ es *saturada* si, para cada grupo G, la condición $G/\Phi(G) \in \mathfrak{F}$ implica que $G \in \mathfrak{F}$, donde $\Phi(G)$ es el subgrupo de Frattini de G.

Una *función formación* f asocia a cada número primo p una formación $f(p)$, que puede ser vacía. Si f es una función formación, entonces la *formación local* $\mathrm{LF}(f)$ definida por f es la clase de todos los grupos G que satisfacen la siguiente propiedad:

- Si H/K es un factor principal de G, entonces $G/\mathrm{C}_G(H/K) \in f(p)$ para todo $p \in \pi(H/K)$.

Una formación $\mathfrak{F}$ es *local* si existe una función formación f tal que $\mathfrak{F} = \mathrm{LF}(f)$. Gaschütz introdujo este concepto en [Gas63], lo que le permitía construir una gran familia de formaciones saturadas. De hecho, el teorema de Gaschütz-Lubeseder-Schmid, que es uno de los resultados más relevantes en teoría de formaciones, afirma que con el método anterior se pueden obtener todas las formaciones saturadas. Este teorema fue probado por Gaschütz y Lubeseder en el universo resoluble. Posteriormente, Schmid probó que esta restricción es innecesaria.

Baer generalizó de distinta manera el teorema anterior en el universo finito. Usó un concepto diferente de formación local, tratando los factores principales no abelianos con más flexibilidad que los abelianos. Esto lo llevó a encontrar una familia de formaciones, la de las formaciones Baer-locales, que contiene a las locales. Estas dos familias coinciden en el universo resoluble. Una *función de Baer* asocia a cada grupo simple J una clase de grupos $f(J)$, de forma que $f(C_p)$ es una formación para todo número primo p. Si f es una función de Baer, entonces la *formación Baer-local* o *formación de Baer* $\mathrm{BLF}(f)$ definida por f es la clase de todos los grupos G que satisfacen la siguiente propiedad:

- Si H/K es un factor principal de G, entonces $G/\mathrm{C}_G(H/K) \in f(J)$, donde J es el factor de composición de H/K.

Las formaciones Baer-locales también fueron estudiadas por Shemetkov (véase [She75]), bajo el nombre de *formaciones de composición* (*composition formations*). Decimos que una formación $\mathfrak{F}$ es *resolublemente saturada* si, para cada grupo G, la condición $G/\Phi(G_{\mathfrak{S}}) \in \mathfrak{F}$ implica que $G \in \mathfrak{F}$, donde $G_{\mathfrak{S}}$ es el radical resoluble de G. Baer probó que una formación es resolublemente saturada si, y sólo si, es Baer-local.

Con propósito de presentar una generalización común de los teoremas de Gaschütz-Lubeseder-Schmid y de Baer, Förster intodujo en [För85] el concepto de formación $\mathfrak{X}$-local, donde $\mathfrak{X}$ es una clase de grupos simples que cumple que $\pi(\mathfrak{X}) = \operatorname{char} \mathfrak{X}$, donde

$$\pi(\mathfrak{X}) := \{p \in \mathbb{P} \mid \text{existe } G \in \mathfrak{X} \text{ tal que } p \in \pi(G)\}$$

y

$$\operatorname{char} \mathfrak{X} := \{p \in \mathbb{P} \mid C_p \in \mathfrak{X}\}.$$

Llamamos $\mathfrak{J}$ a la clase de todos los grupos simples. Si $\mathfrak{Y}$ es una subclase de $\mathfrak{J}$, $\mathfrak{Y}'$ denota la clase $\mathfrak{J} \setminus \mathfrak{Y}$. Utilizamos $\mathrm{E}\,\mathfrak{Y}$ para denotar la clase de todos los grupos cuyos factores de composición están en $\mathfrak{Y}$. Todo grupo G posee un subgrupo normal que pertenece a $\mathrm{E}\,\mathfrak{Y}$ y que contiene a todos los subgrupos normales de G en $\mathrm{E}\,\mathfrak{Y}$, es decir, un $\mathrm{E}\,\mathfrak{Y}$-radical. Para referirnos a él escribimos $\mathrm{O}_{\mathfrak{Y}}(G)$ o bien $G_{\mathrm{E}\,\mathfrak{Y}}$. Si H/K es un factor principal de un grupo G y $H/K \in \mathrm{E}\,\mathfrak{Y}$, decimos que H/K es un $\mathfrak{Y}$*-factor principal* de G. Escribimos $\mathfrak{Y}_p$ para denotar la clase de los grupos simples de $\mathfrak{Y}$ cuyo orden es divisible por p, donde p es un número primo. Si $\mathfrak{Y}$ es una clase de grupos, la menor formación que contiene a $\mathfrak{Y}$ se denota por $\operatorname{form}(\mathfrak{Y})$.

