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DPTO. ESTADÍSTICA E INVESTIGACIÓN OPERATIVA

Multivariate Comparisons of Random Vectors with Applications

(Comparaciones Multivariantes de Vectores Aleatorios con Aplicaciones)

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A mi familia.

Contents

Agradecimientos

El paso del tiempo nos ha ido cambiando. Éramos pequeños cuando decidimos estar aquí y ahora, sin saber por qué. Nada es por casualidad. No hay nada que no provenga de nada. Tampoco para mí ni para esta tesis, memoria escrita a lo largo de un tiempo formidablemente enriquecedor.

El paso del tiempo nos hace ser conscientes de pequeños detalles que nos hacen comprendernos un poco mejor, sin afán de descubrir las curiosas estrategias del destino. Es así como escribo estas palabras, aceptando humildemente los extraños caprichos de la vida.

El paso del tiempo comenzó hace muchos años cuando la señorita Rosa, una profesora del colegio Santa Florentina de La Palma (Cartagena), generó en mí, sin saberlo, el gusto por las matemáticas. Exigente, rigurosa y delicadamente precisa, me hizo confiar en mí y en mi capacidad para con las matemáticas. Pero si bien es cierto que ella fue, sin duda, la que puso en marcha el tiempo inexorable de los números, no menos cierto es que mis primeras "cuentas" las hice con mi abuelo Leandro, el mismo que años después se sentía más orgulloso que nadie imaginando que su nieto podría llegar a ser catedrático. Ojalá hubiera compartido durante más tiempo este tiempo con él.

Después llegó el tiempo del instituto y de sus pequeños descubrimientos, de sentirse mayor, de las primeras curas de humildad y los cortes de mangas, de la rebeldía, de llegar tarde a casa y, por fin, de sentarse y decir: Mamá, Papá, quiero estudiar matemáticas. Si bien es cierto que no era lo que habrían imaginado, acabaron confiando en mí y en mi decisión. Tanto que, a día de hoy, tengo que darles las gracias por dejar que me dejara llevar a este mundo de sueños y pequeñas locuras, por haberme proporcionado no sólo el soporte económico, sino también por haber configurado un espacio donde yo podía dedicar horas y horas para divagar, preocuparme y sentirme orgulloso de los pequeños logros, por dejarme crecer y por haber entendido que yo siempre elijo el camino complicado. Y que seguirá siendo así. Caiga quien caiga.

Y cabe mencionar el apoyo indiscutible de mi hermana, tan diferente a mí y, a la vez, tan cómplice, a la cual agradezco haberme mostrado otros caminos diferentes a los que yo, en principio, habría elegido, el afán y la pasión por la que hay que luchar por las cosas que uno quiere, sobre todo, en esta época tan difícil en la que nos toca apostar por nosotros mismos.

El tiempo como estudiante de la Licenciatura en Matemáticas en la Universidad de Murcia fue la primera toma de contacto con este mundo de pruebas por ensayo y error, de números irracionales, de cuerpos, categorías y funtores, de distancias, de series infinitas e integrales impropias, y también el descubrir que las matemáticas eran algo más que un mundo de números y operaciones, el mundo de la razón me dio la bienvenida con todo lo duro y lo satisfactorio que puede llegar a ser. Fue entonces cuando el profesor de Métodos Estadísticos, José María Ruíz Gómez, vio en mí un futuro candidato a alcanzar este momento. Y aquí estamos. Muchas gracias José María por haberme dado esta oportunidad.

Acabé la licenciatura y José María me presentó al que iba a ser otro de mis directores de tesis, el profesor Félix Belzunce Torregrosa y lo que empezó siendo una relación exclusivamente profesional, derivó en lo que es a día de hoy, una amistad. A él tengo que darle las gracias por enseñarme el mundo de las esperanzas, los intervalos de confianza y los contrastes de hipótesis desde un punto de vista humano. Gracias Félix por estar ahí no sólo para aligerar las cargas, sino también por apretarme las tuercas cuando lo he necesitado.

Por último, conocí al profesor Alfonso Suárez Llorens de la Universidad de Cádiz durante una visita que realizó a la Facultad de Matemáticas de la Universidad de Murcia. Aún no sabíamos que íbamos a compartir trabajos y preocupaciones, pero la conexión que hubo desde el primer momento dio paso a una colaboración en la distancia que ha fructificado en diferentes trabajos siendo Alfonso no sólo sentimentalmente, sino formalmente parte de este equipo.

El paso del tiempo nos hace conscientes de que no hay nada que no devenga en nada y que no provenga de nada. Tengo que dar las gracias aquí a los miembros del Departamento de Estadística e Investigación Operativa por haberme hecho sentir como en casa. En particular, quisiera mencionar a la profesora Pilar del Pino Arabolaza por haberme ayudado en los momentos más débiles, por haberme empujado a ser yo mismo y por haber supuesto para mí el punto de referencia maternal en el mundo profesional.

Si hay algo de lo que me siento orgulloso es la oportunidad de conocer personas de diferentes lugares que han sido claves para alcanzar el objetivo final. Entre ellos, quisiera mencionar al profesor Franco Pellerey, pieza fundamental en esta memoria, que no sólo me ayudó durante los seis meses que pasé en Torino, sino que compartió conmigo su manera de ver las matemáticas, su intuición y su capacidad de trabajo.

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Finalmente, agradezco también a mis compañeros del Departamento de Estadística e Investigación Operativa de la Universidad de Alicante por confiar en mí, por mostrarme otros caminos y por haberme hecho un hueco donde saborear la que considero mi vocación: la docencia.

El tiempo que queda descrito en estas páginas es fiel reflejo y la culminación de un proceso largo y complicado, de un camino estrecho en ocasiones que forma ya parte de mí y de mi historia y que me quedaré conmigo para siempre.

Resumen en castellano

Tanto en la vida cotidiana como en el campo científico estamos habituados a observar fenómenos aleatorios cuyos resultados se expresan mediante números; por ejemplo el voltaje de salida en una fuente de alimentación, el número de personas en la cola del cine, la velocidad de conexión a la red, el tiempo de funcionamiento de un determinado aparato, etc. Incluso en problemas de naturaleza puramente cualitativa es muy frecuente recurrir a la codificación numérica; en situaciones tales como el diagnóstico de un paciente sano o enfermo, preguntas del tipo ¿estudias o trabajas?, donde las respuestas son usualmente codificadas con 0 y 1, aunque podría emplearse cualquier pareja de símbolos con igual precisión.

Este proceso de cuantificación nos lleva de manera natural a considerar ciertas funciones especiales con las que estudiar de una manera más efectiva cuestiones probabilísticas acerca de estos valores aleatorios, de esta forma aparecen las variables aleatorias y, por ende, los vectores aleatorios.

La teoría de ordenaciones estocásticas es actualmente una teoría bien establecida y promisoria. La necesidad de comparar dichos fenómenos aleatorios en base a diferentes criterios queda fructificada en estas relaciones de orden parcial entre variables o vectores aleatorios.

La utilidad y aplicabilidad de los órdenes estocásticos es un hecho contrastado. A lo largo de los últimos años son muchos los ámbitos que han encontrado en este rincón de las Matemáticas una herramienta con la que comparar, escoger y decidir en un momento dado.

Lo cierto es que puede ser realmente fácil comparar dos variables aleatorias, pero el interés real de estas comparaciones reside en la información que extraemos de ellas. La manera más sencilla de comparar variables aleatorias es a través de simples coeficientes, por ejemplo, las varianzas para comparar las variables aleatorias en términos de su dispersión o los coeficientes de Gini y los coeficientes de variación para la concentración, pero en la mayoría de ocasiones estos coeficientes no contemplan toda la información de la que disponemos, de manera que en realidad esta comparación da lugar a comparaciones poco significativas.

Bien sea que nos interese la magnitud, la dispersión o la concretración, o cualquier propiedad de un fenómeno aleatorio, los órdenes estocásticos nos permiten obtener beneficios importantes del conocimiento que tengamos de las funciones de distribución o de sus propiedades, incluso en el caso en el que la complejidad del modelo estocástico nos impida disponer de expresiones analíticas. El objetivo de la investigación matemática consiste entonces en encontrar órdenes bien definidos que permitan hallar cotas o soluciones aproximadas. Otras veces se requiere comparar sistemas desde ciertos puntos de vista y optimizarlos de acuerdo a distintos criterios.

Los órdenes estocásticos también son útiles en situaciones donde sólo se conocen algunas propiedades de las distribuciones del problema considerado. En Economía constituyen una herramienta valiosa en la teoría de decisiones bajo incertidumbre. También se encuentran muchas aplicaciones en diversas disciplinas como la confiabilidad, la física estadística, los seguros, la teoría de colas, la epidemiología, las ciencias actuariales, etc.

Los órdenes estocásticos constituyen un campo abierto con muchos problemas nuevos y un inmenso potencial teórico. En los últimos años han aparecido textos fundamentales que sirven de base para desarrollar aplicaciones en las más diversas disciplinas. Algunas referencias como Müller y Stoyan (2002) ó Shaked y Shanthikumar (2007) resultan convenientes para una introducción general a la teoría de órdenes estocásticos.

La clasificación y ordenación de distribuciones viene siendo una herramienta fundamental en Teoría de Fiabilidad, donde una variable aleatoria representa el tiempo aleatorio de funcionamiento de una componente o sistema de componentes. En este contexto la clasificación de distribuciones permite determinar el proceso de envejecimiento de la componente o sistema y la ordenación permite comparar componentes o sistemas para seleccionar aquéllas con un mayor tiempo de funcionamiento en algún sentido estadístico o probabilístico. En la práctica, esta teoría se relaciona con problemas de fatiga de los materiales, desgaste de piezas, oxidación, mortandad, etc. Cada componente, sistema o modelo de fiabilidad enevejece de una manera diferente y es entonces cuando nacen las diferentes clases de distribuciones de vida que quedan descritas en base a diferentes propiedades de dicha distribución.

La clasificación de distribuciones de vida es de una importancia enorme en Teoría de Fiabilidad especialmente durante las primeras etapas del diseño del producto. Muchas de las cotas de fiabilidad obtenidas a través de diferentes propiedades de las clases proporcionan interesantes estimaciones acerca de la fiabilidad del sistema bajo ciertas hipótesis. Este tipo de resultados relativos a la clasificación de distribuciones juega también un papel importante en la selección y validación de modelos estocásticos. Como referencias principales tenemos Barlow y Proschan (1975), Marshal y Olkin (2007) o Lai y Xie (2006) para un estudio más detallado.

La comparación multivariante de vectores aleatorios se puede llevar a cabo desde numerosos puntos de vista, la mayoría de ellos inspirados en el correspondiente caso univariante. A lo largo de esta memoria, comenzamos definiendo nuevos órdenes multivariantes relacionados con el estudio de diferentes propiedades asociadas a un fenómeno aleatorio tales como la dispersión, la concentración, la kurtosis y el sesgo, a través de tres procedimientos diferentes e inspirados en ordenaciones univariantes. Asímismo, dedicamos sendos capítulos al estudio de distintas nociones de envejecimiento en diferentes casos de vectores bivariantes con componentes dependientes y a la comparación de vidas residuales de vectores y, finalmente, estudiamos el modelo de tiempo exponencial transformado y su aplicación a los modelos de fragilidad, más conocidos como modelos frailty. A continuación, describiremos el contenido de cada uno de los capítulos con un mayor detalle.

En el Capítulo 1, recordamos las nociones básicas referentes a variables y vectores aleatorios que utilizaremos posteriormente. En primer lugar, se definen las funciones asociadas a variables y vectores aleatorios que resultan de importancia en la definición de órdenes estocásticos y nociones de envejecimiento. En particular, resultan de importancia a lo largo de la memoria el orden estocástico usual, dispersivo, right-spread, convexo, estrella, superaditivo, dmrl, nbue y algunos otros juegan papeles secundarios en las relaciones y propiedades. Por otra parte, nociones de envejecimientototales como IFR, IFRA, NBU, DMRL y NBUE son relevantes en el estudio desarrollado.

Resulta adecuado destacar también que las aplicaciones tratadas con posteridad en la memoria están referidas a modelos de vectores aleatorios con componentes dependientes. En estos últimos tiempos, el concepto de cópula ha tomado una relevancia significativa puesto que describe con gran claridad la estructura de dependencia de las distintas componentes. En este primer capítulo, recuperamos también el concepto de cópula el cual aparece relacionado con los modelos de tiempo exponencial transformado y de fragilidad que son tratados en los capítulos posteriores.

En el Capítulo 2, se proponen nuevas extensiones al caso multivariante de diferentes órdenes univariantes. A lo largo de este capítulo, se estudian los llamados órdenes de transformación, llamados así porque están definidos adjudicando propiedades a una transformación entre las dos variables o vectores aleatorios y propuestos por van Zwet (1964).

Roy (2002) plantea la extensión de dichos órdenes al caso multivariante dotando de unas propiedades especiales a una cierta transformación, procedimiento que resulta ser equivalente a la ordenación de variables aleatorias condicionadas truncadas en datos de supervivencia. En la Sección 2.2, proponemos la extensión de órdenes en dispersión y concentración en el mismo sentido, desarrollamos sus propiedades de clausura y preservación y relaciones entre ellos y exponemos ejemplos de modelos analíticos y aplicaciones en teoría de riesgos y desigualdad.

Khaledi y Kochar (2005) y Mercader et al. (2007) proponen la extensión del orden dispersivo y right-spread, respectivamente, teniendo en cuenta la ordenación de cada componente truncado cuando el resto de los componentes son mayores que sus respectivos cuantiles. Este orden multivariante encuentra similitudes con los órdenes multivariantes tratados en la sección anterior, pero cuenta con la ventaja de que la ordenación de vectores con la misma cópula queda reducida a la comparación de las distribuciones marginales. En la Sección 2.3, generalizamos los órdenes de transformación en este sentido y desarrollamos propiedades, relaciones y aplicaciones.

Por otra parte, Fernández-Ponce y Suárez-Llorens (2003) proponen una ordenación multivariante en dispersión basada en las propiedades de una transformación entre los dos vectores aleatorios. En la Sección 2.4, se define un orden multivariante dotando a dicha transformación de la propiedad de convexidad en dos sentidos diferentes.

El estudio de las nociones de envejecimiento surge junto con el desarrollo de nuevas ordenaciones, puesto que éstas son una buena herramienta para describirlas y caracterizarlas. Son muchas las nociones de envejecimiento multivariante que han sido propuestas a lo largo de la literatura basadas en sus correspondientes definiciones univariantes. En particular, Bassan y Spizzichino (1999), Bassan et al. (2002) ó Pellerey (2008) han propuesto diferentes extensiones de algunas nociones de envejecimiento univariantes. Dedicamos el Capítulo 3 al estudio de estas tres extensiones multivariantes.

En la Sección 3.2, estudiamos generalizaciones de dos clases univariantes basadas en comparaciones estocásticas de vidas residuales de vectores con componentes intercambiables para los clases IFR y DMRL cuyas caracterizaciones univariantes son estudiadas por Cao y Wang (1991), Belzunce et al. (2003) y Belzunce et al. (2004) en la línea de las propuestas por Bassan y Spizzichino (1999) y Bassan et al. (2002).

En la Sección 3.3, estudiamos el envejecimiento de vectores bivariantes no intercambiables cuando están descritos por cópulas arquimedianas, en particular de tipo Clayton, y basados en comparaciones de los vectores con sus vidas residuales a través del orden estocástico usual. Se presentan generalizaciones para el caso de vectores con componentes no intercambiables del envejecimiento para componentes intercambiables que se pueden consultar en Pellerey (2008).

Finalmente, en la Sección 3.4 estudiamos condiciones para la comparación de vectores definidos según el modelo de fragilidad y damos condiciones que aseguran un envejecimiento negativo en el sentido de Pellerey (2008).

El orden estocástico usual es quizás el más importante de todos los órdenes estocásticos. En el Capítulo 4, estudiaremos condiciones para asegurar el orden estocástico usual entre dos vectores con cópulas arquimedianas.

En la Sección 4.2, estudiamos condiciones para la comparación de las vidas residuales en el mismo momento $t \geq 0$ entre dos vectores aleatorios no intercambiables con cópulas de Clayton.

En la Sección 4.3, estudiamos condiciones para la comparación de dos vectores definidos mediante un modelo de tiempo exponencial transformado. Estas condiciones están basadas en comparaciones entre los generadores de sus cópulas, de manera que, en contraste con los resultados dados por Scarsini (1998), Müller y Scarsini (2001), Belzunce et al. (2011) quienes comparan estocásticamente vectores con la misma cópula, establecemos condiciones para ordenaciones estocásticas entre vectores con diferente cópula. A continuación, se exponen diferentes aplicaciones, en particular, cotas para valores esperados, modelos de fragilidad, optimización de portfolios y la comparación estocástica de mixturas.

Finalmente, en la Sección 4.4, estudiamos comparaciones en el orden estocástico usual de vectores aleatorios definidos según el modelo de fragilidad.

Los trabajos que han derivado del desarrollo de estos estudios son los siguientes:

- Belzunce, F., Mulero, J. y Ruíz, J.M. (2008). New multivariate IFR and DMRL notions for exchangeable dependent components in Advances in Mathematical Modeling for Reliability, 158-164, by T. Bedford, J. Quigley, L. Walls, B. Alkali, A. Daneshkhah and G. Hardman (Eds.), IOS Press, Amsterdam.

- Belzunce, F., Mulero J., Ruíz, J.M. y Suárez-Llorens, A. (2012a). New multivariate orderings based on conditional distributions. *Applied Stochastic Models in Business and Industry*, doi: 10.1002/asmb.924.

- Belzunce, F., Mulero J., Ruíz, J.M. y Suárez-Llorens, A. (2012b). Multivariate transform orders. Technical Report. Departamento de Estadística e Investigación Operativa, Universidad de Murcia.

- Belzunce, F., Mulero J., Ruíz, J.M. y Suárez-Llorens, A. (2012c). Multivariate convex transform orders. Technical Report. Departamento de Estadística e Investigación Operativa, Universidad de Murcia.

- Mulero J. y Pellerey F. (2010). Bivariate aging properties under archimedean dependence structures. *Communications in Statistics: Theory and Methods* **39**, 3108-3121.
- Mulero, J., Pellerey, F. y Rodríguez-Griñolo, R. (2010a). Negative aging and stochastic comparisons of residual lifetimes in multivariate frailty models. *Journal of Statistical Planning and Inference* **140**, 1594-1600.
- Mulero, J., Pellerey, F. y Rodríguez-Griñolo, R. (2010b). Stochastic comparisons for time transformed exponential models. *Insurance: Mathematics and Economics* **46** (2), 328-333.

Aún así, durante el desarrollo de todos estos trabajos, quedan abiertos diferentes aspectos que aseguran el trabajo futuro que explicamos en las conclusiones.

Por un lado, el estudio pormenorizado de las diferentes propiedades del orden en transformación multivariante convexo definido en la Sección 2.4, su interpretación y aplicaciones, así como la relación de este orden con los órdenes multivariantes existentes en la literatura referidos a ordenaciones en cuanto a kurtosis y sesgo.

Por otro lado, el estudio de las cópulas arquimedianas y el modelo de tiempo exponencial transformado en particular en el ámbito de las nuevas nociones de envejecimiento propuestas en la Sección 3.2, mediante condiciones en el generador de la cópula o las funciones características del modelo, respectivamente.

Como se observa, estas líneas de investigación futuras quedan enmarcadas y apoyadas por el trabajo ya realizado.

1
Preliminaries

Abstract. The outcome of an experiment or game is random and is modeled by random variables and vectors. The comparisons between these random values are mainly based on the comparison of some measures associated to random quantities. Stochastic orders and aging notions are a growing field of research in applied probability and statistics. In this chapter, we present the main definitions and properties that we will use along this memory.

1.1 Introduction

The randomness of events and phenomena has troubled mankind since the earliest times; its hidden essence has been a point of interest for philosophy and the methodology of science, whereas its intrinsic regularities have been brought to light by mathematics and empirical sciences. Various kinds of chance are well-known to every one of us from our everyday experience: the outcome of a coin-toss or die-roll, the length of time spent waiting in line, how meteorological phenomena will proceed. In all such situations, we are unable to predict the outcome of an experiment or the future course of a process. The cause of the difficulty is generally our incomplete information about all the factors at work driving the phenomenon at hand, although it may also lie in the unknown degree of accuracy in the empirical data we possess about it. Randomness may also stem from the excessively complex nature of a phenomenon, which thus precludes a clear-cut deterministic description.

In this chapter, we introduce the main definitions related to random variables and vectors and present the state-of-the-art in univariate and multivariate stochastic orders and aging notions. In particular, some works in which stochastic comparisons are used to characterize these notions are recalled.

Throughout this memory, "increasing" means "nondecreasing" and "decreasing" means "nonincreasing". We will denote by $=_{st}$, the equality in law, and by $\leq_{a.s.}$, the almost surely inequality. For any random vector **X**, or random variable, we will denote by $(\mathbf{X}|A)$ a random vector, or random variable, whose distribution is the conditional distribution of **X** given A.

1.2 On the univariate case

Two notions lie at the base of the probability theory: the random event and probability. The probability theory defines these concepts, studies the relationships between the probabilities of various random events, identifies when events are independent of one another, introduces rules for transforming probabilities, etc. In this context, random variables measure quantities that are subject to variations due to chance, i.e. randomness, in a mathematical sense.

1.2.1 Basic definitions

Let Ω be a **probability space** and $\mathcal{F}$ a **sigma–algebra** in Ω and let $X : \Omega \to \mathbb{R}$ be a random variable, i.e., a measurable function mapping Ω to the real numbers, that is, $X^{-1}((-\infty, x]) \in \mathcal{F}$ for any $x \in \mathbb{R}$.

Many of the properties of random variables are not concerned with the specific random variable X given above, but rather depends on the way X distributes its values. The **distribution function**, denoted by F, of a random variable X, is defined by

$$F(x) = P(X \leq x) \text{ for } x \in \mathbb{R}.$$

Random variables can generally be classified in two categories depending on the set of possible values: discrete and continuous. In particular, along this memory we will work with absolutely continuous random variables which is a subtype of the continuous random variables. In particular, X is an **absolutely continuous** random variable, if there exists a nonnegative function f on $\mathbb{R}$, called the **probability density function** of X such that

$$F(x) = \int_{-\infty}^{x} f(z)dz, \text{ for all } x \in \mathbb{R}.$$

It is clear that if X is an absolutely continuous random variable with distribution function F, then its **density function** is given by

$$f(x) = F'(x)$$

at those points x where F is differentiable.

The **support** of the distribution F is defined as the set of all the points x where F is strictly increasing.

Next we describe some functions of interest for this memory.

Definition 1.2.1 *Let X be a random variable, its* ***survival function*** *is given by*

$$\overline{F}(x) = 1 - F(x) = P(X > x), \text{ for } x \in \mathbb{R}.$$

In some situations, for example survival analysis or reliability, random variables used to be nonnegative because they represent the time until a specified event, not necessarily the end of a life. In these cases, the survival function of X is more meaningful than the distribution function. This function satisfies the following properties:

(i) $\overline{F}$ is a decreasing function.

(ii) $\overline{F}(\infty) = \lim_{x\to\infty} P(X > x) = 0.$

(iii) $\overline{F}(-\infty) = \lim_{x\to-\infty} P(X > x) = 1.$

Note that for any random variable, the distribution and survival function always exist.

It is well known that a real–valued, continuous and strictly monotone function of a single variable has an inverse on its range. It is also known that one can drop the hypothesis of continuity and strict monotonicity to obtain the notion of generalized inverse. Generalized inverses of increasing functions which are not necessarily distribution functions also frequently appear, for example, as transformations of random variables. These functions play an important role, in general, in the probability theory and statistics in terms of quantile functions and, in particular, for example, in risk and insurance theory it is known as Value–at–Risk.

Definition 1.2.2 *Let X be a random variable with distribution function F. For $p \in (0,1)$, the **quantile function** of X is given by the inverse generalized of F, that is,*

$$F^{-1}(p) = \inf\{x : F(x) \geq p\}$$

with $\inf\{\emptyset\} = \infty$.

Note that if F is continuous and strictly increasing, the generalized inverse F^{-1} coincides with the ordinary inverse of F. In literature, the quantile function of F, or X, is also denoted by F^{-} or Q_X.

In the practice, the most useful properties of the quantile functions are presented in the following proposition:

Proposition 1.2.3 *Let X be a random variable with distribution function F. Then,*

(i) $F^{-1}(p)$ *is an increasing and left–continuous function in* $p \in [0,1]$.

(ii) If X is an absolutely continuous random variable, F^{-1} is strictly increasing.

(iii) $FF^{-1}(p) \geq p$ *for all* $p \in [0,1]$. *Furthermore, if F is continuous in $F^{-1}(p)$, the equality holds.*

(iv) If Y is a random variable with distribution function G, then $F(x) \leq G(x)$ for all $x \in \mathbb{R}$ if, and only if,

$$F^{-1}(p) \geq G^{-1}(p),$$

for all $p \in [0,1]$.

An important application of quantile functions is the inverse transform method for generating random variables from univariate distributions in general which is often applied in Monte Carlo simulations.

Proposition 1.2.4 *Let X be a random variable with distribution function F.*

(i) If F is continuous, $F(X) =_{st} U$ where U is a uniformly distributed random variable on $(0, 1)$.

(ii) If U is an uniformly distributed random variable on $(0, 1)$, then $X =_{st} F^{-1}(U)$.

Quantiles of random variables are frequently used in the construction of descriptive statistics, for example, the median, the interquartile range and several measures of skewness and kurtosis based on percentiles. Perhaps, the most important particular case of the quantile is the median, given that the median is the central point which minimizes the average of the absolute deviations.

As mentioned before, we can find quantile functions in risk theory and insurance as Value–at–Risk, denoted by VaR, as a widely used risk measure of the risk of loss on a specific portfolio of financial assets. For a given portfolio, probability and time horizon, VaR is defined as a threshold value such that the probability that the mark-to-market loss on the portfolio over the given time horizon exceeds this value is the given probability level.

The theory of lifetime distributions plays an important role in fields like reliability theory, survival analysis and insurance mathematics. Assume that X is a random variable such that $\overline{F}(t) > 0$ or, equivalently $F(t) \leq 1$, for $x \in \mathbb{R}$.

Therefore,

$$P(t < X \leq t + \Delta t | X > t)$$

can be thought of as the "conditional probability" given survival up to time t of death or failure in the next small increment Δt of time.

Consequently, if we suppose that X is a nonnegative random variable with an absolutely continuous distribution function F and density function f, the "probability" of instantaneous failure at time t is given by

$$\lim_{\Delta t \to 0} \frac{P(t < X \leq t + \Delta t | X > t)}{\Delta t} = \frac{f(t)}{\overline{F}(t)}. \tag{1.1}$$

Definition 1.2.5 *Let X be an absolutely continuous random variable with distribution function F and density functions f. The function*

$$r(t) = \frac{f(t)}{\overline{F}(t)}, \text{ for } t \text{ such that } \overline{F}(t) > 0,$$

is called the ***hazard, or failure, rate function*** *of* X.

The interpretation given by (1.1) makes the notion of hazard rate so useful, both in theory and applications.

In case of nonnegative and absolutely continuous random variables, the hazard rate characterizes its distribution. In fact, given a hazard rate function r, it is possible to know the survival function

$$r(t) = -\frac{d}{dt}\log\left[\overline{F}(t)\right] = \frac{d}{dt}[-\log\overline{F}(t)] = \frac{dR(t)}{dt} \tag{1.2}$$

where $\log$ denotes the natural logarithm and $R(x) = -\log\overline{F}(x)$ is called the **hazard function** of F, or X.

In reliability or survival analysis, given that a unit is of age t, the remaining life after time t is a random variable defined as follows.

Definition 1.2.6 *Let* X *be a random variable and* $t \in \mathbb{R}$ *such that* $\overline{F}(t) > 0$*. The conditioned random variable*

$$X_t = (X - t|X > t)$$

is called the ***residual lifetime*** *at time* t*.*

Note that the hazard rate function can be also introduced by using the residual lifetime. In particular, we have that

$$\lim_{\Delta t \to 0}\frac{P(X_t \leq \Delta t)}{\Delta t} = r(t).$$

The corresponding survival of a residual lifetime X_t, denoted by $\overline{F}_t$, is given by

$$\overline{F}_t(x) = \tfrac{\overline{F}(t+x)}{\overline{F}(t)} \text{ for every } x \geq 0.$$

Given a random variable X, one can consider the expectation of its residual lifetime $\mathbf{E}[X_t]$ and it can be interpreted as the expected remaining life of a unit which has survived up to time t.

Definition 1.2.7 *Let X be a random variable and $t \in \mathbb{R}$ such that $\overline{F}(t) > 0$. The function*

$$m_X(t) = \mathbf{E}[X_t] = \mathbf{E}[X - t | X > t], \text{ for } t \text{ such that } \overline{F}(t) > 0,$$

*is called the **mean residual life function** of X.*

Note that given a random variable X, the mean residual life function is given by

$$m_X(t) = \frac{1}{\overline{F}(t)} \int_t^{\infty} \overline{F}(z) dz$$

for all t such that $\overline{F}(t) > 0$.

In fact, a lifetime distribution may be characterized by its survival function $\overline{F}$, the survival function of the residual lifetime $\overline{F}_t$, the hazard rate function r or the mean residual lifetime function m_X.

Finally, for a continuous random variable X, Fernández–Ponce et al. (1998) and, independently, Shaked and Shanthikumar (1998) defined the right–spread, or excess-wealth, function.

Definition 1.2.8 *Let X be a random variable with finite mean. The function*

$$RS_X(p) = \mathbf{E}[(X - F^{-1}(p))_+] = \int_{F^{-1}(p)}^{\infty} \overline{F}(x) dx, \text{ for } p \in [0,1],$$

*is called the **right spread function** of X where $x_+ = \max\{x, 0\}$.*

Note that, from the definition it is clear that this function can be interpreted as a measure of dispersion to the right of each quantile function F. In risk theory, it is also called **expected shortfall**. The expected shortfall measures the thickness of the upper tail from a fixed quantile. In the actuarial context, when random variables denote the random risk for an insurance company, $(X - F^{-1}(p))_+$ can be interpreted as the amount that reinsurances companies take over and the right spread functions as the expected amount.

It is interesting to note that when X is a continuous random variable, and from Proposition 1.2.3 (iii), it holds

$$RS_X(p) = (1-p) m_X(F^{-1}(p)), \text{ for all } p \in [0,1). \tag{1.3}$$

1.2.2 Univariate stochastic orders

Stochastic orders are considered to be more informative than the comparison of two single numbers (see Preface in Shaked and Shanthikumar, 2007). The comparison of random quantities has been developed in the last 40 years and for a detailed discussion on the subject the reader can look at Shaked and Shanthikumar (2007).

It is often believed that science advances through the identification of important concepts, which are first classified, compared and then quantified. Descriptive Statistics offers the theory and methods to identify statistical quantities associated with such concepts.

Characteristics of distributions or densities such as location, dispersion, skewness and kurtosis have been studied for descriptive purposes. Early on, measures of such characteristics were proposed, but sometimes these single coefficients do not correspond with the *measure they measure*.

In many situations, one has more information than just the single coefficients. The stochastic orders take into account the different types of information related to the random variables from a more general and descriptive point of view.

The comparison of random variables in terms of a descriptive characteristic can be done simply by using single coefficients, but sometimes it is not very informative or they may not be the appropriate quantities in some contexts. Stochastic orders let us to compare random variables in a more general way and seem to be more appropriate.

Next we recall several definitions of different stochastic orders of interest along this work. See Shaked and Shanthikumar (2007) and Müller and Stoyan (2001) for a comprehensive discussion of these and other stochastic orders, as well as their properties and the relationships between them.

The concept of dispersion, i.e., how spread out data are, is fundamental for univariate and multivariate distributions, in fact, a statistical research is unthinkable for a phenomenon without variability. However, there are a lot of ways to study the dispersion of a random variable.

Given two random variables X and Y, a first approach to compare dispersion is by using the comparison between their variances $Var(X)$ and $Var(Y)$. Theoretically, X will be less dispersed than Y if $Var(X) \leq Var(Y)$ because the higher the variance, the more spread out the data. But when we have a more complete description of the data or when the variance is not so descriptive of them, it is more useful to compare X and Y in terms of the univariate dispersive order. A formal definition of this order is based on the notion of quantile. See Lewis and Thompson (1981) for further details.

Definition 1.2.9 *Let X and Y be two random variables with distributions functions F and G, respectively. X is said to be smaller than Y in the* ***dispersive order****, denoted by $X \leq_{disp} Y$, if any pair of quantiles of G are at least as widely separated as the corresponding quantiles of F, i.e, if*

$$F^{-1}(q) - F^{-1}(p) \leq G^{-1}(q) - G^{-1}(p) \text{ for all } 0 < q \leq p < 1.$$

Intuitively it is clear that the dispersive order corresponds with a comparison between the random variables X and Y in terms of its variability because we are comparing intervals where X and Y accumulate the same probability $q - p$.

The dispersive order is location–free, i.e., given $c \in \mathbb{R}$, then $X \leq_{disp} Y$ if, and only if, $X + c \leq_{disp} Y$, as well as reflexive and transitive, but it is not antisymmetric.

This notion has several characterizations of different nature, some of them are collected in the following proposition.

Proposition 1.2.10 *Let X and Y be two random variables with distribution functions F and G, respectively. The following conditions are equivalent:*

(i) $X \leq_{disp} Y$.

(ii) $G^{-1}(p) - F^{-1}(p)$ *increases in* $p \in (0, 1)$.

(iii) $\overline{G}^{-1}(p) - \overline{F}^{-1}(p)$ *decreases in* $p \in (0, 1)$.

(iv) $G^{-1}F(x) - x$ *increases in* x.

When X and Y are absolutely continuous and have densities f and g, respectively, last condition can be differentiated and it holds the following characterization.

Proposition 1.2.11 *Let X and Y be two absolutely continuous random variables with density functions f and g, respectively. Then, $X \leq_{disp} Y$ if, and only if,*

$$g(G^{-1}(p)) \leq f(F^{-1}(p)), \text{ for all } p \in (0,1). \tag{1.4}$$

Note that if r and s are the hazard rate functions of X and Y, respectively, then, from Proposition 1.2.3 (iii), (1.4) is equivalent to

$$s(G^{-1}(p)) \leq r(F^{-1}(p)), \text{ for all } p \in (0,1). \tag{1.5}$$

The dispersive order can be characterized also by dispersive transformations among the two random variables.

Proposition 1.2.12 *Let X and Y be two continuous random variables. Then,*

$$X \leq_{disp} Y$$

if, and only if, $Y =_{st} \phi(X)$ such that $\phi(x') - \phi(x) \geq x' - x$ whenever $x \leq x'$. Moreover, $\phi = G^{-1}F$.

The dispersive order satisfies several desiderable properties, some of them are collected in the following proposition.

Proposition 1.2.13 *Let X and Y be two random variables such that $X \leq_{st} Y$. Then,*

(i) $X \leq_{disp} aX$ *whenever* $a \geq 1$.

(ii) $\phi(X) \leq_{disp} \psi(X)$ *whenever* $\phi \leq_{disp} \psi$*, i.e.,* $\phi' \leq \psi'$ *where* ϕ' *and* ψ' *are the derivatives of* ϕ *and* ψ*, respectively.*

(iii) $X \leq_{disp} Y$ *if, and only if,* $-X \geq_{disp} -Y$.

(iv) *If* $X \leq_{disp} Y$*, then* $\phi(X) \leq_{disp} \phi(Y)$ *for all increasing convex and all decreasing concave function* ϕ.

From Definition 1.2.8, the right spread function is considered as a tool to compare two distributions in terms of their variability.

Definition 1.2.14 *Let X and Y be two random variables with distributions functions F and G, respectively. X is said to be smaller than Y in the* ***right–spread order,*** *denoted by $X \leq_{rs} Y$, if*

$$RS_X(p) \leq RS_Y(p) \text{ for all } p \in (0,1).$$

From (1.3), in the continuous case, it can be characterized using the mean residual life function evaluated in quantiles.

Proposition 1.2.15 *Let X and Y be two absolutely continuous random variables with mean residual life functions m_X and m_Y respectively. Then, $X \leq_{rs} Y$ if, and only if,*

$$m_X(F^{-1}(p)) \leq m_Y(G^{-1}(p)) \text{ for all } p \in (0,1). \tag{1.6}$$

Finally, it is known that the right spread order is weaker than the dispersive order and both of them imply the inequality between the variances:

$$X \leq_{disp} Y \quad \Rightarrow \quad X \leq_{rs} Y \quad \Rightarrow \quad Var(X) \leq Var(Y). \tag{1.7}$$

The observation of the variability of a data set is sometimes supplemented by the study of its kurtosis and skewness. Kurtosis is a measure of whether the data are peaked or flat relative to a normal distribution while skewness is a measure of symmetry, or more precisely, the lack of symmetry.

The notion of skewness intended to represent departure of a density from simmetry (or sometimes even departure from normality), where one tail of the density is "stretched out" more than the other. Though simmetry is natural for distributions with support $(-\infty, \infty)$, the notions of asymmetry have a place also in describing distributions with support $[0, \infty)$.

An alternative to measuring skewness via single quantities is to find an stochastic order which captures the essence of what "F is less skewed than G" should mean. Surprisingly, there is a simple way to arrive at such an orders.

Imagine that the density f of a random variable X is graphed on a sheet of rubber that becomes thinner and thinner toward the right, and hence, more and more wasily stretched toward the right. Then, grasp the right–hand edge of the rubber sheet, stretch it out, and watch the density change shape. If F was simmetryc and unimodal before stretching, then after stretching f has become a new density g which is also unimodal, but which has a relatively long right–hand tail, i.e., g is "skewed to the right". The flexibility requirement of the rubber sheet simply means that the horizontal axis has been transformed by an increasing function ϕ which increments $\phi(x+\Delta)\phi(x)$ increasing as one move to the right (see Marshall and Olkin, 2007).

In fact, several skewness coefficients have been defined as quotient os quantile differences. See, for example, Groeneveld and Meeden (1984) or Balanda and MacGillivray (1990).

The origin of this approach is in van Zwet (1964) who formalized these ideas and proposed three orders whose definitions are the following:

Definition 1.2.16 *Let X and Y be two nonnegative random variables with an interval supports (finite or infinite) and distribution functions F and G, respectively. Then,*

*(i) X is said to be smaller than Y in the **convex order**, denoted by $X \leq_c Y$, if $G^{-1}F(x)$ is convex in the support of F.*

*(ii) X is said to be smaller than Y in the **starshaped order**, denoted by $X \leq_* Y$, if $G^{-1}F(x)$ is starshaped in the support of F, that is, $\frac{G^{-1}F(x)}{x}$ is increasing in $x \geq 0$.*

*(iii) X is said to be smaller than Y in the **superadditive order**, denoted by $X \leq_{su} Y$, if $G^{-1}F(x)$ is superadditive in the support of F. Recall that a function $\phi : \mathbb{R} \mapsto \mathbb{R}$ is superadditive if $\phi(x+y) \geq \phi(x) + \phi(y)$ for any $x, y \in \mathbb{R}$.*

For a survival function $\overline{G}$, which is a decreasing function, its generalized inverse is given by

$$\overline{G}^{-1}(p) = \sup\{x : \overline{G}(x) \geq p\}.$$

Along this memory, it is sometimes convenient to make use of the fact of

$$\begin{aligned} G^{-1}F(x) &= \inf\{y : G(y) \geq F(x)\} \\ &= \sup\{y : \overline{G}(y) \geq \overline{F}(x)\} = \overline{G}^{-1}\overline{F}(x). \end{aligned}$$

These orders are known as transform orders due to its definition are based on properties of the transformation $G^{-1}F(x)$, or $\overline{G}^{-1}\overline{F}(x)$, which maps X onto Y, that is, $Y =_{st} G^{-1}F(X)$, see Proposition 1.2.4.

Proposition 1.2.17 *Let X and Y be two continuous random variables. Then,*

$$X \leq_{c[*,su]} Y$$

if, and only if, $Y =_{st} \phi(X)$ where ϕ is a convex [starshaped, superadditive] transformation. Moreover, $\phi = G^{-1}F$.

Note that, without assuming continuity, $X \leq_{c[*,su]} Y$ if, and only if, $X =_{st} \phi(Y)$ where ϕ is a concave [antistarshaped, subadditive] transformation (see Müller and Stoyan, 2002).

These orders, as well as the dispersive order, can be define by considering properties of the quantile–quantile plot, denoted by Q–Q plot, which is equivalent to the graph of the function $G^{-1}F$. In fact, Müller and Stoyan (2002) defined a family of univariate orders $\leq_{\mathcal{F}}^{qq}$ based on the Q–Q plot, explicitly, $F \leq_{\mathcal{F}}^{qq} G$ if $G^{-1}F \in \mathcal{F}$ where $\mathcal{F}$ is an arbitrary class of increasing functions.