Una $\mathfrak{X}$*-función formación* f es una aplicación que asocia una formación, posiblemente vacía, a cada grupo simple del conjunto $(\operatorname{char} \mathfrak{X}) \cup \mathfrak{X}'$. Si f es una $\mathfrak{X}$-función formación, $\mathrm{LF}_{\mathfrak{X}}(f)$ se define como la clase de todos los grupos G que satisfacen las dos siguientes propiedades:

1. Si $p \in \operatorname{char}(\mathfrak{X})$ y H/K es un $\mathfrak{X}_p$-factor principal de G, entonces $G/\operatorname{C}_G(H/K) \in f(p)$.

2. Si $L \trianglelefteq G$, G/L es monolítico y $\operatorname{Soc}(G/L)$ es un $\mathfrak{X}'$-factor principal de G, entonces $G/L \in f(E)$, donde E es el factor de composición de $\operatorname{Soc}(G/L)$.

Förster demuestra en [För85] que la clase $\mathrm{LF}_{\mathfrak{X}}(f)$ es una formación. Una formación $\mathfrak{F}$ de grupos finitos es $\mathfrak{X}$-*local* si existe una $\mathfrak{X}$-función formación f tal que $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$. Si $\mathfrak{Y}$ es una clase de grupos, la menor formación $\mathfrak{X}$-local que contiene a $\mathfrak{Y}$ se denota por $\operatorname{form}_{\mathfrak{X}}(\mathfrak{Y})$.

Si $\mathfrak{X}$ es la clase de todos los grupos simples, las formaciones $\mathfrak{X}$-locales son exactamente las formaciones locales. Si $\mathfrak{X}$ es la clase de los grupos simples abelianos, se demuestra que las formaciones $\mathfrak{X}$-locales coinciden con las Baer-locales. En el capítulo 1 de la tesis, presentamos resultados básicos de formaciones $\mathfrak{X}$-locales y caracterizamos estas formaciones como las clases de grupos con factores principales centrales generalizados. Si f es una $\mathfrak{X}$-función formación, definimos el concepto de factor principal f-central y probamos el siguiente teorema:

Teorema ([BBCSss]). *Consideremos una formación $\mathfrak{X}$-local $\mathfrak{F}$ y una $\mathfrak{X}$-función formación f que defina a $\mathfrak{F}$. Entonces $\mathfrak{F}$ coincide con la clase de los grupos cuyos factores principales son f-centrales.*

Förster también introdujo en [För85] un subgrupo $\mathfrak{X}$-Frattini $\Phi^*_{\mathfrak{X}}(G)$ para cada grupo G. Definió el concepto de $\mathfrak{X}$-saturación de la manera natural y probó que las formaciones $\mathfrak{X}$-saturadas son exactamente las $\mathfrak{X}$-locales. A pesar de que los teoremas de Gaschütz-Lubeseder-Schmid y Baer se deducen del teorema de Förster, la definición que utiliza de $\mathfrak{X}$-saturación no es la más natural y la demostración del teorema de Baer no se deduce fácilmente de su resultado. En la segunda sección del capítulo 1, introducimos un nuevo subgrupo de tipo Frattini $\Phi_{\mathfrak{X}}(G)$ para cada grupo G asociado a la clase $\mathfrak{X}$. Se define de la siguiente manera:

- Consideremos un número primo p. Decimos que un grupo G pertenece a la clase $\mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ si se cumple:

  1. $\Phi(G)$ es p-elemental abeliano.
  2. $G/\Phi(G)$ es un grupo primitivo con un único subgrupo normal minimal no abeliano, es decir, $G/\Phi(G)$ es un grupo primitivo de tipo 2.
  3. $\operatorname{Soc}\big(G/\Phi(G)\big) \in \mathrm{E}\,\mathfrak{X} \setminus \mathfrak{E}_{p'}$

4. $\mathrm{C}_G^h\big(\Phi(G)\big) \le \Phi(G)$, donde

$$\mathrm{C}_G^h\big(\Phi(G)\big) := \bigcap \{\mathrm{C}_G(H/K) \mid H/K \text{ es un factor principal de } G \text{ tal que } H \le \Phi(G)\}$$

- El subgrupo $\mathfrak{X}$-Frattini de G, $\Phi_{\mathfrak{X}}(G)$, se define como

$$\Phi_{\mathfrak{X}}(G) := \begin{cases} \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) & \text{si } G \notin \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2) \text{ para todo } p \in \mathrm{char}(\mathfrak{X}), \\ \Phi(G) & \text{en otro caso.} \end{cases}$$

- Una formación $\mathfrak{F}$ se dice *$\mathfrak{X}$-saturada* si $G/\Phi_{\mathfrak{X}}(G) \in \mathfrak{F}$ siempre implica que $G \in \mathfrak{F}$.

Este subgrupo $\mathfrak{X}$-Frattini proporciona una definición de $\mathfrak{X}$-saturación más sencilla que la de Förster y más cercana a la definición natural. Al final del capítulo probamos que las formaciones $\mathfrak{X}$-saturadas, con esta nueva definición de $\mathfrak{X}$-saturación, también coinciden con las formaciones $\mathfrak{X}$-locales de Förster (véase [BBCER05]).

Shemetkov y Skiba introdujeron el concepto de formación ω-local, donde ω es un conjunto de números primos. Un *ω-satélite local* f asocia a cada elemento de $\omega \cup \{\omega'\}$ una formación, que puede ser vacía. El símbolo $G_{\omega d}$ denota el mayor subgrupo normal N de G tal que $\omega \cap \pi(H/K) \neq \emptyset$ para todo factor de composición H/K de N (si $\omega \cap \pi\big(\mathrm{Soc}(G)\big) = \emptyset$, definimos $G_{\omega d} = 1$). Si f es un ω-satélite local, entonces $\mathrm{LF}_\omega(f)$ denota la clase de grupos G que satisfacen las dos siguientes condiciones:

1. Si H/K es un factor principal de G y $p \in \pi(H/K) \cap \omega$, entonces $G/\mathrm{C}_G(H/K) \in f(p)$.

2. $G/G_{\omega d} \in f(\omega')$.

Se dice que una formación $\mathfrak{F}$ es *ω-local* si existe un ω-satélite local f tal que $\mathfrak{F} = \mathrm{LF}_\omega(f)$. En este caso, se dice que f es un ω-satélite local de $\mathfrak{F}$. Si p es un número primo, se dice que una formación $\mathfrak{F}$ es *p-saturada* si $G \in \mathfrak{F}$ siempre que $G/\mathrm{O}_p(G) \cap \Phi(G) \in \mathfrak{F}$. Si ω es un conjunto de primos, decimos que $\mathfrak{F}$ es *ω-saturada* si $\mathfrak{F}$ es p-saturada para cada primo $p \in \omega$.

En [SS00] se prueba el siguiente teorema:

Teorema. *Una formación $\mathfrak{F}$ es ω-saturada si, y sólo si, $\mathfrak{F}$ es ω-local.*

Si ω es el conjunto de todos los números primos, las formaciones ω-locales son exactamente las locales, por lo que la idea de formación ω-local es otro acercamiento al concepto de formación local. En el segundo capítulo estudiamos la relación entre las formaciones $\mathfrak{X}$-locales y las formaciones ω-locales. En él se incluyen algunos resultados de [BBCER03]. Si ω es un conjunto de números primos y $\mathfrak{F}$ una formación ω-saturada, se sigue que $\mathfrak{F}$ es $\mathfrak{X}_\omega$-saturada, donde $\mathfrak{X}_\omega$ es la clase de los ω-grupos simples. Sin embargo, la familia de las formaciones $\mathfrak{X}_\omega$-saturadas no coincide en general con la de las formaciones ω-saturadas. Esto se sigue del hecho de que existen formaciones de Baer que no son ω-saturadas para ningún conjunto no vacío ω de primos. La siguiente pregunta surge de forma natural:

> *Consideremos un conjunto de números primos ω. ¿Se puede asegurar la existencia de una clase $\mathfrak{X}(\omega)$ de grupos simples tal que $\operatorname{char}\big(\mathfrak{X}(\omega)\big) = \pi\big(\mathfrak{X}(\omega)\big)$ que cumpla que una formación es ω-saturada si, y sólo si, es $\mathfrak{X}(\omega)$-saturada?*

Probamos mediante un ejemplo que la respuesta es negativa. En el siguiente resultado, que aparece en [BBE06; Chapter 3] y [BBCSss], probamos que una formación $\mathfrak{X}$-local $\mathfrak{F}$ siempre contiene una formación ω-local, donde $\omega = \operatorname{char}\mathfrak{X}$, que tiene la propiedad de ser la mayor formación ω-local contenida en $\mathfrak{F}$.