Given an arbitrary class of increasing functions $\mathcal{F}$, they also defined a family of univariate orders $\leq_{\mathcal{F}}^{pp}$ based on the P–P plot, explicitly, $F \leq_{\mathcal{F}}^{pp} G$ if $GF^{-1} \in \mathcal{F}$.

It is clear that such a relation is reflexive and transitive, but it is antysimmetric. Therefore, the convex, starshaped and superadditive transform orders are partial orders.

If X and Y are two nonnegative random variables, it has been accepted (see, e.g. Arnold and Groeneveld, 1995) that any measure γ of skewness should satisfy that

(i) $\gamma(X) = \gamma(aX + b)$ for all $a > 0$ and all b,

(ii) $\gamma(X) = -\gamma(-X)$,

(iii) If $X \leq_c Y$, then $\gamma(X) \leq \gamma(Y)$.

For example, when X is a random variable with uniquely defined modus y_M, Arnold and Groeneveld (1995) considered

$$\alpha_X = P(X < y_M) - P(y \geq y_M) \in [-1, 1]$$

as a measure of skewness. As Ferreira and Steel (2006) pronounced: this measure "is fairly intuitive for unimodal distributions with negative (positive) values for left (right) skewed distributions and 0 for symmetric distributions". Arnold and Groeneveld (1995) showed that α_X verifies conditions (i), (ii) and (iii) and, therefore, is an example of skewness measure.

Hence, convex order is closely connected with the notion of skewness. It is interesting to recall the following characterization in terms of the quantile function which shows the relevance of the convex order in reliability theory.

Proposition 1.2.18 *Let X and Y be two absolutely continuous random variables with density functions f and g respectively. Then, $X \leq_c Y$ if, and only if,*

$$\frac{f(F^{-1}(p))}{g(G^{-1}(p))} \text{ is an increasing function in } p \in (0, 1). \tag{1.8}$$

Observe that if r and s are the hazard rate functions of X and Y, respectively, this last condition (1.8) is equivalent to the following one.

$$\frac{r(F^{-1}(p))}{s(G^{-1}(p))} \text{ is an increasing function in } p \in (0, 1). \tag{1.9}$$

In case that the support of X is an interval, the starshaped order admits a similar characterization by using the quantile function, i.e., $X \leq_* Y$ if, and only if,

$$\frac{F^{-1}(p)}{G^{-1}(p)} \text{ is an increasing function in } p \in (0, 1). \tag{1.10}$$

The starshaped and superadditive orders can be considered as concentration orders, in fact, an important application is in the context of inequality as we will see next. But first it is necessary to introduce some new concepts.

In literature, one can find several orders that can be interpreted in terms of reliability. For example, Kochar and Wiens (1987), motivated by characterization (1.9), proposed the decreasing in mean residual life and new better than used in expectation orders whose definitions are the following.

Definition 1.2.19 *Let X and Y be two nonnegative random variables with mean residual life functions m_X and m_Y, respectively. Then,*

*(i) X is said to be smaller than Y in the **decreasing in mean residual life**, denoted by $X \leq_{dmrl} Y$, if $\frac{m_X(F^{-1}(p))}{m_Y(G^{-1}(p))}$ is decreasing in $p \in (0,1)$.*

*(ii) X is said to be smaller than Y in the **new better than used in expectation order**, denoted by $X \leq_{nbue} Y$, if $\frac{m_X(F^{-1}(p))}{m_Y(G^{-1}(p))} \leq \frac{\mathbf{E}[X]}{\mathbf{E}[Y]}$ for all $p \in (0,1)$.*

The dmrl and nbue orders have also an interpretation in the context of income distributions. In this context, the Lorenz curve and the Gini index are the most popular tools to analyze and compare income inequality.

The Lorenz curve is a simple way to describe income distribution using a graph and it appears in the context of income distributions, where the incomes of the individuals of a population are considered (nonnegative) random quantities. Given a nonnegative random variable X, its Lorenz curve $L_X : [0,1] \to [0,1]$ is an increasing and convex function and is given by

$$L_X(p) = \frac{\int_0^p F^{-1}(u)du}{\mathbf{E}[X]} \text{ for } p \in (0,1).$$

The closer the Lorenz curve of a country is to the 45–degree line, the more equal the distribution of income is. In an economic framework, when X models the income of the individuals in some population, L_X maps $p \in [0,1]$ to the proportion of the total income of the population which accrues to the poorest $100p\%$ of the population. In insurance, L_X can be thought of as being the fraction of the aggregate claims caused by the $100p\%$ of the treaties with the lowest claim size.

Given a nonnegative random variable X, the Gini coefficient G_X is usually defined mathematically based on the Lorenz curve as

$$GI_X = 1 - 2\int_0^1 L_X(p)dp \text{ for } p \in (0,1).$$

A low Gini coefficient indicates a more equal distribution, with 0 corresponding to complete equality, while higher Gini coefficients indicate more unequal distribution, with 1 corresponding to complete inequality.

Definition 1.2.20 *Let X and Y be two nonnegative random variables with finite means. Then, X is said to be smaller than Y in the* ***Lorenz order****, denoted by $X \leq_L Y$, if*

$$L_X(p) \geq L_Y(p) \text{ for all } p \in (0,1).$$

It is known that if $X \leq_L Y$, then the Gini coefficient of X is smaller than the corresponding coefficient for Y and it clearly shows that the Lorenz order is a more complete comparison between the inequality related to two different distributions.

Among the previous orders we have the following relationships:

$$\begin{array}{ccccccc} X \leq_c Y & \Rightarrow & X \leq_* Y & \Rightarrow & X \leq_{su} Y & & \\ \Downarrow & & & & \Downarrow & & \\ X \leq_{dmrl} Y & & \Rightarrow & & X \leq_{nbue} Y & \Rightarrow & X \leq_L Y \Rightarrow GI_X \leq GI_Y \end{array} \quad (1.11)$$

Along this memory, we also use some stochastic orders whose definitions are the following.

Definition 1.2.21 *Let X and Y be two random variables with distribution functions F and G, survival functions $\overline{F}$ and $\overline{G}$, and density functions f and g, in the absolutely continuous case, respectively. Then,*

- *(i) X is said to be smaller than Y in the* ***usual stochastic order****, denoted by $X \leq_{st} Y$, if $F(x) \geq G(x)$ for all x.*
- *(ii) X is said to be smaller than Y in the* ***likelihood ratio order****, denoted by $X \leq_{lr} Y$, if the ratio $\frac{f(x)}{g(x)}$ is an increasing function over the union of the supports of X and Y.*
- *(iii) X is said to be smaller than Y in the* ***hazard rate order****, denoted by $X \leq_{hr} Y$, if the ratio $\frac{\overline{F}(x)}{\overline{G}(x)}$ is an increasing function over $(-\infty, \max\{u_X, u_Y\})$, where u_X*

and u_Y denote the corresponding right endpoints of the supports of X and Y, respectively, or, equivalently, $r(t) \geq s(t)$ for all $t \in \mathbb{R}$.

(iv) *X is said to be smaller than Y in the* ***increasing convex [increasing concave] order,*** *denoted by $X \leq_{icx\ [icv]} Y$, if $\mathbf{E}[\phi(X)] \leq \mathbf{E}[\phi(Y)]$ for all increasing convex [increasing concave] functions $\phi : \mathbb{R} \to \mathbb{R}$ for which the expectations exist.*

(v) *X is said to be smaller than Y in the* ***Laplace transform order,*** *denoted by $X \leq_{Lt} Y$, if $\mathbf{E}[e^{-sX}] \geq \mathbf{E}[e^{-sY}]$, for all $s \geq 0$.*

Perhaps, the usual stochastic order is one of the best known orders. It satifies a lot of properties that make it a *magnitude* order. Several characterizations for stochastic order are worth noting.

Proposition 1.2.22 *Let X and Y be two random variables, the following conditions are equivalent:*

(i) $X \leq_{st} Y$.

(ii) $\overline{F}(x) \leq \overline{G}(x)$ *for all x.*

(iii) $\mathbf{E}[\phi(X)] \leq \mathbf{E}[\phi(Y)]$ *for all increasing function $\phi : \mathbb{R} \to \mathbb{R}$ for which the expectations exist.*

(iv) $\phi(X) \leq_{st} \phi(Y)$ *for all increasing function ϕ.*

(v) *There exist random variables $\widetilde{X}$ and $\widetilde{Y}$ defined on the same probability space such that X and $\widetilde{X}$ have the same distribution, Y and $\widetilde{Y}$ have the same distribution, and*

$$P(\widetilde{X} \leq \widetilde{Y}) = 1.$$

(vi) $F^{-1}(p) \leq G^{-1}(p)$ *for all $p \in (0,1)$ (see Proposition 1.2.3 (iv)).*

The stochastic orders presented along these section satisfy the following relationships:

$$\begin{array}{ccccccccc} X \leq_{lr} Y & \Rightarrow & X \leq_{hr} Y & \Rightarrow & X \leq_{st} Y & \Rightarrow & X \leq_{icv} Y \Rightarrow & & X \leq_{Lt} Y \\ & & \Downarrow & & \Downarrow & & & & \Downarrow \\ & & X \leq_{mrl} Y & \Rightarrow & X \leq_{icx} Y & & & \Rightarrow & \mathbf{E}[X] \leq \mathbf{E}[Y] \end{array} \tag{1.12}$$

1.2.3 Univariate aging notions

Notions of aging play an important role in reliability theory, survival analysis and other fields. *No aging* means that the age of a unit or component of a system has no effect on the distribution of its residual lifetime, *positive aging* describes the situation where residual lifetime tends to decrease in some probabilistic sense, while *negative aging* means that residual lifetime tends to increase. Attending to these three types of aging, distributions can be classified.

In this way, an abundance of classes of distributions describing aging have been considered in the literature; see, e.g. Barlow and Proschan (1975), Kijima (1997) or Lai and Xie (2007) for an overview. Next we recall some notions of aging.

Definition 1.2.23 *Let X be a nonnegative continuous random variable with survival function $\overline{F}$. Then,*

*(i) X is **increasing [decreasing] in failure rate,** denoted by IFR [DFR], if $-\log \overline{F}$ is a convex [concave] function in $\{t : \overline{F}(t) > 0\}$.*

*(ii) X is **increasing [decreasing] in failure rate in average,** denoted by IFRA [DFRA], if $-\log \overline{F}$ is a starshaped [antistarshaped] function in $\{t : \overline{F}(t) > 0\}$.*

*(iii) X is **new better [worse] than used,** denoted by NBU [NWE], if $-\log \overline{F}$ is a superadditive [subadditive] function in $\{t : \overline{F}(t) > 0\}$.*

*(iv) X is **decreasing [increasing] in mean residual life,** denoted by DMRL [IMRL], if $m_X(t)$ is an increasing [decreasing] function in $\{t : \overline{F}(t) > 0\}$.*

(v) X *is* ***new better [worst] than used in expectation****, denoted by NBUE [NWUE], if* $m_X(t) \leq [\geq] m_X(0)$ *in* $\{t : \overline{F}(t) > 0\}$.

Stochastic orders are related to the study of aging (see, for example, Belzunce and Shaked, 2008a, 2008b). In particular, in all these well known criteria, we are comparing the relative performance of a unit under consideration with that of an exponential distribution. Remember that a unit with an exponential life distribution suffers no aging. In this way, they can be characterized via transform orders by using an exponential distribution, denoted by $\mathcal{E}(\lambda)$, for any $\lambda > 0$.

Proposition 1.2.24 *Let* X *be a nonnegative random variable and* $Y \sim \mathcal{E}(\lambda)$ *for any* $\lambda > 0$. *Then,*

(i) X *is IFR if, and only if,* $X \leq_c Y$.

(ii) X *is IFRA if, and only if,* $X \leq_* Y$.

(iii) X *is NBU if, and only if,* $X \leq_{su} Y$.

(iv) X *is DMRL if, and only if,* $X \leq_{dmrl} Y$.

(v) X *is NBUE if, and only if,* $X \leq_{nbue} Y$.

Note that in the previous proposition there is no condition for $\lambda > 0$. Therefore, a random variable has an aging behavior if, and only if, it can be ordered with an exponential distribution (with any parameter) in the corresponding stochastic order.

These aging notions admit some other characterizations. In particular, in the prosecution we will use the following characterizations of the IFR [DFR] and DMRL notions.

Proposition 1.2.25 *Let* X *be a nonnegative random variable. Then,* X *is IFR [DFR] if, and only if, one of the following equivalent conditions holds:*

(i) $\frac{\overline{F}(t+x)}{\overline{F}(t)}$ *is decreasing [increasing] in* $t \geq 0$ *for all* $x \geq 0$.

(ii) $r(t)$ *is increasing [decreasing] in* $t \geq 0$.

(iii) $X_t \geq_{st} [\leq_{st}] X_{t'}$ *for all* $t \leq t'$ *or, equivalently,* $X_t \geq_{st} [\leq_{st}] X_{t+s}$ *for all* $t, s \geq 0$.

(iv) $X_t \geq_{hr} [\leq_{hr}] X_{t'}$ *for all* $t \leq t'$ *or, equivalently,* $X_t \geq_{hr} [\leq_{hr}] X_{t+s}$ *for all* $t, s \geq 0$.

(v) $X_t \geq_{icv} [\leq_{icv}] X_{t'}$ *for all* $t \leq t'$.

(vi) $X_t \geq_{Lt} [\leq_{Lt}] X_{t'}$ *for all* $t \leq t'$.

(vii) $X_t \geq_{disp} [\leq_{disp}] X_{t'}$ *for all* $t \leq t'$.

Proposition 1.2.26 *Let X be a random a nonnegative random variable. Then, X is DMRL if, and only if, one of the following equivalent conditions holds:*

(i) $X_t \geq_{mrl} X_{t'}$ *for all* $t \leq t'$.

(ii) $X_t \geq_{icx} X_{t'}$ *for all* $t \leq t'$.

1.3 On the multivariate case

Many times, data are collected on a number of units, and on each unit not just one, but many variables are measured. For example, in a psychological experiment, many tests are used, and each individual is subjected to all these tests. Since these are measurements on the same unit (an individual), these measurements (or variables) are correlated. The subject of the multivariate anaylisis deals with the statistical analysis of the data collected on more than one variable.

1.3.1 Basic definitions

A random vector is a function $\mathbf{X} = (X_1, \ldots, X_n) : \Omega \to \mathbb{R}^n$ such that

$$\mathbf{X}^{-1}((-\infty, x_1], \ldots, (-\infty, x_n]) \in \mathcal{F}$$

for any $x_1, \ldots, x_n \in \mathbb{R}$. Roughly speaking, $\mathbf{X}$ a collection of n random variables on a probability space $(\Omega, \mathcal{F}, P)$.

Recall that the usual componentwise partial order on $\mathbb{R}^n$ will be denoted by $\leq$ and it means that, for $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ it holds $\mathbf{x} \leq \mathbf{y}$ if

$x_i \leq y_i$ for all $i = 1, \ldots, n$. A function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^k$ is said to be *increasing* if $f(\mathbf{x}) \leq f(\mathbf{y})$ for $\mathbf{x} \leq \mathbf{y}$.

Most of the properties of random variables can be studied using the **joint distribution function** of the random vector **X**, denoted by F, which, for $\mathbf{x} = (x_1, \ldots, x_2) \in \mathbb{R}^n$, is given by

$$F_{\mathbf{X}}(\mathbf{x}) = P(\mathbf{X} \leq \mathbf{x}) = P(X_1 \leq x_1, \ldots, X_n \leq x_n).$$

In some contexts, the analysis of the random vector requires a closely connected tool. Given a random vector **X**, and for $\mathbf{x} = (x_1, \ldots, x_2) \in \mathbb{R}^n$, its **joint survival function** is given by

$$\overline{F}_{\mathbf{X}}(\mathbf{x}) = P(\mathbf{X} > \mathbf{x}) = P(X_1 > x_1, \ldots, X_n > x_n).$$

The distribution function of the margins X_i, for $i = 1, \ldots, n$, is given by

$$F_i(x) = P(X_i \leq x)$$

and they are called the **marginals distributions** of **X**.

The problem of finding suitable multivariate extensions of univariate quantiles has a long history in statistics. In fact, there have been several approaches to quantile functions for multivariate distributions. The simplest one defines a quantile vector as one that has the marginal classical quantiles as its components, but this does not take into account correlations between the components of the vectors of observations (Chakraborty, 2001). As usually, each extension of the multivariate quantiles is focused on some aspects or properties of the corresponding univariate quantile.

Suppose for example that we are interested in a generalization of the Propostion 1.2.4, it means that the transformation of a uniform distributed random vector via a "multivariate quantile" of a multivariate distribution function should be a random vector with the original distribution function as its distribution function.

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ with absolutely continuous distribution function and let $\mathbf{p} = (p_1, \ldots, p_n) \in [0, 1]^n$. The **standard construction** for **X** proposed by O'Brien (1975), Arjas and Lehtonen (1978) and Ruschendorf

(1981) and denoted by

$$\widehat{x}(\mathbf{p}) = (\widehat{x}_1(p_1), \widehat{x}_2(p_1, p_2), \ldots, \widehat{x}_n(p_1, p_2, \ldots, p_n))$$

where

$$\begin{aligned}
\widehat{x}_1(p_1) &= F^{-1}_{X_1}(p_1) \\
\widehat{x}_2(p_1, p_2) &= F^{-1}_{(X_2|X_1=\widehat{x}_1(p_1))}(p_2) \\
&\vdots \\
\widehat{x}_n(p_1, p_2, \ldots, p_n) &= F^{-1}_{\left(X_n|\bigcap_{j=1}^{n-1} X_j=\widehat{x}_j(p_1,p_2,\ldots,p_j)\right)}(p_n).
\end{aligned} \tag{1.13}$$

This known construction, also called **p–quantile** of **X**, is widely used in simulation theory and plays the role of the quantile in the multivariate case.

By "inverting" the standard construction, we can express the independent uniform random variables U_i as functions of the X_i. Let us denote

$$\overset{\star}{x}(\mathbf{x}) = (\overset{\star}{x}_1(x_1), \overset{\star}{x}_2(x_1, x_2), \ldots, \overset{\star}{x}_n(x_1, x_2, \ldots, x_n))$$

where

$$\begin{aligned}
\overset{\star}{x}_1(x_1) &= F_{X_1}(x_1) \\
\overset{\star}{x}_2(x_1, x_2) &= F_{(X_2|X_1=x_1)}(x_2) \\
&\vdots \\
\overset{\star}{x}_n(x_1, x_2, \ldots, x_n) &= F_{\left(X_n|\bigcap_{j=1}^{n-1} X_j=x_j\right)}(x_n).
\end{aligned} \tag{1.14}$$

By considering these two functions, it is possible to generalize the Proposition 1.2.4.

Proposition 1.3.1 *Let* **X** *be a random vector with distribution function* $F_{\mathbf{X}}$.

(i) If $F_{\mathbf{X}}$ is continuous, $\overset{\star}{x}(\mathbf{X}) =_{st} \mathbf{U}$ where $\mathbf{U} = (U_1, \ldots, U_n)$ is a random vector with n independent uniform distributed components on $(0, 1)$

(ii) If $\mathbf{U} = (U_1, \ldots, U_n)$ is a random vector with n independent uniform distributed components on $(0, 1)$, then $\mathbf{X} =_{st} \widehat{x}(\mathbf{U})$.

Now, from Proposition 1.3.1, we have that the transformation

$$\phi(x_1, \ldots, x_n) = (\phi_1(x_1), \phi_2(x_1, x_2), \ldots, \phi_n(x_1, \ldots, x_n)) \tag{1.15}$$

where, for $i = 1, \ldots, n$,

$$\phi_i(x_1, \ldots, x_i) = \widehat{y}_n(\overset{\star}{x}_1 (x_1), \ldots, \overset{\star}{x}_i (x_1, \ldots, x_i)), \tag{1.16}$$

maps $\mathbf{X}$ onto $\mathbf{Y}$, that is $\mathbf{Y} =_{st} \phi(\mathbf{X})$.

We recall now some multivariate functions that will be used along this work. The first two are a kind of *multivariate hazard rate* and *multivariate failure rate in average* and they were introduced by Johnson and Kotz (1975).

Definition 1.3.2 *Let $\mathbf{X} = (X_1, \ldots, X_n)$ be an absolutely continuous random vector in $\mathbb{R}^n$. For each $\mathbf{x} = (x_1, \ldots, x_n) \in \{\mathbf{x} : \overline{F}(\mathbf{x}) > 0\} \subseteq \mathbb{R}^n$, the **hazard function** of $\mathbf{X}$ is given by*

$$R(x_1, \ldots, x_n) = -\log \overline{F}(x_1, \ldots, x_n),$$

*and the **hazard gradient** of $\mathbf{X}$, by*

$$r(x_1, \ldots, x_n) = (r_1(x_1, \ldots, x_n), r_2(x_1, \ldots, x_n), \ldots r_n(x_1, \ldots, x_n)),$$

where, for $i = 1, \ldots, n$,

$$r_i(x_1, \ldots, x_n) = \frac{\partial}{\partial x_i} R(x_1, \ldots, x_n)$$

*is called the i**th multivariate hazard rate**.*

Definition 1.3.3 *Let $\mathbf{X} = (X_1, \ldots, X_n)$ be an absolutely continuous random vector in $\mathbb{R}^n$. For each $\mathbf{x} = (x_1, \ldots, x_n) \in \{\mathbf{x} : \overline{F}(\mathbf{x}) > 0\} \subseteq \mathbb{R}^n$, the **multivariate failure rate in average vector** of $\mathbf{X}$ is given by*

$$A(x_1, \ldots, x_n) = (A_1(x_1, \ldots, x_n), \ldots A_n(x_1, \ldots, x_n)),$$

where, for $i = 1, \ldots, n$,

$$A_i(x_1, \ldots, x_n) = \frac{1}{x_i} \int_0^{x_i} r_i(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) dy$$

*is called the i**th multivariate failure rate in average**.*

Finally, we need an extension of the mean residual life function in the multivariate case which was proposed by Arnold and Zahedi (1988).

Definition 1.3.4 *Let $\mathbf{X} = (X_1, \ldots, X_n)$ be an absolutely continuous random vector in $\mathbb{R}^n$. For each $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the **multivariate mean residual life** of $\mathbf{X}$ is given by*

$$m(x_1, \ldots, x_n) = (m_1(x_1, \ldots, x_n), \ldots, m_n(x_1, \ldots, x_n)),$$

where, for $i = 1, \ldots, n$,

$$m_i(x_1, \ldots, x_n) = \mathbf{E}[X_i - x_i | X_1 > x_1, \ldots, X_n > x_n]$$

*is called the i**th multivariate mean residual life**.*

Note that we can consider these functions as univariate functions only by fixing $n-1$ variables. In particular, for $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, let

$$\mathbf{x}^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$$

and let $r_i(\cdot|\mathbf{x}^i)$, $A_i(\cdot|\mathbf{x}^i)$ and $m_i(\cdot|\mathbf{x}^i)$, the ith multivariate hazard rate, failure rate in average and mean residual life fixed $\mathbf{x}^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$. In fact, it is easy to check that this functions are the hazard rate, failure rate in average and mean residual life functions of the following univariate random variable

$$\left(X_i \middle| \bigcap_{j \neq i} \{X_j > x_j\} \right).$$

In general, there are many multivariate distributions with the same margins F_i for $i = 1, 2, \ldots, n$, which exhibit different kinds of dependence between the components. Note that in the case of independence, the random vector can be studied just in terms of the margins. In literature, dependence has been analyzed taking into account some coefficients like the correlation or the Kendall tau coefficients, but there are situations in which this study is not complete or descriptive enough.

The notion of copula was introduced by Sklar (1959), and studied by, among others, Kimeldorf and Sampson (1975) under the name of uniform representation and by Deheuvels (1978) under the name of dependence function. The term “copula” was first used in the work of Sklar (1959) and is derived from the latin word copulare, to connect or to join. The main purpose of copulas is to describe the interrelation of several random variables.

Copulas have become a popular multivariate modeling tool in many fields where multivariate dependence is of interest because these functions describe in a more complete way the connection between the components of a random vector.

In actuarial science, copulas are used in modeling dependent mortality and losses (Frees et al., 1996; Frees and Valdez, 1998; Frees and Wang, 2005). In finance, copulas are used in asset allocation, credit scoring, default risk modeling, derivative pricing, and risk management (Bouyè et al., 2000; Embrechts et al., 2003; Cherubini et al., 2004). In biomedical studies, copulas are used in modeling correlated event times and competing risks (Wang and Wells, 2000; Escarela and Carriére, 2003). In engineering, copulas are used in multivariate process control and hydrological modeling (Yan, 2006; Genest and Favre, 2007).

Definition 1.3.5 *A **copula** $C : [0,1]^n \to [0,1]$ is a cumulative distribution function with uniform margins on* $[0,1]$.

The condition that C is a distribution function immediately leads to the following properties:

(i) $C(p_1, \ldots, p_n)$ is increasing in each $p_i \in [0,1]$.

(ii) $C(1, \ldots, 1, p_i, 1, \ldots, 1) = p_i$ for all $1 \le i \le n$.

(iii) For $a_i \le b_i$, $1 \le i \le n$, C satisfies the rectangle inequality

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(p_{1,i_1}, \ldots, p_{n,i_n}) \ge 0,$$

where $p_{j,1} = a_j$ and $p_{j,2} = b_j$.

On the other side, every function satisfying (i), (ii), (iii) is a copula. Furthermore, $C(1, p_1, \ldots, p_{n-1})$ is again a copula and so are all k-dimensional marginals with $2 \le k < d$.

Sklar (1959) showed that for any multivariate distribution function inherently embodies a copula function. On the other hand, any copula evaluated with some margins in the right way, leads to a multivariate distribution function. This is the important contribution of Sklar's Theorem.

Proposition 1.3.6 *(Sklar's Theorem) Given a copula C and n distribution functions F_i, if one defines*

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)), \tag{1.17}$$

then F is a multivariate distribution function with margins $F_1, \ldots, F_n$.

Furthermore, if F is the joint distribution function of a random vector with absolutely continuous marginals $F_1, \ldots, F_n$, then C is unique and can be constructed as follows,

$$C(p_1, \ldots, p_n) = F(F_1^{-1}(p_1), \ldots, F_n^{-1}(p_n)) \textit{ for } (p_1, \ldots, p_n) \in [0,1]^n, \tag{1.18}$$

for more details about copulas see Nelsen (1999).

Analogously, the dependence structure of a random vector **X** can be usefully described by its survival copula. A survival copula K is also a copula, that is, a distribution function with uniform margins on $[0,1]$. Given a survival copula K and n survival functions $\overline{F}_1, \ldots, \overline{F}_n$, if one defines

$$\overline{F}(x_1, \ldots, x_n) = K(\overline{F}_1(x_1), \ldots, \overline{F}_n(x_n)). \tag{1.19}$$

then $\overline{F}$ is a multivariate survival function with survival margins $\overline{F}_1, \ldots, \overline{F}_n$.

Furthermore, it is shown that for any joint survival function $\overline{F}$ with survival margins $\overline{F}_1, \ldots, \overline{F}_n$, there exists a survival copula K such that (1.19) holds. If $\overline{F}_1, \ldots, \overline{F}_n$ are absolutely continuous, then C is unique and can be constructed as follows,

$$K(p_1, \ldots, p_n) = \overline{F}(\overline{F}_1^{-1}(p_1), \ldots, \overline{F}_n^{-1}(p_n)) \text{ for } (p_1, \ldots, p_n) \in [0,1]^n. \tag{1.20}$$

Survival copulas, instead of ordinary copulas, are in particular considered in reliability and actuarial sciences, where survival functions, instead of distributions functions, are commonly studied.

Particularly interesting is the class of Archimedean survival copulas. Let us restrict our attention to the bivariate case.

Definition 1.3.7 *A copula C, or a survival copula, is said to be **Archimedean** if it can be written as*

$$C(u,v) = W(W^{-1}(u) + W^{-1}(v)) \text{ for all } u,v \in [0,1] \tag{1.21}$$

for a suitable one-dimensional, continuous, strictly positive and strictly decreasing and convex survival function $W : \mathbb{R}^+ \to [0,1]$ such that $W(0) = 1$.

*The inverse W^{-1} of the function W is called the **generator** of the Archimedean copula K.*

Along this memory, sometimes we will need relate the generator of the copula W with its random vector **X** and then we will denote it as $W_{\mathbf{X}}$.

As pointed out in Nelsen (1999), many standard survival copulas (such as the ones in Gumbel, Frank, Clayton and Ali-Mikhail-Haq families) are special cases of this class. Vectors of lifetimes having Archimedean survival copulas are of great interest in reliability and actuarial sciences, but also in many other applied contexts, being of this kind the dependence structure of frailty models (see Oakes, 1989). We refer the reader to Müller and Scarsini (2005) or Bassan and Spizzichino (2005), and references therein, for details, properties and recent applications of Archimedean survival copulas.

It is important to observe that when the vector $\mathbf{X} = (X,Y)$ has an Archimedean survival copula, its joint survival distibution can be written as in the following proposition:

Proposition 1.3.8 *Let $\boldsymbol{X} = (X,Y)$ be a bivariate random vector. If **X** has an Archimedean survival copula, then its joint survival function $\overline{F}$ can be written in the form*

$$\overline{F}_{\mathbf{X}}(x,y) = W_{\mathbf{X}}(R_X(x) + R_Y(y)) \tag{1.22}$$

for two suitable continuous and strictly increasing functions

$$R_X, R_Y : [0,+\infty) \to [0,+\infty)$$

such that $R_X(0) = R_Y(0) = 0, \lim_{x\to\infty} R_X(x) = \lim_{y\to\infty} R_Y(y) = \infty$*, where* $W_{\mathbf{X}}$ *is the survival function appearing in* (1.21).

Note that when $\overline{F}$ is defined as in (1.22) then $\overline{G}_X(x) = \overline{F}(x, 0) = W(R_X(x))$, $\overline{G}_Y(y) = \overline{F}(0, y) = W_{\mathbf{X}}(R_Y(y))$ and $W_{\mathbf{X}}^{-1}(x) = R_X(\overline{G}_X^{-1}(x)) = R_Y(\overline{G}_Y^{-1}(x))$.

A useful property of vectors having Archimedean survival copulas is one related to the residual lifetime of a random vector. Assume that $\mathbf{X} = (X_1, \ldots, X_n)$ has joint survival function defined as in (1.22). Then, as one can prove with straightforward calculation, the corresponding vector

$$\mathbf{X}_t = [X_1 - t, X_2 - t | X_1 > t, X_2 > t]$$

of residual lifetimes at time t has joint survival distribution function given by

$$\overline{F}_t(x, y) = W_{\mathbf{X},t}(R_{X_t}(x) + R_{Y_t}(y)) \tag{1.23}$$

where

$$W_{\mathbf{X},t}(x) = \frac{W_{\mathbf{X}}(R_X(t) + R_Y(t) + x)}{W_{\mathbf{X}}(R_X(t) + R_Y(t))},$$

and where

$$R_{X_t}(x) = R_X(t + x) - R_X(t), \qquad R_{Y_t}(y) = R_Y(t + y) - R_Y(t)$$

for $t, x \geq 0$.

Thus, the survival copula of $\mathbf{X}_t$ is defined as

$$K_{\mathbf{X},t}(u, v) = W_{\mathbf{X},t}(W_{\mathbf{X},t}^{-1}(u) + W_{\mathbf{X},t}^{-1}(v)),$$

while its univariate marginal survival functions are given by

$$\overline{G}_{X_t}(x) = W_{\mathbf{X},t}(R_{X_t}(x)) \text{ and } \overline{G}_{Y_t}(y) = W_{\mathbf{X},t}(R_{Y_t}(y)).$$

It should be also observed that both K and K_t are bivariate distribution functions. In the subsequent sections, we will denote with

$$\tilde{\mathbf{X}} = (\tilde{X}, \tilde{Y}) \text{ and } \tilde{\mathbf{X}}_t = (\tilde{X}_t, \tilde{Y}_t)$$

the two bivariate vectors having marginals uniformly distributed on $[0, 1]$ and joint distributions K and K_t, respectively. Obviously, it holds

$$\mathbf{X} =_{st} (\overline{G}_X^{-1}(\tilde{X}), \overline{G}_Y^{-1}(\tilde{Y}))$$

and

$$\tilde{\mathbf{X}} =_{st} (\overline{G}_X(X), \overline{G}_Y(Y)),$$

and similarly for $\mathbf{X}_t$ and $\tilde{\mathbf{X}}_t$.

Along this memory, particular attention will be given to the case of Clayton survival copulas which is an special case of Archimedean copulas.

Definition 1.3.9 *An Archimedean survival copula K is said to be a **Clayton copula** when*

$$W(x) = (x+1)^{-\theta} \textit{ for } \theta \in [0, +\infty),$$

or, equivalently, when it can be written as

$$K(u, v) = \max\{(u^{-\frac{1}{\theta}} + v^{-\frac{1}{\theta}} - 1)^{-\theta}, 0\} \textit{ for } \theta \in [0, +\infty).$$

This copula was introduced by Clayton (1978), who applied them in epidemiology, and further considered in hydrology and credit risks problems by Cook and Johnson (1981) and Charpentier and Juri (2006), for example. Recently, properties and characterizations of Clayton copulas have been also studied in Sungur (2002), Javid (2009) or Arias-Nicolás, Mulero, Núñez-Barrera and Suárez-Llorens (2010).

It should be recalled here that equality in law between $\tilde{\mathbf{X}}_{t_1}$ and $\tilde{\mathbf{X}}_{t_2}$ is satisfied only in case the survival copula is of Clayton type, as shown in recent works by Charpentier (2003 and 2006) and Oakes (2005). This property of Clayton copulas will be extensively used along this memory.

Another application of the Archimedean copulas is the Time Transformed Exponential models, shortly TTE, which have been recently considered in literature as an appropriate way to describe bivariate lifetimes (see Bassan and Spizzichino, 2005, and references therein).

Definition 1.3.10 *Let* $\mathbf{X} = (X_1, X_2)$ *be a pair of exchangeable lifetimes. The vector* $\mathbf{X}$ *is said to be defined via a* ***Time Transformed Exponential Model,*** *denoted by TTE model, if its joint survival function* $\overline{F}$ *can be written as*

$$\overline{F}(t,s) = W(R(t) + R(s)),\ t, s \geq 0, \tag{1.24}$$

for a suitable one-dimensional, continuous, convex and strictly decreasing survival function W *and for a suitable continuous and strictly increasing function*

$$R : [0, +\infty) \to [0, +\infty)$$

such that $R(0) = 0$ *and* $\lim_{t\to\infty} R(t) = \infty$*. We will write in this case that* $\mathbf{X} \sim TTE(W, R)$*.*

Therefore, if $\mathbf{X} \sim TTE(W, R)$, then its survival copula is given by

$$K(u, v) = W(W^{-1}(u) + W^{-1}(v)),$$

that is, its survival copula K is Archimedean with generator W^{-1}. Viceversa, bivariate survival functions $\overline{F}$ that admit an Archimedean survival copula can be written in the form as in (1.24), i.e., they can be defined via a $TTE(W, R)$ model for suitable functions W and R.

Observe that, for a Time Transformed Exponential model $\mathbf{X} = (X_1, X_2)$, the marginals X_1 and X_2 are exchangeable, that is, the joint distribution function F, or the joint survival function $\overline{F}$, is permutation-invariant. The main characteristic is that they "separate", in a sense, dependence properties (based on W) and aging (based on R).

Time Transformed Exponential models include relevant cases of dependent bivariate lifetimes, like independent or Schur constant laws (that is, $W(x) = W_\lambda(x) = \exp(-\lambda x)$ and $R(t) = t$, respectively), and can be derived for example from frailty models (see Marshall and Olkin, 1988, or Oakes, 1989).

The frailty approach is commonly used in reliability theory and survival analysis to model the dependence between subjects or components; according to this model the frailty (an unobservable random variable that describes environmental factors) acts simultaneously on the hazard functions of the lifetimes.

A vector $\mathbf{X}_k = (X_{k,1}, \ldots, X_{k,n})$ of non independent lifetimes is said to be described by a multivariate frailty model if its joint survival function is defined as

$$\overline{F}_k(t_1, \ldots, t_n) = P(X_{k,1} > t_1, \ldots, X_{k,n} > t_n) = \mathbf{E}\left[\left(\Pi_{i=1}^n \overline{G}_{k,i}(t_i)\right)^{\Theta_k}\right], \ t_i \in \mathbb{R}^+, \tag{1.25}$$

where Θ_k is an environmental random frailty taking values in $\mathbb{R}^+$ and $\overline{G}_{k,i}$ is the survival function of lifetime $X_{k,i}$ given $\Theta_k = 1$. For a detailed description of frailty models and their applications we refer the reader to Hougaard (2000).

For example, for the bivariate case,

$$\begin{aligned}\overline{F}(t_1, t_2) &= \mathbf{E}\left[\overline{G}_1(t)^{\Theta}\overline{G}_2(s)^{\Theta}\right] = \mathbf{E}\left[\exp(\Theta(\ln \overline{G}_1(t))) \exp(\Theta(\ln \overline{G}_2(s)))\right] \\ &= W(-\ln \overline{H}_X(t) - \ln \overline{H}_Y(s)) = W(R_X(t) + R_Y(s)), \ \ t, s \geq 0,\end{aligned}$$

that is, **X** is defined via a Time Transformed Exponential model where

$$W(x) = \mathbf{E}[e^{-x\Theta}],$$

and $R_X(t) = -\ln \overline{G}_1(t)$ (and similarly for R_Y). In this context, the survival copula is of Clayton type when the random parameter Θ has distribution in the Gamma family.

1.3.2 Multivariate stochastic orders

As we said before, the study of a random variables or vectors usually involves the analysis of dispersion. Based on the properties and characterizations of the univariate dispersive order, several attempts have been made to extend some of the previous orders to the multivariate case. For example, Shaked and Shanthikumar (1998) and Fernández-Ponce and Suárez-Llorens (2003) propose multivariate dispersive orders based on the standard construction. Other important contributions in this case have been made by Oja (1983), Giovagnoli and Wynn (1995) and Khaledi and Kochar (2005).

From now on, we will assume that the multivariate distribution function is absolutely continuous. Shaked and Shanthikumar (1998) and Fernández-Ponce and Suárez-Llorens (2003) consider the standard construction as a "multivariate quantile" of $\mathbf{X}$. The following definition, proposed by Shaked and Shanthikumar (1998), is the natural generalization of the univariate dispersive order, see Definition 1.2.9.

Definition 1.3.11 *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors. Then, $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the **multivariate variability order,** denoted by $\mathbf{X} \leq_{var} \mathbf{Y}$, if, and only, if*

$$\widehat{y}(\mathbf{p}) - \widehat{x}(\mathbf{p}) \text{ is increasing in } \mathbf{p} \in (0,1)^n.$$

Another multivariate generalization based on the standard construction was given by Fernández-Ponce and Suárez-Llorens (2003).

Definition 1.3.12 *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors. Then, $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the **multivariate dispersive order,** denoted by $\mathbf{X} \leq_{disp} \mathbf{Y}$, if, and only, if*

$$||\widehat{x}(\mathbf{q}) - \widehat{x}(\mathbf{p})||_2 \leq ||\widehat{y}(\mathbf{q}) - \widehat{y}(\mathbf{p})||_2$$

for all $\mathbf{p}, \mathbf{q} \in (0,1)^n$ where $||\cdot||_2$ means the Euclidean distance.

Taking into account the characterization in Proposition 1.2.12, Belzunce et al. (2007) proposed the following extension.

Definition 1.3.13 *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors. Let ϕ be the function defined in (2.15) which maps $\mathbf{X}$ onto $\mathbf{Y}$. Then, $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the **conditional dispersive order,** denoted by $\mathbf{X} \leq_{c-disp} \mathbf{Y}$, if, for all $i = 1, \ldots, n$,*

$$\phi_i(x_1, \ldots, x_i) \text{ is an expansion function in } x_i,$$

where ϕ_i is as in (1.16).

Important contributions in this way have been made by Oja (1983) and Giovagnoli and Wynn (1995). Clearly inspired in Proposition 1.2.12, these authors defined a multivariate dispersion order, denoted by $\mathbf{X} \leq_{\Delta} \mathbf{Y}$, through the existence of a multivariate function $k : \mathbb{R}^n \to \mathbb{R}^n$ which maps stochastically a random vector $\mathbf{X}$ to another one $\mathbf{Y}$, that is, $\mathbf{Y} =_{st} k(\mathbf{X})$, and, for all $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n+1}\}$, it holds that

$$\Delta(k(\mathbf{x}_1), \ldots, k(\mathbf{x}_{n+1})) \geq \Delta(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1})$$

where $\Delta(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1})$ is the volume of the "simplex" with vertices at $\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}$.

Finally, by using a conditioned random variables associated to the corresponding random vectors, Khaledi and Kochar (2005) introduced the following multivariate extension of the dispersive order, which they called upper orthant dispersive order.