Teorema. *Consideremos una clase de grupos simples $\mathfrak{X}$ tal que $\omega = \pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$. Consideremos una formación $\mathfrak{X}$-local $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$. Entonces la formación ω-local $\mathfrak{F}_\omega = \mathrm{LF}_\omega(f)$, donde $f(p) = F(p)$ para todo $p \in \omega$ y $f(\omega') = \mathfrak{F}$, es la mayor formación ω-local contenida en $\mathfrak{F}$.*

Como corolario, obtenemos los siguientes resultados:

Corolario. *Consideremos un conjunto de primos ω y una formación $\mathfrak{F}$ compuesta de grupos ω-separables. Entonces $\mathfrak{F}$ es ω-local si, y sólo si, $\mathfrak{F}$ es $\mathfrak{X}(\omega)$-local.*

Corolario. *Toda formación Baer-local contiene una formación local maximal respecto a la inclusión.*

Además, damos condiciones para asegurar que una formación $\mathfrak{X}$-local es ω-local para $\omega = \operatorname{char}\mathfrak{X}$, como muestra el siguiente resultado. En el enunciado, $\underline{f}$ denota la menor $\mathfrak{X}$-función formación que define $\mathfrak{F}$.

Teorema ([BBCSss])**.** *Sea $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$ una formación $\mathfrak{X}$-local y $\omega = \operatorname{char}\mathfrak{X}$. Las siguientes condiciones son equivalentes:*

1. *$\mathfrak{F}$ es ω-local.*
2. *$G/\mathrm{c}_G(H/K) \in \underline{f}(p)$ para cada $G \in \mathfrak{F}$ y cada $\mathfrak{X}'$-factor principal H/K de G tal que $p \in \pi(H/K) \cap \omega$.*
3. *$\underline{f}(S) \subseteq \underline{f}(p)$ para cada $S \in \mathfrak{X}'$ y $p \in \pi(S) \cap \omega$.*
4. *$\mathfrak{S}_p\underline{f}(S) \subseteq \mathfrak{F}$ para cada $S \in \mathfrak{X}'$ y $p \in \pi(S) \cap \omega$.*

Dadas dos clases de grupos $\mathfrak{Y}$ y $\mathfrak{Z}$, se puede definir la clase producto de la siguiente manera:

$$\mathfrak{Y}\mathfrak{Z} = (G \in \mathfrak{E} \mid \text{existe un subgrupo normal } N \text{ de } G \text{ tal que } N \in \mathfrak{Y} \text{ y } G/N \in \mathfrak{Z}).$$

Sin embargo, esta clase no es en general una formación cuando $\mathfrak{Y}$ y $\mathfrak{Z}$ son formaciones. Se puede modificar la definición anterior para asugurar que esto sí ocurre, es decir, que la clase producto de dos formaciones es de nuevo una formación. Si $\mathfrak{F}$ y $\mathfrak{G}$ son formaciones, el *producto formación* o *producto de Gaschütz* de $\mathfrak{F}$ y $\mathfrak{G}$ es la clase $\mathfrak{F} \circ \mathfrak{G}$ así definida:

$$\mathfrak{F} \circ \mathfrak{G} := (G \in \mathfrak{E} \mid G^{\mathfrak{G}} \in \mathfrak{F}).$$

Es conocido el hecho de que el producto formación de dos formaciones locales es de nuevo una formación local (véase [DH92; IV, 3.13 y 4.8]). Sin embargo, el producto formación de dos formaciones $\mathfrak{X}$-locales no es en general una formación $\mathfrak{X}$-local. Por tanto, es natural preguntarse lo siguiente:

> *¿Qué condiciones podemos exigir a dos formaciones $\mathfrak{F}$ y $\mathfrak{G}$ para asegurar que $\mathfrak{F} \circ \mathfrak{G}$ es una formación $\mathfrak{X}$-local?*

Esta cuestión, que ya fue estudiada por Salomon en [Sal83] para formaciones Baer-locales, es el centro del capítulo 3 de la tesis. Nuestros resultados han sido publicados en [BBCER06].

Si $\mathfrak{K}$ es una clase de grupos y $p \in \operatorname{char}\mathfrak{X}$, definimos la clase

$$K_{\mathfrak{X}}(p) := \mathfrak{S}_p \operatorname{form}\bigl(G/\mathrm{C}_G(H/K) \mid G \in \mathfrak{K} \text{ y } H/K \text{ es un } \mathfrak{X}_p\text{-factor principal de } G\bigr).$$

Tengamos en cuenta que $K_{\mathfrak{X}}(p) = \emptyset$ si no existe nungún grupo $G \in \mathfrak{K}$ que tenga algún $\mathfrak{X}_p$-factor principal.

El primer resultado del capítulo afirma que si $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, donde $\mathfrak{F}$ y $\mathfrak{G}$ son formaciones no vacías y $\mathfrak{F}$ es una formación $\mathfrak{X}$-local, entonces la menor formación $\mathfrak{X}$-local $\mathrm{form}_{\mathfrak{X}}(\mathfrak{H})$ que contiene a $\mathfrak{H}$ se puede definir mediante la $\mathfrak{X}$-función formación h dada por:

$$
\begin{aligned}
h(p) &= \begin{cases} F_{\mathfrak{X}}(p) \circ \mathfrak{G} & \text{if } \mathfrak{S}_p \subseteq \mathfrak{F}, \\ G_{\mathfrak{X}}(p) & \text{if } \mathfrak{S}_p \nsubseteq \mathfrak{F}, \end{cases} && \text{si } p \in \operatorname{char}\mathfrak{X}; \\
h(S) &= \mathfrak{H} && \text{si } S \in \mathfrak{X}'.
\end{aligned}
$$

Consideremos $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, donde $\mathfrak{F}$ y $\mathfrak{G}$ son formaciones no vacías. Decimos que la frontera $\mathrm{b}(\mathfrak{H})$ de $\mathfrak{H}$ es $\mathfrak{X}\mathfrak{G}$-*libre* si cada grupo $G \in \mathrm{b}(\mathfrak{H})$ tal que $\mathrm{Soc}(G)$ es un p-grupo para algún primo $p \in \operatorname{char}\mathfrak{X}$ cumple que $G/\mathrm{C}_G\big(\mathrm{Soc}(G)\big) \notin G_{\mathfrak{X}}(p)$.

El siguiente teorema es uno de los resultados centrales del capítulo.

Teorema. *Consideremos* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$*, donde* $\mathfrak{F}$ *y* $\mathfrak{G}$ *son formaciones no vacías y* $\mathfrak{F}$ *es* $\mathfrak{X}$*-local. Entonces* $\mathfrak{H}$ *es una formación* $\mathfrak{X}$*-local si, y sólo si,* $\mathrm{b}(\mathfrak{H})$ *es* $\mathfrak{X}\mathfrak{G}$*-libre.*

Como consecuencia de este resultado obtenemos condiciones bajo las que la formación producto $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ es $\mathfrak{X}$-local.

Teorema. *Consideremos* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$*, donde* $\mathfrak{F}$ *y* $\mathfrak{G}$ *son formaciones no vacías,* $\mathfrak{F}$ *es* $\mathfrak{X}$*-local y se cumple una de las dos condiciones siguientes:*

1. $\mathfrak{G}$ *es* $\mathfrak{X}$*-local.*

2. $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ *si* $p \in \operatorname{char}\mathfrak{X}$ *y* $F_{\mathfrak{X}}(p) = \emptyset$*.*

Supongamos además que si $p \in \operatorname{char}\mathfrak{X}$*,* $F_{\mathfrak{X}}(p) = \emptyset$ *y* $\mathfrak{S}_p \subseteq \mathfrak{G}$*, entonces* $\mathfrak{F} \subseteq \mathfrak{E}_{p'}$*. Entonces* $\mathfrak{H}$ *es* $\mathfrak{X}$*-local.*

Como las formaciones locales son las formaciones $\mathfrak{X}$-locales cuando $\mathfrak{X}$ es la clase de todos los grupos simples, obtenemos los siguientes resultados como corolarios al teorema anterior:

Corolario. *Supongamos que se cumple alguna de las siguientes condiciones:*

1. $\mathfrak{F}$ *es local y* $\mathfrak{G}$ *es* $\mathfrak{X}$*-local.*

2. $\mathfrak{F}$ *es local y* $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ *para todo* $p \in \operatorname{char}\mathfrak{X}$ *tal que* $F_{\mathfrak{X}}(p) = \emptyset$*.*

Entonces $\mathfrak{H}$ *es una formación* $\mathfrak{X}$*-local.*

Corolario ([DH92; IV, 3.13 y 4.8])**.** *La formación* $\mathfrak{H}$ *es local si se cumple alguna de las siguientes condiciones*

1. *$\mathfrak{F}$ y $\mathfrak{G}$ son locales.*

2. *$\mathfrak{F}$ es local y $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ para todo primo p tal que $F_{\mathfrak{J}}(p) = \emptyset$.*

En la siguiente sección nos planteamos la siguiente pregunta:

> *¿Qué condiciones hay que imponer a las formaciones $\mathfrak{F}$ y $\mathfrak{G}$ para poder asegurar que $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ es $\mathfrak{X}$-local?*

Con el siguiente resultado respondemos a la cuestión:

Teorema. *Consideremos una formación no vacía $\mathfrak{K}$. Se tiene que $\mathfrak{K}$ es $\mathfrak{X}$-local si, y sólo si, se cumplen las siguientes condiciones:*

1. *$K_{\mathfrak{X}}(p) \subseteq \mathfrak{K}$ para todo $p \in \operatorname{char} \mathfrak{X}$.*

2. *Si $G \in \mathrm{b}(\mathfrak{K})$, $N = \operatorname{Soc}(G) \in \mathfrak{S}_p$ donde $p \in \operatorname{char} \mathfrak{X}$ y K es el producto semidirecto natural $[N]\big(G/\operatorname{C}_G(N)\big)$, entonces $K \in \mathrm{b}(\mathfrak{K})$.*

Por último, estudiamos cuándo el producto de dos formaciones es una formación p-local. Obtenemos el siguiente teorema:

Teorema. *Consideremos $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, donde $\mathfrak{F}$ y $\mathfrak{G}$ son dos formaciones no vacías, y un primo p. Las siguientes afirmaciones son equivalentes:*

1. *$\mathfrak{H}$ es una formación p-local.*

2. *$H_{\mathfrak{J}}(p) \subseteq \mathfrak{G}$ o $\mathfrak{S}_p H_{\mathfrak{J}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$.*

Shemetkov propuso la siguiente cuestión en *The Kourovka Notebook* ([MK90]):

> Cuestión 10.72 (Shemetkov). *Probar la indescomponibilidad de la formación de los p-grupos, $\mathfrak{S}_p$, donde p es un primo, como producto de dos subformaciones no triviales.*

Shemetkov y Skiba probaron la conjetura en [SS89]. Como consecuencia del teorema anterior, demostramos al final del capítulo 3 una versión más general de la misma.

Decimos que una formación $\mathfrak{F}$ es una *formación $\mathfrak{X}$-local 1-generada* si existe un grupo G tal que $\mathfrak{F}$ es la menor formación $\mathfrak{X}$-local que contiene a G. El punto de partida del capítulo 4 es la siguiente pregunta, planteada por Skiba en [MK92].

> *Si $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ es una formación de Baer 1-generada, donde $\mathfrak{F}$ y $\mathfrak{G}$ son formaciones no triviales, ¿es $\mathfrak{F}$ una formación de Baer?*

El propio Skiba dio una respuesta negativa a la pregunta, pero la siguiente cuestión todavía quedaba abierta:

> *Supongamos que $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ es la formación de Baer generada por un grupo G, donde $\mathfrak{F}$ y $\mathfrak{G}$ son dos formaciones no triviales. ¿Es $\mathfrak{F}$ una formación de Baer si se cumple que $\mathfrak{H} \neq \mathfrak{G}$ o $\mathfrak{H} \neq \mathfrak{S}_p\mathfrak{H}$ para todo primo p?*

En este capítulo damos respuesta afirmativa a una pregunta más general, ya que planteamos el problema en términos de formaciones $\mathfrak{X}$-locales. El primer resultado es el siguiente:

Teorema ([BBCER03; Teorema 1]). *Supongamos que $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ es una formación $\mathfrak{X}$-local generada por un grupo G. Si $\mathfrak{F}$ y $\mathfrak{G}$ son no triviales y $\mathfrak{H} \neq \mathfrak{G}$ o $\mathfrak{S}_p\mathfrak{H} \neq \mathfrak{H}$ para todo primo $p \in \operatorname{char}\mathfrak{X}$, entonces $\mathfrak{F}$ es $\mathfrak{X}$-saturada.*

A continuación damos una descripción completa de las factorizaciones de una formación $\mathfrak{X}$-local 1-generada. Los resultados aparecen en [BBC].

Teorema. *Consideremos una formación $\mathfrak{X}$-local $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ tal que $\pi(\mathfrak{F}) \subseteq \pi$. Supongamos que $\mathfrak{F}$ y $\mathfrak{G}$ son formaciones no triviales y $\mathfrak{H} \neq \mathfrak{G}$ o $\mathfrak{S}_p\mathfrak{H} \neq \mathfrak{H}$ para todo primo $p \in \pi$.*

Entonces $\mathfrak{H}$ es una formación $\mathfrak{X}$-local 1-generada si, y sólo si, se cumplen las siguientes propiedades:

1. *$\mathfrak{F}$ es una formación local 1-generada metanilpotente.*
2. *$\mathcal{K}_{\mathfrak{A}}(\mathfrak{G}) \cap \mathfrak{X} \subseteq \mathfrak{F}$, donde $\mathcal{K}_{\mathfrak{A}}(\mathfrak{G})$ es la clase de todos los factores de composición de los grupos de $\mathfrak{G}$.*
3. *$H_{\mathfrak{X}}(p) = F_{\mathfrak{X}}(p) \circ \mathfrak{G}$ para todo primo $p \in \pi$.*
4. *Si $A \in \mathfrak{F}$ y $B \in \mathfrak{G}$, entonces $\pi\big(A/\operatorname{C}^{\mathfrak{X}_p}(A)\big) \cap \pi(B) = \emptyset$ para todo $p \in \pi$.*
5. *Si $\mathfrak{F}$ no es π-nilpotente, entonces $\mathfrak{G}$ es abeliana y 1-generada.*
6. *Si $\mathfrak{F}$ es π-nilpotente, entonces $\mathfrak{G}$ es 1-generada o*

   $$\mathfrak{G}/\operatorname{O}_p(\mathfrak{G}) := \operatorname{form}(G/\operatorname{O}_p(G) \mid G \in \mathfrak{G})$$

   es 1-generada para todo $p \in \pi$.

Si $\mathfrak{X}$ es la clase de todos los grupos simples, obtenemos el principal resultado de [Ski83] como corolario a nuestro teorema. Para la clase $\mathfrak{X}$ de todos los grupos simples abelianos, se deduce el teorema central de [GS01].

Una factorización de una formación no trivial $\mathfrak{H}$ como producto $\mathfrak{H} = \mathfrak{H}_1 \circ \cdots \circ \mathfrak{H}_t$, donde $\mathfrak{H}_1, \ldots, \mathfrak{H}_t$ son formaciones, se llama *irreducible* si $\mathfrak{H} \neq \mathfrak{H}_1 \circ \cdots \circ \mathfrak{H}_{i-1}\mathfrak{H}_{i+1} \circ \cdots \circ \mathfrak{H}_t$ para todo $i = 1, 2, \ldots, t$. Como culminación al capítulo, estudiamos cómo son las factorizaciones irreducibles de una formación $\mathfrak{X}$-local 1-generada. Obtenemos el siguiente teorema y extendemos los principales resultados de [Ski92] y [GS03].

Teorema. *Si $\mathfrak{H} = \mathfrak{H}_1 \circ \cdots \circ \mathfrak{H}_t$ es una factorización irreducible de una formación $\mathfrak{X}$-local 1-generada y $\pi(\mathfrak{H}_i) \subseteq \pi$ para todo $i = 1, 2, 3$, entonces $t \leq 3$, y si $t = 3$, entonces $\mathfrak{H}_1$ y $\mathfrak{H}_2$ son nilpotentes, $\mathfrak{H}_3$ es abeliana y $\pi(\mathfrak{H}_2) \cap \pi(\mathfrak{H}_3) = \emptyset$.*