Definition 1.3.14 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors. Then,* $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the* ***upper orthant dispersive order****, denoted by* $\mathbf{X} \leq_{uo-disp} \mathbf{Y}$*, if, for all* $\mathbf{p} = (p_1, \ldots, p_n) \in (0,1)^n$ *and each* $i = 1, 2, \ldots, n$*, holds*

$$\left(X_i \left| \bigcap_{j \neq i} \{X_j > F_j^{-1}(p_j)\} \right. \right) \leq_{disp} \left(Y_i \left| \bigcap_{j \neq i} \{Y_j > G_j^{-1}(p_j)\} \right. \right)$$

Clearly inspired in the previous definition, Mercader (2007) proposed a generalization of the right–spread order.

Definition 1.3.15 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors. Then,* $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the* ***upper orthant right–spread order****, denoted by* $\mathbf{X} \leq_{uo-rs} \mathbf{Y}$*, if, for all* $\mathbf{p} = (p_1, \ldots, p_n) \in (0,1)^n$ *and each* $i = 1, 2, \ldots, n$*, holds*

$$\left(X_i \left| \bigcap_{j \neq i} \{X_j > F_j^{-1}(p_j)\} \right. \right) \leq_{rs} \left(Y_i \left| \bigcap_{j \neq i} \{Y_j > G_j^{-1}(p_j)\} \right. \right)$$

for $i = 1, \ldots, n$*.*

The study of dispersion is complemented by skewness, kurtosis or concentration. As we showed in the previous section, the convex, starshaped and superadditive orders provide an interesting way to study it.

Next we describe some multivariate extensions provided by Roy (2002).

Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be two random vectors with joint distribution functions F and G and joint survival functions $\overline{F}$ and $\overline{G}$, respectively, and such that $\overline{F}(0, \ldots, 0) = \overline{G}(0, \ldots, 0) = 1$. Following Roy (2002), we consider the following notation, let

$$\overline{G}_i(\mathbf{x}) = \frac{\overline{G}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)}{\overline{G}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)}$$

and

$$\overline{F}_i(\mathbf{x}) = \frac{\overline{F}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)}{\overline{F}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)}$$

for $i = 1, 2, \ldots, n$, where we consider $\overline{G}_i(\mathbf{X})$ and $\overline{F}_i(\mathbf{X})$ as functions of x_i with x_j's, $j = 1, 2, \ldots, n$ ($\neq i$), as given parameters. In that case, a solution of u_i, where

$$\overline{G}_i(x_1, \ldots, x_{i-1}, u_i, x_{i+1}, \ldots, x_n) = \overline{F}_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \tag{1.26}$$

can be viewed as $u_i = \overline{G}_i^{-1}\overline{F}_i(\mathbf{x})$, where u_i is a function of x_i given x_j's for $j = 1, 2, \ldots, n$ ($j \neq i$).

Roy (2002) proposed the following definitions of the multivariate convex, star-shaped and superadditive orders.

Definition 1.3.16 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be nonnegative two random vectors and let* u_i *be a solution of* (1.26) *as before. Then,*

(i) $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the* ***multivariate convex order,*** *denoted by* $\mathbf{X} \leq_{mc} \mathbf{Y}$*, if, for any* $(x_1, \ldots, x_n) \in (0, \infty)^n$ *and for all* $i = 1, 2, \ldots, n$*,* u_i *is convex in* x_i*, when* x_j *holds fixed for* $j \neq i$*.*

(ii) $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the* ***starshaped order,*** *denoted by* $\mathbf{X} \leq_{m*} \mathbf{Y}$*, if, for any* $(x_1, \ldots, x_n) \in (0, \infty)^n$ *and for all* $i = 1, 2, \ldots, n$*,* u_i *is star-shaped in* x_i*, when* x_j *holds fixed for* $j \neq i$*.*

(iii) $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the* ***superadditive order,*** *denoted by* $\mathbf{X} \leq_{msu} \mathbf{Y}$*, if, for any* $(x_1, \ldots, x_n) \in (0, \infty)^n$ *and for all* $i = 1, 2, \ldots, n$*,* u_i *is supperadditive in* x_i*, when* x_j *holds fixed for* $j \neq i$*.*

Among the different extensions of the hazard rate order, we consider the one proposed by Hu et al. (2003).

Definition 1.3.17 *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ *be two random vectors. Then,* $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the weak multivariate hazard rate order, denoted by* $\mathbf{X} \leq_{whr} \mathbf{Y}$*, if*

$$\frac{\overline{G}(\mathbf{x})}{\overline{F}(\mathbf{x})} \text{ is increasing in } \mathbf{x} \in \{\mathbf{x} : \overline{G}(\mathbf{x}) > \mathbf{0}\}. \tag{1.27}$$

where $\frac{a}{0} = \infty$ *when* $a > 0$.

Finally, given $\mathbf{X}$ and $\mathbf{Y}$ two random vectors, the equivalent conditions for the univariate stochastic order in (1.2.22), suggest the following extensions to the multivariate case.

(i) $F(\mathbf{x}) \geq G(\mathbf{x})$ for all $\mathbf{x}$.

(ii) $\overline{F}(\mathbf{x}) \leq \overline{G}(\mathbf{x})$ for all $\mathbf{x}$.

(iii) $\mathbf{E}[\phi(\mathbf{X})] \leq \mathbf{E}[\phi(\mathbf{Y})]$ for all increasing function $\phi : \mathbb{R}^n \to \mathbb{R}$ for which the expectations exist.

(iv) $\phi(\mathbf{X}) \leq_{st} \phi(\mathbf{Y})$ for all increasing function $\phi : \mathbb{R}^n \to \mathbb{R}$.

(v) There exist random vectors $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{Y}}$ on the same probability space such that $\mathbf{X}$ and $\widetilde{\mathbf{X}}$ have the same distribution, $\mathbf{Y}$ and $\widetilde{\mathbf{Y}}$ have the same distribution, and

$$P(\widetilde{\mathbf{X}} \leq \widetilde{\mathbf{Y}}) = 1.$$

However, these five equivalent conditions define three different multivariate orders. Conditions (i) and (ii) define two orders called upper orthant order and lower orthant order whose definitions are the following.

Definition 1.3.18 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors with distribution functions* F *and* G *and survival functions* $\overline{F}$ *and* $\overline{G}$*, respectively. Then,*

(i) $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the* ***lower orthant order****, denoted by* $\mathbf{X} \leq_{lo} \mathbf{Y}$*, if* $F(\mathbf{x}) \geq G(\mathbf{x})$ *for all* $\mathbf{x}$.

(ii) $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the* ***upper orthant order****, denoted by* $\mathbf{X} \leq_{uo} \mathbf{Y}$*, if* $\overline{F}(\mathbf{x}) \leq \overline{G}(\mathbf{x})$ *for all* $\mathbf{x}$.

On the other hand, conditions (iii), (iv) and (v) are equivalent and they define the usual multivariate stochastic order.

Definition 1.3.19 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors. Then,* $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the* ***usual multivariate stochastic order****, denoted by* $\mathbf{X} \leq_{st} \mathbf{Y}$*, if* $\mathbf{E}[\phi(\mathbf{X})] \leq \mathbf{E}[\phi(\mathbf{Y})]$ *for all increasing function* $\phi : \mathbb{R}^n \to \mathbb{R}$ *for which the expectations exist.*

Note that $\mathbf{X} \leq_{st} \mathbf{Y}$ strictly implies both $\mathbf{X} \leq_{lo} \mathbf{Y}$ and $\mathbf{X} \leq_{uo} \mathbf{Y}$.

Finally, one can consider stochastic orders related with positive dependence. Notions of positive dependence of two random variables X_1 and X_2 have been introduced in the literature to describe the property that large (or small) values of X_1 tend to go together with large (or small) values of X_2. Such comparison can be generalized to general pairs of bivariate distributions with given marginals. In particular, we need the definition of the positive quadrant dependence order.

Definition 1.3.20 *Let* $\mathbf{X} = (X_1, X_2)$ *and* $\mathbf{Y} = (Y_1, Y_2)$ *be two bivariate random vectors with the same marginal distributions and joint distribution functions* $F_{\mathbf{X}}$ *and* $F_{\mathbf{Y}}$*, respectively. Then,* $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the* ***positive quadrant dependence order****, denoted by* $\mathbf{X} \leq_{PQD} \mathbf{Y}$*, if, and only if,* $F_{\mathbf{X}}(x_1, x_2) \leq F_{\mathbf{Y}}(x_1, x_2)$ *for all* $(x_1, x_2) \in \mathbb{R}^2$.

Note that the positive quadrant dependence order between the random vectors also holds if, and only if, $\overline{F}_{\mathbf{X}}(x_1, x_2) \leq \overline{F}_{\mathbf{Y}}(x_1, x_2)$ for all $(x_1, x_2) \in \mathbb{R}^2$ where $\overline{F}_{\mathbf{X}}$ and $\overline{F}_{\mathbf{Y}}$ are the joint survival functions of $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$, respectively. Moreover, if $K_{\mathbf{X}}$ and $K_{\mathbf{Y}}$ are its survival copulas, then $\mathbf{X} \leq_{PQD} \mathbf{Y}$, if, and only if, $K_{\mathbf{X}}(u, v) \leq K_{\mathbf{Y}}(u, v)$ for all $(u, v) \in [0, 1] \times [0, 1]$.

Given two bivariate random vectors $\mathbf{X}$ and $\mathbf{Y}$, we have that

$$\mathbf{X} \leq_{PQD} \mathbf{Y} \quad \Leftrightarrow \quad \mathbf{X} \leq_{uo} \mathbf{Y}$$

and

$$\mathbf{X} \leq_{PQD} \mathbf{Y} \quad \Leftrightarrow \quad \mathbf{X} \leq_{lo} \mathbf{Y}.$$

It is important to note that while for the upper and lower orthant orders it is not necessary that $\mathbf{X}$ and $\mathbf{Y}$ share the same margins, the comparison in the positive quadrant dependence order requires the same margins. In fact, whereas the upper and lower orthant orders measure the size (or the location) od the random vectors, the positive quadrant dependence order measure the positive dependence of them.

1.3.3 Multivariate aging notions

Notions of multivariate aging can be defined from several points of view. Along this memory, we will use three of them, those due to Roy (1994), Bassan and Spizzichino (1999), Bassan et al. (2002) and Pellerey (2008).

Roy (1994) defined the multivariate classes of life distributions MIFR, MIFRA, MNBU, MDMRL, MNBUE and its corresponding duals MDFR, MDFRA, MNWU, MIMRL and MNWUE, as follows.

Definition 1.3.21 *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *be a nonnegative random vector. Then,*

(i) $\mathbf{X}$ *is said to be* ***multivariate increasing [decreasing] in failure rate****, denoted as MIFR [MDFR], if* $r_i(x_1, \ldots, x_n)$ *is an increasing [decreasing] function in* x_i *for all* $(x_1, \ldots, x_n) \in \mathbb{R}^n_+$ *and* $i = 1, \ldots, n$*.*

(ii) $\mathbf{X}$ *is said to be* ***multivariate increasing [decreasing] in failure rate in average****, denoted as MIFRA [MDFRA], if* $A_i(x_1, \ldots, x_n)$ *is an increasing [decreasing] function in* x_i *for all* $(x_1, \ldots, x_n) \in \mathbb{R}^n_+$ *and* $i = 1, \ldots, n$*.*

(iii) $\mathbf{X}$ *is said to be* ***multivariate new better [worse] than used****, denoted as MNBU [MNWU], if, for all* $(x_1, \ldots, x_n) \in \mathbb{R}^n_+$*,* $y_i \geq 0$ *and* $i = 1, \ldots, n$*, holds*

$$\overline{F}(x_1, \ldots, x_{i-1}, x_i + y_i, x_{i+1}, \ldots, x_n)\overline{F}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$
$$\leq [\geq] \overline{F}(x_1, \ldots, x_i, \ldots, x_n)\overline{F}(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n).$$

(iv) $\mathbf{X}$ *is said to be* ***multivariate decreasing [increasing] in mean residual life****, denoted as MDMRL [MIMRL], if* $m_i(x_1, \ldots, x_n)$ *is a decreasing [increasing] function in* x_i *for all* $(x_1, \ldots, x_n) \in \mathbb{R}^n_+$ *and* $i = 1, \ldots, n$*.*

(v) $\mathbf{X}$ *is said to be* ***multivariate new better [worse] than used in expectation****, denoted as MNBUE [MNWUE], if*

$$m_i(x_1, \ldots, x_n) \leq [\geq] m_i(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n_+$ *and* $i = 1, \ldots, n$.

Roy (2002) showed that some of these notions, in particular MIFR, MIFRA and MNBU, can be characterized by using a multivariate exponential distribution in the same way as in Proposition 1.2.24.

Let us denote as $G(\{\lambda_i\}_{i=1}^n, \{\lambda_{ij}\}_{i>j}, \ldots, \lambda_{1,\ldots,n})$ a random vector $\mathbf{Y}$ with exponential multivariate distribution type Gumbel, that is, a random vector with survival function $\overline{G}(x_1, \ldots, x_n)$ given by the following expression:

$$\overline{G}(x_1, \ldots, x_n) = \exp\left[-\sum_i \lambda_i x_i - \sum_{i>j}\sum \lambda_{ij} x_i x_j - \cdots - \lambda_{1,2,\ldots,n} \prod_{j \neq i} x_j\right], \quad (1.28)$$

where $(x_1, \ldots, x_n) \in \mathbb{R}^n_+$.

Proposition 1.3.22 *Let* $\mathbf{X}$ *be a nonnegative random vector and*

$$\mathbf{Y} \sim G(\{\lambda_i\}_{i=1}^n, \{\lambda_{ij}\}_{i>j}, \ldots, \lambda_{1,\ldots,n}),$$

i.e., $\mathbf{Y}$ *is a random vector type Gumbel with survival function given by* (1.28) *for any set of parameters* $\{\lambda_i\}_{i=1}^n, \{\lambda_{ij}\}_{i>j}, \ldots, \lambda_{1,\ldots,n}$. *Then,*

(i) $\mathbf{X}$ *is MIFR if, and only if,* $\mathbf{X} \leq_{mc} \mathbf{Y}$.

(ii) $\mathbf{X}$ *is MIFRA if, and only if,* $\mathbf{X} \leq_{m*} \mathbf{Y}$.

(iii) $\mathbf{X}$ *is MNBU if, and only if,* $\mathbf{X} \leq_{msu} \mathbf{Y}$.

Analogously as in the univariate case, note that the parameters do not affect the aging behavior of the random vector, that is, a random vector has an aging behavior if, and only if, it is smaller than a multivariate exponential distribution of type Gumbel in the corresponding order. The key point of this property is that the conditional distributions of these particular case of multivariate exponential distribution are again exponential distributed.

Bassan and Spizzichino (1999) and Bassan et al. (2002) proposed extensions of some of these notions in the multivariate case, in particular MIFR and MDMRL, when the lifetimes of the components have exchangeable joint probability distributions. Recall that a joint probability distribution F is said to be exchangeable if F is permutation-invariant. The starting point for their proposal is the following. Based on Proposition 1.2.25, it is possible to prove that given two independent and identically distributed random lifetimes X and Y, then X (and Y) is IFR if, and only if, for all $x \leq y$

$$(X - x|X > x, Y > y) \geq_{st} (Y - y|X > x, Y > y).$$

A similar characterization holds for the IFR holds replacing the stochastic order by the hazard rate order and a similar one holds for the DMRL notion replacing the stochastic order by the comparisons of expectations or by the mean residual life order (see Proposition 1.2.26). Dropping the independence, but keeping exchangeability, leads Bassan and Spizzichino (1999) and Bassan et al. (2002) to propose new multivariate notions of ageing, as follows.

Definition 1.3.23 *Let (X, Y) be an exchangeable random vector. Then*

(i) *(X, Y) is said to be **bivariate increasing in failure rate**, denoted by BIFR, if, for $x \leq y$,*

$$(X - x|X > x, Y > y) \geq_{st} (Y - y|X > x, Y > y).$$

(ii) *(X, Y) is said to be **bivariate increasing in failure rate in the strong sense**, denoted by s-BIFR, if, for $x \leq y$,*

$$(X - x|X > x, Y > y) \geq_{hr} (Y - y|X > x, Y > y).$$

(iii) *(X, Y) is said to be **bivariate decreasing in mean residual life in the weak sense**, denoted by w-BDMRL, if, for $x \leq y$,*

$$\mathbf{E}[X - x|X > x, Y > y] \geq \mathbf{E}[Y - y|X > x, Y > y].$$

(iv) *(X, Y) is said to be **bivariate decreasing in mean residual life in the strong sense**, denoted by s-BDMRL, if, for $x \leq y$,*

$$(X - x|X > x, Y > y) \geq_{mrl} (Y - y|X > x, Y > y).$$

Pellerey (2008) and Mulero and Pellerey (2010) consider multivariate generalizations of the IFR and DFR notions can be defined considering multivariate versions of the stochastic comparisons given in Proposition 1.2.25.

Definition 1.3.24 *Let* **X** *be a random vector. Then,*

(i) **X** *is said to be* ***multivariate increasing [decreasing] failure rate,*** *denoted by* $\mathcal{A}^{+}_{FR}[\mathcal{A}^{-}_{FR}]$*, if*

$$\mathbf{X}_{t+s} \leq_{st} [\geq_{st}]\mathbf{X}_t \text{ for all } t, s \geq 0. \tag{1.29}$$

(ii) **X** *is said to be* ***multivariate increasing [decreasing] failure rate in the weak sense,*** *denoted by* $\mathcal{A}^{w+}_{FR}[\mathcal{A}^{w-}_{FR}]$*, if*

$$\mathbf{X}_{t+s} \leq_{lo} [\geq_{lo}]\mathbf{X}_t \text{ for all } t, s \geq 0. \tag{1.30}$$

Also, one can consider the class $\mathcal{A}^0$ of bivariate lifetimes such $\mathbf{X}_{t+s} =_{st} \mathbf{X}_t$ where $=_{st}$ (equality in law) holds for every $t, s \geq 0$ (see, e.g., Ghurye and Marshall, 1984).

Definition 1.3.25 *Let* **X** *be a random vector. Then* **X** *is said to have the* ***multivariate lack of memory property,*** *denoted by* $\mathcal{A}^0$*, if*

$$\mathbf{X}_{t+s} =_{st} \mathbf{X}_t \text{ for all } t, s \geq 0. \tag{1.31}$$

Conditions (3.2) and (3.3) are of course of interest in different fields of applied probability, like reliability and actuarial sciences. In reliability theory, in particular, they provide sufficient conditions for the usual stochastic comparison of two systems having the same coherent life function τ, but built using used components: in fact, for example, for every $t, s \geq 0$ one has $\tau(\mathbf{X}_{t+s}) \leq_{st} \tau(\mathbf{X}_t)$ if (3.2) holds, as follows from the fact that coherent functions are non–decreasing in their arguments (see also Theorem 6.B.16(a) in Shaked and Shanthikumar, 2007).

2

New multivariate orders

Abstract. In this chapter, new multivariate extensions of several univariate orders are proposed. To start with, we define some new multivariate orders based on the comparison of univariate marginal distributions conditioned on survival data for the rest of the components. Additionally, we define a new multivariate convex order based on the multivariate convexity of a certain transformation between two random vectors. Relationships among multivariate orders and applications to some multivariate random vectors are provided. Another possible extensions of these univariate orders for the future research are proposed as well.

2.1 Introduction

As we said in the previous chapter, a multivariate order is an interesting tool to analyze the data properties and to obtain extensions for multivariate data of univariate order concepts such as dispersion or concentration. Generalization of these univariate concepts to the multivariate case is not easy due to the the lack of a unique criterion for ordering multivariate observations in $\mathbb{R}^n$.

However, there are several attempts to generalize univariate orders attending to different criteria. In this chapter, we consider and study three alternatives to define some multivariate orders.

In particular, in Section 2.2, we consider multivariate extensions for the dispersive, right–spread, decreasing in mean residual life and new better than used in expectation orders based on comparisons of truncated random variables. In Section 2.3, we consider extensions for the convex, starshaped and superadditive orders based on conditioning on quantiles. To finish, in Section 2.4, we consider a multivariate extension of the convex order based on the convexity of a proper transformation among two random vectors.

2.2 New multivariate orders based on truncation

2.2.1 Why truncation?

Given a bivariate random vector (X, Y), we can find several notions of positive dependence based on conditional random variables like $(X|Y > x)$, $(X|Y \leq x)$ or $(X|Y = x)$. Some of these notions are of interest in the context of reliability (see Harris, 1970, Basu, 1971, Block, 1977, Roy, 1994, Roy, 2001, Spizzichino, 2001, Roy, 2002 and Lai and Xie, 2006) and in some other contexts like inequality theory (see Muliere and Petrone, 1992) and risk theory (see Denuit et al., 2005). Many of these notions are defined by comparison of the conditional distributions with the marginal distribution under the theoretical assumption that X and Y are independent.

These comparisons can be extended to compare general pairs of random vectors. For example, given two bivariate random vectors (X_1, X_2) and (Y_1, Y_2), and based on the PQD order notion, it is possible to provide comparisons of the conditional random variables $(X_2|X_1 > x)$ and $(Y_2|Y_1 > x)$, and also for $(X_1|X_2 > x)$ and $(Y_1|Y_2 > x)$. These comparisons are made in terms of the distribution functions and expected values of the conditional random variables (see Section 9.A in Shaked and Shanthikumar, 2007).

This topic is of interest in the context of reliability and risk theory where the detection of concordant behaviour (the components tend to be all large together or small together) is specially important and in particular to compare the degree of concordance among to financial assets or two sets of components of a system. It is natural to consider variability measures to compare random quantities and, especifically, these conditional random variables.

In risk theory the comparison can be made in terms of the variances in order to avoid situation of great uncertainty. Let us consider the following real example in which we consider two bivariate random vectors of returns. In the first case we consider the returns of two Spanish energetic companies, Endesa (E) and Iberdrola (I), and let us denote by (R_E, R_I) the bivariate random vector of the corresponding returns. In the other case, we consider the returns of two Spanish bank companies, Santander (S) and BBVA (B), and let us denote the corresponding vector of returns by (R_S, R_B). Data are of public access and can be easily obtained from the Yahoo! Finance site. We have considered bivariate samples of size 400 where the share value is measured from March 2009 until December 2010. If we denote by x_t the share value at time t, the rate of return at time t is defined by computing the rate

$$\frac{x_t - x_{t-1}}{x_{t-1}}.$$

In figure 2.2.1, we give a plot of the bivariate samples which indicates, in both cases, a positive association between the bivariate returns. This is a typical behaviour given that similar securities usually have some positive correlation with each other.

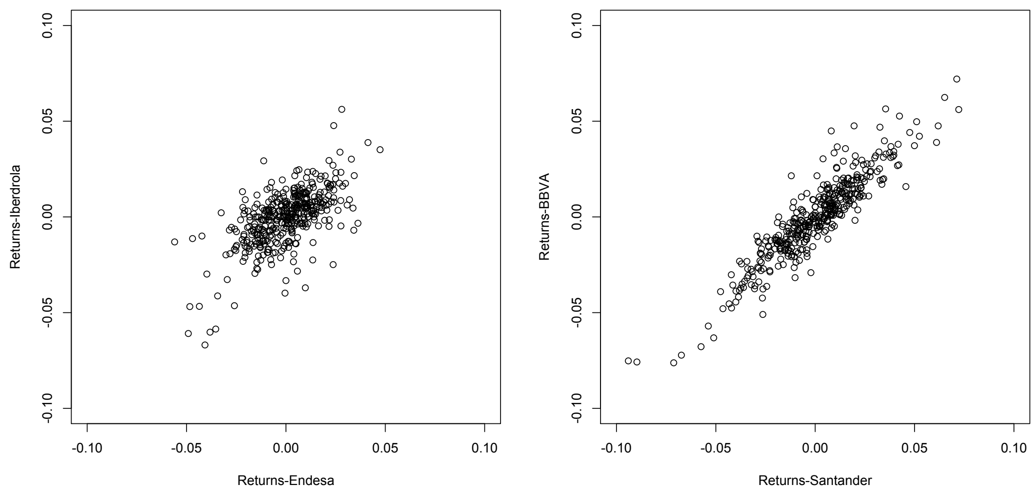

Figure 1: *Plots of samples values for* (R_E, R_I) *and* (R_S, R_B).

Additionally we can ask ourselves: how does affect the values of one component to the uncertainty or variability of the other component? That is, we can ask about the behaviour, for example of $Var(R_E|R_I > x)$ and $Var(R_S|R_B > x)$ (or $Var(R_I|R_E > x)$ and $Var(R_B|R_S > x)$). This is of interest if we want to avoid high uncertainty or variability. In figure 2, we provide empirical values for these quantities (we have computed the conditional variances up to a point x where the sample size is not big enough to provide a good estimation of the variance). As we can see the conditional variances for the energetic companies are smaller than the corresponding for the bank companies which shows that the uncertainty is smaller for the case of the energetic companies than that for the bank companies. Therefore, if we are interested to avoid uncertainty, we should chose the energetic companies. It is worth to mention that Arias-Nicolás et al. (2009) and Arias-Nicolás et al. (2010) used a similar graphical tool to detect empirically multivariate aging properties by plotting some classical multivariate dispersion measures associated with multivariate residual lifetimes.

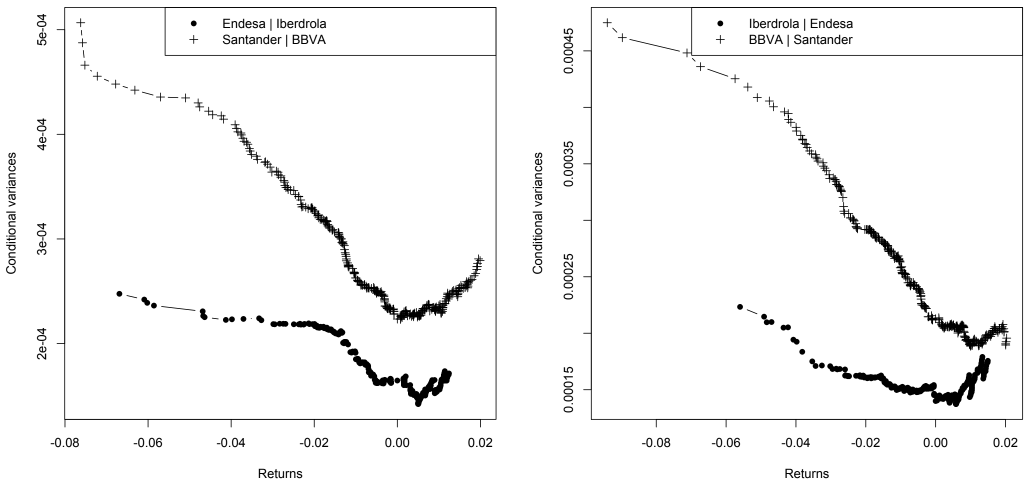

Figure 2: *Estimation of conditional variances for* $(R_E|R_I > x)$ *and* $(R_S|R_B > x)$.

This situation appears also in some parametric models. Let us consider the case of a bivariate Pareto distribution, denoted as $\mathbb{P}(\alpha_1, \alpha_2, a)$, with joint survival function

$$\overline{F}(x_1, x_2) = \left(\frac{x_1}{\alpha_1} + \frac{x_2}{\alpha_2} - 1\right)^{-a}$$

where $x_i \geq \alpha_i > 0$ and $a > 2$.

It is not difficult to see that

$$Var(X_i|X_j > x_j) = \frac{\alpha_i^2}{\alpha_j^2} \frac{a}{(a-1)^2(a-2)} x_j^2.$$

Let us consider now $(X_1, X_2) \sim \mathbb{P}(\alpha_1, \alpha_2, a)$ and $(Y_1, Y_2) \sim \mathbb{P}(\alpha_1, \alpha_2, b)$, clearly if $a > b$, then $Var(X_i|X_j > x_j) \leq Var(Y_i|Y_j > x_j)$ for $i \neq j \in \{1, 2\}$.

The aim of this chapter is to propose and study comparisons of random vectors in terms of comparisons of conditional and truncated random variables. In particular, we consider the comparison in terms of dispersion and concentration.

2.2.2 Definition and first properties

In the univariate case, comparison of dispersion is related to the dispersive and right–spread orders, while concentration can be related to the convex, starshaped, superadditive, Lorenz, DMRL and NBUE orders.

Roy (2002) proposed definitions of the multivariate convex, starshaped and superadditive orders, denoted by $\leq_{mc}$, $\leq_{m*}$ and $\leq_{msu}$, respectively, and Hu, Khaledi and Shaked (2003) proposed a definition for the multivariate (weak) hazard rate order, denoted by $\leq_{whr}$ (see Section 1.3.2).

From now on, let us denote as $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and

$$\mathbf{x}^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1},$$

for $i = 1, \ldots, n$. Moreover, given a random vector $\mathbf{X} = (X_1, \ldots, X_n)$, we will denote as

$$\mathbf{X}^i = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n). \tag{2.1}$$

Along this section, we consider that

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) = \left(X_i \left| \bigcap_{j \neq i} \{X_j > x_j\} \right.\right)$$

and

$$\left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right) = \left(Y_i \left| \bigcap_{j \neq i} \{Y_j > x_j\} \right.\right)$$

have an interval support, finite or infinite.

We note that the orders proposed by Roy (2002) and Hu et al. (2003) share an important property: they can be characterized in terms of the conditional distributions. In fact, it is easy to see that these orders can be characterized as follows.

Proposition 2.2.1 *Let* **X** *and* **Y** *be two random vectors (nonnegative in the case of the starshaped order). Then,*

$$\mathbf{X} \leq_{mc[m*,msu,whr]} \mathbf{Y}$$

if, and only if, for all $\mathbf{x} \in \mathbb{R}^n$ *and each* $i = 1, \ldots, n$*, holds*

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{c[*,su,hr]} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right). \tag{2.2}$$

Proof. Let us consider the multivariate convex order as in Definition 1.3.16. By definition, $\mathbf{X} \leq_{mc} \mathbf{Y}$, if, and only if, u_i is convex in x_i when x_j is fixed for $j \neq i$, where u_i is a solution of

$$\overline{G}_i(x_1, \ldots, x_{i-1}, u_i, x_{i+1}, \ldots, x_n) = \overline{F}_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$$

It is easy to see that $u_i = \overline{G}_i^{-1}\overline{F}_i(\mathbf{X})$ for $j \neq i$ where

$$\overline{F}_i(\mathbf{X}) = \frac{\overline{F}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)}{\overline{F}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)}$$

and

$$\overline{G}_i(\mathbf{X}) = \frac{\overline{G}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)}{\overline{G}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)}$$

are obviously the survival functions of the following random variables

$$(X_i|\mathbf{X}^i > \mathbf{x}^i) \text{ and } (Y_i|\mathbf{Y}^i > \mathbf{x}^i),$$

respectively, where $\mathbf{x}^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

This proves the required result. The cases of the starshaped and superadditive orders can be proved in a similar way.

In the other case, let us consider the weak multivariate hazard rate order as in Definition 1.3.17. Then, $\mathbf{X} \leq_{whr} \mathbf{Y}$, if, and only if,

$$\frac{\overline{G}(\mathbf{x})}{\overline{F}(\mathbf{x})} \text{ is increasing in } \mathbf{x} \in \{\mathbf{x} : \overline{G}(\mathbf{x}) > \mathbf{0}\},$$

or, equivalently, if $x_i \leq x_i'$, then

$$\frac{\overline{F}(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)}{\overline{F}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)} \leq \frac{\overline{G}(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)}{\overline{G}(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)}$$

for all $\mathbf{x}^i \in \mathbb{R}^n$ and each $i = 1, \ldots, n$. Therefore, the previous inequality can be written as

$$P\left(X_i > x_i' \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq P\left(Y_i > x_i' \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right),$$

which is equivalent to (see Proposition 1.2.22)

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{st} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right).$$

Taking into account that the stochastic order is closed under translations,

$$\left(X_i - x_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{st} \left(Y_i - x_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right),$$

and this is equivalent to the hazard rate order between the conditioned random variables, see Proposition 1.2.25 (ii). ■

These characterizations show that the previous orders are based on comparisons of univariate marginal distributions given some information for the rest of the components. In particular, if the components of the random vector denote the time to fail or to death of some units, systems or organisms, then we compare the univariate times to fail when we have the survival data information for the rest of the components.

Given a random vector $\mathbf{X} = (X_1, \ldots, X_n)$, for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1 \ldots, n$, let $F_i(\cdot|\mathbf{x}^i)$, $F_i^{-1}(\cdot|\mathbf{x}^i)$, $\overline{F}_i(\cdot|\mathbf{x}^i)$, $r_i(\cdot|\mathbf{x}^i)$, $A_i(\cdot|\mathbf{x}^i)$ and $m_i(\cdot|\mathbf{x}^i)$ be the distribution, quantile, survival, hazard rate, hazard rate in average and mean residual life function, respectively, of the conditional random variable

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right).$$

Note that the hazard rate, the hazard rate in average and the mean residual life functions, respectively, of this conditional random variable are equivalent to the ith multivariate hazard rate, ith multivariate failure rate in average and the ith multivariate mean residual life functions of **X**, respectively, as in Definitions 1.3.2, 1.3.3 and 1.3.4.

Analogously, for a random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$, and for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$, we denote by $G_i(\cdot|\mathbf{x}^i)$, $G_i^{-1}(\cdot|\mathbf{x}^i)$, $\overline{G}_i(\cdot|\mathbf{x}^i)$, $s_i(\cdot|\mathbf{x}^i)$, $B_i(\cdot|\mathbf{x}^i)$ and $l_i(\cdot|\mathbf{x}^i)$ the distribution, quantile, survival, hazard rate, hazard rate in average and mean residual life function, respectively, of the conditional random variable

$$\left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right).$$

Based on previous characterizations, we consider the following generalizations of some univariate orders.

Definition 2.2.2 *Let* **X** *and* **Y** *be two random vectors. Then,*

(i) **X** *is said to be smaller than* **Y** *in the* ***multivariate dispersive order****, denoted by* $\mathbf{X} \leq_{mdisp} \mathbf{Y}$*, if, for all* $\mathbf{x} \in \mathbb{R}^n$ *and each* $i = 1, \ldots, n$*, holds*

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{disp} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right).$$

(ii) **X** *is said to be smaller than* **Y** *in the* ***multivariate right spread order****, denoted by* $\mathbf{X} \leq_{mrs} \mathbf{Y}$*, if, for all* $\mathbf{x} \in \mathbb{R}^n$ *and each* $i = 1, \ldots, n$*, holds*

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{rs} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right).$$

Let **X** *and* **Y** *be two nonnegative random vectors. Then,* *(iii)* $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the* ***multivariate decreasing in mean residual life order****, denoted by* $\mathbf{X} \leq_{mdmrl} \mathbf{Y}$*, if, for all* $\mathbf{x} \in \mathbb{R}^n$ *and each* $i = 1, \ldots, n$*, holds*

$$\left(X_i \left|\mathbf{X}^i > \mathbf{x}^i\right.\right) \leq_{dmrl} \left(Y_i \left|\mathbf{Y}^i > \mathbf{x}^i\right.\right).$$

(iv) $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the* ***multivariate new better than used in expectation order****, denoted by* $\mathbf{X} \leq_{mnbue} \mathbf{Y}$*, if, for all* $\mathbf{x} \in \mathbb{R}^n$ *and each* $i = 1, \ldots, n$*, holds*

$$\left(X_i \left|\mathbf{X}^i > \mathbf{x}^i\right.\right) \leq_{mbue} \left(Y_i \left|\mathbf{Y}^i > \mathbf{x}^i\right.\right).$$

These orders are transitive, reflexive and closed under marginalization and under conjunctions, as we show in the next results.

Proposition 2.2.3 *(i) Let* $\mathbf{X}$ *be a random vector (nonnegative in the case of the mdmrl and mnbue orders), then* $\mathbf{X} \leq_{mdisp[mrs,mdmrl,mnbue]} \mathbf{X}$*.*

(ii) Let $\mathbf{X}$*,* $\mathbf{Y}$ *and* $\mathbf{Z}$ *be three random vectors (nonnegative in the case of the mdmrl and mnbue orders). If*

$$\mathbf{X} \leq_{mdisp[mrs,mdmrl,mnbue]} \mathbf{Y} \text{ and } \mathbf{Y} \leq_{mdisp[mrs,mdmrl,mnbue]} \mathbf{Z},$$

then $\mathbf{X} \leq_{mdisp[mrs,mdmrl,mnbue]} \mathbf{Z}$*.*

(iii) Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ *be a set of independent random vectors (nonnegative in the case of the mdmrl and mnbue orders) where* $\mathbf{X}_i = (X_i^1, \ldots, X_i^{d_i})$*,* $i = 1, \ldots, n$*. Let* $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ *be a set of independent random vectors (nonnegative in the case of the mdmrl and mnbue orders) where* $\mathbf{Y}_i = (Y_i^1, \ldots, Y_i^{d_i})$*,* $i = 1, \ldots, n$*. If, for* $i = 1, \ldots, n$*,*

$$\mathbf{X}_i \leq_{mdisp[mrs,mdmrl,mnbue]} \mathbf{Y}_i,$$

then

$$(\mathbf{X}_1, \ldots, \mathbf{X}_n) \leq_{mdisp[mrs,mdmrl,mnbue]} (\mathbf{Y}_1, \ldots, \mathbf{Y}_n)$$

That is, the mdisp, mrs, mdmrl and mnbue orders are closed under conjunctions.

Proof. We give the proof for the dispersive order, the proof for the rest of orders is similar.

(i) Let $\mathbf{X}$ be a random vector, then, for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$,

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{disp} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right),$$

or, equivalently, $\mathbf{X} \leq_{mdisp} \mathbf{X}$.

(ii) Let $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ be three random vectors. Suppose that $\mathbf{X} \leq_{mdisp} \mathbf{Y}$ and $\mathbf{Y} \leq_{mdisp} \mathbf{Z}$, then

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{disp} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right),$$

and

$$\left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right) \leq_{disp} \left(Z_i \left| \mathbf{Z}^i > \mathbf{x}^i \right.\right).$$

By combining the last two inequalities, $\mathbf{X} \leq_{mdisp} \mathbf{Z}$.

(iii) Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ and $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ be two sets of independent random vectors where the dimension of $\mathbf{X}_i$ and $\mathbf{Y}_i$ is d_i, $i = 1, \ldots, n$. Let us assume that $\mathbf{X}_i \leq_{mdisp} \mathbf{Y}_i$ for $i = 1, \ldots, n$, then we have to prove that, for all $\mathbf{x}_i \in \mathbb{R}^n$, and each $i = 1, \ldots, n$, $k = 1, \ldots, d_i$,

$$\left(X_i^k \left| \mathbf{X}_i^k > \mathbf{x}_i^k, \bigcap_{j \neq i} \{\mathbf{X}_j > \mathbf{x}_j\} \right.\right) \leq_{disp} \left(Y_i^k \left| \mathbf{Y}_i^k > \mathbf{x}_i^k, \bigcap_{j \neq i} \{\mathbf{Y}_j > \mathbf{x}_j\} \right.\right).$$

By hypothesis, X_i^k and $\mathbf{X}_j$ (and Y_i^k and $\mathbf{Y}_j$) are independent if $j \neq i$, therefore

$$\begin{aligned} \left(X_i^k \left| \mathbf{X}_i^k > \mathbf{x}_i^k, \bigcap_{j \neq i} \{\mathbf{X}_j > \mathbf{x}_j\} \right.\right) &= \left(X_i^k \left| \mathbf{X}_i^k > \mathbf{x}_i^k \right.\right) \\ &\leq_{disp} \left(Y_i^k \left| \mathbf{Y}_i^k > \mathbf{x}_i^k \right.\right) \\ &= \left(Y_i^k \left| \mathbf{Y}_i^k > \mathbf{x}_i^k, \bigcap_{j \neq i} \{\mathbf{Y}_j > \mathbf{x}_j\} \right.\right). \end{aligned}$$

■

Next we show that, under some regularity conditions, previous orders are closed under marginalization.

Proposition 2.2.4 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors (nonnegative in the case of the mdmrl and nbue orders). Let* $I = (i_1, \ldots, i_r) \subseteq (1, \ldots, n)$*,* $1 \leq r \leq n$ *and* $\mathbf{X}_I = (X_{i_1}, \ldots, X_{i_r})$ *and* $\mathbf{Y}_I = (Y_{i_1}, \ldots, Y_{i_r})$*.*

(i) If $\mathbf{X} \leq_{mdisp} \mathbf{Y}$*, then* $\mathbf{X}_I \leq_{mdisp} \mathbf{Y}_I$*.*

(ii) If $\mathbf{X} \leq_{mrs[mdmrl,mnbue]} \mathbf{Y}$*, and if, for all* $t \notin I$*,*

$$\mathbf{E}\left[\left(X_r \middle| \bigcap_{s\in I, s\neq r} \{X_s > x_s\}, \bigcap_{t\notin I}\{X_t > x_t\}\right)_+\right] \to \mathbf{E}\left[\left(X_r \middle| \bigcap_{s\in I, s\neq r} \{X_s > x_s\}\right)_+\right],$$

and

$$\mathbf{E}\left[\left(Y_r \middle| \bigcap_{s\in r, s\neq r} \{Y_s > y_s\}, \bigcap_{t\notin I}\{Y_t > x_t\}\right)_+\right] \to \mathbf{E}\left[\left(Y_r \middle| \bigcap_{s\in r, s\neq r} \{Y_s > y_s\}\right)_+\right],$$

when $x_t \to -\infty$*, then* $\mathbf{X}_I \leq_{mdisp} \mathbf{Y}_I$*.*

Proof.

(i) Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors such that $\mathbf{X} \leq_{mdisp} \mathbf{Y}$. It is not difficult to see that, for all $r \in I$,

$$\left(X_r \middle| \bigcap_{s\in I, s\neq r} \{X_s > x_s\}, \bigcap_{t\notin I}\{X_t > x_t\}\right)$$

converges in distribution when $x_t \to -\infty$ to

$$\left(X_r \middle| \bigcap_{s\in I, s\neq r} \{X_s > x_s\}\right).$$

And similarly for

$$\left(Y_r \middle| \bigcap_{s\in r, s\neq r} \{Y_s > y_s\}, \bigcap_{t\notin I}\{Y_t > x_t\}\right)$$

and
$$\left(Y_r \left| \bigcap_{s \in r, s \neq r} \{Y_s > y_s\}\right.\right).$$
The result follows by the preservation of the dispersive order under convergence in distribution, see Lemma 7 in Lewis and Thompson (1981).

(ii) The proof follows under similar arguments that in case (i). In this case, we need also, from (1.3), the additional result for the convergence of the corresponding right spread function. This follows from Theorem 1.5.11 in Müller and Stoyan (2002).

■

We observe that, in previous theorem, the additional condition on the convergence of the expected values can be dropped in the case that the left extremes of the supports of all the random variables are finite.

We provide characterizations of the multivariate convex order and the new multivariate dispersive, right–spread, dmrl and nbue orders, in terms of the hazard gradient and multivariate mean residual life.

Proposition 2.2.5 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors (nonnegative in the case of mdmrl and mnbue orders). Then,*

(i) $\mathbf{X} \leq_{mc} \mathbf{Y}$ *if, and only if,*
$$\frac{r_i(F_i^{-1}(p_i|\mathbf{x}^i)|\mathbf{x}^i)}{s_i(G_i^{-1}(p_i|\mathbf{x}^i)|\mathbf{x}^i)}$$
is an increasing function in p_i*, for all* $\mathbf{x} \in \mathbb{R}^n$ *and each* $i = 1, \ldots, n$*.*

ii) $\mathbf{X} \leq_{mdisp} \mathbf{Y}$ *if, and only if,*
$$r_i(F_i^{-1}(p_i|\mathbf{x}^i)|\mathbf{x}^i) \geq s_i(G_i^{-1}(p_i|\mathbf{x}^i)|\mathbf{x}^i),$$
for all $p_i \in (0,1)$*,* $\mathbf{x} \in \mathbb{R}^n$ *and each* $i = 1, \ldots, n$*.*

(iii) $\mathbf{X} \leq_{mrs} \mathbf{Y}$ *if, and only if,*
$$m_i(F_i^{-1}(p_i|\mathbf{x}^i)|\mathbf{x}^i) \leq l_i(G_i^{-1}(p_i|\mathbf{x}^i)|\mathbf{x}^i),$$

for all $p_i \in (0,1)$*,* $\mathbf{x} \in \mathbb{R}^n$ *and each* $i = 1, \ldots, n$*.*

Let $\mathbf{X}$ *and* $\mathbf{Y}$ *be two nonnegative random vectors. Then,*

(iv) $\mathbf{X} \leq_{mdmrl} \mathbf{Y}$ *if, and only if,*

$$\frac{m_i(F_i^{-1}(p_i|\mathbf{x}^i)|\mathbf{x}^i)}{l_i(G_i^{-1}(p_i|\mathbf{x}^i)|\mathbf{x}^i)}$$

is a decreasing function in $p_i \in (0,1)$*, for all* $\mathbf{x}^i \in \mathbb{R}_+^{n-1}$ *and each* $i = 1, \ldots, n$*.*

(v) $\mathbf{X} \leq_{mnbue} \mathbf{Y}$ *if, and only if,*

$$\frac{m_i(F_i^{-1}(p_i|\mathbf{x}^i)|\mathbf{x}^i)}{l_i(G_i^{-1}(p_i|\mathbf{x}^i)|\mathbf{x}^i)} \leq \frac{\mathbf{E}\left[X_i \left|\mathbf{X}^i > \mathbf{x}^i\right.\right]}{\mathbf{E}\left[Y_i \left|\mathbf{Y}^i > \mathbf{x}^i\right.\right]},$$

for all $p_i \in (0,1)$*,* $\mathbf{x}^i \in \mathbb{R}_+^{n-1}$ *and each* $i = 1, \ldots, n$*.*

Proof. Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors. Then,

i) $\mathbf{X} \leq_{mc} \mathbf{Y}$, if, and only if, for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$,

$$\left(X_i \left|\mathbf{X}^i > \mathbf{x}^i\right.\right) \leq_c \left(Y_i \left|\mathbf{Y}^i > \mathbf{x}^i\right.\right),$$

which, from (1.9), is equivalent to

$$\frac{r_i(F_i^{-1}(p_i|\mathbf{x}^i)|\mathbf{x}^i)}{s_i(G_i^{-1}(p_i|\mathbf{x}^i)|\mathbf{x}^i)}$$

is an increasing function in $p_i \in (0,1)$, for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$. The proofs for the multivariate dispersive, right–spread, decreasing in mean residual life and new better than used in expectation are similar and based in the corresponding univariate characterizations (see (1.5), Proposition 1.2.15 and Definition 1.2.19).

■

Next we provide a result for the preservation under transformations of the new multivariate dispersive and right-spread orders.

Proposition 2.2.6 *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *be two random vectors and let* $\phi_1, \ldots, \phi_n : \mathbb{R} \to \mathbb{R}$ *be univariate strictly increasing convex or strictly decreasing concave functions. If* $\mathbf{X} \leq_{mdisp[mrs]} \mathbf{Y}$*, then*

$$(\phi_1(X_1), \ldots, \phi_n(X_n)) \leq_{mdisp[mrs]} (\phi_1(Y_1), \ldots, \phi_n(Y_n)).$$

Proof. Let $\phi_1, \ldots, \phi_n$ be n univariate strictly increasing convex functions. It holds

$$(\phi_1(X_1), \ldots, \phi_n(X_n)) \leq_{mdisp} (\phi_1(Y_1), \ldots, \phi_n(Y_n))$$

if, and only if, for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$,

$$\left(\phi_i(X_i) \left| \bigcap_{j \neq i} \{\phi_j(X_j) > x_j\} \right.\right) \leq_{disp} \left(\phi_i(Y_i) \left| \bigcap_{j \neq i} \{\phi_j(Y_j) > x_j\} \right.\right). \quad (2.3)$$

From $\mathbf{X} \leq_{mdisp} \mathbf{Y}$, it follows, for all $\mathbf{x}^i \in \mathbb{R}^{n-1}$ and each $i = 1, \ldots, n$, that

$$\left(X_i \left| \bigcap_{j \neq i} \{X_j > x_j\} \right.\right) \leq_{disp} \left(Y_i \left| \bigcap_{j \neq i} \{Y_j > x_j\} \right.\right), \quad (2.4)$$

therefore

$$\left(X_i \left| \bigcap_{j \neq i} \{X_j > \phi_j^{-1}(x_j)\} \right.\right) \leq_{disp} \left(Y_i \left| \bigcap_{j \neq i} \{Y_j > \phi_j^{-1}(x_j)\} \right.\right).$$

and from Proposition 1.2.13 (iv), it holds

$$\left(\phi_i(X_i) \left| \bigcap_{j \neq i} \{X_j > \phi_j^{-1}(x_j)\} \right.\right) \leq_{disp} \left(\phi_i(Y_i) \left| \bigcap_{j \neq i} \{Y_j > \phi_j^{-1}(x_j)\} \right.\right). \quad (2.5)$$

Clearly, condition (2.5) is equivalent to condition (2.3).

The case dealing with the multivariate right-spread order is similar by using Theorem 3.C.4 in Shaked and Shanthikumar (2007). ■

Roy (2002) characterized the classes MIFR, MIFRA and MNBU given in Definition 1.3.21 using the exponential multivariate distribution type Gumbel and we observe that some of the new orders, in particular mdmrl and mnbue, can be used to characterize the dmrl and nbue classes of distributions.

Proposition 2.2.7 *Let* **X** *be a nonnegative random vector and*

$$\mathbf{Y} \sim G(\{\lambda_i\}_{i=1}^n, \{\lambda_{ij}\}_{i>j}, \ldots, \lambda_{1,\ldots,n}),$$

that is, **Y** *is a random vector type Gumbel with survival function given by* (1.28). *Then,*

(i) **X** *is MDMRL if, and only if,* $\mathbf{X} \leq_{mdmrl} \mathbf{Y}$.

(ii) **X** *is MNBUE if, and only if,* $\mathbf{X} \leq_{mnbue} \mathbf{Y}$.

Proof.

(i) $\mathbf{X} \leq_{mdmrl} \mathbf{Y}$ holds if, and only if, for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{dmrl} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right).$$

Besides, the random variables

$$\left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right)$$

are exponential distributed, hence, by the characterization of the univariate class DMRL, see Proposition 1.2.24 (iv), we have that

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right)$$

are DMRL random variables for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$ and this implies, by definition 1.3.21 (iv), that the random vector **X** is MDMRL.

(ii) In this case, let $\mathbf{X} \leq_{mnbue} \mathbf{Y}$. As for the previous case, by Proposition 1.2.24, the random variables

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right)$$

are NBUE, which, from Definition 1.3.21 (v), means that XX is MNBUE.

■

2.2.3 Relationships among multivariate orders

In this subsection we provide relationships among the new orders and previous stochastic orders.

First, we observe the following relationships that follow from (1.11) and (1.7).

$$\begin{array}{ccccc} \mathbf{X} \leq_{mc} \mathbf{Y} & \Rightarrow & \mathbf{X} \leq_{m*} \mathbf{Y} & \Rightarrow & \mathbf{X} \leq_{msu} \mathbf{Y} \\ \Downarrow & & & & \Downarrow \\ \mathbf{X} \leq_{mdmrl} \mathbf{Y} & & \Rightarrow & & \mathbf{X} \leq_{mnbue} \mathbf{Y}. \end{array}$$

and

$$\mathbf{X} \leq_{mdisp} \mathbf{Y} \quad \Rightarrow \quad \mathbf{X} \leq_{mrs} \mathbf{Y}$$

Next we describe some other relationships.

Proposition 2.2.8 *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *be two nonnegative random vectors, then* $\mathbf{X} \leq_{m*} \mathbf{Y}$ *if, and only if,* $(\log X_1, \ldots, \log X_n) \leq_{mdisp} (\log Y_1, \ldots, \log Y_n)$. *Equivalently,* $\mathbf{X} \leq_{mdisp} \mathbf{Y}$ *if, and only if,* $(e^{X_1}, \ldots, e^{X_n}) \leq_{m*} (e^{Y_1}, \ldots, e^{Y_n})$.

Proof. It holds $\mathbf{X} \leq_{m*} \mathbf{Y}$ if, and only if, for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$,

$$\left(X_i \,\middle|\, \mathbf{X}^i > \mathbf{x}^i\right) \leq_* \left(Y_i \,\middle|\, \mathbf{Y}^i > \mathbf{x}^i\right). \tag{2.6}$$

From Theorem 4.B.1 in Shaked and Shanthikumar (2007), it follows that (2.6) is equivalent to

$$\left(\log X_i \,\middle|\, \bigcap_{j \neq i} \{X_j > x_j\}\right) \leq_{disp} \left(\log Y_i \,\middle|\, \bigcap_{j \neq i} \{Y_j > x_j\}\right) \tag{2.7}$$

for all $\mathbf{x} \in \mathbb{R}^n_+$ and each $i = 1, \ldots, n$.

If we consider $x'_j = \log x_j$ for $j \neq i$ and $\mathbf{x}' = (x'_1, \ldots, x'_n)$, then (2.7) becomes

$$\left(\log X_i \,\middle|\, \bigcap_{j \neq i} \{\log X_j > x'_j\}\right) \leq_{disp} \left(\log Y_i \,\middle|\, \bigcap_{j \neq i} \{\log Y_j > x'_j\}\right) \tag{2.8}$$

for all $\mathbf{x}' \in \mathbb{R}^n$ and each $i = 1, \ldots, n$.

The last inequality is equivalent to $(\log X_1, \ldots, \log X_n) \leq_{mdisp} (\log Y_1, \ldots, \log Y_n)$. Thus, the assertion is proved. ■

Under the condition that the conditional random variables are ordered in the usual stochastic order, the multivariate superadditive order implies the multivariate dispersive order as follows.

Proposition 2.2.9 *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *be two random vectors such that*

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{st} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right)$$

for all $\mathbf{x} \in \mathbb{R}^n$ *and each* $i = 1, \ldots, n$*. If* $\mathbf{X} \leq_{msu} \mathbf{Y}$*, then* $\mathbf{X} \leq_{mdisp} \mathbf{Y}$*.*

Proof. Since

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{st} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right)$$

and

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{su} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right)$$

for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$, then, from Proposition 4.B.2 in Shaked and Shanthikumar (2007),

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{disp} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right)$$

or, equivalently, $\mathbf{X} \leq_{mdisp} \mathbf{Y}$. ■

Another condition, under which $\mathbf{X} \leq_{msu} \mathbf{Y}$ implies $\mathbf{X} \leq_{mdisp} \mathbf{Y}$, is given in the next proposition.

Proposition 2.2.10 *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *be two random vectors. If* $\mathbf{X} \leq_{whr} \mathbf{Y}$ *and* $\mathbf{X} \leq_{msu} \mathbf{Y}$*, then* $\mathbf{X} \leq_{mdisp} \mathbf{Y}$*.*

Proof. The condition $\mathbf{X} \leq_{whr} \mathbf{Y}$ is equivalent to (as shown Proposition 2.2.1)

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{hr} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right)$$

for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$. By (1.12), it holds

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{st} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right)$$

for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$. The stated comparison now follows from Proposition 2.2.9. ■

If we suppose that $(\mathbf{X}|\mathbf{X} > \mathbf{x}) \leq_{st} (\mathbf{Y}|\mathbf{Y} > \mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n_+$, we obtain the following corollary:

Proposition 2.2.11 *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *be two random vectors. If* $(\mathbf{X}|\mathbf{X} > \mathbf{x}) \leq_{st} (\mathbf{Y}|\mathbf{Y} > \mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{X} \leq_{msu} \mathbf{Y}$*, then* $\mathbf{X} \leq_{mdisp} \mathbf{Y}$.

Proof. First, we observe that the condition $(\mathbf{X}|\mathbf{X} > \mathbf{x}) \leq_{st} (\mathbf{Y}|\mathbf{Y} > \mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^n$, implies, from the preservation under marginalization of the multivariate stochastic order, the condition

$$(X_i \left| \mathbf{X} > \mathbf{x} \right.) \leq_{st} (Y_i \left| \mathbf{Y} > \mathbf{x} \right.),$$

for each all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$, which is equivalent to

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{hr} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right)$$

for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$, that is, $\mathbf{X} \leq_{whr} \mathbf{Y}$, and the result follows from Proposition 2.2.10. ■

In contrast to the previous conditions, assuming the next limiting property, we have the following proposition.

Proposition 2.2.12 *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *be two random vectors such that* $\mathbf{X} \leq_{msu} \mathbf{Y}$*. If it holds, for all* $\mathbf{x} \in \mathbb{R}^n$ *and each* $i = 1, \ldots, n$*,*

$$\lim_{x_i \to 0} \frac{G_i^{-1}(F_i(x_i|\mathbf{x}^i)|\mathbf{x}^i)}{x_i} \geq 1,$$

then $\mathbf{X} \leq_{mdisp} \mathbf{Y}$.

Proof. The proof is straightforward from Theorem 4.B.3 in Shaked and Shanthikumar (2007). ■

The next proposition lists some relations between some of the multivariate stochastic orders under multivariate aging conditions.

Proposition 2.2.13 *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *be two nonnegative random vectors. We assume, in (iv) and (v), that for any* $\mathbf{x} \in \mathbb{R}^n$ *and for each* $i = 1, \ldots, n$*, the left extreme of the support of*

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right)$$

is smaller than the left extreme of the support of

$$\left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right).$$

Then,

(i) If $\mathbf{X} \leq_{whr} \mathbf{Y}$ *and* $\mathbf{X}$ *or* $\mathbf{Y}$ *is MDFR, then* $\mathbf{X} \leq_{mdisp} \mathbf{Y}$*.*

(ii) If $\mathbf{X} \leq_{mdisp} \mathbf{Y}$ *and* $\mathbf{X}$ *or* $\mathbf{Y}$ *is MIFR, then* $\mathbf{X} \leq_{whr} \mathbf{Y}$*.*

(iii) If $\mathbf{X}$ *is MNBU and* $\mathbf{Y}$ *is MNWU, then* $\mathbf{X} \leq_{mdisp} \mathbf{Y}$ *if, and only if,* $XX \leq_{whr} \mathbf{Y}$*.*

(iv) If $\mathbf{X} \leq_{mrs} \mathbf{Y}$ *and* $\mathbf{X}$ *or* $\mathbf{Y}$ *is MDMRL then* $\mathbf{X} \leq_{wmrl} \mathbf{Y}$*.*

(v) If $m_i(x_i|\mathbf{x}^i) \leq l_i(x_i|\mathbf{x}^i)$ *for all* $\mathbf{x} \in \mathbb{R}^n_+$ *and each* $i = 1, \ldots, n$ *and* $\mathbf{X}$*, or* $\mathbf{Y}$ *is MIMRL, then* $\mathbf{X} \leq_{mrs} \mathbf{Y}$*.*

Proof.

(i) Let us suppose that $\mathbf{X}$ is MDFR, this means that $r_i(x_i|\mathbf{x}^i)$ is a decreasing function in x_i for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$. Condition $\mathbf{X} \leq_{whr} \mathbf{Y}$ is equivalent to

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{hr} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right).$$

By Theorem 3.B.20 in Shaked and Shanthikumar (2007) we can state that

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) \leq_{disp} \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}^i \right.\right),$$

and, by definition, this is equivalent to $\mathbf{X} \leq_{mdisp} \mathbf{Y}$. The case when $\mathbf{Y}$ is MDFR is proved in a similar way.

(ii) Let us suppose that $\mathbf{X}$ is MIFR, which that $r_i(x_i|\mathbf{x}^i)$ is a decreasing function in x_i for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$. Besides, $\mathbf{X} \leq_{mdisp} \mathbf{Y}$ which is equivalent to

$$\left(X_i \left|\mathbf{X}^i > \mathbf{x}^i\right.\right) \leq_{disp} \left(Y_i \left|\mathbf{Y}^i > \mathbf{x}^i\right.\right).$$

By Theorem 3.B.20 in Shaked and Shanthikumar (2007) we can state that

$$\left(X_i \left|\mathbf{X}^i > \mathbf{x}^i\right.\right) \leq_{hr} \left(Y_i \left|\mathbf{Y}^i > \mathbf{x}^i\right.\right),$$

and, by definition, this is equivalent to $\mathbf{X} \leq_{whr} \mathbf{Y}$. The case when $\mathbf{Y}$ is MIFR is proved in a similar way.

(iii) This statement can be proved in a similar way by using Theorem 3.B.20 c) in Shaked and Shanthikumar (2007).

(iv) Analogously using Theorem 3.C.5 in Shaked and Shanthikumar (2007).

(v) Analogously using Theorem 3.C.6 in Shaked and Shanthikumar (2007).

■

2.2.4 Applications and examples

In this subsection, we provide several situations where these multivariate stochastic orders can be applied and examples of parametric families of random vectors ordered according to some of the multivariate orders previously considered in this section and that satisfy some of the multivariate aging notions considered.

Recall that these multivariate aging notions can be characterized in terms of some multivariate stochastic orders and, therefore, can be seen as examples of comparisons in some multivariate orders when one of the random vectors follows a multivariate Gumbel distribution.

Comparisons of risks

In the context of actuarial theory, a non-negative random variable X represents the random amount that an insurance company will pay to a policyholder, in case of claim. The comparison of risks is carried out through the comparison of some measures of interest. Two common measures are the value at risk and expected shortfall notions, which are evaluated at any point $p \in (0,1)$. The value at risk is simply the quantile function, that is, the value at risk is given by $VaR[X,p] \equiv F^{-1}(p)$ and the expected shortfall is the right–spread function, that is $RS_X(p) \equiv \mathbf{E}[(X - F^{-1}(p))_+]$.

The dispersive and right-spread orders provide comparisons of risks based on the value at risk and the expected shortfall. According to this, two risky situations A and B with random risks X_A and X_B respectively, can be ranked in terms of the dispersive or the right-spread orders. If $X_A \leq_{\text{disp}} X_B$ or $X_A \leq_{\text{rs}} X_B$, then we can say that B is more risky than A (see Denuit et al, 2005, for notation and discussion on this topic).

In some situations insurance companies do not have only one policy for some policyholders, but two policies. For example, a policyholder can have a policy to insure the car and another one to ensure the house with random claims X and Y. It is clear that X and Y should exhibit some kind of dependence, for example positive dependence. So let us consider two risky situations A and B for policyholders in the situation described above, with random risks (X_A, Y_A) and (X_B, Y_B). If we can compare these two random variables in terms of the mdisp or mrs orders, that is, if $(X_A, Y_A) \leq_{mdisp} (X_B, Y_B)$ or $(X_A, Y_A) \leq_{mrs} (X_B, Y_B)$, then, from Proposition 2.2.3, we compare the marginal risks in the dispersive or right-spread order, but also we provide a more detailed and illustrative comparison of the risks taking into account the dependence among risks. In particular, from the definition, we can provide comparisons of the type

$$(X_A|Y_A > x) \leq_{disp[rs]} (X_B|Y_B > x)$$

or

$$(Y_A|X_A > x) \leq_{disp[rs]} (Y_B|X_B > x),$$

which compares risks with the additional information that the level of claim of the other risk is above some certain threshold. Another example of this type is provided in Frees and Valdez (1998), in this paper the authors consider the following situation of insurance company indemnity claims. Each claim consists of an indemnity payment (the loss, X) and an allocated loss adjustment expense (ALAE, Y). Similar comments can be done for this case.

Comparisons of inequality and deprivation

Another application can be found in the context of inequality. First we recall that the nbue order implies the Lorenz order (see (1.11)). In particular, if $X \leq_{\mathrm{L}} Y$, then the coefficient of variation of X is smaller than the corresponding coefficient for Y. Also it preserves the comparison of the Gini index. In the context of income distributions, the Lorenz curve and the Gini index are the most popular tools to analyze and compare income inequality. However, they can be used only to compare random variables, but in some situations the interest can be focused on more than one random variable to describe the inequality in a population. Consider for example the following situation. In order to study the inequality in a population researchers can be interested not only in the household's income per year (X) but also in the household's properties (Y), clearly these two random variables are related. How can we compare now the inequality among two different populations, taking into account these two random variables? Let us consider two countries A and B with random vectors (X_A, Y_A) and (X_B, Y_B), respectively, as described above. The comparison of the marginal Gini indexes is an incomplete study that does not take into account the dependence structure of the two random variables. Another possibility is to compare not only the marginal Gini indexes but also the Gini index of the incomes given that the value of the properties is above any threshold x that is, to compare $GI_{X_A}(x) = GI_{(X_A|Y_A>x)}$ with $GI_{X_B}(x) = GI_{(X_B|Y_B>x)}$ and a similar comparison for the Gini indexes $GI_{Y_A}(x) = GI_{(Y_A|X_A>x)}$ with $GI_{Y_B}(x) = GI_{(Y_B|X_B>x)}$. From previous discussion and previous results we have that if $(X_A, Y_A) \leq_{mdmrl} (X_B, Y_B)$ or $(X_A, Y_A) \leq_{mnbue} (X_B, Y_B)$, then

$$GI_{X_A}(x) \leq GI_{X_B}(x) \text{ and } GI_{Y_A}(x) \leq GI_{Y_B}(x) \text{ for all } x > 0.$$

Recall that (X_A, Y_A) is MNBUE [MNWUE] if, and only if,

$$(X_A, Y_A) \leq_{mnbue} [\geq_{mnbue}](X_B, Y_B),$$

where (X_B, Y_B) has a joint survival function given by (1.28). In this case it is not difficult to see that $GI_{X_B}(x) = 0.5 = GI_{Y_B}(x)$. Therefore $GI_{X_A}(x)$ and $GI_{X_B}(x)$ are bounded from above (below) by 0.5. It is very interesting to identify this situation because Gini indexes greater than 0.5 suggest problems of inequality.

Let us see another example of this situation. Policy makers and researchers are often interested in assessing the changes in inequality and poverty over time. An example is the case studied in Vinh et al. (2010) which consider several parametric models for the joint distribution of incomes for different years across the same population. In particular they found that the best bivariate approximation for incomes in Australia for 2001 and 2005, denoted by (AI_{2001}, AI_{2005}), is the one provided by a Gumbel copula with Singh-Maddala margins (see Vinh et al., 2010, for details of the estimated parameters). Following previous discussion, we consider the conditional Gini indexes $GI_{AI_{2001}}(x) = GI_{(AI_{2001}|AI_{2005}>x)}$ and $GI_{AI_{2005}}(x) = GI_{(AI_{2005}|AI_{2001}>x)}$. Given that there is no closed expression for these values, we have considered the estimated values for these quantities based on bivariate samples for this model of size 1000. The estimated values can be found in figure 3, which shows that $GI_{AI_{2001}}(x)$ and $GI_{AI_{2005}}(x)$ are less than 0.5.

The comparison in the mdisp and mrs orders is related to the comparison of deprivation in the context of incomes and wealth. As can be seen in Duclos and Araar (2006) an indicator of relative deprivation of an individual with income $F^{-1}(p)$, when comparing himself to another individual with income $F^{-1}(q)$, is given by

$$\delta(p, q) = (F^{-1}(q) - F^{-1}(p))_+,$$

and the expected relative deprivation of an individual at rank p is given by

$$\overline{\delta}(p) = \int_0^1 \delta(p, q)dq = E[(X - F^{-1}(p))_+].$$

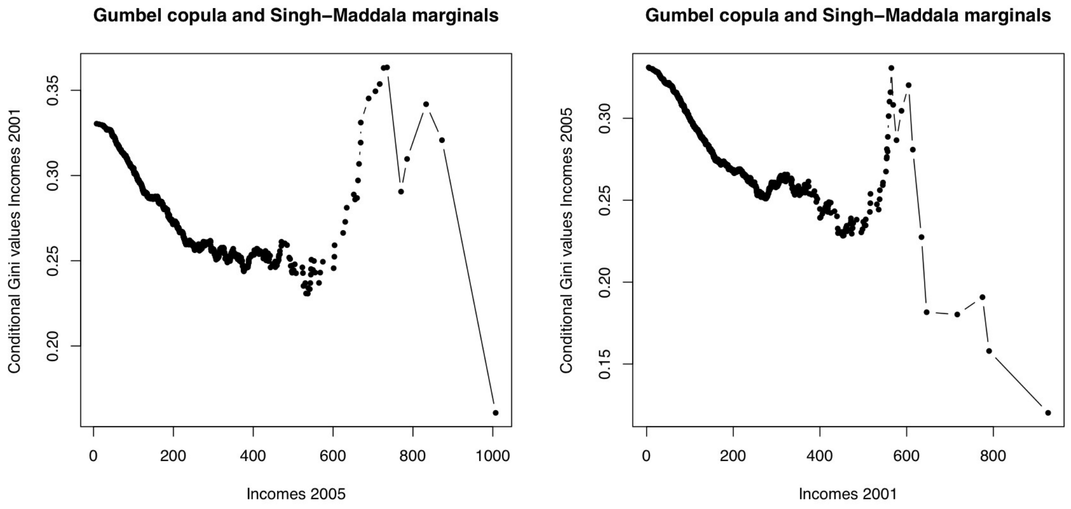

Figure 3: *Plots of estimated conditional Gini values for* (AI_{2001}, AI_{2005}).

Therefore, the comparison in the dispersive order and the right spread order are comparisons of relative deprivation among two populations and previous comments are valid for the mdisp and mrs orders in the context of income distributions.

Finally, we consider several parametric examples where some of the previous notions can be studied.

Farlie-Gumbel-Morgenstern Distributions

The survival function of the bivariate Farlie-Gumbel-Morgenstern distribution is given by

$$\overline{F}(x_1, x_2) = \overline{F}_1(x_1)\overline{F}_2(x_2)[1 + \alpha F_1(x_1)F_2(x_2)],$$

where F_1 and F_2 are the marginal distributions and $-1 \leq \alpha \leq 1$. Let us denote by FGM_α a bivariate random vector with joint survival function provided by the previous expression. For this model, Hu et al. (2003) show that $FGM_\alpha \leq_{whr} FGM_{\alpha'}$ for $\alpha \leq \alpha'$. Also Johnson and Kotz (1975) show that, under certain conditions, this model is MDFR. If this is the case for FGM_α or $FGM_{\alpha'}$, then from Proposition 2.2.13 we get that $FGM_\alpha \leq_{mdisp[mrs]} FGM_{\alpha'}$. Recall also that, if FGM_α is MIFR [MDFR], then $FGM_\alpha \leq [\geq]_{mc}\mathbf{Y}$ where $\mathbf{Y}$ follows a Gumbel type distribution, see (1.28). Johnson and Kotz (1975) describe several cases of MIFR and MDFR distributions when the marginals are Weibull distributed.

Applications of this model can be found in the context of health care usage (see Prieger, 2002).

Conditionally specified models

Arnold et al. (1999) provide several examples of bivariate random vectors where the conditional marginal distributions are given explicitly. Let us consider some examples.

(i) Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be two bivariate random vectors and suppose that for each $x_1, x_2 > 0$, the conditional random variables X_1 given $X_2 > x_2$ and X_2 given $X_1 > x_1$ are Weibull distributed with survival functions as follows

$$P(X_1 > x_1 | X_2 > x_2) = \exp\{-x_1^{\gamma^1_{\mathbf{X}}}(\alpha + \gamma_{\mathbf{X}} x_2^{\gamma^2_{\mathbf{X}}})\}$$

and

$$P(X_2 > x_2 | X_1 > x_1) = \exp\{-x_2^{\gamma^2_{\mathbf{X}}}(\beta + \gamma_{\mathbf{X}} x_1^{\gamma^1_{\mathbf{X}}})\},$$

respectively. The Gumbel bivariate exponential distribution corresponds to the choice $\gamma^1_{\mathbf{X}} = \gamma^2_{\mathbf{X}} = 1$.

Analogously, suppose that for each $x_1, x_2 > 0$, the conditional random variables Y_1 given $Y_2 > x_2$ and Y_2 given $Y_1 > x_1$ are Weibull distributed with survival functions as follows

$$P(Y_1 > x_1 | Y_2 > x_2) = \exp\{-x_1^{\gamma^1_{\mathbf{Y}}}(\delta + \gamma_{\mathbf{Y}} x_2^{\gamma^2_{\mathbf{Y}}})\}$$

and

$$P(Y_2 > x_2 | Y_1 > x_1) = \exp\{-x_2^{\gamma^2_{\mathbf{Y}}}(\nu + \gamma_{\mathbf{Y}} x_1^{\gamma^1_{\mathbf{Y}}})\},$$

respectively.

For fixed $x_2 > 0$, the hazard rate and quantile functions of the conditional distribution $(X_1 | X_2 > x_2)$ are given, respectively, by

$$r_{(X_1|X_2>x_2)}(x_1) = \gamma^1_{\mathbf{X}}(\alpha + \gamma_{\mathbf{X}} x_2^{\gamma^2_{\mathbf{X}}}) x_1^{\gamma^1_{\mathbf{X}}-1} \tag{2.9}$$

and

$$F^{-1}_{(X_1|X_2>x_2)}(p) = \left[\frac{-\log\{1-p\}}{\alpha + \gamma_{\mathbf{X}} x_2^{\gamma^2_{\mathbf{X}}}}\right]^{1/\gamma^1_{\mathbf{X}}}$$

Analogously, we can compute the hazard rate and the quantile functions for the rest of the above conditional distributions.

From Proposition 2.2.5 (ii), it is not difficult to see that all the following conditions, for all $x_1, x_2 > 0$, are sufficient in order to obtain $\mathbf{X} \leq_{mdisp} \mathbf{Y}$:

1. $\gamma^1_{\mathbf{X}} \geq \gamma^1_{\mathbf{Y}}$,
2. $\gamma^2_{\mathbf{X}} \geq \gamma^2_{\mathbf{Y}}$,
3. $(\alpha + \gamma_{\mathbf{X}} x_2^{\gamma^2_{\mathbf{X}}})^{1/\gamma^1_{\mathbf{X}}} \geq (\delta + \gamma_{\mathbf{Y}} x_2^{\gamma^2_{\mathbf{Y}}})^{1/\gamma^1_{\mathbf{Y}}}$
4. $(\alpha + \gamma_{\mathbf{X}} x_2^{\gamma^2_{\mathbf{X}}})^{1/\gamma^1_{\mathbf{X}}} \geq (\delta + \gamma_{\mathbf{Y}} x_2^{\gamma^2_{\mathbf{Y}}})^{1/\gamma^1_{\mathbf{Y}}}$.

It is easy to see that the random vector $\mathbf{X}$ is MIFR in the sense of Roy if $\gamma^1_{\mathbf{X}} \geq 1$ and $\gamma^2_{\mathbf{X}} \geq 1$.

(ii) Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be two bivariate random vectors. Suppose that for each $x_1, x_2 > 0$, the conditional random variables X_1 given $X_2 > x_2$ and X_2 given $X_1 > x_1$ are logistic with survival functions as follows

$$P(X_1 > x_1 | X_2 > x_2) = \left[1 + \exp\left\{\frac{x_1 - \mu 1_{\mathbf{X}}(x_2)}{\sigma 1_{\mathbf{X}}}\right\}\right]^{-1},$$

and

$$P(X_2 > x_2 | X_1 > x_1) = \left[1 + \exp\left\{\frac{x_2 - \mu 2_{\mathbf{X}}(x_1)}{\sigma 2_{\mathbf{X}}}\right\}\right]^{-1}$$

respectively, where $\mu 1_{\mathbf{X}}, \mu 2_{\mathbf{X}} \in \mathbb{R}^+$ and $\theta_{\mathbf{X}} \in [0, 2]$.

Analogously, suppose that for each $x_1, x_2 > 0$, the conditional random variables Y_1 given $Y_2 > x_2$ and Y_2 given $Y_1 > x_1$ are logistic with survival functions as follows

$$P(Y_1 > x_1 | Y_2 > x_2) = \left[1 + \exp\left\{\frac{x_1 - \mu^1_{\mathbf{Y}}(x_2)}{\sigma 1_{\mathbf{Y}}}\right\}\right]^{-1}$$

and

$$P(Y_2 > x_2|Y_1 > x_1) = \left[1 + \exp\left\{\frac{x_2 - \mu^2_{\mathbf{Y}}(x_1)}{\sigma 2_{\mathbf{Y}}}\right\}\right]^{-1}$$

respectively, where $\mu^1_{\mathbf{Y}}, \mu^2_{\mathbf{Y}} \in \mathbb{R}^+$ and $\theta_{\mathbf{Y}} \in [0, 2]$.

After some computations we have that

$$r_{(X_1|X_2>x_2)}(x_1) = \left[1 + \exp\left\{\frac{x_1 - \mu^1_{\mathbf{X}}(x_2)}{\sigma 1_{\mathbf{X}}}\right\}\right]^{-1} \frac{\exp\{\frac{x_1 - \mu^1_{\mathbf{X}}(x_2)}{\sigma^1_{\mathbf{X}}}\}}{\sigma^1_{\mathbf{X}}}$$

and

$$F^{-1}_{(X_1|X_2>x_2)}(p) = \sigma 1_{\mathbf{X}} \log\left\{\frac{p}{1-p}\right\} + \mu 1_{\mathbf{X}}(x_2)$$

From 2.2.5 (iii), the following conditions are sufficient for $\mathbf{X} \leq_{mrs} \mathbf{Y}$:

1. $\sigma^1_{\mathbf{X}} \leq \sigma^1_{\mathbf{Y}}$,
2. $\sigma^2_{\mathbf{X}} \leq \sigma^2_{\mathbf{Y}}$.

for all $x_1, x_2 > 0$.

(iii) Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be two bivariate random vectors. Suppose that for each $x_1, x_2 > 0$, the conditional random variables X_1 given $X_2 > x_2$ and X_2 given $X_1 > x_1$ have survival functions as follows

$$P(X_1 > x_1|X_2 > x_2) = \left[\overline{F}^1_{\mathbf{X}}(x_1)\right]^{\alpha^1_{\mathbf{X}}(x_2)},$$

and

$$P(X_2 > x_2|X_1 > x_1) = \left[\overline{F}^2_{\mathbf{X}}(x_2)\right]^{\alpha^2_{\mathbf{X}}(x_1)},$$

respectively. These conditional survival variables lead to a Gumbel's type I bivariate exponential distribution.

Analogously, suppose that for each $y_1, y_2 > 0$, the conditional random variables Y_1 given $Y_2 > y_2$ and Y_2 given $Y_1 > y_1$ have survival functions as follows

$$P(Y_1 > y_1|Y_2 > y_2) = [\overline{F}^1_{\mathbf{Y}}(y_1)]^{\alpha^1_{\mathbf{Y}}(y_2)},$$

and

$$P(Y_2 > y_2|Y_1 > y_1) = [\overline{F}^2_{\mathbf{Y}}(y_2)]^{\alpha^2_{\mathbf{Y}}(y_1)},$$

respectively. Analogously, these conditional survival variables lead to a Gumbel's type I bivariate exponential distribution.

Suppose that $\overline{F}^1_{\mathbf{X}} = \overline{F}^1_{\mathbf{Y}} \equiv \overline{F}^1$ and $\overline{F}^2_{\mathbf{X}} = \overline{F}^2_{\mathbf{Y}} \equiv \overline{F}^2$ and that the hazard rate relative to $\overline{F}^1$ and $\overline{F}^2$ are decreasing.

From Proposition 2.2.5 (ii), we can state that if $\alpha^1_{\mathbf{X}}(x_2) \geq \alpha^1_{\mathbf{Y}}(x_2)$ and $\alpha^2_{\mathbf{X}}(x_1) \geq \alpha^2_{\mathbf{Y}}(x_1)$, then $\mathbf{X} \leq_{mdisp} \mathbf{Y}$.

Similar results can be given for some of the examples considered in Arnold, Castillo and Sarabia (1999).

Multivariate aging properties for exchangeable random vectors

Let $\mathbf{X} = (X, Y)$ be an exchangeable random vector of non-negative marginal random variables with distribution function $F_X \equiv F_Y \equiv F$ and hazard rate function $r_X \equiv r_Y \equiv r$ and let $\overline{H}(x,y) = P(X > x, Y > y)$ be its corresponding survival function. If $K(u_1, u_2)$ is the bivariate survival copula, it holds that $\overline{H}(x,y) = K(\overline{F}(x), \overline{F}(y))$, see (1.19).

Let (V_1, V_2) be a random vector with uniform margins and satisfying $P(V_1 > v_1, V_2 > v_2) = K(1-v_1, 1-v_2)$ for all $v_1, v_2 \in [0,1]$, i.e., K is its survival copula.

For the sake of simplicity, let us denote by $f(x|y)$, $\overline{F}(x|y)$ and $r(x|y)$, the density, survival and hazard rate functions of $(X|Y > y)$, respectively. Moreover, we consider the notation $K_{u_1} \equiv \frac{\partial}{\partial u_1} K$.

Proposition 2.2.14 *If the marginal random variable X is IFR and $\frac{(1-v_1)K_{u_1}(1-v_1,1-v_2)}{K(1-v_1,1-v_2)}$ is an increasing function in v_1 for all $v_2 \in [0,1]$, then $\mathbf{X}$ is MIFR in the sense of Roy.*

Proof. Note that, in case of exchangeability, $\mathbf{X}$ is MIFR in the sense of Roy if $r(x|y)$ is an increasing function in x for all y. The survival function of $(X|Y > y)$ is given by

$$\overline{F}(x|y) = \frac{\overline{H}(x,y)}{\overline{F}(y)} = \frac{K(\overline{F}(x), \overline{F}(y))}{\overline{F}(y)}$$

then

$$f(x|y) = -\frac{d}{dx}\overline{F}(x|y) = \frac{1}{\overline{F}(y)}\left[-\frac{d}{dx}K(\overline{F}(x), \overline{F}(y))\right] = \frac{f(x)}{\overline{F}(y)}K_{u_1}(\overline{F}(x), \overline{F}(y))$$

Therefore,

$$r(x|y) = f(x)\frac{K_{u_1}(\overline{F}(x), \overline{F}(y))}{K(\overline{F}(x), \overline{F}(y))} = r(x)\frac{\overline{F}(x)K_{u_1}(\overline{F}(x), \overline{F}(y))}{K(\overline{F}(x), \overline{F}(y))}.$$

Let us note that, fixed $y \in \mathbb{R}$, it exists $v_2 \in [0,1]$ such as $F^{-1}(v_2) = y$. From the increasingness of the quantile function, it is easy to see that $r(x|F^{-1}(v_2))$ is an increasing function in x for all $v_2 \in \mathbb{R}$ if, and only if, $r(F^{-1}(v_1)|F^{-1}(v_2))$ is an increasing function in v_1 for all $v_1 \in [0,1]$.

Taking into account these observations, we have that

$$r(F^{-1}(v_1)|F^{-1}(v_2)) = r(F^{-1}(v_1))\frac{(1-v_1)K_{u_1}(1-v_1, 1-v_2)}{K(1-v_1, 1-v_2)}.$$

Since X is IFR, then $r(F^{-1}(v_1))$ is an increasing function in v_1. On the other hand, by hypothesis,

$$\frac{(1-v_1)K_{u_1}(1-v_1, 1-v_2)}{K(1-v_1, 1-v_2)}$$

is an increasing function in v_1 for all $v_2 \in [0,1]$. Putting these results together, we conclude that $r(x|y)$ is an increasing function in x for all $y \in \mathbb{R}$ and the statement is proved. ∎

If we consider that K is the Clayton copula, then

$$\frac{(1-v_1)K_{u_1}(1-v_1, 1-v_2)}{K(1-v_1, 1-v_2)} = \frac{(1-v_1)^{-1/\theta}}{(1-v_1)^{-1/\theta} + (1-v_2)^{-1/\theta} - 1}$$

is increasing in v_1 for all $v_2 \in [0,1]$. In this way, if X and Y are IFR and K is the Clayton copula, then **X** $= (X, Y)$ is a random vector MIFR in the sense of Roy.

Proposition 2.2.15 *If the marginal random variable X is IFRA and $\frac{-\log K(1-u_1, 1-u_2)}{-\log(1-u_1)}$ is an increasing function in v_1 for all $v_2 \in [0,1]$, then **X** is MIFRA in the sense of Roy.*

Proof. In this case, **X** is MIFRA in the sense of Roy if, and only if, $(X|Y > y)$ is IFRA for all y, that is, **X** is MIFRA if, and only if,

$$\frac{-\log \overline{F}(x|y)}{x}$$

is an increasing function in y. Taking into account that the survival function of $(X|Y > y)$ is given by (2.2.4), this is equivalent to say that

$$\phi(x) = \frac{-\log K(\overline{F}(x), \overline{F}(y)}{x}$$

is increasing in x for all y. Recall that $\phi(x)$ is increasing in x if, and only if,

$$\phi(F^{-1}(u_1)) = \frac{-\log K(1-u_1, \overline{F}(y))}{F^{-1}(u_1)}$$

is increasing in u_1. Let $u_2 \in [0,1]$ be such that $F^{-1}(u_2) = y$, then

$$\phi(F^{-1}(u_1)) = \frac{-\log K(1-u_1, 1-u_2)}{F^{-1}(u_1)} = \frac{-\log K(1-u_1, 1-u_2)}{-\log(1-u_1)} \frac{-\log(1-u_1)}{F^{-1}(u_1)}.$$

Note that the marginal random variable X is IFRA if, and only if, $\frac{-\log \overline{F}(t)}{t}$ is increasing in t or, equivalently, if, and only if, $\frac{-\log(1-u_1)}{F^{-1}(u_1)}$ is increasing in u_1. On the other hand, we suppose that $\frac{-\log K(1-u_1, 1-u_2)}{-\log(1-u_1)}$ is increasing in v_1. Putting these results together, we can conclude that $\phi(F^{-1}(u_1))$ is an increasing function in u_1 and the proof is obtained. ■

2.3 New multivariate orders based on conditional distributions truncated on quantiles

2.3.1 Why quantiles?

As we see in the previous section, given a bivariate random vector (X, Y), we can find several notions of dependence based on conditional random variables like $(X|Y > x)$, $(X|Y \leq x)$ or $(X|Y = x)$. However, in some situations, we can be interested in cases where the point of truncation is a quantile. This is the case of risk theory. In risk theory the interest is focused on the study of the losses beyond the value-at-risk. Let us consider the example where a policyholder can have a policy to insure the car and another one to ensure the house with random claims

X and Y, with distribution functions F and G, respectively. In this case, we can be interested in studying the dependence of the two random variables, in terms of $(X|Y > G^{-1}(p))$ and $(Y|X > F^{-1}(p))$. Therefore, the comparisons should be made in terms of the conditioned random variables at quantiles.

From now on, for a random vector $\mathbf{X} = (X_1, \ldots, X_n)$, let

$$\mathbf{X}^i = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n).$$

Moreover, given $F_1, \ldots, F_n$ the marginal distribution functions of $\mathbf{X}$, let us denote as $\mathbf{x}(\mathbf{p}) = (F_1^{-1}(p_1), \ldots, F_n^{-1}(p_n)) \in \mathbb{R}^n$ and

$$\mathbf{x}(\mathbf{p})^i = (F_1^{-1}(p_1), \ldots, F_{i-1}^{-1}(p_{i-1}), F_{i+1}^{-1}(p_{i+1}), \ldots, F_n^{-1}(p_n)) \in \mathbb{R}^{n-1},$$

for all $\mathbf{p} \in (0,1)^n$ and each $i = 1, \ldots, n$.

This approach have been considered by Khaledi and Kochar (2005) and Mercader (2007) to provide multivariate generalizations of the dispersive and right spread order, respectively, see definitions 1.3.14 and 1.3.15. Recall that $\mathbf{X} \leq_{uo-disp[uo-rs]} \mathbf{Y}$, if, for all $\mathbf{p} \in (0,1)^n$ and each $i = 1, 2, \ldots, n$, holds

$$\left(X_i \left|\mathbf{X}^i > \mathbf{x}(\mathbf{p})^i\right.\right) \leq_{disp[rs]} \left(Y_i \left|\mathbf{Y}^i > \mathbf{y}(\mathbf{p})^i\right.\right). \tag{2.10}$$

The purpose of this section is to define and study new multivariate extensions of the transform orders, that is, convex, starshaped and superaddtive orders, based on conditional distributions truncated on quantiles in the same way given in (2.10).

2.3.2 Definitions and first properties

Clearly inspired in the orders introduced by Khaledi and Kochar (2005) and Mercader (2007), we propose the following definitions.

Definition 2.3.1 *Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be two random vectors (nonnegative in the case of the starshaped order) and F_i and G_i, the distribution functions for X_i and Y_i, respectively, for $i = 1, \ldots, n$. Then,* *(i)* $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the* ***multivariate upper orthant convex order,*** *denoted by* $\mathbf{X} \leq_{uo-c} \mathbf{Y}$*, if, for all* $\mathbf{p} \in (0,1)^n$ *and each* $i = 1, \ldots, n$ *holds*

$$\left(X_i \left|\mathbf{X}^i > \mathbf{x}(\mathbf{p})^i\right.\right) \leq_c \left(Y_i \left|\mathbf{Y}^i > \mathbf{y}(\mathbf{p})^i\right.\right).$$

(ii) $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the* ***multivariate upper orthant starshaped order,*** *denoted by* $\mathbf{X} \leq_{uo-*} \mathbf{Y}$*, if, for all* $\mathbf{p} \in (0,1)^n$ *and each* $i = 1, \ldots, n$ *holds*

$$\left(X_i \left|\mathbf{X}^i > \mathbf{x}(\mathbf{p})^i\right.\right) \leq_* \left(Y_i \left|\mathbf{Y}^i > \mathbf{y}(\mathbf{p})^i\right.\right).$$

(iii) $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the* ***multivariate upper orthant superadditive order,*** *denoted by* $\mathbf{X} \leq_{uo-su} \mathbf{Y}$*, if, for all* $\mathbf{p} \in (0,1)^n$ *and each* $i = 1, \ldots, n$ *holds*

$$\left(X_i \left|\mathbf{X}^i > \mathbf{x}(\mathbf{p})^i\right.\right) \leq_{su} \left(Y_i \left|\mathbf{Y}^i > \mathbf{y}(\mathbf{p})^i\right.\right).$$

Obviously, from (1.11), these orders satisfy the following relationships

$$\mathbf{X} \leq_{uo-c} \mathbf{Y} \quad \Leftrightarrow \quad \mathbf{X} \leq_{uo-*} \mathbf{Y} \quad \Leftrightarrow \quad \mathbf{X} \leq_{uo-su} \mathbf{Y}$$

and they are transitive, reflexive and closed under marginalization and under conjunctions as we point out in the following result.

Proposition 2.3.2 *(i) Let* $\mathbf{X}$ *be a random vector (nonnegative in the case of the starshaped order), then* $\mathbf{X} \leq_{uo-c[uo-*,uo-su]} \mathbf{X}$*.*

(ii) Let $\mathbf{X}$*,* $\mathbf{Y}$ *and* $\mathbf{Z}$ *be three random vectors. If*

$$\mathbf{X} \leq_{uo-c[uo-*,uo-su]} \mathbf{Y} \text{ and } \mathbf{Y} \leq_{uo-c[uo-*,uo-su]} \mathbf{Z},$$

then $\mathbf{X} \leq_{uo-c[uo-*,uo-su]} \mathbf{Z}$*.*

(iii) Let $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors (nonnegative in the case of the starshaped order). Let* $I = (i_1, \ldots, i_r) \subseteq (1, \ldots, n)$*,* $1 \leq r \leq n$ *and* $\mathbf{X}_I = (X_{i_1}, \ldots, X_{i_r})$ *and* $\mathbf{Y}_I = (Y_{i_1}, \ldots, Y_{i_r})$*. If* $\mathbf{X} \leq_{uo-c[uo-*,uo-su]} \mathbf{Y}$*, then* $\mathbf{X}_I \leq_{uo-c[uo-*,uo-su]} \mathbf{Y}_I$*. That is, the upper orthant convex, upper orthant starshaped and upper orthant superadditive orders are closed under marginalization.*

(iv) *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be a set of independent random vectors (nonnegative in the case of the starshaped order) where $\mathbf{X}_i = (X_i^1, \ldots, X_i^{d_i})$, $i = 1, \ldots, n$. Let $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ be a set of independent random vectors (nonnegative in the case of the starshaped order) where $\mathbf{Y}_i = (Y_i^1, \ldots, Y_i^{d_i})$, $i = 1, \ldots, n$. If, for $i = 1, \ldots, n$,*

$$\mathbf{X}_i \leq_{uo-c[uo-*,uo-su]} \mathbf{Y}_i,$$

then

$$(\mathbf{X}_1, \ldots, \mathbf{X}_n) \leq_{uo-c[uo-*,uo-su]} (\mathbf{Y}_1, \ldots, \mathbf{Y}_n).$$

That is, the upper orthant convex, upper orthant starshaped and upper orthant superadditive orders are closed under conjunctions.

Proof. Let us consider the convex order. The proofs for the starshaped and superadditive orders are similar.

(i) Let **X** be a random vector, then, for all $\mathbf{p} \in (0,1)^n$ and each $i = 1, \ldots, n$,

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}(\mathbf{p})^i \right.\right) \leq_c \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}(\mathbf{p})^i \right.\right),$$

therefore $\mathbf{X} \leq_{uo-c} \mathbf{X}$.

(ii) Let **X**, **Y** and **Z** be three random vectors. Suppose that $\mathbf{X} \leq_{uo-c} \mathbf{Y}$ and $\mathbf{Y} \leq_{uo-c} \mathbf{Z}$, then, each $i = 1, \ldots, n$ and all $\mathbf{p}^i \in (0,1)^{n-1}$,

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}(\mathbf{p})^i \right.\right) \leq_c \left(Y_i \left| \mathbf{Y}^i > \mathbf{y}(\mathbf{p})^i \right.\right)$$

and

$$\left(Y_i \left| \mathbf{Y}^i > \mathbf{y}(\mathbf{p})^i \right.\right) \leq_c \left(Z_i \left| \mathbf{Z}^i > \mathbf{z}(\mathbf{p})^i \right.\right),$$

for all $\mathbf{p} \in (0,1)^n$ and each $i = 1, \ldots, n$. By combining the last two inequalities, $\mathbf{X} \leq_{uo-c} \mathbf{Z}$.

(iii) Let **X** and **Y** be two random vectors such that $\mathbf{X} \leq_{mdisp} \mathbf{Y}$. It is not difficult to see that, for all $t \notin I$,

$$\left(X_r \left| \bigcap_{s \in I, s \neq r} \{X_s > F_s^{-1}(p_s)\}, \bigcap_{t \notin I} \{X_t > F_t^{-1}(p_t)\} \right.\right)$$

converges in distribution when $p_t \to 0$ to

$$\left(X_r \middle| \bigcap_{s \in I, s \neq r} \{X_s > F_s^{-1}(p_s)\} \right).$$

And similarly for

$$\left(Y_r \middle| \bigcap_{s \in r, s \neq r} \{Y_s > G_s^{-1}(p_s)\}, \bigcap_{t \notin I} \{Y_t > G_t^{-1}(p_t)\} \right)$$

and

$$\left(Y_r \middle| \bigcap_{s \in r, s \neq r} \{Y_s > G_s^{-1}(p_s)\} \right).$$

The result follows by the preservation of the multivariate convex, starshaped and superadditive transform orders under convergence in distribution, see Lemma 7 in Lewis and Thompson (1981).

(iv) Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ and $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ be two sets of independent random vectors where the dimension of $\mathbf{X}_i$ and $\mathbf{Y}_i$ is d_i for $i = 1, \ldots, n$. Suppose that $\mathbf{X}_i \leq_{uo-c} \mathbf{Y}_i$ for $i = 1, \ldots, n$, then we have to prove that, for all $\mathbf{p} \in (0,1)^n$ and each $i = 1, \ldots, n$, $k = 1, \ldots, d_i$,

$$\left(X_i^k \middle| \mathbf{X}_i^k > \mathbf{x}_i(\mathbf{p})^k, \bigcap_{j \neq i} \{\mathbf{X}_j > \mathbf{x}(\mathbf{p})_j\} \right) \leq_c \left(Y_i^k \middle| \mathbf{Y}_i^k > \mathbf{y}_i(\mathbf{p})^k, \bigcap_{j \neq i} \{\mathbf{Y}_j > \mathbf{y}(\mathbf{p})_j\} \right).$$

By hypothesis, X_i^k and $\mathbf{X}_j$ (and Y_i^k and $\mathbf{Y}_j$) are independent if $j \neq i$, therefore

$$\begin{aligned} \left(X_i^k \middle| \mathbf{X}_i^k > \mathbf{x}_i(\mathbf{p})^k, \bigcap_{j \neq i} \{\mathbf{X}_j > \mathbf{x}_j(\mathbf{p})\} \right) &= \left(X_i^k \middle| \mathbf{X}_i^k > \mathbf{x}_i(\mathbf{p})^k \right) \\ &\leq_{disp} \left(Y_i^k \middle| \mathbf{Y}_i^k > \mathbf{y}_i(\mathbf{p})^k \right) \\ &= \left(Y_i^k \middle| \mathbf{Y}_i^k > \mathbf{y}_i(\mathbf{p})^k, \bigcap_{j \neq i} \{\mathbf{Y}_j > \mathbf{y}_j(\mathbf{p})\} \right), \end{aligned}$$

which proves the stated result.

■

Let us fix some notation. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random vector, then we will denote as $F_i(\cdot|\mathbf{x}(\mathbf{p})^i)$, $F_i^{-1}(\cdot|\mathbf{x}(\mathbf{p})^i)$, $\overline{F}_i(\cdot|\mathbf{x}(\mathbf{p})^i)$, $r_i(\cdot|\mathbf{x}(\mathbf{p})^i)$, the distribution, quantile, survival and hazard rate function, respectively, of the conditional random variable

$$\left(X_i \,\middle|\mathbf{X}^i > \mathbf{x}(\mathbf{p})^i\right).$$

Analogously, for a random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$, we will denote by $G_i(\cdot|\mathbf{x}(\mathbf{p})^i)$, $G_i^{-1}(\cdot|\mathbf{x}(\mathbf{p})^i)$, $\overline{G}_i(\cdot|\mathbf{x}(\mathbf{p})^i)$, $s_i(\cdot|\mathbf{x}(\mathbf{p})^i)$, the distribution, quantile, survival and hazard rate function, respectively, of the corresponding conditional random variable

$$\left(Y_i \,\middle|\mathbf{Y}^i > \mathbf{y}(\mathbf{p})^i\right).$$

Next we obtain a characterization of the upper orthant convex and upper orthant starshaped orders in terms of the hazard rate and quantile functions of the conditional random variables truncated on the quantiles.

Proposition 2.3.3 *Let* **X** *and* **Y** *be two random vectors (nonnegative in the case of the starshaped order). Then,*

(i) $\mathbf{X} \leq_{uo-c} \mathbf{Y}$ *if, and only if, for each* $i = 1, \ldots, n$ *and all* $\mathbf{p}^i \in (0,1)^{n-1}$*,*

$$\frac{r_i(F_i^{-1}(p_i|\mathbf{x}(\mathbf{p})^i)|\mathbf{x}(\mathbf{p})^i)}{s_i(G_i^{-1}(p_i|\mathbf{y}(\mathbf{p})^i)|\mathbf{y}(\mathbf{p})^i)} \text{ is an increasing function in } p_i \in (0,1).$$

(ii) $\mathbf{X} \leq_{uo-*} \mathbf{Y}$ *if, and only if, for each* $i = 1, \ldots, n$ *and all* $\mathbf{p}^i \in (0,1)^{n-1}$*,*

$$\frac{F_i^{-1}(p_i|\mathbf{x}(\mathbf{p})^i)}{G_i^{-1}(p_i|\mathbf{y}(\mathbf{p})^i)} \text{ is an increasing function in } p_i \in (0,1).$$

Proof.

(i) $\mathbf{X} \leq_{uo-c} \mathbf{Y}$ if, and only if,

$$\left(X_i \,\middle|\mathbf{X}^i > \mathbf{x}(\mathbf{p})^i\right) \leq_c \left(Y_i \,\middle|\mathbf{Y}^i > \mathbf{y}(\mathbf{p})^i\right)$$

for all $\mathbf{p}^i \in (0,1)^{n-1}$ and each $i = 1, \ldots, n$. From (1.9), it is equivalent to, for all $\mathbf{p} \in (0,1)^n$ and each $i = 1, \ldots, n$,

$$\frac{r_i(F_i^{-1}(p_i|\mathbf{x}(\mathbf{p})^i)|\mathbf{x}(\mathbf{p})^i)}{s_i(G_i^{-1}(p_i|\mathbf{y}(\mathbf{p})^i)|\mathbf{y}(\mathbf{p})^i)} \text{ is an increasing function in } p_i \in (0,1).$$

(ii) For the starshaped order, the proof is similar taking into account (1.10).

■

2.3.3 Relationships among multivariate orders

Under the necessary condition $\overline{F}_i(x_i|\mathbf{x}(pp)^i) \leq \overline{G}_i(x_i|\mathbf{y}(pp)^i)$ for all x_i, all $\mathbf{p} \in (0,1)^n$ and each $i = 1, \ldots, n$, which is the usual stochastic order between the conditional random variables truncated in the quantiles, the upper orthant superadditive order implies the dispersive order proposed by Khaledi and Kochar (2005) as we can see in the following result.

Proposition 2.3.4 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two nonnegative random vectors such that*

$$\overline{F}_i(x_i|\mathbf{x}(\mathbf{p})^i) \leq \overline{G}_i(x_i|\mathbf{y}(\mathbf{p})^i)$$

for all $\mathbf{p} \in (0,1)^n$ *and* $i = 1, \ldots, n$*. If* $\mathbf{X} \leq_{uo-su} \mathbf{Y}$*, then* $\mathbf{X} \leq_{uo-disp} \mathbf{Y}$*.*

Proof. If $\mathbf{X}$ and $\mathbf{Y}$ are two nonnegative random vectors such that

$$\overline{F}_i(x_i|\mathbf{x}(\mathbf{p})^i) \leq \overline{G}_i(x_i|\mathbf{y}(\mathbf{p})^i)$$

for all $\mathbf{p} \in (0,1)^n$ and $i = 1, \ldots, n$, then the conditioned random variables satisfy the hypothesis in Proposition 4.B.2 in Shaked and Shanthikumar (2007) and, hence, they are ordered in the upper orthant dispersive sense. ■

We can replace the assumption of the usual stochastic order between the corresponding conditional random variables by the following limiting condition and we can establish a similar result.

Proposition 2.3.5 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two nonnegative random vectors such that* $\mathbf{X} \leq_{uo-su} \mathbf{Y}$*. If, for all* $\mathbf{p} \in (0,1)^n$ *and* $i = 1, \ldots, n$*, holds*

$$\lim_{x_i \to 0} \frac{G_i^{-1}(F_i(x_i|\mathbf{x}(\mathbf{p}^i))|\mathbf{y}(\mathbf{p})^i}{x_i} \geq 1,$$

then $\mathbf{X} \leq_{uo-disp} \mathbf{Y}$.

Proof. The proof is straightforward from Theorem 4.B.3 in Shaked and Shanthikumar (2007). ■

As pointed out in Section 1.3, given a random vector $\mathbf{X}$ with margins $F_1, \ldots, F_n$, there exists a copula C such that

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

Let XX and YY be two random vectors, one interesting situation is to assume that both random vectors share the same structure of dependence, that is, they have the same copula C. Important contributions in this case have been made by Scarsini (1988), Müller and Scarsini (2001), Khaledi and Kochar (2005), Belzunce et al. (2008) and Balakrishnan et al. (2011).

If we assume that particular situation, we can provide the following result.

Proposition 2.3.6 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors (nonnegative in the case of starshaped order) with the same copula* C*. Then,*

(i) $\mathbf{X} \leq_{uo-c} \mathbf{Y}$ *if, and only if,* $X_i \leq_c Y_i$ *for all* $i = 1, \ldots, n$.

(ii) $\mathbf{X} \leq_{uo-*} \mathbf{Y}$ *if, and only if,* $X_i \leq_* Y_i$ *for all* $i = 1, \ldots, n$.

(iii) $\mathbf{X} \leq_{uo-su} \mathbf{Y}$ *if, and only if,* $X_i \leq_{su} Y_i$ *for all* $i = 1, \ldots, n$.

Proof. If we assume that $\mathbf{X}$ and $\mathbf{Y}$ have the same copula, by Lemma 2.1 in Khaledi and Kochar (2005), it is not difficult to see that

$$G_i^{-1}(F_i(x|\mathbf{x}(\mathbf{p})^i)|\mathbf{y}(\mathbf{p}^i)) = G_i^{-1}(F_i(x))$$

for all $\mathbf{p} \in (0,1)^n$ and $i = 1, \ldots, n$, and both composition functions share the same properties. ■

Corollary 2.3.7 *Let* $\mathbf{Y} = (a_1 + b_1 X_1, a_2 + b_2 X_2, \ldots, a_n + b_n X_n)$ *(nonnegative in the case of the starshaped order). Then, for* $a_i \geq 1$ *and* $b_i \in \mathbb{R}$*,* $\mathbf{X} \leq_{uo-c[uo-*,uo-su]} \mathbf{Y}$*.*

Proof. The result follows from Proposition 2.3.6 taking into account that $\mathbf{X}$ and $\mathbf{Y}$ have the same copula. ■

As we showed in the previous chapter, Roy (2002) defined the multivariate convex, starshaped and superadditive order. Recall that $\mathbf{X} \leq_{mc[m*,msu]} \mathbf{Y}$ if, and only if,

$$\left(X_i \left|\mathbf{X}^i > \mathbf{x}^i\right.\right) \leq_{c[*,su]} \left(Y_i \left|\mathbf{Y}^i > \mathbf{x}^i\right.\right)$$

for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$,

Next we provide a relationship between the upper orthant orders defined in this section and the multivariate orders proposed by Roy (2002) in the bivariate case.

Proposition 2.3.8 *Let* $\mathbf{X} = (X_1, X_2)$ *and* $\mathbf{Y} = (Y_1, Y_2)$ *be two bivariate random vectors (nonnegative in the case of the starshaped order) with survival functions* $\overline{F}$ *and* $\overline{G}$*, respectively. If*

(i) $X_1 \leq_{st} Y_1$ *and* $X_2 \leq_{st} Y_2$*,*

(ii) $(Y_i|Y_j > x_j)$ *is an increasing function in* x_j *in the convex [starshaped, superadditive] order for* $i \neq j$*.*

Then, If $\mathbf{X} \leq_{mc[m*,msu]} \mathbf{Y}$*, then* $\mathbf{X} \leq_{uo-c[uo-*,uo-su]} \mathbf{Y}$*.*

Proof. We give the proof in the case of convex order, the starshaped and superadditive cases are similar.

In order to prove that $\mathbf{X} \leq_{uo-disp} \mathbf{Y}$, we have to see that, for all $p \in (0,1)$,

$$\left(X_1|X_2 > F_{X_2}^{-1}(p)\right)\right) \leq_c \left(Y_1|Y_2 > F_{Y_2}^{-1}(p)\right) \tag{2.11}$$

and

$$\left(X_2|X_1 > F_{X_1}^{-1}(p)\right) \leq_c \left(Y_2|Y_1 > F_{Y_1}^{-1}(p)\right). \tag{2.12}$$

From $\mathbf{X} \leq_{mc} \mathbf{Y}$, it holds, for all $x_1, x_2 \in \mathbb{R}$,

$$(X_1|X_2 > x_2) \leq_c (Y_1|Y_2 > x_2) \tag{2.13}$$

and

$$(X_2|X_1 > x_1) \leq_c (Y_2|Y_2 > x_1).$$

Let us prove (2.11). From (2.13), we have that

$$\left(X_1|X_2 > F_{X_2}^{-1}(p)\right) \leq_c \left(Y_1|Y_2 > F_{X_2}^{-1}(p)\right),$$

and from (ii),

$$\left(X_1|X_2 > F_{X_2}^{-1}(p)\right) \leq_c \left(Y_1|Y_2 > F_{Y_2}^{-1}(p)\right).$$

By combining the last two inequalities, it follows the required result. ■

Next we give a characterization of the MIFR, MIFRA and MNBU aging notions given by Roy (1994) based on the upper orthant convex, upper orthant starshaped and upper orthant superaddtive order.

Proposition 2.3.9 *Let* $\mathbf{X}$ *be a nonnegative random vector and*

$$\mathbf{Y} \sim G(\{\lambda_i\}_{i=1}^n, \{\lambda_{ij}\}_{i>j}, \ldots, \lambda_{1,\ldots,n}),$$

that is, $\mathbf{Y}$ *is a random vector type Gumbel with survival function given by* (1.28). *Then,*

(i) $\mathbf{X}$ *is MIFR if, and only if,* $\mathbf{X} \leq_{uo-c} \mathbf{Y}$.

(ii) $\mathbf{X}$ *is MIFRA if, and only if,* $\mathbf{X} \leq_{uo-*} \mathbf{Y}$.

(iii) $\mathbf{X}$ *is MNBU if, and only if,* $\mathbf{X} \leq_{uo-su} \mathbf{Y}$.

Proof. For the shake of simplicity, recall that for $\mathbf{X} = (X_1, \ldots, X_n)$, given $(x_1, \ldots, x_n) \in \mathbb{R}^n$, and $i = 1, \ldots, n$, we denoted by $F_i(\cdot|\mathbf{x}^i)$, $F_i^{-1}(\cdot|\mathbf{x}^i)$, $\overline{F}_i(\cdot|\mathbf{x}^i)$, $r_i(\cdot|\mathbf{x}^i)$, $A_i(\cdot|\mathbf{x}^i)$, the distribution function, quantile function, survival function, hazard rate function and hazard rate in average function of the conditional random variable

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right).$$

Assume that **X** is MIFR, then $r_i(x_i|\mathbf{x}^i)$ is an increasing function in x_i for all $\mathbf{x} \in \mathbb{R}^n$ and each $i = 1, \ldots, n$, or, equivalently, the conditional distribution given by (**??**) is IFR.

Fixed $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $i \in \{1, \ldots, n\}$, there exist p_i such that $x_i = F^{-1}(p_i)$ for $i \neq j$. Let $\mathbf{p} = (p_1, \ldots, p_n)$, therefore,

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}^i \right.\right) = \left(Y_i \left| \mathbf{Y}^i > \mathbf{x}(\mathbf{p})^i \right.\right).$$

From Proposition 1.3.22,

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}(\mathbf{p})^i \right.\right) \leq_c \left(Y_i \left| \mathbf{Y}^i > \mathbf{y}(\mathbf{p})^i \right.\right)$$

for each $i = 1, \ldots, n$. Then, $\mathbf{X} \leq_{uo-c} \mathbf{Y}$.

On the other hand, if we assume that $\mathbf{X} \leq_{uo-c} \mathbf{Y}$, then, for all $\mathbf{p} \in (0,1)^n$ and each $i = 1, \ldots, n$,

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}(\mathbf{p})^i \right.\right) \leq_c \left(Y_i \left| \mathbf{Y}^i > \mathbf{y}(\mathbf{p})^i \right.\right). \tag{2.14}$$

Since,

$$\left(Y_i \left| \mathbf{Y}^i > \mathbf{y}(\mathbf{p})^i \right.\right)$$

is exponential distributed, then

$$\left(X_i \left| \mathbf{X}^i > \mathbf{x}(\mathbf{p})^i \right.\right)$$

is IFR and, therefore, **X** is MIFR.

The proof of (ii) and (iii) is similar. ■

2.3.4 Applications and examples

In this section we study some situations, as well as some parametrical examples, of the orders defined and studied in the previous sections.

Pareto distribution

Let us consider the bivariate Pareto distribution, denoted as $\mathbb{P}(\alpha_1, \alpha_2, a)$, whose joint survival function is given by

$$\overline{F}(x_1, x_2) = \left(\frac{x_1}{\alpha_1} + \frac{x_2}{\alpha_2} - 1\right)^{-a}$$

where $x_i \geq \alpha_i > 0$ and $a > 2$.

It is not difficult to see that

$$\overline{F}_{(X_1|X_2>x_2)}(x_1) = \frac{\left(\frac{x_1}{\alpha_1} + \frac{x_2}{\alpha_2} - 1\right)^{-a}}{\left(\frac{x_2}{\alpha_2}\right)^{-a}}$$

$$f_{(X_1|X_2>x_2)}(x_1) = \frac{a}{\alpha_1}\frac{\left(\frac{x_1}{\alpha_1} + \frac{x_2}{\alpha_2} - 1\right)^{-a-1}}{\left(\frac{x_2}{\alpha_2}\right)^{-a}}$$

$$r_{(X_1|X_2>x_2)}(x_1) = \frac{a}{\alpha_1}\left(\frac{x_1}{\alpha_1} + \frac{x_2}{\alpha_2} - 1\right)^{-1}$$

$$F^{-1}_{(X_1|X_2>x_2)}(u) = \alpha_1\left[\frac{x_2}{\alpha_2}(1-u)^{-1/a} - \frac{x_2}{\alpha_2} + 1\right]$$

Therefore,

$$r_{(X_1|X_2>x_2)}\left(F^{-1}_{(X_1|X_2>x_2)}(u)\right) = \frac{a}{\alpha_1}\left(\frac{x_2}{\alpha_2}\right)^{-1}(1-u)^{1/a}$$

Let us consider $(X_1, X_2) \sim \mathbb{P}(\alpha_1, \alpha_2, a)$ and $(Y_1, Y_2) \sim \mathbb{P}(\beta_1, \beta_2, b)$. When $\alpha_i = \beta_i$, Belzunce et al. (2011) show that, if $a > b$, then

$$Var[X_i|X_j > x_j] \leq Var[Y_i|Y_j > x_j] \text{ for } i \neq j \in \{1, 2\}.$$

The comparisons in terms of the upper orthat convex, starshaped and superadditive orders can be obtain without taking into account the values of α_i and β_i

$i = 1, 2$, that is, they have no influence on the transform orders. Fixed $p \in (0, 1)$,

$$r_{\left(X_1|X_2>F_{X_2}^{-1}(p)\right)}\left(F^{-1}_{\left(X_1|X_2>F_{X_2}^{-1}(p)\right)}(u)\right) = \frac{a}{\alpha_1}(1-u)^{1/a}(1-p)^{1/a}$$

and

$$r_{\left(Y_1|Y_2>F_{Y_2}^{-1}(p)\right)}\left(F^{-1}_{\left(Y_1|Y_2>F_{Y_2}^{-1}(p)\right)}(u)\right) = \frac{b}{\beta_1}(1-u)^{1/b}(1-p)^{1/b}.$$

Since

$$\frac{r_{\left(X_1|X_2>F_{X_2}^{-1}(p)\right)}\left(F^{-1}_{\left(X_1|X_2>F_{X_2}^{-1}(p)\right)}(u)\right)}{r_{\left(Y_1|Y_2>F_{Y_2}^{-1}(p)\right)}\left(F^{-1}_{\left(Y_1|Y_2>F_{Y_2}^{-1}(p)\right)}(u)\right)} = \frac{\frac{a}{\alpha_1}(1-u)^{1/a}(1-p)^{1/a}}{\frac{-b}{\beta_1}(1-u)^{1/b}(1-p)^{1/b}},$$

it holds that, if $a > b$,

$$\frac{r_{\left(X_1|X_2>F_{X_2}^{-1}(p)\right)}\left(F^{-1}_{\left(X_1|X_2>F_{X_2}^{-1}(p)\right)}(u)\right)}{r_{\left(Y_1|Y_2>F_{Y_2}^{-1}(p)\right)}\left(F^{-1}_{\left(Y_1|Y_2>F_{Y_2}^{-1}(p)\right)}(u)\right)} \text{ is an increasing function in } p \in (0,1),$$

or, equivalently,

$$\left(X_1|X_2 > F_{X_2}^{-1}(p)\right) \leq_c \left(Y_1|Y_2 > F_{Y_2}^{-1}(p)\right).$$

Analogously, it is easy to see that, if $a > b$,

$$\left(X_2|X_1 > F_{X_1}^{-1}(p)\right) \leq_c \left(Y_2|Y_1 > F_{Y_1}^{-1}(p)\right)$$

Therefore, independently of the values of α_i and β_i, if $a > b$, then $\mathbf{X} \leq_{uo-c} \mathbf{Y}$ (and hence $\mathbf{X} \leq_{uo-*} \mathbf{Y}$ and $\mathbf{X} \leq_{uo-su} \mathbf{Y}$).

Generalized Order Statistics

Perhaps the most well known model of a random vector with ordered components is the random vector of order statistics. This models arises in natural way when we arrange in increasing order a set of observations from a random variable. Another example is the case of epoch times of a counting process, like the case of a nohomogeneous Poisson process. Epoch times of nohomogeneous Poisson processes can be introduced as record values of a proper sequence of random variables, which is another typical example of ordered data. Given the similarity of several results for order statistics and record values Kamps (1995a) introduces the model of generalized order statistics. This model provides a unified approach to study order statistics and record values, and several other models of ordered data.

First we provide the definition of generalized order statistics following Kamps (1995a) and (1995b).

Definition 2.3.10 *Let $n \in \mathbb{N}$, $k \geq 1$, $m_1, \ldots, m_{n-1} \in \mathbb{R}$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$, be parameters such that $\gamma_r = k + n - r + M_r \geq 1$ for all $r \in 1, \ldots, n-1$, and let $\widetilde{m} = (m_1, \ldots, m_{n-1})$, if $n \geq 2$ ($\tilde{m} \in \mathbb{R}$ arbitrary, if $n = 1$). We call **uniform generalized order statistics** to the random vector $(U_{(1,n,\tilde{m},k)}, \ldots, U_{(n,n,\tilde{m},k)})$ with joint density function*

$$h(u_1, \ldots, u_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{j=1}^{n-1} (1 - u_j)^{m_j} \right) (1 - u_n)^{k-1}$$

on the cone $0 \leq u_1 \leq \ldots \leq u_n \leq 1$. Now given a distribution function F we call generalized order statistics based on F to the random vector

$$(X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)}) \equiv \left(F^{-1}(U_{(1,n,\tilde{m},k)}), \ldots, F^{-1}(U_{(n,n,\tilde{m},k)}) \right).$$

If F is an absolutely continuous distribution with density f, the joint density function of $(X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)})$ is given by

$$f(x_1, \ldots, x_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{j=1}^{n-1} \bar{F}(x_j)^{m_j} f(x_j) \right) \bar{F}(x_n)^{k-1} f(x_n)$$

on the cone $F^{-1}(0) \leq x_1 \leq \ldots \leq x_n \leq F^{-1}(1)$.

For GOS's we have that two random vectors of GOS's with the same set of parameters and possibly based on different distributions have the same copula (see Belzunce et al. (2008)).

Let us see now several models that are included in this model. As we have mentioned previously, order statistics and record values are a particular case of this model.

Taking $m_i = 0$ for all $i = 1, \dots, n-1$ and $k = 1$ we get the random vector of order statistics $(X_{1:n}, X_{2:n}, \cdots, X_{n:n})$ from a set of n independent and identically distributed (i.i.d) observations $X_1, X_2, \dots, X_n$ with common absolutely continuous distribution F, in particular we get that $X_{i:n} =_{\text{st}} X_{(i,n,0,1)}$.

Let us consider the case of record values. Chandler (1952) introduced the mathematical notion of record values to study, from a statistical point of view, sequences of record values that arise in practice. Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables, which can be considered as independent observations from the random variable X. Denote the cumulative distribution function of X by F, and assume that F is absolutely continuous. Also, denote by f the corresponding density function. Record values are defined by means of record times, so first let us recall the definition of record times. Given a sequence of i.i.d. random variables as above, the **record times** are the random variables

$$\begin{aligned} L(1) &= 1, \\ L(n) &= \min\{j > L(n-1) | X_j > X_{L(n-1)}\}, \quad n = 2, 3, \dots . \end{aligned}$$

The sequence of **record values** $X(n)$ is defined by

$$X(n) \equiv X_{L(n)}, \quad n = 1, 2, \dots .$$

Taking $m_i = -1$ for all $i = 1, \dots, n-1$ and $k = 1$ we get that $X(i) =_{st} X_{(i,n,-1,1)}$.

Some additional particular cases of GOS's are the following.

A generalization of the record values, is the case in which $k \in \mathbb{N}$, obtaining what is called k-records.

Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of i.i.d random variables with distribution function F and let $k \in \mathbb{N}$. Denoting by

$$(X_{j-k+1,j})_{j=k,k+1,\dots},$$

the sequence of k-th largest order statistics, the random variables given by

$$L^k(1) = 1$$
$$L^k(n+1) = \min\{j > L(n) : X_{j,j+k-1} > X_{L^k(n),L^k(n)+k-1}\}$$

are called the **record times**, and the quantities

$$X_{L^k(n)} \equiv X_{L^k(n),L^k(n)+k-1}$$

are called k-**th records** or k-records, which are a special case of GOS's taking $m_i = -1$ for all $i = 1, \dots, n-1$ and $k \in \mathbb{N}$.

A life-testing experiment of interest in reliability studies involves N independent and identically distributed random variables placed simultaneously on test and at the time of the m-th failure R_i surviving units are randomly censored from the test. The progressively Type-II censored order statistics arising from such a reliability experiment can be obtained from the model of GOS's by setting $n = m$, $m_i = R_i$ and $k = R_m + 1$.

An interesting model contained in the model of generalized order statistics is that of order statistics under multivariate imperfect repair; see Shaked and Shanthikumar (1986). Suppose n items start to function at the same time 0. Upon failure, an item undergoes a repair. If i items $(i = 0, 1, \cdots, n-1)$ have already been scrapped, then, with probability p_{i+1}, the repair is unsuccessful and the item is scrapped, and with probability $1 - p_{i+1}$, the repair is successful and minimal.

Let us now consider n items with i.i.d. random lifetimes $X_1, \cdots, X_n$, with the same distribution F and density function f. Let $(X_{(1)}, \cdots, X_{(n)})$ be the ordered random lifetimes resulting from $X_1, \cdots, X_n$ under such a minimal repair policy. Then, the joint density function of $(X_{(1)}, \cdots, X_{(n)})$ is given by

$$f(t_1, \cdots, t_n) = n! \prod_{j=1}^{n} p_j f(t_j)(\overline{F}(t_j))^{(n-j+1)p_j-(n-j)p_{j+1}-1} \text{ for } 0 \le t_1 \le \cdots \le t_n.$$

It is evident that this is a particular case of the joint density function of generalized order statistics based on F for the choice of parameters $k = p_n$ and $m_j = (n-j+1)p_j - (n-j)p_{j+1} - 1$.

Next we show a property of the comparisons of random vectors of generalized orders statistics in the upper orthant orders defined in this section.

Theorem 2.3.11 *Let* $(X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)})$ *and* $(Y_{(1,n,\tilde{m},k)}, \ldots, Y_{(n,n,\tilde{m},k)})$ *be the random vectors of generalized order statistics based on distributions* F *and* G *from random variables* X *and* Y*, respectively. If*

$$X \leq_{uo-c[uo-*,uo-su]} Y,$$

then $\mathbf{X} \leq_{uo-c[uo-*,uo-su]} \mathbf{Y}$.

Proof. Since the vectors of order statistics have the same copula, the result follows from Theorem 2.3.6. ■

2.4 Conclusions and future research

The new criteria to compare multivariate distributions considered in this memory suggest a way to compare multivariate distributions in terms of conditional distributions.

First, we have considered conditional distributions of the type

$$\left(X_i \middle| \bigcap_{j \neq i} \{X_j > x_j\} \right).$$

This is in the spirit of previous works like Roy (2002) and Hu, Khaledi and Shaked (2003). We have focused our attention to the comparison of these random variables in terms of dispersion and concentration, and therefore the main fields of applications are reliability, risks and inequality. This approach can be used in other contexts changing the type of conditioning, considering conditional random variables of the type

$$\left(X_i \middle| \bigcap_{j \neq i} \{X_j \leq x_j\} \right) \text{ or } \left(X_i \middle| \bigcap_{j \neq i} \{X_j = x_j\} \right)$$

or the comparison criteria, for example in terms of variability, entropy, etc. Similar results to those obtained in Section 2.2 can be given for these random variables. The obtained results in Section 2.2 are included in Belzunce et al. (2012a).

We have also considered new criteria to compare multivariate distributions considered taking into account the important role of the quantiles and provides a way to compare multivariate distributions in terms of conditional distributions of the type

$$\left(X_i \,\middle|\, \bigcap_{j \neq i} \{X_j > F_j^{-1}(p_j)\} \right).$$

This approach is clearly inspired in previous works Khaledi and Kochar (2005) and Mercader (2007). Again we have focused on the conditioned random variables of the type "greater than", but it is possible to follow a similar approach with conditioned random variables of the type "equal to" and "smaller than" as we have previously discussed. The obtained results in Section 2.3 are included in Belzunce et al. (2012b).

Now we want to point out some future research, in particular for the proposal of new multivariate convex transform orders.

When we have considered the univariate convex order, it is interesting to recall, that $X \leq_c Y$ if, and only if, there exist an increasing convex function ϕ from the support of X to $\mathbb{R}$, such that $Y =_{st} \phi(X)$. That is, there exist an increasing convex transformation that maps X onto Y. In particular, for two random variables X and Y with distribution functions F and G, respectively, if $X \leq_c Y$, the function $\phi(x) = G^{-1}(F(x))$ is an increasing convex transformation which maps X onto Y.

Following this idea and given two n-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$, when can consider a definition of multivariate convex transform order based on the condition that there exist some increasing transformation which maps $\mathbf{X}$ onto $\mathbf{Y}$ that satisfy some kind of convexity. In Chapter 1 we have described the transformation that plays the role of $G^{-1}(F)$ in the multivariate case. Next we recall it.

Let $\mathbf{Y} = (Y_1, \dots, Y_n)$ be a random vector in $\mathbb{R}^n$ with absolutely continuous distribution function and let $\mathbf{p} = (p_1, \dots, p_n) \in [0, 1]^n$. The standard construction for

Y is given by

$$\widehat{y}(\mathbf{p}) = (\widehat{y}_1(p_1), \widehat{y}_2(p_1,p_2), \ldots, \widehat{y}_n(p_1,\ldots,p_n))$$

where

$$\begin{aligned} \widehat{y}_1(p_1) &= F_{Y_1}^{-1}(p_1) \\ \widehat{y}_2(p_1,p_2) &= F^{-1}_{(Y_2|Y_1=\widehat{y}_1(p_1))}(p_2) \\ &\vdots \\ \widehat{y}_n(p_1,\ldots,u_n) &= F^{-1}_{(Y_n|\bigcap_{j=1}^{n-1} Y_j=\widehat{y}_j(p_1,\ldots,p_j))}(p_n) \end{aligned}$$

This transformation plays the role of the quantile in the multivariate case.

Given a random vector **X** we consider the transformation

$$\overset{\star}{x}(\mathbf{x}) = (\overset{\star}{x}_1(x_1), \overset{\star}{x}_2(x_1,x_2), \ldots, \overset{\star}{x}_n(x_1,\ldots,x_n))$$

where

$$\begin{aligned} \overset{\star}{x}_1(x_1) &= F_{X_1}(x_1) \\ \overset{\star}{x}_2(x_1,x_2) &= F_{(X_2|X_1=x_1)}(x_2) \\ &\vdots \\ \overset{\star}{x}_n(x_1,\ldots,x_n) &= F_{(X_n|\bigcap_{j=1}^{n-1} X_j=x_j)}(x_n). \end{aligned}$$

Now, from Proposition 1.3.1, we have that the transformation

$$\phi(x_1,\ldots,x_n) = (\phi_1(x_1), \phi_2(x_1,x_2), \ldots, \phi_n(x_1,\ldots,x_n)) \qquad (2.15)$$

where, for $i = 1,\ldots,n$,

$$\phi_i(x_1,\ldots,x_i) = \widehat{y}_i(\overset{\star}{x}_1(x_1), \ldots, \overset{\star}{x}_i(x_1,\ldots,x_i)),$$

maps **X** onto **Y**, that is $\mathbf{Y} =_{st} \phi(\mathbf{X})$.

Therefore, under some kind of convexity for this transformation we can propose a multivariate definition of the convex transform order. In particular we have considered two possible definitions.

Definition 2.4.1 *Let* **X** *and* **Y** *be two random vectors. Let* $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ *be the function defined in* (2.15) *which maps* **X** *onto* **Y**. *Then,* **X** *is said to be smaller than*

$\mathbf{Y}$ *in the* ***multivariate transform convex order****, denoted by* $\mathbf{X} \leq_{mtc} \mathbf{Y}$*, if, and only, if* $\phi_i(x_1, \ldots, x_i)$ *is convex in* $\mathbb{R}^i$*, for every* $i = 1, \ldots, n$*, that is, if for every* $i = 1, \ldots, n$ *the hessian of* ϕ_i*,*

$$\nabla^2\phi_i(x_1, \ldots, x_i) = \begin{pmatrix} \frac{\partial^2}{\partial x_1^2}\phi_i(x_1, \ldots, x_i) & \frac{\partial^2}{\partial x_1 x_2}\phi_i(x_1, \ldots, x_i) & \cdots & \frac{\partial^2}{\partial x_1 x_i}\phi_i(x_1, \ldots, x_i) \\ \frac{\partial^2}{\partial x_2 x_1}\phi_i(x_1, \ldots, x_i) & \frac{\partial^2}{\partial x_2^2}\phi_i(x_1, \ldots, x_i) & \cdots & \frac{\partial^2}{\partial x_2 x_i}\phi_i(x_1, \ldots, x_i) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial x_i x_1}\phi_i(x_1, \ldots, x_i) & \frac{\partial^2}{\partial x_i x_2}\phi_i(x_1, \ldots, x_i) & \cdots & \frac{\partial^2}{\partial x_i^2}\phi_i(x_1, \ldots, x_i), \end{pmatrix}$$

is semidefinite positive, for every $(x_1, \ldots, x_i)$ *in the support of* $(X_1, \ldots, X_i)$*.*

Definition 2.4.2 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors. Let* ϕ *be the function defined before which maps* $\mathbf{X}$ *onto* $\mathbf{Y}$*. Then,* $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the* ***conditional multivariate transform convex order****, denoted by* $\mathbf{X} \leq_{c-mtc} \mathbf{Y}$*, if*

$$\phi_i(x_1, \ldots, x_i) \text{ is a convex function in } x_i \text{ for all } i = 1, 2, \ldots, n. \tag{2.16}$$

Some preliminary results can be given that we describe next.

First we observe that the $\leq_{mtc}$ is stronger thant the $\leq_{c-mtc}$ order.

Theorem 2.4.3 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors, with absolutely continuous distributions. If* $\mathbf{X} \leq_{mtc} \mathbf{Y}$*, then* $\mathbf{X} \leq_{c-mtc} \mathbf{Y}$*.*

Proof. Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors such that $\mathbf{X} \leq_{mtc} \mathbf{Y}$, then the hessian of ϕ_i, $\nabla^2\phi_i(x_1, \ldots, x_i)$ is semidefinite positive. This implies that the elements of the principal diagonal are nonnegative, in particular,

$$\frac{\partial^2}{\partial x_i^2}\phi_i(x_1, \ldots, x_i) \geq 0,$$

i.e., $\phi_i(x_1, \ldots, x_i)$ is convex in x_i, for each $i = 1, 2 \ldots, n$. ∎

As we see next, in the following example, this is an strict implication.

Example 2.4.4 *Let us consider a bivariate random vector* $\mathbf{Y} = (Y_1, Y_2)$ *with bivariate Gumbel distribution* $G(1,1,1)$*, that is, with joint density function given by*

$$g(y_1, y_2) = (y_1 + y_2 + y_1 y_2)e^{-(y_1+y_2+y_1y_2)} \text{ for } (y_1, y_2) \in \mathbb{R}^2.$$

Let us consider the random vector $\mathbf{X} = (X_1, X_2)$*, with joint density given by*

$$f(x_1, x_2) = (4x_1^5 x_2 + 4x_1^3 x_2^3 + 4x_1^7 x_2^3)e^{-(x_1^2+x_1^2x_2^2+x_1^4x_2^2)} \text{ for } (x_1, x_2) \in \mathbb{R}^2.$$

It is not difficult to see that $\mathbf{Y} =_{st} (\phi_1(X_1), \phi_2(X_1, X_2))$*, where* $\phi_1(x_1) = x_1^2$ *and* $\phi_2(x_1, x_2) = x_1^2 x_2^2$*.*

In this case, the hessian of ϕ_2 *is given by*

$$\nabla^2 \phi_2(x_1, x_2) = \begin{pmatrix} 2x_2^2 & 4x_1x_2 \\ 4x_1x_2 & 2x_1^2 \end{pmatrix},$$

where $\det(\nabla^2\phi_2(x_1, x_2)) = -12x_1^2x_2^2$*, and, hence,* $\nabla^2\phi_2(x_1, x_2)$ *is not semidefinite positive.*

However, $\phi_1(x_1) = x_1^2$ *and* $\phi_2(x_1, x_2) = x_1^2x_2^2$ *are convex in* x_1 *and* x_2*, respectively.*

Therefore, $\mathbf{X} \leq_{c-mtc} \mathbf{Y}$ *and it is not ordered in the mtc order.*

The following is a characterization of the c-mtc order, which justifies the notation of this order.

Theorem 2.4.5 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors, with absolutely continuous distributions. Then,* $\mathbf{X} \leq_{c-mtc} \mathbf{Y}$ *if, and only if,*

$$X_1 \leq_c Y_1 \tag{2.17}$$

and

$$\left(X_i \,\middle|\, \bigcap_{j=1}^{i-1} \{X_j = \widehat{x}_j(u_1, \ldots, u_j)\}\right) \leq_c \left(Y_i \,\middle|\, \bigcap_{j=1}^{i-1} \{Y_j = \widehat{y}_j(u_1, \ldots, u_j)\}\right), \tag{2.18}$$

for $i = 2, \ldots, n$ *and for all* u_i *such that* $0 < u_i < 1$*,* $i = 1, \ldots, n$*.*

Proof. From Definition 2.4.2, $\mathbf{X} \leq_{c-mtc} \mathbf{Y}$ if, and only if,

$$\phi_1(x_1) = \widehat{y}_1(\overset{\star}{x}_1(x_1)) = F_{Y_1}^{-1}(F_{X_1}(x_1))$$

which is equivalent to $X_1 \leq_c Y_1$, and, for $i = 2, \ldots, n$,

$$\phi_i(x_1, \ldots, x_i) = \widehat{y}_i(\overset{\star}{x}_i(x_1, \ldots, x_i)) =$$

$$= F^{-1}_{\left(Y_i \middle| \bigcap_{j=1}^{i-1} \{Y_j = \phi_j(x_1,\ldots,x_j)\}\right)} \left(F_{\left(X_i \middle| \bigcap_{j=1}^{i-1} \{X_j = x_j\}\right)}(x_i) \right)$$

is convex in x_i for all $i = 1, \ldots, n$, which means that

$$\left(X_i \middle| \bigcap_{j=1}^{i-1} \{X_j = \widehat{x}_j(u_1, \ldots, u_j)\} \right) \leq_c \left(Y_i \middle| \bigcap_{j=1}^{i-1} \{Y_j = \widehat{y}_j(u_1, \ldots, u_j)\} \right).$$

■

Another interesting result for these orders is the case of random vectors with the same copula.

Theorem 2.4.6 *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *be two random vectors such that they have the same copula. Then,* $\mathbf{X} \leq_{mtc[c-mtc]} \mathbf{Y}$ *if, and only if,* $X_i \leq_c Y_i$ *for all* $i = 1, \ldots, n$.

Proof. Arias-Nicolás et al. (2005) showed that, for two random vectors with the same copula, the function ϕ can be expressed as

$$\phi_i(x_1, \ldots, x_n) = G_i^{-1}(F_i(x_i))$$

for all $i = 1, \ldots, n$, which, under the assumptions, clearly satisfies the conditions considered in definitions 2.4.1 and 2.4.2. ■

These results are included in Belzunce et al. (2012c).

3

Bivariate aging properties

Abstract. As we have seen in Section 1.3.3, several authors have proposed multivariate extensions of univariate aging notions such as IFR (increasing failure rate) and DMRL (decreasing mean residual life) notions. Bassan and Spizzichino (1999), Bassan et al. (2002) or Pellerey (2008) proposed new multivariate notions when the lifetimes of the components have exchangeable joint probability distributions. In this chapter, we consider new multivariate aging notions, in the sense proposed by Bassan and Spizzichino (1999) and Bassan et al. (2002) and study the role of aging properties proposed by Pellerey (2008) in the context of frailty models.

3.1 Introduction

As we showed in the previous chapters, when X is the lifetime of a device then X_t can be interpreted as the residual lifetime of the device at time t, given that the device is alive at time t.

Several characterizations of aging notions of items, components or individuals by means of stochastic comparisons between the residual lifetimes X_0, X_t and X_{t+s}, with $t, t+s \in \{x : \overline{F}(x) > 0\}$, have been considered and studied in literature. For a review of the topic, see Belzunce and Shaked (2008a) and Belzunce and Shaked (2008b).

These characterizations serve a few purposes. On the one hand, they can be used when one wants to prove analytically that some random variable has an aging property, as well as they also throw a new light of understanding on the intrinsic meaning of the aging notions that are involved. Recall that the *increasing failure rate* and *decreasing in mean residual life* are well known aging notions and, as we showed in Proposition 1.2.25 and 1.2.26, can be characterized by comparisons among residual lifetimes.

On the one hand, as we can see in Definition 1.3.23, Bassan and Spizzichino (1999) and Bassan et al. (2002) proposed extensions of the IFR and DMRL notions in the multivariate case when the lifetimes of the components have exchangeable joint probability distributions. Here, we propose and study new multivariate aging notions for exchangeable random vectors based on comparisons of their residual lifetimes in terms of some univariate stochastic orders. These results are included in Belzunce et al. (2008).

On the other hand, we present some results considering both stochastic comparisons between $\mathbf{X}_t$ and $\mathbf{X}_{t+s}$ for all $t, s \geq 0$ and the case of dependent lifetimes having different distributions in the sense of Definition 1.3.24. These results are included in Mulero and Pellerey (2010).

Finally, we give some results dealing with negative aging of frailty models included in Mulero et al. (2010a).

3.2 New bivariate aging notions in the exchangeable case

In this section, we present some new definitions of bivariate aging notions for exchangeable random vectors that complete some other definitions proposed by Bassan and Spizzichino (1999) and Bassan et al. (2002). For further details, see Section 1.3.3 in Chapter 1.

These bivariate aging notions extend some univariate aging notions in the sense of decreasing monotonicity of residual lives in the increasing convex order, increasing concave order and Laplace transform order. These new notions can be related to some positive dependence notions.

Clearly inspired in Definition 1.3.23 and based on the characterization for the IFR and DMRL notions provided in propositions 1.2.25 and 1.2.26, we propose the following new bivariate aging notions:

Definition 3.2.1 *Let (X, Y) be an exchangeable random vector. Then,*

*(i) (X, Y) is said to be **bivariate increasing in failure rate in the weak sense,** denoted by w-BIFR, if, for $x \leq y$,*

$$(X - x|X > x, Y > y) \geq_{icv} (Y - y|X > x, Y > y).$$

*(ii) (X, Y) is said to be **bivariate increasing in failure rate in the Laplace transform sense,** denoted by BIFR(Lt), if, for $x \leq y$,*

$$(X - x|X > x, Y > y) \geq_{Lt} (Y - y|X > x, Y > y).$$

*(iii) (X, Y) is said to be **bivariate decreasing in mean residual life,** denoted by BDMRL, if, for $x \leq y$,*

$$(X - x|X > x, Y > y) \geq_{icx} (Y - y|X > x, Y > y).$$

From (1.11), we have the following relationships among the new bivariate aging notions and the previous ones:

$$\begin{array}{ccccccc} s-BIFR & \Rightarrow & BIFR & \Rightarrow & w-BIFR \Rightarrow & & BIFR(Lt) \\ \Downarrow & & \Downarrow & & & & \Downarrow \\ s-BDMRL & \Rightarrow & BDMRL & & \Rightarrow & & w-BDMRL. \end{array}$$

Next we give some results that describe sufficient conditions under which a bivariate random vector has the new aging notions.

Proposition 3.2.2 *Let (X, Y) be an exchangeable random vector. If*

(i) $(X - x|X > x, Y > y)$ is increasing in y in the icx order, for all x, and

(ii) $(X|Y > y)$ *is DMRL for all* y*,*

then (X, Y) *is BDMRL.*

Proof. Let $0 \le x \le y$ and let us consider the random variable $(X|Y > y)$. From (ii) and Proposition 1.2.26 (ii), it follows that

$$(X - x|Y > y, X > x) \ge_{icx} (X - y|Y > y, X > y).$$

Now from (i), we have

$$(X - y|Y > y, X > y) \ge_{icx} (X - y|Y > x, X > y) =_{st} (Y - y|Y > y, X > x),$$

where the equality holds from the the exchangeability of the components. By combining the last two inequalities, the required result is proved.

■

Under similar arguments we have the following results.

Proposition 3.2.3 *Let* (X, Y) *be an exchangeable random vector. If*

(i) $(X - x|X > x, Y > y)$ *is increasing in* y *in the icv order, for all* x*, and*

(ii) $(X|Y > y)$ *is IFR for all* y*,*

then (X, Y) *is w-BIFR.*

Proof. Let $0 \le x \le y$ and let us consider the random variable $(X|Y > y)$. From (ii) and Proposition 1.2.25 (v), it follows that

$$(X - x|Y > y, X > x) \ge_{icv} (X - y|Y > y, X > y).$$

Now from (i), we have

$$(X - y|Y > y, X > y) \ge_{icv} (X - y|Y > x, X > y) =_{st} (Y - y|Y > y, X > x),$$

where the equality holds from the the exchangeability of the components. By combining the last two inequalities, the required result is proved. ■

Proposition 3.2.4 *Let (X, Y) be an exchangeable random vector. If*

(i) $(X - x|X > x, Y > y)$ *is increasing in y in the Lt order, for all x, and*

(ii) $(X|Y > y)$ *is IFR for all y,*

then (X, Y) is BIFR(Lt).

Proof. Let $0 \leq x \leq y$ and let us consider the random variable $(X|Y > y)$. From (ii) and Proposition 1.2.25 (vi), it follows that

$$(X - x|Y > y, X > x) \geq_{Lt} (X - y|Y > y, X > y).$$

Now from (i), we have

$$(X - y|Y > y, X > y) \geq_{Lt} (X - y|Y > x, X > y) =_{st} (Y - y|Y > y, X > x),$$

where the equality holds from the the exchangeability of the components. By combining the last two inequalities, the required result is proved. ■

These multivariate aging notions serve also to characterize some univariate aging notions for the margins previously defined in literature. Next we recall these notions. For further details, see Cao and Wang (1991), Deshpande et al. (1986), Belzunce et al. (1999) and Yue and Cao (2001).

Definition 3.2.5 *Let X be a non negative random variable. Then,*

(i) X is said to be ***new better than used in increasing convex order,*** *denoted by NBUC, if $X \geq_{icx} (X - t|X > t)$, for all $t > 0$.*

(ii) X is said to be ***new better than used in increasing concave order,*** *denoted by NBU(2), if $X \geq_{icv} (X - t|X > t)$, for all $t > 0$.*

(iii) X is said to be ***new better than used in Laplace transform order,*** *denoted by NBU_{Lt}, if $X \geq_{Lt} (X - t|X > t)$, for all $t > 0$.*

Now we can give the following results dealing with the aging of the margins.

Proposition 3.2.6 *Let (X, Y) be an exchangeable non negative random vector. If*

(i) (X, Y) *is BDMRL, and*

(ii) $(X|Y > y) \leq_{icx} X$*,*

then X is NBUC.

Proof. Given that (X, Y) is BDMRL and exchangeable, then, for $y \geq x \geq 0$, we have

$$(X - y|X > y, Y > x) \leq_{icx} (X - x|X > x, Y > y).$$

Now letting $x = 0$, we have the following chain of implications

$$(X - y|X > y) \leq_{icx} (X|Y > y) \leq_{icx} X,$$

where the second inequality follows from condition (ii), therefore X is NBUC. ■

Proposition 3.2.7 *Let (X, Y) be an exchangeable non negative random vector. If*

(i) (X, Y) *is w-BIFR, and*

(ii) $(X|Y > y) \leq_{icv} X$*,*

then X is NBU(2).

Proof. Given that (X, Y) is w-BIFR and exchangeable, then, for $y \geq x \geq 0$, we have

$$(X - y|X > y, Y > x) \leq_{icv} (X - x|X > x, Y > y).$$

Now letting $x = 0$, we have the following chain of implications

$$(X - y|X > y) \leq_{icv} (X|Y > y) \leq_{icv} X,$$

where the second inequality follows from condition (ii), therefore X is NBU(2). ■

Proposition 3.2.8 *Let (X, Y) be an exchangeable non negative random vector. If*

(i) (X, Y) *is BIFR(Lt), and*

(ii) $(X|Y > y) \leq_{Lt} X$,

then X is NBU$_{Lt}$.

Proof. Given that (X, Y) is BIFR(Lt) and exchangeable, then, for $y \geq x \geq 0$, we have

$$(X - y|X > y, Y > x) \leq_{Lt} (X - x|X > x, Y > y).$$

Now letting $x = 0$, we have the following chain of implications

$$(X - y|X > y) \leq_{Lt} (X|Y > y) \leq_{Lt} X,$$

where the second inequality follows from condition (ii), therefore X is NBU$_{Lt}$. ■

To finish with, we provide a result about preservation under mixtures.

Let $\mathbf{\Theta}$ be a random vector, with support $\chi \subseteq \mathbb{R}^n$, and Π its distribution. Let us consider that given $\mathbf{\Theta} = \boldsymbol{\theta}$, X and Y are independent with common survival function $\overline{G}(u \mid \boldsymbol{\theta})$, then the joint survival function of (X, Y), is exchangeable and is given by

$$\int_{\chi} \overline{G}(x \mid \boldsymbol{\theta})\overline{G}(y \mid \boldsymbol{\theta})d\Pi(\boldsymbol{\theta}). \tag{3.1}$$

Proposition 3.2.9 *If $(X|\mathbf{\Theta} = \theta)$ (and $(Y|\mathbf{\Theta} = \theta)$) is DMRL for all $\theta \in \chi$, then (X, Y) is BDMRL.*

Proof. First we observe that given two random variables X and Y, with survival functions $\overline{F}$ and $\overline{G}$, then $X \leq_{icx} Y$ if, and only if, $\int_t^{+\infty} \overline{F}(u)du \leq \int_t^{+\infty} \overline{G}(u)du$, for all t. Therefore and from previous characterization, the result follows if we prove that

$$\int_{t+x}^{\infty} \overline{F}(u, y)du - \int_{t+y}^{\infty} \overline{F}(u, x)du \geq 0 \text{ for all } y \geq x \geq 0 \text{ and for all } t \geq 0.$$

Now from (3.1) and assuming that conditions of Fubini's Theorem hold the previous inequality can be rewritten as

$$\int_{\chi}\left(\overline{G}(y \mid \boldsymbol{\theta})\int_{t+x}^{\infty}\overline{G}(u \mid \boldsymbol{\theta}) - \overline{G}(t_1 \mid \boldsymbol{\theta})\int_{t+y}^{\infty}\overline{G}(u \mid \boldsymbol{\theta})\right) d\Pi(\boldsymbol{\theta}) \geq 0,$$

which follows from the hypothesis and from characterization (ii) in Proposition 1.2.26. ■

3.3 Bivariate aging properties for random vectors with Clayton copulas

As we see have seen in Section 1.3.3, another extensions of some univariate aging properties were provided by Pellerey (2008). In this section, we focus on these bivariate notions.

From now on, let us consider a pair $\mathbf{X} = (X, Y)$ of nonnegative random variables with joint survival function

$$\overline{F}(x, y) = P(X > x, Y > y),$$

and marginal univariate survival functions

$$\overline{G}_X(x) = \overline{F}(x, 0) = P(X > x) \text{ and } \overline{G}_Y(x) = \overline{F}(0, x) = P(Y > x),$$

of X and Y, respectively.

Assume that $\overline{F}$ is a continuous survival function which is strictly decreasing on each argument, and that $\overline{G}_X(0) = \overline{G}_Y(0) = 1$.

Let $\mathbf{X}_t = (X - t, Y - t | X > t, Y > t)$ be the pair of the residual lifetimes at time $t \geq 0$, that is, the pair of nonnegative random variables having joint survival function

$$\overline{F}_t(x, y) = P(X > t + x, Y > t + y | X > t, Y > t) = \frac{\overline{F}(x + t, y + t)}{\overline{F}(t, t)}.$$

As we see in Definition 1.3.24, bivariate generalizations of the IFR and DFR notions can be defined considering the stochastic inequalities

$$\mathbf{X}_{t+s} \leq_{st} [\geq_{st}]\mathbf{X}_t \text{ for all } t, s \geq 0, \tag{3.2}$$

and

$$\mathbf{X}_{t+s} \leq_{lo} [\geq_{lo}]\mathbf{X}_t \text{ for all } t, s \geq 0. \tag{3.3}$$

We will denote with $\mathcal{A}^+_{FR}[\mathcal{A}^-_{FR}]$ the class of bivariate lifetimes that satisfy (3.2), and by $\mathcal{A}^{w+}_{FR}[\mathcal{A}^{w-}_{FR}]$ the class of bivariate lifetimes that satisfy (3.3), where w means "weakly". Also, we consider the corresponding multivariate lack of memory property class, $\mathcal{A}^0$, of bivariate lifetimes such that in (3.2) the equality $=_{st}$ (equality in law) holds for every $t, s \geq 0$.

Recall from Proposition 1.3.8 that when a vector $\mathbf{X} = (X, Y)$ has an Archimedean survival copula, then its joint survival function $\overline{F}$ can be written in the form

$$\overline{F}(x, y) = W(R_X(x) + R_Y(y))$$

for two suitable continuous and strictly increasing functions

$$R_X, R_Y : [0, +\infty) \to [0, +\infty)$$

such that $R_X(0) = R_Y(0) = 0, \lim_{x\to\infty} R_X(x) = \lim_{y\to\infty} R_Y(y) = \infty$ and a suitable one-dimensional, continuous, strictly positive, strictly decreasing and convex survival function $W : \mathbb{R}^+ \to [0, 1]$ such that $W(0) = 1$ called the generator of the copula.

In the case of a random vector with a Clayton copula, that is, when $W(x) = (x + 1)^{-\theta}$ for $\theta \in [0, +\infty)$, the survival function of the random vector is given by

$$\overline{F}(x, y) = (R_X(x) + R_Y(y) + 1)^{-\theta}.$$

Multivariate aging notions, defined by means of stochastic comparisons between $\mathbf{X}$ and $\mathbf{X}_t$, with $t \geq 0$ as in Definition 1.3.24, have been studied in Pellerey (2008), who considered pairs of lifetimes having the same marginal distribution. In this section, we present the generalizations of his results, considering both stochastic comparisons between $\mathbf{X}_t$ and $\mathbf{X}_{t+s}$ for all $t, s \geq 0$, when the dependence structure is given by a Clayton copula.

Observing that $\tilde{\mathbf{X}}_t =_{st} \tilde{\mathbf{X}}_{t+s}$ when the underlying survival copula is of the Clayton type, next we give conditions for $\mathbf{X}$ to satisfy the properties $\mathcal{A}^0$, $\mathcal{A}^+_{FR}$ or $\mathcal{A}^-_{FR}$.

Theorem 3.3.1 *Let $\mathbf{X}$ have joint survival function defined as in (1.22), and let $W(x) = (x+1)^{-\theta}$ for some positive constant θ (i.e., let $\mathbf{X}$ have a Clayton survival copula). Then,*

(i) $\mathbf{X}$ satisfies the weak multivariate lack of memory property $\mathcal{A}^0$ if, and only if, R_X and R_Y are defined as

$$R_X(x) = \frac{e^{b_X x} - 1}{\alpha_X} \text{ and } R_Y(y) = \frac{e^{b_Y y} - 1}{\alpha_Y} \tag{3.4}$$

where α_X, α_Y, b_X and b_Y are strictly positive real numbers satisfying

$$\frac{1}{\alpha_X} + \frac{1}{\alpha_Y} = 1 \text{ and } b_X = b_Y.$$

(ii) $\mathbf{X} \in \mathcal{A}^+_{FR}$ $[\mathcal{A}^-_{FR}]$ if, and only if, the functions R_X and R_Y satisfy

$$\frac{R_X(t+u) + R_Y(t) + 1}{R_X(t) + R_Y(t) + 1} \geq [\leq] \frac{R_X(t+s+u) + R_Y(t+s) + 1}{R_X(t+s) + R_Y(t+s) + 1} \tag{3.5}$$

and

$$\frac{R_X(t) + R_Y(t+u) + 1}{R_X(t) + R_Y(t) + 1} \geq [\leq] \frac{R_X(t+s) + R_Y(t+s+u) + 1}{R_X(t+s) + R_Y(t+s) + 1}. \tag{3.6}$$

for every $t, s, u \geq 0$.

Proof.

(i) Recall that in order to satisfy the condition $\mathbf{X} =_{st} \mathbf{X}_t$ the vectors $\mathbf{X}$ and $\mathbf{X}_t$ should necessarily have the same survival copula (which is unique, being F absolutely continuous). This means that the equality

$$W(W^{-1}(u) + W^{-1}(v)) = \frac{W(R_X(t) + R_Y(t) + W_t^{-1}(u) + W_t^{-1}(v))}{W(R_X(t) + R_Y(t))}$$

should be satisfied for all $u, v \in [0,1]$ and $t \geq 0$, i.e., that K should be a Clayton survival copula (with any positive value for the parameter θ). Moreover, it should also be satisfied $\overline{G}_X(s) = \overline{G}_{X_t}(s)$ and $\overline{G}_Y(s) = \overline{G}_{Y_t}(s)$ for every $s, t \geq 0$. Letting $W(x) = (x+1)^{-\theta}$, these equalities are actually verified only when the functions R_X and R_Y are defined as in (3.4) and $\alpha_X, \alpha_Y, b_X, b_Y$ satisfy the equality in the statement (see Aczél, 1966, for details).

Thus, under these assumptions it holds

$$\begin{aligned}\mathbf{X} =_{st} (\overline{G}_X^{-1}(\tilde{X}), \overline{G}_Y^{-1}(\tilde{Y})) &=_{st} (\overline{G}_X^{-1}(\tilde{X}_t), \overline{G}_Y^{-1}(\tilde{Y}_t)) \\ &=_{a.s.} (\overline{G}_{X_t}^{-1}(\tilde{X}_t), \overline{G}_{Y_t}^{-1}(\tilde{Y}_t)) = \mathbf{X}_t,\end{aligned}$$

i.e., $\mathbf{X} =_{st} \mathbf{X}_t$ for all $t \geq 0$, and also $\mathbf{X}_t =_{st} \mathbf{X}_{t+s}$ for all $t, s \geq 0$. The reversed implication follows observing that such equalities are satisfied only by the functions W, R_X and R_Y defined above.

(ii) We give the proof for the case $\mathbf{X} \in \mathcal{A}_{FR}^+$, the case for $\mathbf{X} \in \mathcal{A}_{FR}^-$ is similar.

Let $t, s, u \geq 0$ and $W(x) = (x+1)^{-\theta}$, with $\theta > 0$, the Clayton survival copula. If we assume (3.5) and (3.6), then

$$\left(\frac{R_X(t+u) + R_Y(t) + 1}{R_X(t) + R_Y(t) + 1}\right)^{-\theta} \leq \left(\frac{R_X(t+s+u) + R_Y(t+s) + 1}{R_X(t+s) + R_Y(t+s) + 1}\right)^{-\theta}$$

and

$$\left(\frac{R_X(t) + R_Y(t+u) + 1}{R_X(t) + R_Y(t) + 1}\right)^{-\theta} \leq \left(\frac{R_X(t+s) + R_Y(t+s+u) + 1}{R_X(t+s) + R_Y(t+s) + 1}\right)^{-\theta},$$

or, equivalently,

$$\overline{G}_{X_t}(u) \leq \overline{G}_{X_{t+s}}(u) \text{ and } \overline{G}_{Y_t}(u) \leq \overline{G}_{Y_{t+s}}(u) \text{ for all } t, s, u \geq 0. \tag{3.7}$$

Therefore, we have that

$$\begin{aligned}\mathbf{X}_{t+s} &=_{st} (\overline{G}_{X_{t+s}}^{-1}(\tilde{X}_{t+s}), \overline{G}_{Y_{t+s}}^{-1}(\tilde{Y}_{t+s})) =_{st} (\overline{G}_{X_{t+s}}^{-1}(\tilde{X}_t), \overline{G}_{Y_{t+s}}^{-1}(\tilde{Y}_t)) \\ &\leq_{a.s.} (\overline{G}_{X_t}^{-1}(\tilde{X}_t), \overline{G}_{Y_t}^{-1}(\tilde{Y}_t)) = \mathbf{X}_t,\end{aligned}$$

for all $s, t \geq 0$ where the second equality in law follows the property of the Clayton survival copula, while the inequality from (3.7). Thus, $\mathbf{X}_{t+s} \leq_{st} \mathbf{X}_t$ which means that $\mathbf{X} \in \mathcal{A}^+_{FR}$.

■

It should be observed that the resulting joint survival function and univariate marginal survival functions of the vector **X** satisfying the assertions of Theorem 3.3.1(i) are, respectively,

$$\overline{F}(x, y) = \left(1 + \frac{e^{bx} - 1}{\alpha_X} + \frac{e^{by} - 1}{\alpha_Y}\right)^{-\theta}, \tag{3.8}$$

$$\overline{G}_X(x) = \left(\frac{e^{bx} + (\alpha_X - 1)}{\alpha_X}\right)^{-\theta} \text{ and } \overline{G}_Y(y) = \left(\frac{e^{by} + (\alpha_Y - 1)}{\alpha_Y}\right)^{-\theta}, \tag{3.9}$$

where $\frac{1}{\alpha_X} + \frac{1}{\alpha_Y} = 1$, with $\theta, b, \alpha_X, \alpha_Y \in \mathbb{R}^+$.

It is interesting also to observe that univariate survival functions defined as in (3.9) become exponential distributions when $\alpha = 1$, are the survival functions of DFR lifetimes when $\alpha > 1$, and, viceversa, of IFR lifetimes when $\alpha < 1$. Since assumption $\frac{1}{\alpha_X} + \frac{1}{\alpha_Y} = 1$ should be satisfied for a bivariate vector **X** to be in the $\mathcal{A}^0$ class, this means that a necessary condition for the weak multivariate lack of memory property is that the two marginal distributions should be DFR.

Examples of bivariate vectors **X** that are in the $\mathcal{A}^-_{FR}$ class can be provided reasoning as in the $\mathcal{A}^0$ case. In fact, we can again take the functions R_X and R_Y defined as in (3.4), always letting $b = b_X = b_Y$ to be any strictly positive real number, but this time assuming that $\frac{1}{\alpha_X} + \frac{1}{\alpha_Y} > 1$. In this case, in fact, inequalities (3.5) and (3.6) are satisfied with $\geq$, as one can verify, and the joint survival function and univariate marginal survival functions of the corresponding vector **X** are as in (3.8) and (3.9), respectively, but with $\frac{1}{\alpha_X} + \frac{1}{\alpha_Y} > 1$.

Recalling what have been said previously regarding the univariate survival functions in (3.9), one can observe that four possible cases can happen when $\frac{1}{\alpha_X} + \frac{1}{\alpha_Y} > 1$: if α_X, $\alpha_Y < 1$, then both margins X and Y are DFR; if $\alpha_X > 1$, $\alpha_Y < 1$ then X is DFR and Y is IFR; if $\alpha_X < 1$ and $\alpha_Y > 1$ then X is DFR and

Y is IFR and, unexpectedly, if both α_X and α_Y are greater than 1 then both the marginals X and Y are univariate IFR, even if **X** is in the $\mathcal{A}^-_{FR}$ class.

Viceversa, in case $\frac{1}{\alpha_X} + \frac{1}{\alpha_Y} < 1$, then the bivariate vector **X** results to be in the $\mathcal{A}^+_{FR}$ class, since in this case both the inequalities $\leq$ are satisfied in (3.5) and (3.6).

This happens, for example, assuming again that R_X and R_Y are defined as in (3.4), and thus again in case that the joint survival function and univariate marginal survival functions of the vector **X** are as in (3.8) and (3.9), but this time letting $\frac{1}{\alpha_X} + \frac{1}{\alpha_Y} < 1$. Under this assumption there is only one possible case for the univariate aging: $\alpha_X, \alpha_Y > 1$, i.e., both margins X and Y are univariate IFR.

These considerations can be somehow generalized to the case that the functions R_X and R_Y are different than the ones in (3.4). However, in this case we should assume the same marginal distribution for X and Y (i.e., letting $R_X = R_Y = R$).

Theorem 3.3.2 *Let* **X** *have joint survival function defined as in (1.22), equal marginal distributions and Clayton survival copula. Let R be such that*

$$R(t+s)R(t+u) \geq R(t)R(t+s+u) \;\; \forall t, s, u \geq 0, \tag{3.10}$$

i.e., let $\frac{R(t+s)}{R(t)}$ be decreasing in t for all $s \geq 0$. Then we have the following assertions,

(i) If the marginals of **X** *are DFR then* **X** *is in the $\mathcal{A}^-_{FR}$ class.*

(ii) If **X** *is in the $\mathcal{A}^+_{FR}$ class then its marginal distributions are IFR.*

Proof.

(i) Let $W(x) = (x+1)^{-\theta}$ for some positive constant θ, and observe that, when $R_X = R_Y = R$, the marginal distributions of **X** are DFR if, and only if,

$$R(t+s+u)R(t) \leq R(t+s)+R(t+u)-R(t+s+u)-R(t)+R(t+s)R(t+u) \tag{3.11}$$

for all $t, s, u \geq 0$. Inequality (3.11) clearly implies

$$\begin{aligned} R(t+s+u)R(t) \;\; &\leq \;\; R(t+s)+R(t+u)-R(t+s+u)-R(t)+ \\ &\quad +R(t+s)R(t+u)+[R(t+s)R(t+u)-R(t)R(t+s+u)] \end{aligned}$$

for all $t, s, u \geq 0$, which, in turns, is equivalent to the inequalities (3.5) and (3.6) with $\geq$. Thus, assertion (i) is proved.

(ii) Viceversa, let **X** be in the class $\mathcal{A}^{+}_{FR}$, i.e., let

$$\begin{aligned} R(t+s+u)R(t) \quad \geq \quad & R(t+s)+R(t+u)-R(t+s+u)-R(t)+ \\ & +R(t+s)R(t+u)+[R(t+s)R(t+u)-R(t)R(t+s+u)] \end{aligned}$$

for all $t, s, u \geq 0$ (this follows from (3.5) and (3.6)). Under assumption (3.10) this inequality implies

$$R(t+s+u)R(t) \geq R(t+s)+R(t+u)-R(t+s+u)-R(t)+R(t+s)R(t+u)$$

for all $t, s, u \geq 0$, which, in turns, implies that the marginals of **X** are IFR.

■

Next we give an example where Theorem 3.3.2 can be applied.

Example 3.3.3 *Let H be any absolutely continuous univariate cumulative distribution, with corresponding density h and survival function $\overline{H}$, and let*

$$R(t) = [\overline{H}(t)]^{-1} - 1,$$

so that the marginals of **X** *have the same survival distribution $\overline{G}(t) = W(R(t)) = (R(t)+1)^{-\theta} = \overline{H}(t)^{\theta}$. Referring to univariate distributions, this is what in the literature is usually called proportional hazard model or Lehman's alternative. We have here a bivariate generalization where the proportionality factor is given by the parameter θ that describes the degree of dependence between* **X** *and* **Y**.

In this case the assumption $R(t+s)/R(t)$ decreasing in t for all $s \geq 0$ means that

$$\frac{H(t+s)\overline{H}(t)}{\overline{H}(t+s)H(t)}$$

is decreasing in t for all $s \geq 0$. By deriving, it is equivalent to

$$\frac{h(t)}{\overline{H}(t)}\frac{1}{H(t)} \text{ decreasing in } t.$$

Since $1/H(t)$ is always decreasing, the assumption on R is satisfied if $h(t)/\overline{H}(t)$ is decreasing, i.e., if H is DFR. Thus, for this model, if the underlying distribution H is DFR then it follows that the random vector $\mathbf{X}$ is in the $\mathcal{A}^-_{FR}$ class.

It is interesting to observe that the same statement does not hold for positive aging. In fact, letting $\overline{H}(x) = \frac{\alpha}{e^{bx}+\alpha-1}$, i.e., $R(x) = \frac{e^{bx}-1}{\alpha}$, we have a counterexample where H is IFR but $\mathbf{X}$ is in the $\mathcal{A}^-_{FR}$ (as shown before).

3.4 Negative aging for frailty models

The frailty approach is commonly used in reliability theory and survival analysis to model the dependence between subjects or components; according to this model the frailty (an unobservable random variable that describes environmental factors) acts simultaneously on the hazard functions of the lifetimes.

Recall that a vector $\mathbf{X}_k = (X_{k,1}, \ldots, X_{k,n})$ of non independent lifetimes is said to be described by a multivariate frailty model if its joint survival function is defined as

$$\overline{F}_k(t_1, \ldots, t_n) = \mathbb{P}[X_{k,1} > t_1, \ldots, X_{k,n} > t_n] = \mathbf{E}\left[\left(\Pi_{i=1}^n \overline{G}_{k,i}(t_i)\right)^{\Theta_k}\right],\ t_i \in \mathbb{R}^+,$$

where Θ_k is an environmental random frailty taking values in $\mathbb{R}^+$ and $\overline{G}_{k,i}$ is the survival function of lifetime $X_{k,i}$ given $\Theta_k = 1$, for further details see Section 1.3.

In this section, we provide sufficient conditions for the stochastic comparisons $\mathbf{X}_{i,\mathbf{t}} \leq \mathbf{X}_{i,\mathbf{t+v}}$ where $\mathbf{t} = (t_1, \ldots, t_n)\mathbb{R}^n$ and $\mathbf{v}$ is a vector with nonnegative components.

In order to prove the main results, we need to introduce a theoretical tool which is the n–Laplace ratio orders.

Definition 3.4.1 *Let Θ_1 and Θ_2 be two nonnegative random variables. Then, Θ_2 is said to be smaller than Θ_1 in the n–**Laplace transform ratio order**, denoted by $\Theta_2 \leq_{n-Lt-r} \Theta_1$, with $n \in \mathbb{N}^+$, if*

$$\frac{\mathbf{E}[\Theta_1^{n-1}\exp(-s\Theta_1)]}{\mathbf{E}[\Theta_2^{n-1}\exp(-s\Theta_2)]} \text{ is decreasing in } s \in \mathbb{R}^+.$$

Note that these orders do not imply, nor are implied by, the usual stochastic order, and that $\Theta_2 \leq_{n-Lt-r} \Theta_1$ holds if, and only if,

$$\frac{W_1^{(n-1)}(s)}{W_2^{(n-1)}(s)} \text{ is decreasing in } s,$$

where

$$W_k(s) = \mathbf{E}[\exp(-s\Theta_k)] = \int_0^\infty \exp(-s\theta)dH_k(\theta),\ s \in \mathbb{R}^+, \tag{3.12}$$

where $W_k^{(n-1)}$ is the derivative of order $n-1$ of W_k (with $W_k^{(0)} = W_k$) and H_k is the cumulative distribution of Θ_k, $k = 1, 2$.

The n–Laplace transform ratio orders have been never considered before in the literature in general. However, the particular case of the 1–Laplace transform ratio order is equivalent to the order Laplace transform ratio order defined and studied in Shaked and Wong (1997) and further considered in Bartoszewicz (1999), who derived some of its characterizations and established inequalities for negative moments of ordered random variables. Also,the 2–Laplace transform ratio order is the same as the differentiated Laplace transform ratio order defined in Li et al.(2009), where a complete study on its properties and applications is provided. Like the orders mentioned above, the n–Laplace transform ratio orders do not imply the usual stochastic order. To prove it,it suffices to consider the variables Θ_1 and Θ_2 having discrete densities f_{Θ_1} and f_{Θ_1}, respectively, defined as

$$f_{\Theta_1}(t) = \begin{cases} 0.2 & \text{if } t = 1 \\ 0.4 & \text{if } t = 2 \\ 0.4 & \text{if } t = 2.9 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_{\Theta_2}(t) = \begin{cases} 0.3 & \text{if } t = 1 \\ 0.4 & \text{if } t = 2 \\ 0.3 & \text{if } t = 3 \\ 0 & \text{otherwise.} \end{cases}$$

With some straightforward calculation it is easy to verify that $\Theta_2 \leq_{2-Lt-r} \Theta_1$, while the usual stochastic order between Θ_1 and Θ_2 is not satisfied, since their survival functions do intersect. Moreover, the usual stochastic order does not imply the n–Laplace transform–likelihood ratio orders, since it does not imply the $\leq_{Lt-r}$ order (see Shaked and Wong, 1997).

A second example of variables that are ordered in the 2–Laplace transform ratio order but not in usual stochastic order is for $\Theta_1 \sim U[0,3]$ and $\Theta_2 \sim U[1,2]$. These two variables are not ordered in usual stochastic order because their survival functions do intersect (and neither are ordered in the stronger likelihood ratio order, as one can verify), however, it holds $\Theta_2 \leq_{2-Lt-r} \Theta_1$ being the ratio

$$\frac{W_1^{(1)}(s)}{W_2^{(1)}(s)} = \frac{-3e^{-3s}s + (1 - e^{-3s})}{3[(e^{-s} - 2e^{-2s})s + (e^{-s} - e^{-2s})]} \text{ decreasing in } s \geq 0.$$

An example where two nonnegative variables are ordered in n–Laplace transform ratio order for every value of n is described in the following proposition. Here, $Ga(\alpha, \lambda)$ denotes the gamma distribution with shape parameter α and scale parameter λ.

Proposition 3.4.2 *Let* $\Theta_1 \sim Ga(\alpha_1, \lambda_1)$ *and* $\Theta_2 \sim Ga(\alpha_2, \lambda_2)$. *Then* $\Theta_2 \leq_{n-Lt-r} \Theta_1$ *for every* $n \geq 0$ *whenever* $\alpha_1 \geq \alpha_2$ *and* $\lambda_1 \leq \lambda_2$.

Proof. It is well known that if $\Theta \sim Ga(\alpha, \lambda)$ its associated Laplace transform is given by

$$W(s) = \lambda^{\alpha}(\lambda + s)^{-\alpha}$$

and its derivative of order n is given by

$$W^{(n-1)}(s) = (-1)^{n-1}\frac{(\alpha + n - 1)!}{(\alpha - 1)!}\lambda^{\alpha}(\lambda + s)^{\alpha+n-1} = -(\alpha + n - 1)(\lambda + s)^{-1}W^{(n-2)}(s)$$

Therefore,

$$\frac{W_1^{(n-1)}(s)}{W_2^{(n-1)}(s)} = C_{\lambda_1,\lambda_2,\alpha_1,\alpha_2,n}\frac{(\lambda_1 + s)^{-\alpha_1-n}}{(\lambda_2 + s)^{-\alpha_2-n}} = C_{\lambda_1,\lambda_2,\alpha_1,\alpha_2,n}(\lambda_2 + s)^{\alpha_2-\alpha_1}(\frac{\lambda_2 + s}{\lambda_1 + s})^{n+\alpha_1}$$

where $C_{\lambda_1,\lambda_2,\alpha_1,\alpha_2,n}$ does not depend on s. It is easy to see that this ratio is decreasing in s if, and only if, $\alpha_1 \geq \alpha_2$ and $\lambda_1 \leq \lambda_2$. ∎

Two preliminary results related with the n–Laplace transform ratio orders are needed along this section. The proof of the first two of them easily follows from standard Total Positivity techniques (see Karlin, 1968, for definitions, main properties and details on Total Positivity theory).

Lemma 3.4.3 *Let the survival functions W_k, with $k = 1, 2$, be defined as in* (3.12). *Then,*

$$\frac{W_k^{(n-1)}(s+z)}{W_k^{(n-1)}(s)} \text{ is increasing in } s \in \mathbb{R}^+$$

for every $z \in \mathbb{R}^+$ and $n \geq 1$.

Proof. First observe that the assertion holds if, and only if, for every $n \geq 1$,

$$\frac{W_k^{(n-1)}(s)}{W_k^{(n)}(s)} \text{ is increasing in } s \in \mathbb{R}^+ \tag{3.13}$$

Let us denote

$$W_k^{(n)}(s) = \int_0^\infty a(n, \theta) b(s, \theta) dH_k(\theta),$$

where $a(n, \theta) = \theta^n$ and $b(s, \theta) = \exp(-s\theta)$. It is easy to verify that $a(n, \theta)$ is TP_2 (*totally positive of order 2*), while $b(s, \theta)$ is RR_2 (*reverse regular of order 2*). Thus by the Basic Composition Formula it follows that $W_k^{(n)}(s)$ is RR_2, i.e., that the ratio (3.13) is increasing in s. ■

The second preliminary result describes the relationships among the n–Laplace transform ratio orders.

Lemma 3.4.4 *Let $\Theta_2 \leq_{n-Lt-r} \Theta_1$. Then, $\Theta_2 \leq_{i-Lt-r} \Theta_1$ for every $i = 1, 2, \ldots, n$, and, in particular,*

$$\mathbf{E}[\exp(-s\Theta_2)] \geq \mathbf{E}[\exp(-s\Theta_1)].$$

Proof. Again using the Basic Composition Formula it is easy to verify that when the ratio $W_1^i(s)/W_2^i(s)$ is decreasing, then

$$\frac{\int_s^\infty W_1^{(i)}(z) dz}{\int_s^\infty W_2^{(i)}(z) dz} = \frac{W_1^{(i-1)}(s)}{W_2^{(i-1)}(s)}$$

is decreasing in s.

In particular, $\frac{W_1(s)}{W_2(s)}$ is decreasing in s, thus

$$1 = \frac{W_1(0)}{W_2(0)} \geq \frac{W_1(s)}{W_2(s)} = \frac{\mathbf{E}[\exp(-s\Theta_1)]}{\mathbf{E}[\exp(-s\Theta_2)]}$$

■

In the literature, one can find several characterizations of aging notions of univariate nonnegative variables by means of stochastic comparisons between the residual lifetimes $X_t = (X - t \big| X > t)$ (see, e.g, Barlow and Proschan, 1975). Among others, the following negative aging notion is well known: the random lifetime X is said to be **decreasing in failure rate**, denoted by DFR, if, and only if,

$$X_t \leq_{st} X_{t+v} \text{ for all } t, v \geq 0.$$

As a natural multivariate generalization of the DFR notion, one can in fact consider the stochastic inequality

$$\mathbf{X_t} \leq_{st} \mathbf{X_{t+v}} \tag{3.14}$$

and says that the vector of lifetimes **X** is *multivariate DFR* if (3.14) holds for all nonnegative vectors **t** and **v**.

Using arguments similar to those in the proof of Theorem 4.4.2 in Chapter 4, it is possible to prove also the following result, which describes conditions for this notion of negative multivariate aging. Note that here the vector $\mathbf{X}_1$ does not need to have nonnegative components.

Theorem 3.4.5 *Let* $\mathbf{X}_1$ *be an* n*–dimensional random vector having joint survival function defined as in* (1.25)*. Then,* $\mathbf{X}_{1,\mathbf{t}} \leq_{st} \mathbf{X}_{2,\mathbf{t+v}}$ *holds for every* $\mathbf{t} = (t_1, \ldots, t_n)$ *and every nonnegative* $\mathbf{v} = (v_1, \ldots, v_n)$ *if, for all* $i = 1, \ldots, n$*, the variable* $\tilde{X}_{1,i}$ *has decreasing hazard rate, i.e., if* $\tilde{X}_{1,i,t_i} \leq_{st} \tilde{X}_{1,i,t_i+v_i}$ *for all* $t_i \in \mathbb{R}$ *and* $v_i \in \mathbb{R}^+$.

Proof. Let $\mathbf{u} = (u_1, \ldots, u_n)$ be any vector of non-negative components. Note that, as shown in the proof of Theorem 4.4.2,

$$\overline{F}_{\mathbf{X}_{1,\mathbf{t}}}(\mathbf{u}) = \mathbf{E}\left[\left(\Pi_{i=1}^n \overline{G}_{1,i,t_i}(u_i)\right)^{\widetilde{\Theta}_\mathbf{t}}\right] \text{ and } \overline{F}_{\mathbf{X}_{1,\mathbf{t+v}}}(\mathbf{u}) = \mathbf{E}\left[\left(\Pi_{i=1}^n \overline{G}_{1,i,t_i+v_i}(u_i)\right)^{\widetilde{\Theta}_\mathbf{t+v}}\right]$$

where $\widetilde{\Theta}_\mathbf{t}$ and $\widetilde{\Theta}_\mathbf{t+v}$ have distribution $\widetilde{H}_\mathbf{t}$ and $\widetilde{H}_\mathbf{t+v}$, respectively, defined as

$$\widetilde{H}_\mathbf{t}(\theta) = \frac{\int_0^\theta \exp\{\tau[\sum_{j=1}^n \ln \overline{G}_{1,j}(t_j)]\} dH_\mathbf{t}(\tau)}{\int_0^\infty \exp\{\tau[\sum_{j=1}^n \ln \overline{G}_{1,j}(t_j)]\} dH_k(\tau)}$$

and

$$\widetilde{H}_{\mathbf{t+v}}(\theta) = \frac{\int_0^\theta \exp\{\tau[\sum_{j=1}^n \ln \overline{G}_{1,j}(t_j + v_j)]\} dH_{\mathbf{t+v}}(\tau)}{\int_0^\infty \exp\{\tau[\sum_{j=1}^n \ln \overline{G}_{1,j}(t_j + v_j)]\} dH_k(\tau)}.$$

Thus,

$$\mathbf{E}[\exp(-s\widetilde{\Theta}_{\mathbf{t}})] = \frac{\mathbf{E}[\exp(-(s+\widetilde{t})\Theta_1)]}{\mathbf{E}[\exp(-\widetilde{t}\Theta_1)]}$$

and

$$\mathbf{E}[\exp(-s\widetilde{\Theta}_{\mathbf{t+v}})] = \frac{\mathbf{E}[\exp(-(s+\widetilde{t}_v)\Theta_1)]}{\mathbf{E}[\exp(-\widetilde{t}_v\Theta_1)]},$$

where $\widetilde{t} = -\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)$ and $\widetilde{t}_v = -\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j + v_j)$.

Let us denote with

$$\widetilde{\widetilde{W}}_{1,\mathbf{t}}(s) = \mathbf{E}[\exp(-s\widetilde{\Theta}_{\mathbf{t}})] \text{ and } \widetilde{\widetilde{W}}_{1,\mathbf{t+v}}(s) = \mathbf{E}[\exp(-s\widetilde{\Theta}_{\mathbf{t+v}})]$$

the Laplace transforms of $\widetilde{H}_{\mathbf{t}}$ and $\widetilde{H}_{\mathbf{t+v}}$, respectively.

It holds

$$\frac{\widetilde{\widetilde{W}}_{1,\mathbf{t}}(s)}{\widetilde{\widetilde{W}}_{1,\mathbf{t+v}}(s)} = \frac{\mathbf{E}[\exp(-\widetilde{t}_v\Theta_1)]}{\mathbf{E}[\exp(-\widetilde{t}\Theta_1)]} \cdot \frac{W_{1,\mathbf{t}}(s+\widetilde{t})}{W_{1,\mathbf{t}}(s+\widetilde{t}_v)}.$$

It is easy to verify that the ratio $\frac{W_{1,\mathbf{t}}^{(n-1)}(s+\widetilde{t})}{W_{1,\mathbf{t}}^{(n-1)}(s+\widetilde{t}_v)}$ is decreasing in s because of Lemma 3.4.3 and inequality $\widetilde{t} \leq \widetilde{t}_v$. Thus $\widetilde{\Theta}_{\mathbf{t+v}} \leq_{n-Lt-r} \widetilde{\Theta}_{\mathbf{t}}$.

Moreover, from the assumption on the margins $[X_{1,i}|\ \Theta_1 = 1]$ easily follows that $[X_{1,i,t_i}|\widehat{\Theta}_1 = 1] \leq_{st} [X_{1,i,t_i+v_i}|\widehat{\Theta}_1 = 1]$, i.e., that $\overline{G}_{1,i,t_i}(u_i) \leq_{st} \overline{G}_{1,i,t_i+v_i}(u_i)$ for all $u_i \in \mathbb{R}^+$ and $i = 1, \ldots, n$.

Thus, the assertion follows applying Theorem 4.4.1. ■

This result is not surprising, in particular if compared with similar conditions reported in literature for other notions of negative multivariate aging (see, e.g., Spizzichino and Torrisi, 2001).

4

Stochastic comparisons for some particular cases of Archimedean copulas

Abstract. In this chapter, we will provide conditions to compare, in different stochastic ways, two bivariate lifetimes $\mathbf{X} \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ and $\mathbf{Y} \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$. These conditions are essentially based on stochastic comparisons between $W_{\mathbf{X}}$ and $W_{\mathbf{Y}}$ (or, better, between the univariate variables X^* and Y^* having survival functions $W_{\mathbf{X}}$ and $W_{\mathbf{Y}}$, respectively). Thus, in contrast to the results presented in Müller and Scarsini (2001), we will provide here simple conditions for the stochastic comparison between bivariate lifetimes having different copulas. Some examples of applications will be also provided.

4.1 Introduction

In this chapter, we will consider two different bivariate vectors of lifetimes, $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$, and we provide conditions for comparisons of their residual bivariate lifetimes $\mathbf{X}_t$ and $\mathbf{Y}_t$, i.e.,

$$\mathbf{X}_t = (X_1 - t, X_2 - t | X_1 > t, X_2 > t) \text{ and } \mathbf{Y}_t = (Y_1 - t, Y_2 - t | Y_1 > t, Y_2 > t).$$

Even for the stochastic inequalities between residual lifetimes, it is of course possible to consider multivariate generalizations. For example, one can be interested in comparisons between the residual lifetimes of two vectors of lifetimes $\mathbf{X}$

and **Y** of the kind

$$\mathbf{X}_t \leq_{st} \mathbf{Y}_t \text{ for all } t \geq 0. \tag{4.1}$$

It is important to note that if this comparison holds, then $\phi(\mathbf{X}_t) \leq_{st} \phi(\mathbf{Y}_t)$ for all increasing function $\phi : \mathbb{R}^n \to \mathbb{R}$. This property is the key of the main applications of the comparisons of the residual lifetimes. For example, inequalities as in (4.1) can be used to compare the lifetimes of systems built using used components, that is, if (4.1) holds, then $\tau(\mathbf{X}_t) \leq_{st} \tau(\mathbf{Y}_t)$ for every coherent life function τ and for all $t \geq 0$.

Also given that $(X_1 > t_1, \ldots, X_n > t_n) = (\min\{X_i : i = 1, \ldots, n\})$, previous inequalities can be used to provide comparisons for a coherent system based on two different sets of components when we have additional (survival) information about the first failure of the components.

In insurance theory, they can be obviously used to compare the residual lifetimes of two pairs of ensured persons when the assumption of independence in the couples does not apply.

First, we will consider random vectors with Clayton copula. These results are included in Mulero and Pellrey (2010). Recall that if **X** is a random vector with a Clayton copula, then its survival function can be written as

$$\overline{F}(x, y) = W(R_X(x) + R_Y(y))$$

where

$$W(x) = (x + 1)^{-\theta} \text{ for } \theta \in [0, +\infty),$$

and $R_X, R_Y : [0, +\infty) \to [0, +\infty)$ are two suitable continuous and strictly increasing functions such that $R_X(0) = R_Y(0) = 0$ and $\lim_{x\to\infty} R_X(x) = \lim_{y\to\infty} R_Y(y) = \infty$.

In this case, it is important to take into account that for a random vector with Clayton copula, it holds the equality in law between $\tilde{\mathbf{X}}_{t_1}$ and $\tilde{\mathbf{X}}_{t_2}$. In fact, this equality holds if, and only if, the structure of dependence is given by a Clayton copula.

Secondly, we consider random vectors defined via a Time Transformed Exponential model, denoted by $\mathbf{X} \sim TTE(W, R)$, that is, when the joint survival function $\overline{F}$ can be written as

$$\overline{F}(t,s) = W(R(t) + R(s)),\ t,s \geq 0, \tag{4.2}$$

for further details see Section 1.3. These results are included in Mulero et al. (2010b).

Note that, as a particular case of property (1.23), given $\mathbf{X} \sim TTE(W,R)$ and considering the vector $\mathbf{X}_t = [(X_1 - t, X_2 - t)|X_1 > t, X_2 > t]$ of the residual lifetimes at time $t \geq 0$, then $\mathbf{X}_t \sim TTE(W_t, R_t)$, i.e., it has joint survival function $\overline{F}_t(x,y)$ given by

$$\overline{F}_t(x,y) = W_t(R_t(x) + R_t(y))$$

where

$$W_t(x) = \frac{W(2R(t)+x)}{W(2R(t))}, \tag{4.3}$$

and where

$$R_t(x) = R(t+x) - R(t), \tag{4.4}$$

for $t, x \geq 0$.

Thus, the survival copula of $\mathbf{X}_t$ is defined by

$$K_t(u,v) = W_t(W_t^{-1}(u) + W_t^{-1}(v)),\ u,v \in [0,1],$$

where W_t is defined as in (4.3).

4.2 Stochastic comparisons for random vectors with Clayton copula

In this section we will assume that the dependence structure of both the random pairs is described by Clayton survival copulas.

Next, we give a result assuming the exchangeability for the X_i and for the Y_i, i.e., $R_{X_1} = R_{X_2} = R_{\mathbf{X}}$ and $R_{Y_1} = R_{Y_2} = R_{\mathbf{Y}}$.

Moreover, in Theorem 4.2.1 (i), we assume that $\mathbf{X}$ and $\mathbf{Y}$ have the same Clayton survival copula, so obtaining a comparison in usual stochastic order sense between $\mathbf{X}_t$ and $\mathbf{Y}_t$, while in Theorem 4.2.1 (ii), we remove such assumption, so obtaining the weaker comparison in upper orthant order.

Theorem 4.2.1 *Let $\boldsymbol{X} = (X_1, X_2)$ and $\boldsymbol{Y} = (Y_1, Y_2)$ be two bivariate random vectors vectors having joint survival function defined as in* (1.22) *with Clayton copulas parametrized by $\theta_{\mathbf{X}}$ and $\theta_{\mathbf{Y}}$, respectively. Moreover, let $R_{X_1} = R_{X_2} = R_{\mathbf{X}}$ and $R_{Y_1} = R_{Y_2} = R_{\mathbf{Y}}$, that is, $\mathbf{X}$ and $\mathbf{Y}$ are exchangeable and let F and G be the distribution functions of the margins of $\mathbf{X}$ and $\mathbf{Y}$, respectively.*

(i) If $\theta_{\mathbf{X}} = \theta_{\mathbf{Y}}$, $R'_{\mathbf{Y}}(0) \leq R'_{\mathbf{X}}(0)$ and the ratio $R'_{\mathbf{Y}}(t)/R'_{\mathbf{X}}(t)$ is a decreasing function in t, then $\boldsymbol{X}_t \leq_{st} \boldsymbol{Y}_t$ for all $t \geq 0$.

(ii) If $\theta_{\mathbf{X}} \geq \theta_{\mathbf{Y}}$, $R'_{\mathbf{Y}}(0) \leq R'_{\mathbf{X}}(0)$ and the ratio $R'_{\mathbf{Y}}(t)/R'_{\mathbf{X}}(t)$ is a decreasing function in t, then $\boldsymbol{X}_t \leq_{uo} \boldsymbol{Y}_t$ for all $t \geq 0$.

Proof. Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be two bivariate and exchangeable random vectors as described in the hypothesis.

(i) Let $\overline{F}$ and $\overline{G}$ be the survival functions of X_i and Y_i, respectively, for $i = 1, 2$. It is not hard to verify that when $W(x) = (x+1)^{-\theta}$ for some positive constant θ, then $\overline{F}_t(u) \leq \overline{G}_t(u)$ for all $t, u \geq 0$ if, and only if, $A(t,s) + 2B(t,s) \geq 0$, where

$$A(t,s) = [R_{\mathbf{X}}(t+s) - R_{\mathbf{X}}(t)] - [R_{\mathbf{Y}}(t+s) - R_{\mathbf{Y}}(t)]$$

and

$$B(t,s) = R_{\mathbf{Y}}(t)R_{\mathbf{X}}(t+s) - R_{\mathbf{X}}(t)R_{\mathbf{Y}}(t+s).$$

Since $R'_{\mathbf{Y}}(t)/R'_X(t)$ is a decreasing function in t, by the Basic Composition Formula (see Karlin, 1968), it follows that also $R_{\mathbf{Y}}(t)/R_{\mathbf{X}}(t)$ is a decreasing function in t and therefore that $B(t,s) \geq 0$. Moreover, if $R'_{\mathbf{Y}}(0) \leq R'_{\mathbf{X}}(0)$ and $R'_{\mathbf{Y}}(t)/R'_{\mathbf{X}}(t)$ is a decreasing function in t then $R'_{\mathbf{Y}}(t) \leq R'_{\mathbf{X}}(t)$ for any t, and therefore $A(t,s) \geq 0$. Thus $\overline{F}_t(u) \leq \overline{G}_t(u)$ holds for all $t, u \geq 0$.

It follows

$$\begin{aligned}\mathbf{X}_t &=_{st} (\overline{F}_t^{-1}(U_1^t), \overline{F}_t^{-1}(U_2^t)) =_{st} (\overline{F}_t^{-1}(U_1), \overline{F}_t^{-1}(U_2)) \\ &\leq_{a.s.} (\overline{G}_t^{-1}(U_1), \overline{G}_t^{-1}(U_2)) =_{st} (\overline{G}_t^{-1}(U_1^t), \overline{G}_t^{-1}(U_2^t)) = \mathbf{Y}_t,\end{aligned}$$

where $\mathbf{U} = (U_1, U_2)$ is the vector having as joint distribution the Clayton survival copula, and $\mathbf{U}_t = (U_1^t, U_2^t) =_{st} \mathbf{U}$ is the vector having as distribution the common survival copula of $\mathbf{X}_t$ and $\mathbf{Y}_t$. The assertion follows.

(ii) As shown in (i), from the assumption on $R_{\mathbf{X}}$ and $R_{\mathbf{Y}}$ it follows that $A(t,s) + 2B(t,s) \geq 0$, which can be written as

$$\left(\frac{1 + R_{\mathbf{X}}(t) + R_{\mathbf{X}}(t+s)}{1 + 2R_{\mathbf{X}}(t)}\right)^{-1} \leq \left(\frac{1 + R_{\mathbf{Y}}(t) + R_{\mathbf{Y}}(t+s)}{1 + 2R_{\mathbf{Y}}(t)}\right)^{-1}.$$

Thus,

$$\begin{aligned} \overline{F}_t(s) &= \left(\frac{1 + R_{\mathbf{X}}(t) + R_{\mathbf{X}}(t+s)}{1 + 2R_{\mathbf{X}}(t)}\right)^{-\theta_X} \leq \left(\frac{1 + R_{\mathbf{Y}}(t) + R_{\mathbf{Y}}(t+s)}{1 + 2R_{\mathbf{Y}}(t)}\right)^{-\theta_{\mathbf{X}}} \\ &\leq \left(\frac{1 + R_{\mathbf{Y}}(t) + R_{\mathbf{Y}}(t+s)}{1 + 2R_{\mathbf{Y}}(t)}\right)^{-\theta_{\mathbf{Y}}} = \overline{G}_t(s). \end{aligned}$$

Moreover, from $\theta_{\mathbf{X}} \geq \theta_{\mathbf{Y}}$ it follows that $K_{\theta_{\mathbf{X}}} \leq_{PQD} K_{\theta_{\mathbf{Y}}}$, which is equivalent to $\tilde{X} \leq_{PQD} \tilde{Y}$. So we have, for any $t \geq 0$,

$$\begin{aligned} \mathbf{X}_t &= (\overline{F}_t^{-1}(\tilde{X}_{1,t}), \overline{F}_t^{-1}(\tilde{X}_{2,t})) =_{st} (\overline{F}_t^{-1}(\tilde{X}_1), \overline{F}_t^{-1}(\tilde{X}_2)) \\ &\leq_{PQD} (\overline{F}_t^{-1}(\tilde{Y}_1), \overline{F}_t^{-1}(\tilde{Y}_2)) \leq_{a.s.} (\overline{G}_t^{-1}(\tilde{Y}_1), \overline{G}_t^{-1}(\tilde{Y}_2)) \\ &=_{st} (\overline{G}_t^{-1}(\tilde{Y}_t), \overline{G}_t^{-1}(\tilde{Y}_t)) = \mathbf{Y}_t. \end{aligned}$$

The assertion follows.

■

Example 4.2.2 *An example where the assumptions of Theorem 4.2.1 (i) are satisfied is when* $R_{\mathbf{X}}(t) = \log(1 + b_{\mathbf{X}}t)$*,* $R_{\mathbf{Y}}(t) = \log(1 + b_{\mathbf{Y}}t)$ *with* $b_{\mathbf{X}} \leq b_{\mathbf{Y}}$*. In the context of frailty models, for example, this means that if* $\mathbf{X}$ *and* $\mathbf{Y}$ *are such that* $\overline{F}_{\mathbf{X}}(t,s) = \mathbf{E}[\overline{H}_{\mathbf{X}}(t)^{\Theta}\overline{H}_{\mathbf{X}}(s)^{\Theta}]$ *and* $\overline{F}_{\mathbf{Y}}(t,s) = \mathbf{E}[\overline{H}_{\mathbf{Y}}(t)^{\Theta}\overline{H}_{\mathbf{Y}}(s)^{\Theta}]$*, where* Θ *is Gamma distributed,* $\overline{H}_{\mathbf{X}}(t) = 1/(1 + b_{\mathbf{X}}t)$ *and* $\overline{H}_{\mathbf{Y}}(t) = 1/(1 + b_Y t)$*, then* $\mathbf{X}_t \leq_{st} \mathbf{Y}_t$ *for all* $t \geq 0$ *whenever* $0 \leq b_{\mathbf{X}} \leq b_{\mathbf{Y}}$*.*

Next, we provide conditions for $\mathbf{X}_t \leq_{st} \mathbf{Y}_t$ for all $t \geq 0$ in the case of different margins.

Theorem 4.2.3 *Let* $\boldsymbol{X} = (X_1, X_2)$ *and* $\boldsymbol{Y} = (Y_1, Y_2)$ *be two bivariate random vectors vectors having joint survival function defined as in* (1.22) *with Clayton copulas parametrized by* $\theta_{\mathbf{X}}$ *and* $\theta_{\mathbf{Y}}$*, respectively.*

If $\theta_{\mathbf{X}} = \theta_{\mathbf{Y}}$ *and, for every* $t, s \geq 0$*, it holds*

(i) $(X_i - t\,|(X_1, X_2) > (t,t)) \leq_{st} (Y_i - t\,|(Y_1, Y_2) > (t,t))$*,* $i = 1, 2$*,*

(ii)

$$\begin{vmatrix} R_{X_1}(t+s) - R_{X_1}(t) & R_{Y_1}(t+s) - R_{Y_1}(t) \\ & \\ R_{X_2}(t) & R_{Y_2}(t) \end{vmatrix} \geq 0, \tag{4.5}$$

(iii)

$$\begin{vmatrix} R_{X_2}(t+s) - R_{X_2}(t) & R_{Y_2}(t+s) - R_{Y_2}(t) \\ & \\ R_{X_1}(t) & R_{Y_1}(t) \end{vmatrix} \geq 0, \tag{4.6}$$

then $\boldsymbol{X}_t \leq_{st} \boldsymbol{Y}_t$ *for every fixed* $t \geq 0$*.*

Proof. Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be two bivariate vectors having joint survival function defined as in (1.22), and let the survival copulas of $\mathbf{X}$ and $\mathbf{Y}$ be Clayton copulas parametrized by $\theta_{\mathbf{X}}$ and $\theta_{\mathbf{Y}}$, respectively.

Let us consider the univariate residual lifetimes $X_{i,t} = (X_i - t\,|(X_1, X_2) > (t,t))$ and $Y_{i,t} = (Y_i - t\,|(Y_1, Y_2) > (t,t))$ of the marginal random variables X_i and Y_i, and let $\overline{F}_t^{X_i} = \frac{W(R_{X_i}(t+s))}{W(R_{X_i}(t))}$ and $\overline{G}_t^{Y_i} = \frac{W(R_{Y_i}(t+s))}{W(R_{Y_i}(t))}$ be their survival functions, for $i = 1, 2$. It is easy to verify that for every fixed t and $i = 1, 2$ it holds $X_{i,t} \leq_{st} Y_{i,t}$ if, and only if,

$$R_{X_i}(t+s) - R_{X_i}(t) - (R_{Y_i}(t+s) - R_{Y_i}(t)) + R_{X_i}(t+s)R_{Y_i}(t) - R_{X_i}(t)R_{Y_i}(t+s) \geq 0 \tag{4.7}$$

for any $s \geq 0$.

On the other hand, let $\mathbf{X}_t = (X_1 - t, X_2 - t | X_1 > t, X_2 > t)$ be the vector of the residual lifetimes of $\mathbf{X}$ at time $t \geq 0$. Its marginal residual lifetimes $X_{t,1}$

and $X_{t,2}$ have the survival functions $\overline{G}_1^{X_t}(s) = \frac{W(R_{X_1}(t+s)+R_{X_2}(t))}{W(R_{X_1}(t)+R_{X_2}(t))}$ and $\overline{G}_2^{X_t}(s) = \frac{W(R_{X_1}(t)+R_{X_2}(t+s))}{W(R_{X_1}(t)+R_{X_2}(t))}$, respectively.

Similarly, let us call $\overline{G}_1^{Y_t}$ and $\overline{G}_2^{Y_t}$ the survival functions of $Y_{t,1}$ and $Y_{t,2}$, which are the marginals of $\mathbf{Y}_t$.

For every $t, s \geq 0$, it holds $\overline{G}_1^{X_t}(s) \leq \overline{G}_1^{Y_t}(s)$ and $\overline{G}_2^{X_t}(s) \leq \overline{G}_2^{Y_t}(s)$ if, and only if,

$$\begin{aligned}&R_{X_1}(t+s) - R_{X_1}(t) - (R_{Y_1}(t+s) - R_{Y_1}(t)) + R_{X_1}(t+s)R_{Y_1}(t) - \\ &- R_{X_1}(t)R_{Y_1}(t+s) + R_{X_1}(t+s)R_{Y_2}(t) - R_{X_1}(t)R_{Y_2}(t) + \\ &+ R_{X_2}(t)R_{Y_1}(t) - R_{X_2}(t)R_{Y_1}(t+s) \geq 0 \end{aligned} \tag{4.8}$$

and

$$\begin{aligned}&R_{X_2}(t+s) - R_{X_2}(t) - (R_{Y_2}(t+s) - R_{Y_2}(t)) + R_{X_2}(t+s)R_{Y_2}(t) - + \\ &- R_{X_2}(t)R_{Y_2}(t+s) + R_{X_2}(t+s)R_{Y_1}(t) - R_{X_1}(t)R_{Y_1}(t) + \\ &+ R_{X_1}(t)R_{Y_2}(t) - R_{X_1}(t)R_{Y_2}(t+s) \geq 0 \end{aligned} \tag{4.9}$$

From (4.5), (4.6) and (4.7) it follows that (4.8) and (4.9) are satisfied and therefore

$$\begin{aligned}\mathbf{X}_t &=_{st} ((\overline{G}_t^{X_1})^{-1}(U_1^t), (\overline{G}_t^{X_2})^{-1}(U_2^t)) =_{st} ((\overline{G}_t^{X_1})^{-1}(U_1), (\overline{G}_t^{X_2})^{-1}(U_2)) \\ &\leq_{a.s.} ((\overline{G}_t^{Y_1})^{-1}(U_1), (\overline{G}_t^{Y_2})^{-1}(U_2)) =_{st} ((\overline{G}_t^{Y_1})^{-1}(U_1^t), (\overline{G}_t^{Y_2})^{-1}(U_2^t)) = \mathbf{Y}_t.\end{aligned}$$

■

Example 4.2.4 *An example where $\boldsymbol{X} = (X_1, X_2)$ and $\boldsymbol{Y} = (Y_1, Y_2)$ are vectors of lifetimes satisfying the assumptions in Theorem 4.2.3 is when $R_{X_i}(t) = \frac{e^{bt}-1}{\alpha_{X_i}}$, $R_{Y_i}(t) = \frac{e^{bt}-1}{\alpha_{Y_i}}$, for $i = 1, 2$, where $\alpha_{X_1} \cdot \alpha_{Y_2} = \alpha_{Y_1} \cdot \alpha_{X_2}$ and $\alpha_{X_i} \leq \alpha_{Y_i}$.*

4.3 Stochastic comparisons for Time Transformed Exponential models

In this section, we state and prove the stochastic comparisons between Time Transformed Exponential models. From now on, let $\mathbf{X} = (X_1, X_2) \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ and $\mathbf{Y} = (Y_1, Y_2) \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$ be two bivariate lifetimes described by two different Time Transformed Exponential models as in (1.24).

Note that, in this case, $\overline{G}_{\mathbf{X}}(t) = W_{\mathbf{X}}(R_{\mathbf{X}}(t))$ and $\overline{G}_{\mathbf{Y}}(t) = W_{\mathbf{Y}}(R_{\mathbf{Y}}(t))$ are the univariate marginal survival functions of **X** and **Y**, respectively, i.e., the survival functions of X_i and Y_i, respectively.

Let us denote with X^* and Y^* the univariate lifetimes whose survival functions are $W_{\mathbf{X}}$ and $W_{\mathbf{Y}}$, respectively.

Next, we recall some definitions and properties of several stochastic orders considered in the prosecution, see Chapter 7 in and Shaked and Shanthikumar, 2007.

Definition 4.3.1 *Let* **X** *and* **Y** *be two random vectors. Then,*

ii) **X** *is said to be smaller than* **Y** *in the* ***upper orthant-convex order****, denoted by* $\mathbf{X} \leq_{uo-cx} \mathbf{Y}$*, if*

$$\mathbf{E}[\prod_{i=1}^{n} g_i(X_i)] \leq \mathbf{E}[\prod_{i=1}^{n} g_i(Y_i)]$$

for every collection $\{g_1, g_2, \ldots, g_n\}$ *of univariate increasing convex functions, for which the expectations exist.*

iii) **X** *is said to be smaller than* **Y** *in the* ***lower orthant-concave order****, denoted by* $\mathbf{X} \leq_{lo-cv} \mathbf{Y}$*, if*

$$\mathbf{E}[\prod_{i=1}^{n} h_i(X_i)] \leq \mathbf{E}[\prod_{i=1}^{n} h_i(Y_i)]$$

for every collection $\{h_1, h_2, \ldots, h_n\}$ *of univariate increasing functions such that* h_i *is concave on the union of the supports of* X_i *and* Y_i*,* $i = 1, 2, \ldots, n$*, for which the expectations exist.*

We also recall a notion that will be mentioned along this section, see Chapter 6 in Shaked and Shanthikumar, 2007, for details.

Definition 4.3.2 *A random vector* $\mathbf{X} = (X_1, \ldots, X_n)$ *is said to be* ***conditionally increasing in sequence****, denoted by CIS, if, for* $i = 2, \ldots, n$*,*

$$(X_i|X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) \leq_{st} (X_i|X_1 = x_1', \ldots, X_{i-1} = x_{i-1}')$$

for all $x_j \leq x_j'$*,* $j = 1, 2, \ldots, i-1$*, where* $(X_i|X_1 = x_1, \ldots, X_{i-1} = x_{i-1})$ *denotes the conditional distribution of* X_i *given* $X_1 = x_1, \ldots, X_{i-1} = x_{i-1}$ *for all* $x_1, \ldots, x_{i-1} \in \mathbb{R}$*.*

Note that, in the bivariate case, $\mathbf{X} = (X_1, X_2)$ is CIS if $(X_2|X_1 = u_1) \leq_{st} (X_2|X_1 = u_2)$ for all $u_1 \leq u_2$ (i.e., if X_2 is stochastically increasing in X_1). One of the reason of interest in the CIS property is due to the following statement (Shaked and Shanthikumar, 2007, Theorem 6.B.4).

Lemma 4.3.3 *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *be two random vectors. If either* $\mathbf{X}$*, or* $\mathbf{Y}$*, is CIS and*

(i) $X_1 \leq_{st} Y_1$*,*

(ii) $(X_i|X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) \leq_{st} (Y_i|Y_1 = x_1, \ldots, Y_{i-1} = x_{i-1})$*, for all* x_j*,* $j = 1, \ldots, i-1$*,*

then $\mathbf{X} \leq_{st} \mathbf{Y}$*.*

Conditions for a random vector $\mathbf{X}$ having an Archimedean copula (or survival copula) to be CIS may be found in Müller and Scarsini (2005).

Finally, we remark that random variables having log-convex densities play a crucial role in the next section. Log-convexity and log-concavity are popular concepts both in reliability and in economics (see, for example, Shaked and Shanthikumar, 1987, or An, 1998).

From now on, we will consider two bivariate random vectors $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$. A first immediate sufficient condition one can prove to get stochastic comparisons between $\mathbf{X}$ and $\mathbf{Y}$ is the following.

Proposition 4.3.4 *Let* $\mathbf{X} \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ *and* $\mathbf{Y} \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$*. If*

(i) $R_{\mathbf{X}} = R_{\mathbf{Y}} \equiv R$*, and*

(ii) $X^* \leq_{st} Y^*$*,*

then $\mathbf{X} \leq_{uo} \mathbf{Y}$.

Proof. It is enough to observe that

$$\overline{F}_{\mathbf{X}}(t,s) = W_{\mathbf{X}}(R(t)+R(s)) \leq W_{\mathbf{Y}}(R(t)+R(s)) = \overline{F}_{\mathbf{Y}}(t,s)$$

for all $s, t \geq 0$, where the inequality follows from $X^* \leq_{st} Y^*$. ■

An immediate question one can consider is if, under the same assumptions, it is possible to get stronger comparisons between **X** and **Y**. Actually, the answer to this question is negative, as shown in Counterexample 4.3.6 below. However, under some additional assumptions it is possible to get $\mathbf{X} \leq_{st} \mathbf{Y}$, as shown in the following statement.

For it, let us denote with $w_{\mathbf{X}}$ and $w_{\mathbf{Y}}$ the density functions of the random variables X^* and Y^*, i.e., let $w_{\mathbf{X}}(x) = -W'_{\mathbf{X}}(x)$ and $w_{\mathbf{Y}}(x) = -W'_{\mathbf{Y}}(x)$ (non-negativity of this densities is due to monotonicity of $W_{\mathbf{X}}$ and $W_{\mathbf{Y}}$).

Theorem 4.3.5 *Let* $\mathbf{X} \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ *and* $\mathbf{Y} \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$. *If*

(i) $R_{\mathbf{X}} = R_{\mathbf{Y}} \equiv R$*,*

(ii) $X^* \leq_{lr} Y^*$ *and*

(iii) X^**, or* Y^**, has log-convex density,*

then $\mathbf{X} \leq_{st} \mathbf{Y}$.

Proof. Let us suppose that X^* has density $w_{\mathbf{X}}$ that is log-convex. Then by Proposition 1 in Averous and Dortet-Bernadet (2004), it follows that **X** is CIS. Thus, by Lemma 1.1 in order to prove that $\mathbf{X} \leq_{st} \mathbf{Y}$ it is sufficient to verify that

((i) $X_1 \leq_{st} Y_1$ and

(ii) $(X_2|X_1 = u) \leq_{st} (Y_2|Y_2 = u)$ for all $u \in \mathbb{R}^+$.

By assumption (ii) and (1.11), we have $X^* \leq_{st} Y^*$, and therefore, clearly, $W_{\mathbf{X}}(R(t)) \leq W_{\mathbf{Y}}(R(t))$ for all $t \geq 0$, i.e., $X_1 \leq_{st} Y_1$.

With straightforward computations, from the density function, it is easy to verify that, for all $u, t \geq 0$,

$$\overline{F}_{(X_2|X_1=u)}(t) = \frac{w_{\mathbf{X}}(R(t)+R(u))}{w_{\mathbf{X}}(R(u))}$$

and similarly for the density and survival function of $(Y_2|Y_1 = u)$.

Now observe that $(X_2|X_1 = u) \leq_{st} (Y_2|Y_2 = u)$ for all u if, and only if,

$$\overline{F}_{(X_2|X_1=)]}(t) \leq \overline{F}_{(Y_2|Y_1=u)}(t) \; \forall t \geq 0$$

i.e., if, and only if,

$$\frac{w_{\mathbf{X}}(R(t)+R(u))}{w_{\mathbf{X}}(R(u))} \leq \frac{w_{\mathbf{Y}}(R(t)+R(u))}{w_{\mathbf{Y}}(R(u))} \; \forall t \geq 0.$$

This inequality is clearly verified from the assumption (ii). ■

Note that the assumption (ii) in the above theorem can not be replaced by a weaker one, like a comparison between X^* and Y^* in the usual stochastic or the hazard rate order, as shown in the following counterexample.

Example 4.3.6 *Let* $\mathbf{X} \sim TTE(W_{\mathbf{X}}, R)$ *and* $\mathbf{Y} \sim TTE(W_{\mathbf{Y}}, R)$*, with*

$$W_{\mathbf{X}}(x) = \exp\{1 - e^x\}, \; W_{\mathbf{Y}}(x) = \frac{1}{1+x},$$

(inverses of the generators of the copula numbered as 9 in Nelsen (1999) and the Clayton copula, respectively) and

$$R(t) = \frac{e^t - 1}{2}.$$

It can be easily verified with straightforward calculations that Y^* *has log-convex density and that inequality* $X^* \leq_{hr} Y^*$ *holds, while* $X^* \leq_{lr} Y^*$ *does not hold. It can also be observed that it holds* $\mathbf{X} \leq_{uo} \mathbf{Y}$*, but not* $\mathbf{X} \leq_{lo} \mathbf{Y}$*. In fact, for example, taking* $t_0 = 0.1$ *and* $s_0 = 0.2$*, it is* $F_{\mathbf{X}}(t_0, s_0) \leq F_{\mathbf{Y}}(t_0, s_0)$ *(as one can verify with a direct* *computation). Thus, it can not be* $\mathbf{X} \leq_{st} \mathbf{Y}$*. Moreover, since the hazard rate order implies the usual stochastic order, this example also shows that the usual univariate stochastic order between* X^* *and* Y^* *does not imply the usual multivariate stochastic order between the corresponding bivariate lifetimes.*

The previous statement can be generalized to the case where R_X and R_Y are different.

Theorem 4.3.7 *Let* $\mathbf{X} \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ *and* $\mathbf{Y} \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$*. If*

(i) $R_{\mathbf{X}}(t) \geq R_{\mathbf{Y}}(t)$ *for all* $t \geq 0$*,*

(ii) $X^* \leq_{lr} Y^*$ *and*

(iii) X^**, or* Y^**, has log-convex density,*

then $\mathbf{X} \leq_{st} \mathbf{Y}$*.*

Proof. Let $\mathbf{Z}$ be a bivariate random vector with survival function $\overline{F}_{\mathbf{Z}}(t,s) = W_{\mathbf{X}}(R_{\mathbf{Y}}(t) + R_{\mathbf{Y}}(s))$. By Theorem 4.3.5, it follows that $\mathbf{Z} \leq_{st} \mathbf{Y}$.

On the other hand, $\mathbf{X}$ and $\mathbf{Z}$ have the same copula, and marginals ordered in the usual stochastic order. Thus, it follows $\mathbf{X} \leq_{st} \mathbf{Z}$ (see Müller and Scarsini, 2001).

Combining $\mathbf{X} \leq_{st} \mathbf{Z}$ and $\mathbf{Z} \leq_{st} \mathbf{Y}$, it follows $\mathbf{X} \leq_{st} \mathbf{Y}$. ∎

Under a quite stronger assumption on the functions $R_{\mathbf{X}}$ and $R_{\mathbf{Y}}$ it is possible to obtain the comparison between the residual lifetimes of $\mathbf{X}$ and $\mathbf{Y}$ at any time $t \geq 0$.

Theorem 4.3.8 *Let* $\mathbf{X} \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ *and* $\mathbf{Y} \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$*. If*

(i) $R'_{\mathbf{X}}(t) \geq R'_{\mathbf{Y}}(t)$ *for all* $t \geq 0$*,*

(ii) $X^* \leq_{lr} Y^*$ *and*

(iii) X^**, or* Y^**, has log-convex density,*

then $\mathbf{X}_t \leq_{st} \mathbf{Y}_t$ *for all* $t \geq 0$*.*

Proof. As pointed out in the introduction, for any fixed t it holds $\mathbf{X}_t \sim TTE(W_{\mathbf{X},t}, R_{\mathbf{X},t})$ and $\mathbf{Y}_t \sim TTE(W_{\mathbf{Y},t}, R_{\mathbf{Y},t})$, where

$$R_{\mathbf{X},t}(x) = R_{\mathbf{X}}(t+x) - R_{\mathbf{X}}(t),\ x \geq 0,$$

and $W_{\mathbf{X},t}$ is the survival function of a variable $X^*_{\tilde{t}} = [X^* - \tilde{t} \mid X^* > \tilde{t}]$ where $\tilde{t} = 2R_{\mathbf{X}}(t)$, being

$$W_{\mathbf{X},t}(x) = \frac{W_{\mathbf{X}}(2R(t)+x)}{W_{\mathbf{X}}(2R(t))} = \frac{W_{\mathbf{X}}(\tilde{t}+x)}{W_{\mathbf{X}}(\tilde{t})}$$

(and similarly for $R_{\mathbf{Y},t}$ and $W_{\mathbf{Y},t}$).

Now observe that, since $X^* \leq_{lr} Y^*$, by theorems 1.C.6 and 1.C.8 in Shaked and Shanthikumar (2007) it follows that $X^*_{\tilde{t}} \leq_{lr} Y^*_{\tilde{t}}$. On the other hand, since X^*, or Y^*, has log-convex density then also $X^*_{\tilde{t}}$, or $Y^*_{\tilde{t}}$ has log-convex density (since clearly log-convexity of $w_{\mathbf{X}}(x)$ implies log-convexity of $w_{\mathbf{X},t}(x)$). Moreover, since $R'_{\mathbf{X}}(t) \geq R'_{\mathbf{Y}}(t)$ for all $t \geq 0$, then

$$R_{\mathbf{X}}(t+x) - R_{\mathbf{X}}(t) \geq R_{\mathbf{Y}}(t+x) - R_{\mathbf{Y}}(t)$$

for all $t, x \geq 0$, i.e., $R_{\mathbf{X},t}(x) \geq R_{\mathbf{Y},t}(x)$ for all $x \geq 0$. Thus, by Theorem 4.3.7, the assertion follows. ■

In a similar manner, it is possible to get conditions for negative bivariate aging of the bivariate lifetime **X**, in the sense described in the following statement.

Corollary 4.3.9 *Let* $\mathbf{X} \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$. *Suppose that* X^* *has log-convex density and* $R_{\mathbf{X}}$ *is concave. Then* $\mathbf{X}_t \leq_{st} \mathbf{X}_{t+s}$ *for all* $t, s > 0$.

Proof. Fix $t, s > 0$, and denote $\widetilde{t+s} = 2R_{\mathbf{X}}(t+s)$ and $\tilde{t} = 2R_{\mathbf{X}}(t)$. Since X^* has log-convex density, it follows that $X^*_{\widetilde{t+s}} \geq_{lr} X^*_{\tilde{t}}$, where $X^*_{\widetilde{t+s}}$ has survival function

$$W_{\mathbf{X},t+s}(x) = \frac{W_{\mathbf{X}}(2R(t+s)+x)}{W_{\mathbf{X}}(2R(t+s))} = \frac{W_{\mathbf{X}}(\widetilde{t+s}+x)}{W_{\mathbf{X}}(\widetilde{t+s})},\ x \geq 0,$$

while $X^*_{\tilde{t}}$ has survival function $W_{\mathbf{X},t}$ defined as in the previous proof. Moreover, since X^* has log-convex density, then also $X^*_{\tilde{t}}$ has log-convex density.

On the other hand, if $R_{\mathbf{X}}$ is a concave function, then

$$R_{\mathbf{X},t+s}(x) = R(t+s+x) - R(t+s) \leq R(t+x) - R(t) = R_{\mathbf{X},t}(x)$$

for all $x \geq 0$. Thus, by Corollary 4.3.7, the assertion follows. ■

Note that the statement of this corollary generalizes Theorem 3.3.1, where conditions for bivariate aging are considered.

Some additional results can be provided for comparisons of expectations. These results were initially stated in Mulero et al. (2010), however their proofs were not right at all. For a corrected proof, we refer the reader to Li and Lin (2011). We state next without proof the two results.

Theorem 4.3.10 *Let* $\mathbf{X} \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ *and* $\mathbf{Y} \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$. *Let* $g_1(x)$ *and* $g_2(x)$ *be two nonnegative increasing functions where* g_1, *or* g_2, *is convex [concave]. If*

(i) $R_{\mathbf{X}} = R_{\mathbf{Y}} \equiv R$ *is a concave [convex] function, and*

(ii) $X^* \leq_{icx} [\leq_{icv}] \, Y^*$,

then $\mathbf{E}[g_1(X_1)g_2(X_2)] \leq \mathbf{E}[g_1(Y_1)g_2(Y_2)]$, *provided the expectations exist.*

Theorem 4.3.11 *Let* $\mathbf{X} \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ *and* $\mathbf{Y} \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$. *Let* $g_1(x)$ *and* $g_2(x)$ *be two nonnegative strictly decreasing functions where* $g_1(x)$ *or* $g_2(x)$ *is convex [concave]. If*

(i) $R_{\mathbf{X}} = R_{\mathbf{Y}} \equiv R$ *is a convex [concave] function, and*

(ii) $X^* \geq_{icv} [\geq_{icx}] \, Y^*$,

then $\mathbf{E}[g_1(X_1)g_2(X_2)] \geq \mathbf{E}[g_1(Y_1)g_2(Y_2)]$, *provided the expectations exist.*

Some immediate consequences follow from Theorem 4.3.10.

Corollary 4.3.12 *Let* $\mathbf{X} \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ *and* $\mathbf{Y} \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$. *If*

(i) $R_{\mathbf{X}} = R_{\mathbf{Y}} \equiv R$ *is a concave [convex] function, and*

(ii) $X^* \leq_{icx} \ [\leq_{icv}] \, Y^*$,

then $\mathbf{E}[X_1 X_2] \leq \mathbf{E}[Y_1 Y_2]$.

Proof. By taking $g_1(x)$ and $g_2(x)$ both the identity function, the result follows immediately from Theorem 4.3.10. ■

Corollary 4.3.13 *Let* $\mathbf{X} \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ *and* $\mathbf{Y} \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$. *If*

(i) $R_{\mathbf{X}}(t) \geq R_{\mathbf{Y}}(t)$ *for all* $t \geq 0$ *and* $R_{\mathbf{Y}}$ *is a concave [convex] function, and*

(ii) $X^* \leq_{icx} \ [\leq_{icx}] \, Y^*$,

then $\mathbf{X} \leq_{uo-cx} \ [\leq_{lo-cv}] \, \mathbf{Y}$.

Proof. Let $\mathbf{Z}$ be a bivariate random vector with survival function

$$\overline{F}_{\mathbf{Z}}(t, s) = W_{\mathbf{X}}(R_{\mathbf{Y}}(t) + R_{\mathbf{Y}}(s)).$$

Since $\mathbf{X}$ and $\mathbf{Z}$ have the same copula and marginals ordered in the usual stochastic order, by Theorem 4.1 in Müller and Scarsini (2001), it follows that $\mathbf{X} \leq_{st} \mathbf{Z}$, which in turns implies $\mathbf{X} \leq_{uo-cx} \mathbf{Z}$ and $\mathbf{X} \leq_{lo-cv} \mathbf{Z}$. Moreover, from Theorem 4.3.10 follows easily that

$$\mathbf{E}[g_1(X_1) g_2(X_2)] \leq \mathbf{E}[g_1(Y_1) g_2(Y_2)]$$

for all univariate nonnegative increasing convex [concave] functions g_1 and g_2, i.e., that $\mathbf{X} \leq_{uo-cx} \ [\leq_{lo-cv}] \, \mathbf{Y}$ holds. Combining the two stochastic inequalities the assertion is obtained. ■

Note that all the results stated in this section can be easily generalized to the case where $\mathbf{X}$ and $\mathbf{Y}$ are vectors of non exchangeable variables, as well as to the case of conditioned residual lifetimes of the kind $\mathbf{X}_{(t_1,t_2)} = [(X_1 - t_1, X_2 - t_2) | \, X_1 > t_1, X_2 > t_2]$ instead of $\mathbf{X}_t = \mathbf{X}_{(t,t)} = [(X_1 - t, X_2 - t) | \, X_1 > t, X_2 > t]$.

4.3.1 Some applications

Some applications of the results presented in Section 4.3 follow immediately from the properties of the usual stochastic order.

For example, in Theorem 4.3.5 we will state conditions to compare random vectors in the usual stochastic order, it follows that, if its assumptions are satisfied, then $h(\mathbf{X}) \leq_{st} h(\mathbf{Y})$ for every increasing function h. Thus, in particular, the sums $X_1 + X_2$ and $Y_1 + Y_2$, or the maximum and the minimum of $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$, are stochastically ordered in ST sense.

Also, for example, an order between the Values-at-Risk, of any order $\alpha \in (0, 1)$, for two risk positions aX_1+bX_2 and aY_1+bY_2, $a, b \in \mathbb{R}$, follows from Theorem 4.3.5 (see Embrechts et. al., 2003, for applications in risk management of comparisons between Values-at-Risk of this kind).

Next, we give several applications of these results.

Bounds for expected values

Bounds for expected values, based on comparisons with respect to the independent case, can be provided making use of the results described in the previous section. Usefulness of these bounds is of course due to the fact that, in general, expectations are easier to compute under independence.

Let for example $\mathbf{X} \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ be such that $X^* \leq_{lr} Z_\lambda$, where Z_λ is exponentially distributed with mean $1/\lambda$, i.e., let $w_{\mathbf{X}}(x) \leq w_{\mathbf{X}}(0)\exp(-\lambda x)$ for all $x \geq 0$. Since Z_λ has log-convex density, then for every increasing function h one has $\mathbf{E}[h(\mathbf{X})] \leq \mathbf{E}[h(\mathbf{Z})]$, where $\mathbf{Z} = (Z_1, Z_2)$ has independent components having survival functions $\overline{G}_{\mathbf{Z}}(t) = \exp(-\lambda R_{\mathbf{X}}(t))$. Note that under the same assumptions we also have

$$\overline{F}_{X_1+X_2}(t) \leq \int_0^s \overline{G}_{\mathbf{Z}}(t-s)dG_{\mathbf{Z}}(s), \text{ for all } t \geq 0,$$

since $\mathbf{X} \leq_{st} \mathbf{Z}$.

Assume now that X^* possesses the HNBUE (*harmonically new better than used in expectation*) aging property (see Klefsjö, 1983, or Pellerey, 2000, for properties and applications of this aging notion and its dual notion HNWUE). Then $X^* \leq_{icx} Z_\lambda$, where Z_λ is exponentially distributed with mean $\mathbf{E}[X^*] = 1/\lambda$. Moreover, let $R_{\mathbf{X}}$ be a concave function. Then for every pair of functions g_1 and g_2 that are increasing and convex we have $\mathbf{E}[g_1(X_1)g_2(X_2)] \le \mathbf{E}[g_1(Z_1)]\mathbf{E}[g_2(Z_2)]$, where $\mathbf{Z} = (Z_1, Z_2)$ is defined as before (with $\lambda = 1/\mathbf{E}[X^*]$).

Frailty models

As we see in Chapter 1, in the frailty approach, $W_{\mathbf{X}}$ is a Laplace transform. Thus, its corresponding density $w_{\mathbf{X}}$ is always log-convex (see, e.g., An, 1998) and therefore assumption (iii) of Theorem 4.3.5 is always satisfied. Let now $\mathbf{X}$ and $\mathbf{Y}$ be two vectors defined as before, mixtures of conditionally independent variables with respect to two environmental random parameters $\Theta_{\mathbf{X}}$ and $\Theta_{\mathbf{Y}}$, respectively, i.e., let

$$\overline{F}_{\mathbf{X}}(t,s) = \mathbf{E}[\overline{H}(t)^{\Theta_{\mathbf{X}}}\overline{H}(s)^{\Theta_{\mathbf{X}}}] \text{ and } \overline{F}_{\mathbf{Y}}(t,s) = \mathbf{E}[\overline{H}(t)^{\Theta_{\mathbf{Y}}}\overline{H}(s)^{\Theta_{\mathbf{Y}}}]$$

for some survival function $\overline{H}$. By Theorem 4.3.5 and Theorem 1 in Bartoszewicz and Skolimowska (2006), it follows that a sufficient condition for $\mathbf{X} \ge_{st} \mathbf{Y}$ is the inequality $\Theta_{\mathbf{X}} \le_{lr} \Theta_{\mathbf{Y}}$.

Moreover, let $\Theta_{\mathbf{Y}} = \theta$ a.s., so that $\mathbf{Y} = (Y_1, Y_2)$ has independent components, with survival function $\overline{G}_{\mathbf{Y}}(t) = \overline{H}(t)^{\theta}$. If $\overline{H}$ is DFR, so that $R_{\mathbf{X}}$ is concave, from Theorem 4.3.10 it follows $\mathbf{E}[g_1(X_1)g_2(X_2)] \ge \mathbf{E}[g_1(Y_1)]\mathbf{E}[g_2(Y_2)]$ for all increasing and convex functions g_1 and g_2, being X^* with log-convex density and therefore also HNWUE.

Portfolio optimization

In actuarial and financial literature it is a common assumption that utility functions are increasing and concave. In particular, exponential utilities are often considered in portfolio theory (see, e.g., Kaas et al., 2001). Thus, let us consider the case of an exponential utility u defined as $u(t) = c(1 - e^{-\alpha t})$, with $c, \alpha > 0$. Let $\mathbf{X} \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ and $\mathbf{Y} \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$ be two different pairs of assets. Assume that $R_{\mathbf{X}} = R_{\mathbf{Y}} = R$, where R is convex and consider the two portfolios $S_{\mathbf{X}} = X_1 + X_2$ and $S_{\mathbf{Y}} = Y_1 + Y_2$. By Theorem 4.3.11 it follows that if $X^* \ge_{icv} Y^*$

then $\mathbf{E}[u(S_{\mathbf{X}})] \leq \mathbf{E}[u(S_{\mathbf{Y}})]$. In fact:

$$\begin{aligned}\mathbf{E}[u(S_{\mathbf{X}})] &= c(1 - \mathbf{E}[\exp(-\alpha X_1)\exp(-\alpha X_2)]) \\ &\leq c(1 - \mathbf{E}[\exp(-\alpha Y_1)\exp(-\alpha Y_2)]) = \mathbf{E}[u(S_{\mathbf{Y}})]\end{aligned}$$

being $g(t) = \exp(-\alpha t)$ decreasing and convex.

Let now $\mathbf{Z} = (Z_1, Z_2) \sim TTE(W_{\mathbf{Z}}, R_{\mathbf{Z}})$, where $W_{\mathbf{Z}}(x) = \exp(-x/\mathbf{E}[X^*])$, so that Z_1 and Z_2 are independent, and let $R_{\mathbf{X}} = R_{\mathbf{Y}} = R$ be concave. Assume that X^* is HNBUE, which can be rewritten as $X^* \geq_{icv} Z_\lambda$ where Z_λ is exponentially distributed with mean $\mathbf{E}[X^*]$. Then one can get the following upper bound for the expected utility $\mathbf{E}[u(S_{\mathbf{X}})]$:

$$\mathbf{E}[u(S_{\mathbf{X}})] \leq \mathbf{E}[u(S_{\mathbf{Z}})] = c[1 - (\mathbf{E}[e^{-\alpha Z_1}])^2]$$

where Z_1 has survival function $\overline{G}_{\mathbf{Z}}(t) = \exp(-R(t)/\mathbf{E}[X^*])$.

Stochastic ordering of mixtures

Consider a family $\{\mathbf{X}_\theta, \theta \in \mathcal{T} \subseteq \mathbb{R}\}$ of bivariate lifetimes where $\mathbf{X}_\theta \sim TTE(W_\theta, R)$ with

$$W_\theta(x) = [W(x)]^\theta,\ \forall x \in \mathbb{R}^+, \theta \in \mathcal{T}, \tag{4.10}$$

for some suitable convex survival function W. Nelsen (1997) call the families of Archimedean copulas of this kind as α *families*, and this is the case, for example, of Clayton copulas (where $W(x) = (1+x)^{-1}$ and $\mathcal{T} = (0, +\infty)$) and Gumbel-Barnett copulas (where $W(x) = \exp(1 - e^x)$ and $\mathcal{T} = (0, +\infty)$).

Let now W be such that the corresponding variable X^* has log-convex density (like for the two examples above), then X^*_θ also has log-convex density. It is easy to verify that $X^*_{\theta_1} \leq_{lr} X^*_{\theta_2}$ whenever $\theta_1 \geq \theta_2$ (here $X^*_{\theta_i}$ has survival function $W_{\theta_i}(x) = [W(x)]^{\theta_i}$, $i = 1, 2$). It immediately follows, by Theorem 4.3.5, that in this case $\mathbf{X}_\theta$ is stochastically decreasing in θ, which means that $\mathbf{X}_{\theta_1} \leq_{st} \mathbf{X}_{\theta_2}$ whenever $\theta_1 \geq \theta_2$.

Assume now that the parameter θ describes some environmental factors, related to the degree of dependence between the components of $\mathbf{X}_\theta$, and consider two different random environmental parameters Θ_1 and Θ_2, assuming values in

$\mathcal{T}$. Thus, consider the two vectors $\mathbf{X}_{\Theta_1}$ and $\mathbf{X}_{\Theta_2}$ defined as mixtures of an α family $\{\mathbf{X}_\theta, \theta \in \mathcal{T} \subseteq \mathbb{R}\}$, defined as above, and the random parameters Θ_1 and Θ_2. The following holds.

Proposition 4.3.14 *Let $\{\mathbf{X}_\theta, \theta \in \mathcal{T} \subseteq \mathbb{R}\}$ be an α family having the survival function W as inverse of the basic generator of the copula, and let $-W$ have log-convex derivative. Then $\Theta_1 \geq_{st} \Theta_2$ implies $\mathbf{X}_{\Theta_1} \leq_{st} \mathbf{X}_{\Theta_2}$.*

Proof. As pointed out before, from the assumptions and Theorem 4.3.5 follows that $\mathbf{X}_\theta$ is stochastically decreasing in θ. Therefore, for all increasing functions ϕ it holds $\mathbf{E}[\phi(\mathbf{X}_{\theta_1})] \geq \mathbf{E}[\phi(\mathbf{X}_{\theta_2})]$ whenever $\theta_1 \leq \theta_2$, provided the expectations exist. Thus $\Psi(\theta) = \mathbf{E}[\phi(\mathbf{X}_\theta)]$ is an decreasing function in θ. On the other hand, from the condition $\Theta_1 \leq_{st} \Theta_2$, it follows that $\mathbf{E}[\phi(\Theta_1)] \leq \mathbf{E}[\phi(\Theta_2)]$ for all increasing functions ϕ, provided the expectations exist. In particular, since $\Psi(\theta)$ is an increasing function in θ, then

$$\mathbf{E}[\phi(\mathbf{X}_{\Theta_1})] = \mathbf{E}[\Psi(\Theta_1)] \leq \mathbf{E}[\Psi(\Theta_2)] = \mathbf{E}[\phi(\mathbf{X}_{\Theta_1})]$$

for all increasing functions ϕ, and this yields the stated result. ■

As already stated, Clayton and Gumbel-Barnett copulas satisfy the assumptions of Proposition 4.3.14. Furthermore, it is possible to prove also the above result dealing with the Frank family of copulas which is not an α family.

Proposition 4.3.15 *Let $\mathbf{X}_\theta \sim TTE(W_\theta, R)$ with $W_\theta(x) = -\frac{1}{\theta}\log[e^{-x}(e^{-\theta}-1)+1]$ (inverse of the generator of the Frank copula) and let Θ_1 and Θ_2 be two nonnegative random variables. If $\Theta_1 \geq_{st} \Theta_2$, then $\mathbf{X}_{\Theta_1} \leq_{st} \mathbf{X}_{\Theta_2}$.*

4.4 Stochastic comparisons for frailty models

Next we provide conditions between two multivariate frailty models $\mathbf{X}_k = (X_{k,1}, \ldots, X_{k,n})$ for $k = 1, 2$, that is, two random vectors with non independent lifetimes and joint survival function defined as

$$\overline{F}_k(t_1, \ldots, t_n) = \mathbb{P}[X_{k,1} > t_1, \ldots, X_{k,n} > t_n] = \mathbf{E}\left[\left(\Pi_{i=1}^n \overline{G}_{k,i}(t_i)\right)^{\Theta_k}\right],\ t_i \in \mathbb{R}^+,$$

where Θ_k is an environmental random frailty taking values in $\mathbb{R}^+$ and $\overline{G}_{k,i}$ is the survival function of lifetime $X_{k,i}$ given $\Theta_k = 1$, for further details see sections 1.3 and 3.4.

The first result describes new conditions for the usual stochastic comparison between two multivariate frailty models. Recall that $\tilde{X}_{k,i}$ denotes the random variable whose survival function is the baseline survival function $\overline{G}_{k,i}$.

Theorem 4.4.1 *Let* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *be two* $n-$*dimensional random vectors with survival functions defined as in* (1.25). *If*

(i) $\Theta_2 \leq_{n-Lt-r} \Theta_1$*, and*

(ii) $\tilde{X}_{1,i} \leq_{st} \tilde{X}_{2,i}$ *for all* $i = 1, 2, \ldots, n$*,*

then $\mathbf{X}_1 \leq_{st} \mathbf{X}_2$.

Proof. Let us consider a vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ having joint survival function

$$\overline{F}_{\mathbf{Y}}(t_1, t_2, \ldots, t_n) = \mathbf{E}\left[\left(\Pi_{i=1}^{n}\overline{G}_{2,i}(t_i)\right)^{\Theta_1}\right],\ t_i \in \mathbb{R}^+.$$

First we prove that $\mathbf{Y} \leq_{st} \mathbf{X}_2$. For it, observe that the joint survival function of the random vector $\mathbf{X}_2$ can be written as

$$\overline{F}_{\mathbf{X}_2}(t_1, t_2, \ldots, t_n) = W_2\left(-\Sigma_{i=1}^{n} ln\overline{G}_{2,i}\right).$$

Since the survival functions of the margins $X_{2,i}$ are

$$\overline{F}_{X_{2,i}}(t_i) = W_2(ln\overline{G}_{2,i}(t_i)),$$

while their inverses are

$$\overline{F}^{-1}_{X_{2,i}}(u_i) = \overline{G}^{-1}_{2,i}(exp(-W_2^{-1}(u_i))),$$

one can verify that the survival copula of X_2 is Archimedean, that is,

$$\overline{F}_{\mathbf{X}_2}(\overline{F}^{-1}_{X_{2,1}}(u_1), \ldots, \overline{F}^{-1}_{X_{2,n}}(u_n)) = W_2\left(\Sigma_{i=1}^{n} W_2^{-1}(u_i)\right).$$

for all $u_1, \ldots, u_n \in [0, 1]$.

It should be observed that the survival copula of $\mathbf{X}_2$ does not depend on the baseline distributions $G_{2,i}$, but only on the random frailty $\mathbf{Y}_2$. Similarly, the survival copulas of vectors $\mathbf{X}_1$ and $\mathbf{Y}$ depend only on the random frailty $\mathbf{Y}_1$, and therefore $\mathbf{X}_1$ and $\mathbf{Y}$ have the same survival copula.

Since by Lemma 3.4.3, $\frac{W_2^{(n-1)}(s+z)}{W_2^{(n-1)}(s)}$ is decreasing in s, we can apply Theorem 2.8 in MÃijller and Scarsini (2005) which states that in this case $\mathbf{X}_2$ satisfies the CIS property. Thus, in order to prove that $\mathbf{Y} \leq_{st} \mathbf{X}_2$ it suffices to verify that assumptions (i) and (ii) in Lemma 4.3.3 are satisfied (letting $\mathbf{Y} :=_{st} \mathbf{Y}_1$ and $\mathbf{X}_2 := \mathbf{Y}_2$). Note that, for all $t_1 \in \mathbb{R}$,

$$
\begin{aligned}
\overline{F}_{Y_1}(t_1) = \mathbf{E}[\overline{G}_{2,1}(t_1)^{\Theta_1}] &= \mathbf{E}[\exp(\Theta_1 \ln \overline{G}_{2,1}(t_1))] \\
&\leq \mathbf{E}[\exp(\Theta_2 \ln \overline{G}_{2,1}(t_1))] = \mathbf{E}[\overline{G}_{1,1}(t_1)^{\Theta_2}] = \overline{F}_{2,1}(t_1),
\end{aligned}
$$

where the inequality follows from assumption (a) and Lemma 3.4.4. Thus (i) in Lemma 4.3.3 holds.

Moreover, for all $i = 1, \dots, n$ and $t_j \geq 0, j = 1, \dots, i$, it holds

$$
\begin{aligned}
\overline{F}_{Y_i|Y_1=t_1,\dots,Y_{i-1}=t_{i-1}}(t_i) &= \int_{t_i}^{\infty} f_{Y_i|Y_1=t_1,\dots,Y_{i-1}=t_{i-1}}(u)du \\
&= \int_{t_i}^{\infty} \frac{\int_0^\infty \theta^i g_{2,i}(u)\overline{G}_{2,i}^{\theta-1}(u) \prod_{j=1}^{i-1} g_{2,j}(t_j)\overline{G}_{2,j}^{\theta-1}(t_j)dH_1(\theta)}{\int_0^\infty \theta^{i-1} \prod_{j=1}^{i-1} g_{2,j}(t_j)\overline{G}_{2,j}^{\theta-1}(t_j)dH_1(\theta)} du \\
&= \frac{\int_0^\infty \theta^{i-1}\overline{G}_{2,i}^{\theta}(t_i) \prod_{j=1}^{i-1} \overline{G}_{2,j}^{\theta}(t_j)dH_1(\theta)}{\int_0^\infty \theta^{i-1} \prod_{j=1}^{i-1} \overline{G}_{2,j}^{\theta}(t_j)dH_1(\theta)} \\
&= \frac{W_1^{(i-1)}(-\ln \overline{G}_{2,i}(t_i) - \sum_{j=1}^{i-1} \ln \overline{G}_{2,j}(t_j))}{W_1^{(i-1)}(-\sum_{j=1}^{i-1} \ln \overline{G}_{2,j}(t_j))} \\
&\leq \frac{W_2^{(i-1)}(-\ln \overline{G}_{2,i}(t_i) - \sum_{j=1}^{i-1} \ln \overline{G}_{2,j}(t_j))}{W_2^{(i-1)}(-\sum_{j=1}^{i-1} \ln \overline{G}_{2,j}(t_j))} \\
&= \dots \\
&= \overline{F}_{X_{2,i}|X_{2,1}=t_1,\dots,X_{2,i-1}=t_{i-1}}(t_i),
\end{aligned}
$$

where, again, the inequality follows from assumption (i). Thus, also assumption (ii) in Lemma 4.3.3 is satisfied. We can then assert that $\mathbf{Y} \leq_{st} \mathbf{X}_2$.

Now observe that, by Theorem 6.B.14 in Shaked and Shanthikumar (2007), it holds $\mathbf{X}_1 \leq_{st} \mathbf{Y}$, having the vectors $\mathbf{X}_1$ and $\mathbf{Y}$ the same copula and stochastically ordered margins (by assertion (ii) and closure of usual stochastic order with respect to mixtures).

The main assertion now follows from $\mathbf{X}_1 \leq_{st} \mathbf{Y} \leq_{st} \mathbf{X}_2$. ■

Under a stronger assumption than (ii) of Theorem 4.4.1, it is possible to get a stronger comparison between $\mathbf{X}_1$ and $\mathbf{X}_2$, which involves the vectors of their residual lifetimes.

Theorem 4.4.2 *Let* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *be two* n*–dimensional random vectors with survival functions defined as in* (1.25)*. If*

(i) $\Theta_2 \leq_{n-Lt-r} \Theta_1$*, and*

(ii) $\tilde{X}_{1,i} =_{st} \tilde{X}_{2,i}$ *for all* $i = 1, 2, \ldots, n$*,*

then $\mathbf{X}_{1,\mathbf{t}} \leq_{st} \mathbf{X}_{2,\mathbf{t}}$ *for every vector* $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$*.*

Proof. Let $\mathbf{u} = (u_1, \ldots, u_n)$ be an arbitrary vector of nonnegative components. Note that

$$\begin{aligned}
\overline{F}_{\mathbf{X}_{k,\mathbf{t}}}(\mathbf{u}) &= \frac{\overline{F}_k(\mathbf{t}+\mathbf{u})}{\overline{F}_k(\mathbf{t})} = \frac{\int_0^\infty \left(\Pi_{i=1}^n \overline{G}_{k,i}(t_i+u_i)\right)^\theta dH_k(\theta)}{\int_0^\infty \left(\Pi_{i=1}^n \overline{G}_{k,i}(t_i)\right)^\theta dH_k(\theta)} \\
&= \frac{\int_0^\infty \exp\{\theta[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j+u_j)]\} dH_k(\theta)}{\int_0^\infty \exp\{\theta[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\theta)} \\
&= \int_0^\infty \exp\{\theta[\sum_{j=1}^n \ln(\frac{\overline{G}_{k,j}(t_j+u_j)}{\overline{G}_{k,j}(t_j)}]\} \frac{\exp\{\theta[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\theta)}{\int_0^\infty \exp\{\theta[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\theta)} \\
&= \int_0^\infty \exp\{\theta[\sum_{j=1}^n \ln(\frac{\overline{G}_{k,j}(t_j+u_j)}{\overline{G}_{k,j}(t_j)}]\} d\widetilde{H}_k(\theta).
\end{aligned}$$

Thus, $\mathbf{X}_{k,\mathbf{t}}$ has joint survival function which can be expressed as

$$\overline{F}_{\mathbf{X}_{k,\mathbf{t}}}(\mathbf{u}) = \mathbf{E}\left[\left(\Pi_{i=1}^n \overline{G}_{k,i,t_i}(u_i)\right)^{\widetilde{\Theta}_k}\right]$$

where

$$\overline{G}_{k,i,t_i}(u_i) = \frac{\overline{G}_{k,j}(t_j + u_j)}{\overline{G}_{k,j}(t_j)}$$

and where $\widetilde{\Theta}_k$ has distribution $\widetilde{H}_k$ defined as

$$\widetilde{H}_k(\theta) = \frac{\int_0^\theta \exp\{\tau[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\tau)}{\int_0^\infty \exp\{\tau[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\tau)}\,.$$

Thus, also,

$$\mathbf{E}[\exp(-s\widetilde{\Theta}_k)] = \frac{\mathbf{E}[\exp(-(s+\widetilde{t}_k)\Theta_k)]}{\mathbf{E}[\exp(-\widetilde{t}_k\Theta_k)]},$$

where $\widetilde{t}_k = -\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)$. Note that $\widetilde{t}_1 = \widetilde{t}_2$ by assumption (b).

Let us denote $\widetilde{W}_{k,\mathbf{t}}(s) = \mathbf{E}[\exp(-s\widetilde{\Theta}_k)]$. It holds

$$\begin{aligned} \frac{\widetilde{W}_{1,\mathbf{t}}^{(n-1)}(s)}{\widetilde{W}_{2,\mathbf{t}}^{(n-1)}(s)} &= \frac{\mathbf{E}[\exp(-\widetilde{t}_2\Theta_2)]}{\mathbf{E}[\exp(-\widetilde{t}_1\Theta_1)]} \cdot \frac{W_{1,\mathbf{t}}^{(n-1)}(s+\widetilde{t}_1)}{W_{2,\mathbf{t}}^{(n-1)}(s+\widetilde{t}_2)} \\ &= \frac{\mathbf{E}[\exp(-\widetilde{t}_2\Theta_2)]}{\mathbf{E}[\exp(-\widetilde{t}_1\Theta_1)]} \cdot \frac{W_{1,\mathbf{t}}^{(n-1)}(s+\widetilde{t}_1)}{W_{2,\mathbf{t}}^{(n-1)}(s+\widetilde{t}_1)}. \end{aligned}$$

Since $\frac{W_{1,\mathbf{t}}^{(n-1)}(s+\widetilde{t}_1)}{W_{2,\mathbf{t}}^{(n-1)}(s+\widetilde{t}_1)}$ is decreasing in s by assumption (i), it holds $\widetilde{\Theta}_2 \leq_{n-Lt-r} \widetilde{\Theta}_1$. Moreover, from assumption (ii) obviously follows that $\overline{G}_{1,i,t_i}(u_i) \leq \overline{G}_{2,i,t_i}(u_i)$ for all $u_i \in \mathbb{R}^+$ and $i = 1, \dots, n$, i.e., $\tilde{X}_{1,i} \leq_{st} \tilde{X}_{2,i}$ for all $i = 1, 2, \dots, n$.

Thus one can apply Theorem 4.4.1 to $\mathbf{X}_{1,\mathbf{t}}$ and $\mathbf{X}_{2,\mathbf{t}}$ getting the assertion. ■

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